Almost commuting matrices with respect to normalized Hilbert-Schmidt norm.

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Abstract

Almost-commuting matrices with respect to the normalized Hilbert-Schmidt norm are considered. Normal almost commuting matrices are proved to be near commuting.

1 Introduction

Let $\mathbb{C}_{n\times n}$ be the set of complex $n \times n$-matrices. Let $\mathcal{H}_n, \mathcal{U}_n, \mathcal{N}_n \subset \mathbb{C}_{n\times n}$ be the sets of self-adjoin (Hermitian), unitary and normal matrices, correspondingly. For $X, Y \in \mathbb{C}_{n\times n}$ let $[X, Y] = XY - YX$. The following problem is classic

Problem 1. ***(Must almost commuting matrices be nearly commuting?)***
Let $S_n = \mathbb{C}_{n\times n}, \mathcal{H}_n, \mathcal{U}_n$ or $\mathcal{N}_n$.
For each $\epsilon > 0$, is there a $\delta = \delta(\epsilon) > 0$ such that for each positive integer n, if $A, B \in S_n$ with $\|A\|, \|B\| \le 1$ and $\|AB - BA\| \le \delta$, then there exist $\tilde{A}, \tilde{B} \in S_n$ with $[\tilde{A}, \tilde{B}] = 0$ and $\|A - \tilde{A}\|, \|B - \tilde{B}\| \le \epsilon$?

Here $\delta = \delta(\epsilon)$ is independent of n; the non-uniform version of the problem ($\delta = \delta(n, \epsilon)$) has affirmative answers [1, 7, 9]. In spite of the equivalence of norms for finite dimensional spaces, the answer on the uniform problem depends on the norms $\|\cdot\|_n : \mathbb{C}_{n\times n} \to \mathbb{R}$. Indeed, the equivalence may be non-uniform with respect to n. In the series of works [2, 4, 5, 6, 11] the complete answer on Problem 1 have been found for $\|\cdot\| = \|\cdot\|_{op}$. Where

$$\|A\|_{op} = \sup\{\|Ax\| \ : \ \|x\| = 1\}.$$[1]

The answer on Problem 1 is affirmative for $S_n = \mathcal{H}_n$ and negative for all other cases ($\|\cdot\| = \|\cdot\|_{op}$). We didn't know any results for other norms. In the present paper we consider Problem 1 for $\|\cdot\| = \|\cdot\|_{tr}$, the normalized Hilbert-Schmidt norm:

$$\|A\|_{tr} = \sqrt{\frac{1}{n}\sum_{j,k=1}^{n} |A_{j,k}|^2}.$$

Our interest in normalized Hilbert-Schmidt norm arises from its use in factor II von Neumann algebras and hyperlinear groups, [8, 10]. It turns out that the normalized Hilbert-Schmidt norm is more friendly for Problem 1. We manage to prove that the answer is affirmative for $S_n = \mathcal{H}_n, \mathcal{U}_n, \mathcal{N}_n$, even if we speak about several almost-commuting matrices. The reason why it is true is the following. Transform the matrix A into its diagonal form. In this basis one can approximate $\|\cdot\|_{tr}$-almost commuting matrices A, B by block diagonal matrices $\tilde{A}, \tilde{B}$ such that all blocks of $\tilde{A}$ are multiples of unit matrices.

[1] We consider $\mathbb{C}^n$ as a Hilbert space with scalar product $(x, y) = \sum x_i^* y_i$. It defines the Hilbert norm on $\mathbb{C}^n$, $\|x\| = \sqrt{(x, x)}$.

The technique of the paper is elementary. We systematically use that the squire of the normalized Hilbert-Schmidt norm of a block diagonal matrix is a convex combination of the squires of the norms of its blocks and concavity of some estimates, see Section 3 for details.

All estimates in the theorems are given in the form "$\|\cdot\| < C\epsilon^{\alpha}$", where C is an integer. We do not try to optimize the values of C, we just have decided that using some proper numbers is less awkward than the use of expression of the type "there exists $C > 0$ such that...".

2 Notations and inequalities

We consider $\mathbb{C}^n$ as a Hilbert space with the scalar product $(x, y) = \sum x_i^* y_i$. It defines the Hilbert norm on $\mathbb{C}^n$, $\|x\| = \sqrt{(x, x)}$. Set $\mathbb{C}_{n\times n}$ of complex $n \times n$ matrices naturally acts on $\mathbb{C}^n$. As usual, we include $\mathbb{C} \subset \mathbb{C}_{n\times n}$ by constant diagonal matrices. So, $1 \in C_{n\times n}$, some times $1_n \in C_{n\times n}$ denotes the unit matrix. For $A = \{A_{i,j}\} \in \mathbb{C}_{n\times n}$ we define the normalized trace

$$tr(A) = 1/n \sum_{i=1}^{n} A_{i,i}.$$

It defines a scalar product on $\mathbb{C}_{n\times n}$:

$$\langle A, B\rangle = tr(A^* B) = \frac{1}{n} \sum_{i,j} A_{ij}^* B_{ij}$$

and the normalized trace norm (normalized Hilbert-Schmidt norm)

$$\|A\|_{tr} = \sqrt{\langle A, A\rangle} = \sqrt{\sum_{i,j} |A_{ij}|^2}$$

We also need the uniform operator norm

$$\|A\|_{op} = \sup\{\|Ax\| \ : \ \|x\| = 1\}$$

We list some useful well-known inequalities, see [8], in the following:

Lemma 1. *1. $|\langle A, B\rangle| \le \|A\|_{tr}\|B\|_{tr}$ (the Cauchy-Schwarz inequality); substituting $1 \to B$ gives $|\,tr(A)| \le \|A\|_{tr}$;*

2. $\|A + B\|_{tr} \le \|A\|_{tr} + \|B\|_{tr}$

3. $\|AB\|_{tr} \le \|A\|_{op}\|B\|_{tr}$ and $\|BA\|_{tr} \le \|A\|_{op}\|B\|_{tr}$

4. $\|A\|_{tr} \le \|A\|_{op} \le \sqrt{n}\|A\|_{tr}$

5. If P is an orthogonal projector on k-dimensional subspace, then $\|P\|_{tr} = \frac{\sqrt{k}}{\sqrt{n}}$.

6. If a matrix A is of rank k, then there exists an orthogonal projector P of rank k such that $A = PA$ and, by items 3,5 $\|A\|_{tr} \le \sqrt{\frac{k}{n}}\|A\|_{op}$.

7. $\|A\|_{op}^2 = \|A^*A\|_{op}$; $\|A\|_{tr}^2 = \langle A, A\rangle = \langle 1, A^*A\rangle \le \|A^*A\|_{tr}$

Remark 1. *$\|\cdot\|_{op}$ is an algebraic norm: $\|AB\|_{op} \le \|A\|_{op}\|B\|_{op}$, but normalized trace norm is not a good algebraic norm. We have only $\|AB\|_{tr} \le \sqrt{n}\|A\|_{tr}\|B\|_{tr}$ (it follows from 3,4 of Lemma 1).*

Remark 2. *It is well known and easy to check that for unitary matrices U, V $\|UXV\|_i = \|X\|_i$, where $i = op$ or tr. So, the norms $\|\cdot\|_i$ define norm on the set of linear operators from a Hilbert space to another Hilbert space.*

The following lemma says that if $\|A\|_{tr}$ is small then there is an orthogonal projector P of a large rank, such that $\|PA\|_{op}$ is small. Precisely,

Lemma 2. *For any $A \in C_{n\times n}$ there exists an orthogonal projector P such that*

- $\|E-P\|_{tr} < \sqrt{\|A\|_{tr}}, \quad \|PA\|_{op} < \sqrt{\|A\|_{tr}}$
- *If A is normal, then, in addition, $AP = PA$.*

Proof. Observe that $\|A\|_{op} = \sqrt{\|A^*A\|_{op}} = \sqrt{\|AA^*\|_{op}}$ and AA^* is positive. Let $0 \le \lambda_1 \le \lambda_2 \le ... \le \lambda_n$ be the eigenvalues of AA^*. So,

$$\|A\|_{tr}^2 = tr(AA^*) = \frac{1}{n}\sum_i \lambda_i \ge \frac{\delta^2}{n}|\{i \mid \lambda_i \ge \delta^2\}|.$$

So, we have $|\{i \mid \lambda_i \ge \delta^2\}| \le n\|A\|_{tr}^2/\delta^2$. Let P_δ be the orthogonal projector on the space spanned by all eigenvectors of AA^* with $\lambda_i < \delta^2$. Then $\|PA\|_{op} = \sqrt{\|PAA^*P\|_{op}} < \delta$ and $\|(E-P)\|_{tr} \le \|A\|_{tr}/\delta$. Putting $\delta = \sqrt{\|A\|_{tr}}$ proves the first part of the lemma. The second part easily follows by construction of P. □

3 The concave estimate principle

We will need the following

Claim 1. *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave increasing function with $\phi(0) = 0$. Then $x \to \phi^2(\sqrt{x})$ is a concave increasing function.*

Proof. On the set of functions $\mathbb{R}^+ \to R^+$ we define an operation T: $(T\phi)(x) = \phi^2(\sqrt{x})$. It is clear that if $\phi_1 \le \phi_2$ then $T\phi_1 \le T\phi_2$. Here $\phi_1 \le \phi_2$ if $\phi_1(x) \le \phi_2(x)$ for all $x \in \mathbb{R}^+$. For all $x_0 \in \mathbb{R}^+$ there exists $\alpha, \beta \ge 0$ such that $\phi(x_0) = \alpha x_0 + \beta$ and $\phi(x) \le \alpha x + \beta$. (We have used here that $\phi(0) = 0$.) Observe that $T(\alpha x + \beta) = \alpha^2 x + \beta^2 + 2\alpha\beta\sqrt{x}$ is concave. So, for any $x \in \mathbb{R}^+$ there exists a concave f_x, such that $T\phi(x) = f_x(x)$ and $T\phi \le f_x$. We deduce that $T\phi$ is concave. □

Let $P \in \mathbb{C}[x_1, x_2, \ldots, x_k, x_1^*, x_2^*, \ldots, x_k^*]$ (polynomial with complex coefficients, where x_i^* is interpreted as complex conjugate to x_i).

Definition 1. 1. *We say that matrices $A_1, .., A_k$ are an ϵ-solution of P (ϵ-satisfy P) if*

$$\|P(A_1, ..., A_k, A_1^*, \ldots, A_k^*)\|_{tr} \le \epsilon$$

2. *Let $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $\lim_{\epsilon\to 0} \delta_\epsilon = 0$. Polynomial P is called δ_ϵ-stable if for any ϵ-solution $A_1, A_2, ..., A_k \in \mathbb{C}_{n\times n}$ of P there exists an exact solution $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_k \in \mathbb{C}_{n\times n}$ of P δ_ϵ-close to $A_1, \ldots A_k$, that is $\|A_i - \tilde{A}_i\|_{tr} < \delta_\epsilon$ for $i = 1, ..., k$. Polynomial P is called stable if it is δ_ϵ-stable for some δ_ϵ, $\lim_{\epsilon\to 0} \delta_\epsilon = 0$. (Note that δ_ϵ is independent of n.)*

Let $\mathcal{I} = \langle I_1, I_2, ..., I_r\rangle$ be an ordered partition of $\{1, ..., n\}$. A matrix A is said to be **$\mathcal{I}$-block diagonal** (or just **$\mathcal{I}$-diagonal**) if nonzero elements of A appear on $I_j \times I_j$ places only. (It is clear that this is a usual block-diagonal matrix, after conjugation with a permutation.) Similarly, we call matrix A **cyclically $\mathcal{I}$-three-diagonal** if nonzero elements of A appear on $I_{j\oplus 1} \times I_j$, $I_j \times I_j$ and $I_j \times I_{j\oplus 1}$ places only. Here $\oplus$ is the sum $\mod r$.

Solution (ϵ-solution) $A_1, ..., A_k$ of a polynomial P is called $\mathcal{I}$-diagonal solution ($\mathcal{I}$-diagonal ϵ-solution), if all matrices $A_1, ..., A_k$ are $\mathcal{I}$-diagonal. Under our assumptions, $P(A_1, \ldots, A_k^*)$ is $\mathcal{I}$-diagonal if all A_i are.

Lemma 3. *Suppose, that a polynomial P is $\delta(\epsilon)$-stable with a concave $\delta(\epsilon)$. Then for any $\mathcal{I}$-diagonal ϵ-solution of P there exist an $\mathcal{I}$-diagonal solution of P that is $\delta(\epsilon)$-close to this ϵ-solution.*

Proof. The proof uses the following facts:

1. $\sum_j \alpha_j \delta(x_j) \leq \delta(\sum_j \alpha_j x_j)$ for $x_j, \alpha_j \geq 0$, $\sum_j \alpha_j = 1$ and concave δ.
2. Let A be an $\mathcal{I}$-diagonal matrix and $A^1, A^2, \cdots, A^r$ its diagonal components of dimensions $d_1, \ldots, d_r$, correspondingly. Then

$$\|A\|_{tr}^2 = \sum_j \frac{d_j}{n} \|A^j\|_{d_j}^2,$$

where $n = d_1 + d_2 + \cdots + d_r$ and $\|\cdot\|_d$ is the normalized Hilbert-Schmidt norm on $\mathbb{C}_{d\times d}$.

Let $A_1, \ldots, A_k$ be an $\mathcal{I}$-diagonal solution of P with diagonal components A_j^l. Let $\|P(A_1^l, ..., A_k^l)\| = \epsilon_l$. We have $\epsilon^2 \geq \sum \frac{d_l}{n}\epsilon_l^2$. There exists a solution $\tilde{A}_1^l, \ldots, \tilde{A}_k^l$ of P with

$$\|\tilde{A}_j^l - A_j^l\|_{tr}^2 \leq \delta^2(\sqrt{\epsilon^2}).$$

Solution $\tilde{A}_j$ is constructed by blocks $\tilde{A}_j^l$. Now

$$\|\tilde{A}_j - A_j\|_{tr}^2 = \sum_l \frac{d_l}{n} \|\tilde{A}_j^l - A_j^l\|^2 \leq \sum_l \frac{d_l}{n} \delta^2(\sqrt{\epsilon_l^2}) \leq \delta^2\left(\sqrt{\sum_l \frac{d_l}{n}\epsilon_l^2}\right) \leq \delta^2(\epsilon).$$

Here we use concavity of $\delta^2(\sqrt{x})$ by Claim 1. ∎

4 Almost unitary matrices are near unitary

Lemma 4. *Let $B : L_1 \to L_2$ be an unitary operator from a Hilbert space L_1 to a Hilbert space L_2 such that $\|B^*B - 1_{L_1}\|_{op} \leq \epsilon \leq 1/3$. Then there exists an unitary operator $V : L_1 \to L_2$ such that $\|B - V\|_{op} < 2\epsilon$.*

Proof. Just take $V = B(B^*B)^{-1/2}$, where $(B^*B)^{-1/2} = (1_{L_1} - X)^{-1/2} = \sum_{j=0}^{\infty} \frac{(2j)!X^j}{2^{2j}(j!)^2}$. (We have denoted $X = B^*B - 1$.) Make the following estimates

$$\|(B^*B)^{-1/2} - 1\|_{op} \leq \sum_{j=1}^{\infty} \frac{(2j)!\epsilon^j}{2^{2j}(j!)^2} < \sum_{j=1}^{\infty} \epsilon^j = \frac{\epsilon}{1-\epsilon}.$$

Now,

$$\|V - B\|_{op} \leq \|B\|_{op} \|(B^*B)^{-1/2} - 1\|_{op} < \frac{\epsilon(\epsilon+1)}{1-\epsilon} \leq 2\epsilon,$$

for $\epsilon \leq 1/3$. (We were using the fact that $\|B\|_{op} = \sqrt{\|B^*B\|_{op}} \leq \sqrt{\|B^*B - 1\|_{op} + 1} \leq \|B^*B - 1\|_{op} + 1$.) ∎

Lemma 5. *Let $\|A^*A - 1\|_{tr} \leq \epsilon \leq 1/3$ for a matrix A. Then there exists a unitary U, such that $\|A - U\|_{tr} \leq 5\epsilon^{\frac{1}{4}}$ and $\|A - U\|_{tr} \leq (3 + \|A\|_{op})\sqrt{\epsilon}$.*

Proof. By Lemma 2 there exists orthogonal projector P, $\|1 - P\|_{tr} \leq \sqrt{\epsilon}$ such that $\|PA^*AP - P\|_{op} \leq \sqrt{\epsilon}$. Let $X = Im\, P$. Consider the restriction $B = A|_X : X \to Y = A(X)$, then $B^* : Y \to X$. Observe that $B^* = PA^*|_Y$. So, $\|B^*B - 1_X\|_{op} \le \sqrt{\epsilon}$ and, by Lemma 4, there exists a unitary $V : X \to Y$ with $\|V - B\|_{op} < 2\sqrt{\epsilon}$. Let $\tilde{V}$ be any unitary operator from $X^\perp$ to $Y^\perp$. Take $U = V \oplus \tilde{V}$. We estimate:

$$\|A - V \oplus \tilde{V}\|_{tr} \le \|P_Y A P_X - P_Y V P_X\|_{tr} + \|AP_X^\perp\|_{tr} + \|P_Y^\perp \tilde{V} P_X^\perp\|_{tr} \le$$

$$\|P_Y A P_X - P_Y V P_X\|_{op} + \|\tilde{V}\|_{op}\|P_X^\perp\|_{tr} + \|AP_X^\perp\|_{tr} \le 3\sqrt{\epsilon} + \|AP_X^\perp\|_{tr}.$$

Now, the first inequality of the lemma follows from

$$\|AP_X^\perp\|_{tr}^2 \le \|P_X^\perp A^* A P_X^\perp\|_{tr} \le \|P_X^\perp\|_{tr} + \|P_X^\perp A^* A P_X^\perp - P_X^\perp\|_{tr} \le$$

$$\sqrt{\epsilon} + \|P_X\|_{op}^2 \|A^*A - 1\|_{tr} \le \sqrt{\epsilon} + \epsilon.$$

The second inequality of the lemma follows from

$$\|AP_X^\perp\|_{tr} \le \|A\|_{op}\|P_X^\perp\|_{tr} \le \|A\|_{op}\sqrt{\epsilon}.$$

□

We will work with $\mathcal{I}$-diagonal matrices, so we need a global concave estimate.

Corollary 1. *Let $\|A\|_{op} \le 3$. Then there exists a unitary matrix V such that $\|A - V\|_{tr} \le 6\sqrt{\|A^*A - 1\|_{tr}}$.*

Proof. For $\|A^*A - 1\|_{tr} \le 1/3$ it is Lemma 5. Further, $\|A - V\|_{tr} \le \|A\|_{tr} + 1$ and $\|A\|_{tr}^2 \le \|A^*A\|_{tr} \le \|A^*A - 1\|_{tr} + 1$. So, $\|A - V\|_{tr} \le \sqrt{\|A^*A - 1\|_{tr} + 1} + 1$. It remains to check that $6\sqrt{x} > \sqrt{x+1} + 1$ for $x > 1/3$. □

5 Almost commuting unitary matrices are near commuting

Theorem 1. *Let U_1 and U_2 be unitary matrices. Then there exists unitary matrices A_1, A_2, $[A_1, A_2] = 0$ such that $\|U_1 - A_1\|_{tr} \le 30\left(\|[U_1, U_2]\|_{tr}\right)^{1/9}$ and $\|U_2 - A_2\|_{tr} \le 30\left(\|[U_1, U_2]\|_{tr}\right)^{1/9}$. In addition, $[A_1, U_1] = 0$.*

Before the proof of the theorem we consider an example where U_2 is a cyclic permutation and U_1 its diagonal form: $U_1 = diag(w, w^2, \ldots, w^n = 1)$ with $w = \exp(\frac{2\pi i}{n})$ and $U_2 = P_n$ with

$$P_{j,k} = \begin{cases} 1 & \text{if} \qquad j = k + 1 \mod n \\ 0 & \text{overwise} \end{cases}.$$

This is a counterexample to Problem 1 for $\|\cdot\| = \|\cdot\|_{op}$ and $S_n = \mathcal{U}_n$ found by Voiculescu, [6] (he proves that it is indeed a counterexample). One has $\|[U_1, U_2]\|_{op} = \|[U_1, U_2]\|_{tr} = |1 - w| \to 0$ when $n \to \infty$. Suppose, for simplicity, that $n = md$ for large m and d. Then we can take as A_1 and A_2 the following block-diagonal matrices: $A_1 = diag(\tilde{w}1_d, \tilde{w}^2 1_d, \ldots, \tilde{w}^m 1_d)$, where $\tilde{w} = \exp(\frac{2\pi i}{m})$ and $A_2 = diag(P_d, P_d, \ldots, P_d)$.

Proof. After transformation $U_1 \to V^{-1}U_1V$, $U_2 \to V^{-1}U_2V$, assume that $U_1 = diag(\alpha_1, \alpha_2, ..., \alpha_n)$. The main idea of the proof is the following. Change some elements of U_2 by 0 and approximate U_1 by a diagonal matrix with spectrum $\exp(2\pi i\frac{j}{m})$ for proper m in such a way that U_1 and U_2 become block diagonal matrices with all blocks of U_1 being multiples of unit matrices. New U_1 and U_2 are commuting, but now U_2 is not unitary. Approximate U_2 by an unitary matrix, conserving its block structure. It can be done using Corollary 1 and Lemma 3. Let us describe this procedure in details.

Let $\|[U_1, U_2]\|_{tr} = \epsilon$. Take a positive integer $t \ge 6$ that will be optimized latter. Let $w = \exp(\frac{2\pi i}{t})$. Let $|1 - w| = \Delta$. One has

$$6/t \le \Delta \le 2\pi/t < 7/t.$$

1. Let $U_2 = \{u_{jk}\}$. Define $\tilde{U}_2 = \{\tilde{u}_{jk}\}$ by the following rule:

$$\tilde{u}_{jk} = \begin{cases} u_{jk} & \text{if} \quad |\alpha_j - \alpha_k| < \Delta \\ 0 & \text{if} \quad |\alpha_j - \alpha_k| \geq \Delta \end{cases}$$

   One has

$$\|\tilde{U}_2 - U_2\|_{tr} \leq \epsilon/\Delta$$

   Indeed,

$$n\epsilon^2 \geq n\|[U_1, U_2]\|^2 = \sum_{j=1,k=1}^{n,n} |\alpha_k - \alpha_j|^2 |u_{jk}|^2 \geq \Delta^2 \sum_{|\alpha_k - \alpha_j| \geq \Delta} |u_{jk}|^2 = \Delta^2 n \|\tilde{U}_2 - U_2\|_{tr}^2$$

   One can check that $\|[U_1, \tilde{U}_2]\| \leq \|[U_1, U_2]\| \leq \epsilon$.

2. Approximate U_1 by a diagonal matrix $\tilde{U}_1$ with spectrum $\{w^j \ : \ j = 0, 1, \dots, t-1\}$. Precisely, let $\mathcal{I} = \{I_0, I_1, \dots, I_{t-1}\}$ with $I_j = \{l \ : \ \alpha_l \in (w^{j-1/2}, w^{j+1/2}]\}$, where $(x, y]$ is a semiopen arc of the unit circle in $\mathbb{C}$. Then $\tilde{U}_1$ is an $\mathcal{I}$-diagonal matrix with j block $U_1^j = w^j$. One has that

$$\|\tilde{U}_1 - U_1\|_{tr} \leq \Delta$$

   and $\|[\tilde{U}_1, \tilde{U}_2]\|_{tr} \leq \epsilon + 2\Delta$. Observe that $\tilde{U}_2$ is a cyclically $\mathcal{I}$-three-diagonal matrix.

3. Fix another parameter $a \in \mathbf{N}$ that will be optimized latter. One can find $S = \{s_0, s_2, \dots, s_{c-1}\} \subset \{0, 1, 2, \dots, t-1\}$ such that
   - $a \leq |s_{r \oplus 1} - s_r| \leq 3a$, for any $r = 0, 1, ..., c-1$. Here $\oplus$ is the sum $\mod c$.
   - $|I_j| \leq \frac{n}{a}$ for any $j \in S$.
   - $|S| = c \leq t/a$

4. In order to construct A_1 we make a more rough partition $\tilde{\mathcal{I}} = \{\tilde{I}_0, \tilde{I}_2, \dots, \tilde{I}_{c-1}\}$. Where $\tilde{I}_j = I_{s_j} \cup I_{s_j+1} \cup \dots \cup I_{s_{j\oplus 1}-1}$, where $\pm$ is $\mod t$ and $\oplus$ is $\mod c$. Now, A_1 is a cyclically $\tilde{\mathcal{I}}$-diagonal matrix with the blocks $A_1^j = w^{\frac{1}{2}(s_j + s_{j\oplus} - 1)}$. By the first item of 3) we have

$$\|\tilde{U}_1 - A_1\|_{tr} \leq |1 - w^{\frac{3}{2}a}| \leq \frac{3}{2} a\Delta$$

   and $\|[A_1, \tilde{U}_2]\|_{tr} \leq \epsilon + 2\Delta + 3a\Delta$.

5. We give our construction of A_2 in two steps. Recall that $\tilde{U}_2$ is $\mathcal{I}$-three-diagonal. We construct B by removing from $\tilde{U}_2$ the blocks $I_{j-1} \times I_j$ and $I_j \times I_{j-1}$ for each $j \in S$. The resulting matrix B is $\tilde{\mathcal{I}}$-diagonal and, consequently, $[A_1, B] = 0$. We estimate:

$$\|\tilde{U}_2 - B\|_{tr} = \sum_{j \in S} (\|U_2^{j-1,j}\|_{tr} + \|U_2^{j,j-1}\|_{tr}) \leq 2\frac{|S|}{\sqrt{a}} \leq 2\frac{t}{a^{3/2}}.$$

   For the first inequality we use $\|U_2^{j-1,j}\|_{tr} \leq \sqrt{\frac{|I_j|}{n}} \|U_2^{j-1,j}\|_{op}$ (the item 6. of Lemma 1) and $\|U_2^{j-1,j}\|_{op} \leq \|U_2\|_{op} = 1$ (the operator norm of a submatrix is less than the operator norm of the matrix). The same inequalities are valid for $\|U_2^{j,j-1}\|$. The second inequality is a property of S.

6. The matrix B, A_1 are $\tilde{\mathcal{I}}$-diagonal and each block of A_1 is a multiple of the unit matrix, so $[A_1, B] = 0$. The problem is that B is not unitary. $\|U_2 - B\|_{tr} \le \frac{\epsilon}{\Delta} + 2\frac{t}{a^{3/2}} \le \epsilon t/6 + 2ta^{-3/2} = \gamma$. and $\|B\|_{op} \le 3$ (B is $\mathcal{I}$-three-diagonal with the operator norm of each block less than 1 as submatrices of a unitary matrix.) It follows that

$$\|B^*B - 1\|_{tr} = \|B^*B - U_2^*U_2\|_{tr} \le \|B^*B - B^*U_2\|_{tr} + \|B^*U_2 - U_2^*U_2\|_{tr}$$

$$\le \|B^*\|_{op}\|B - U_2\|_{tr} + \|U_2\|_{op}\|B^* - U_2^*\|_{tr} \le 4\gamma.$$

The matrix B is an $\tilde{\mathcal{I}}$-diagonal matrix. By Lemma 3 and Corollary 1 there exists a unitary $\tilde{I}$-diagonal matrix A_2 with

$$\|B - A_2\|_{tr} \le 12\sqrt{\gamma} = 12\sqrt{\frac{\epsilon t}{6} + 2ta^{-3/2}}$$

It is clear that $[A_1, A_2] = 0$.

7. We only need to choose a, t and estimate $\|U_i - A_i\|_{tr}$. Suppose, for a moment[2], that $\epsilon \le 6^{-9/7}$, choose $a, t \in \mathbf{N}$ such that $\epsilon^{-7/9} \le t \le 2\epsilon^{-7/9}$ and $\epsilon^{-2/3} \le a \le 2\epsilon^{-2/3}$. We have:

$$\|U_1 - A_1\|_{tr} \le \Delta + \frac{3}{2}a\Delta \le \frac{7}{t} + \frac{21a}{2t} \le 7\epsilon^{7/9} + 21\epsilon^{1/9} < 30\epsilon^{1/9}.$$

Further,

$$\|U_2 - A_2\|_{tr} \le \frac{1}{6}\epsilon t + 2ta^{-3/2} + 12\sqrt{\frac{1}{6}\epsilon t + 2ta^{-3/2}} \le \frac{13}{3}\epsilon^{2/9} + 12\sqrt{\frac{13}{3}\epsilon^{1/9}} < 30\epsilon^{1/9}.$$

For $\epsilon \ge 6^{-9/7}$ we have

$$\|A_i - U_i\|_{tr} \le 2 \le 30 \cdot 6^{-1/7} \le 30\epsilon^{1/9}$$

The pair A_1, A_2 satisfies the statement of the theorem. $\square$

We need the following

Claim 2. *Let $\|A\|_{op}, \|B\|_{op}, \|\tilde{A}\|_{op}, \|\tilde{B}\|_{op} \le 1$. Then*

$$\|[\tilde{A}, \tilde{B}]\|_{tr} \le \|[A, B]\|_{tr} + 2(\|A - \tilde{A}\|_{tr} + \|B - \tilde{B}\|_{tr}).$$

Proof. $\|AB - \tilde{A}\tilde{B}\|_{tr} = \|AB - A\tilde{B} + A\tilde{B} - \tilde{A}\tilde{B}\|_{tr} \le \|A\|_{op}\|B - \tilde{B}\|_{tr} + \|A - \tilde{A}\|_{tr}\|\tilde{B}\|_{op} \le \|B - \tilde{B}\|_{tr} + \|A - \tilde{A}\|_{tr}$. Combining it with the same estimate for $\|BA - \tilde{B}\tilde{A}\|_{tr}$ we get the claim. $\square$

Theorem 2. *There exists $\delta(\epsilon, k)$, $\delta(\epsilon, k) \to 0$ when $\epsilon \to 0$ for any $k \in \mathbf{N}$, such that if $\|[U_i, U_j]\|_{tr} \le \epsilon$ for unitary $U_1, U_2, \ldots, U_k$, then there exist pairwise commuting unitary matrices $A_1, \ldots, A_k$ such that $\|U_j - A_j\|_{tr} \le \delta(\epsilon, k)$.*

Proof. Let $\psi(x) = 30x^{1/9}$ and $\phi_j(\cdot)$ be defined by the relation:

$$\phi_0(x) = x, \quad \phi_{j+1}(x) = 4\psi(\phi_j(x)) + x.$$

For $r = 1, \ldots k-1$ we prove by induction the following statement

There exist a unitary matrix V, a partition $\mathcal{I}_r$ of $\{1, \ldots, n\}$, and $\mathcal{I}_r$-diagonal matrices $\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_k$, such that

- *All blocks of $\tilde{U}_1, \ldots \tilde{U}_r$ are multiples of the unit matrix.*

[2] The condition on ϵ is to guarantee $t \ge 6$.

- $\|\tilde{U}_j - V^{-1}U_jV\|_{tr} \le \psi\left(\phi_{j-1}(\epsilon)\right)$, *for* $j \le r$ *and* $\|\tilde{U}_j - V^{-1}U_jV\|_{tr} \le \psi\left(\phi_{r-1}(\epsilon)\right)$, *for* $j > r$.

The theorem follows from the Statement for $r = k-1$ and $\delta(\epsilon, k) = \psi\left(\phi_{k-1}(\epsilon)\right)$. Let us proof the Statement.

$r = 1$**.** In the proof of Theorem 1 matrix A_1 and partitions $\mathcal{I}$ and $\tilde{\mathcal{I}}$ is independent of U_2. The construction of A_2 depends on partitions $\mathcal{I}$ and $\tilde{\mathcal{I}}$ only. So, we may construct $\tilde{\mathcal{I}}$-diagonal $\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_k$ satisfying the Statement for $r = 1$.

$r \to r+1$. Let $\tilde{U}_1, \ldots, \tilde{U}_k$ be as in the Statement. Then $\|[\tilde{U}_i, \tilde{U}_j]\| = 0$ for $i < r$ and, by Claim 2, $\|[\tilde{U}_i, \tilde{U}_j]\| \le \phi_r(\epsilon)$ for $i, j \ge r$. Let $I_l \in \mathcal{I}_r$. Work with $\tilde{U}_r^l, \ldots, \tilde{U}_k^l$ as in the proof for $r = 1$. Then apply Lemma 3. □

6 Self-adjoint matrices.

For every almost-commuting self-adjoint matrices A, B we construct commuting self-adjoint matrices with the same operator norm and close to A, B by the normalized Hilbert-Schmidt norm. In order to preserve the operator norm we need

Lemma 6. *Let A, B, $C = A + B$ be self adjoint matrices. Let $D(B)$ and $D(C)$ be the decreasing diagonal form of B and C, correspondingly. Then* $\|D(C) - D(B)\|_{op} \le \|A\|_{op}$

Proof. Let $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n$, $\beta_1 \ge \cdots \ge \beta_n$ and $\gamma_1 \ge \cdots \ge \gamma_n$ be (ordered) eigenvalues of A,B and C, correspondingly. The H.Weyl inequality [3, 12] states:

$$\gamma_{j+k-1} \le \alpha_j + \beta_k.$$

Writing $-C = -A - B$ and reordering the eigenvalues we get:

$$\gamma_{k-j+1} \ge \alpha_{n-j+1} + \beta_k.$$

Putting $j = 1$ in the both inequalities and using the fact that $\alpha_1, -\alpha_n \le \|A\|_{op}$ we get

$$-\|A\|_{op} + \beta_k \le \gamma_k \le \|A\|_{op} + \beta_k.$$

□

Corollary 2. *Let A, C be self-adjoint and C be $\mathcal{I}$-diagonal. Then there exists $\mathcal{I}$-diagonal self-adjoint matrix $\tilde{C}$ such that $\|\tilde{C}\|_{op} \le \|A\|_{op}$ and $\|\tilde{C} - C\|_{tr} \le \|C - A\|_{tr}$*

Proof. We can choose $\tilde{C}$ such that $D(\tilde{C}) = D(C) - D(C - A)$. □

Theorem 3. *Let H_1 and H_2 be self-adjoint matrices, such that $\|H_i\|_{op} \le 1$, $i = 1, 2$. Then there exists self-adjoint matrices A_1, A_2, $[A_1, A_2] = 0$ such that $\|H_1 - A_1\|_{tr} \le 12\left(\|[H_1, H_2]\|_{tr}\right)^{1/6}$ and $\|H_2 - A_2\|_{tr} \le 12\left(\|[H_1, H_2]\|_{tr}\right)^{1/6}$, $\|A_i\|_{op} \le 1$. In addition, $[A_1, H_1] = 0$.*

Proof. We follow the same routine as in the proof of Theorem 1. Instead of Lemma 5 we use Corollary 2 to keep the operator norm.

Let $\|[H_1, H_2]\|_{tr} = \epsilon$ We suppose that $H_1 = diag(\alpha_1, \alpha_2, ..., \alpha_n)$, $-1 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1$. Take a positive integer t that will be optimized latter.

1. Let $H_2 = \{h_{jk}\}$. Define $\tilde{H}_2 = \{\tilde{h}_{jk}\}$ by the following rule:

$$\tilde{h}_{jk} = \begin{cases} h_{jk} & \text{if} \quad |\alpha_j - \alpha_k| < 1/t \\ 0 & \text{if} \quad |\alpha_j - \alpha_k| \ge 1/t \end{cases}$$

As in the proof of Theorem 1 one has

$$\|\tilde{H}_2 - H_2\|_{tr} \le \epsilon t.$$

Clearly, $\tilde{H}_2$ is self-adjoin.

2. Approximate H_1 by a diagonal matrix $\tilde{H}_1$ with spectrum $\{\frac{j}{t} \ : \ j = -t, \dots, t\}$. Precisely, let $\mathcal{I} = \{I_{-t}, I_{-t+1}, \dots, I_t\}$ with $I_j = \{l \ : \ \alpha_l \in (\frac{2j-1}{2t}, \frac{2j+1}{2t}]\}$, where $(x, y]$ is a semiopen interval. Then $\tilde{H}_1$ is an $\mathcal{I}$-diagonal matrix with the j-th block $H_1^j = \frac{j}{t}$. One has that

$$\|\tilde{H}_1 - H_1\|_{tr} \leq \frac{1}{t}$$

and that $\tilde{H}_2$ is an $\mathcal{I}$-three-diagonal matrix (not cyclically $\mathcal{I}$-three-diagonal).

3. Fix another parameter $a \in \mathbf{N}$ that will be optimized latter. One can find $S = \{s_0, s_2, \dots, s_{c-1}\}$, $-t \leq s_1 < s_1 < \dots s_{c-1} \leq t$ such that
   - $a \leq |s_{r+1} - s_r| \leq 2a$, for any $r = 0, 1, ..., c-1$.
   - $|I_j| \leq \frac{n}{a}$ for any $j \in S$.
   - $|S| = c \leq (2t+1)/a$

4. In order to construct A_1 we make more rough partition $\tilde{\mathcal{I}} = \{\tilde{I}_0, \tilde{I}_2, \dots, \tilde{I}_{c-1}\}$. Where $\tilde{I}_j = I_{s_j} \cup I_{s_j+1} \cup \dots \cup I_{s_{j+1}-1}$. Now, A_1 is cyclically $\tilde{\mathcal{I}}$-diagonal matrix with block $A_1^j = \frac{s_j + s_{j+1} - 1}{2t}$. By the first item of 3) we have

$$\|\tilde{H}_1 - A_1\|_{tr} \leq \frac{a}{t}$$

5. We give construction of A_2 in two steps. Recall that $\tilde{H}_2$ $\mathcal{I}$-three-diagonal. We construct B by removing from $\tilde{H}_2$ blocks $I_{j-1} \times I_j$ and $I_j \times I_{j-1}$ for each $j \in S$. The resulting matrix B is $\tilde{\mathcal{I}}$-diagonal and, consequently, $[A_1, B] = 0$. We estimate:

$$\|\tilde{H}_2 - B\|_{tr} = \sum_{j \in S} (\|H_2^{j-1,j}\|_{tr} + \|H_2^{j,j-1}\|_{tr}) \leq 2\frac{|S|}{\sqrt{a}} \leq 2\frac{2t+1}{a^{3/2}}.$$

For the first inequality we use $\|H_2^{j-1,j}\|_{tr} \leq \sqrt{\frac{|I_j|}{n}} \|H_2^{j-1,j}\|_{op}$ (the item 6. of Lemma 1) and $\|H_2^{j-1,j}\|_{op} \leq \|H_2\|_{op} \leq 1$ (the operator norm of a submatrix is less than the operator norm of the matrix). The same inequalities are valid for $\|H_2^{j,j-1}\|$. The second inequality is a property of S.

6. The matrix B is $\tilde{\mathcal{I}}$-diagonal, self-adjoint, and

$$\|H_2 - B\|_{tr} \leq \epsilon t + 2\frac{2t+1}{a^{3/2}}.$$

So by Corollary 2 there exists $\tilde{\mathcal{I}}$-diagonal self-adjoint A_2 with

$$\|H_2 - A_2\|_{tr} \leq 2\epsilon t + 4\frac{2t+1}{a^{3/2}}.$$

It is clear that $[A_1, A_2] = 0$.

7. We only need to choose a, t and estimate $\|H_i - A_i\|_{tr}$. Suppose, for a moment, that $\epsilon \leq 4^{-1}$, choose $a, t \in \mathbf{N}$ such that $\frac{1}{2}\epsilon^{-5/6} \leq t \leq \epsilon^{-5/6}$ and $\epsilon^{-2/3} \leq a \leq 2\epsilon^{-2/3}$. We have:

$$\|H_1 - A_1\|_{tr} \leq \frac{1}{t} + \frac{a}{t} \leq 2\epsilon^{5/6} + 4\epsilon^{1/6} < 12\epsilon^{1/6}$$

Further,

$$\|H_2 - A_2\|_{tr} \leq 2\epsilon^{1/6} + 8\epsilon^{1/6} + 4\epsilon < 12\epsilon^{1/6}$$

For $\epsilon \geq 4^{-1}$ we have

$$\|A_i - H_i\|_{tr} \leq \|A_i - H_i\|_{op} \leq 2 < 12(4^{-1/6}) < 12\epsilon^{1/6}$$

The pair A_1, A_2 satisfies the statement of the theorem. $\square$

Theorem 4. *There exists $\delta(\epsilon, k)$, $\delta(\epsilon, k) \to 0$ when $\epsilon \to 0$ for any $k \in \mathbf{N}$, such that if $\|[H_i, H_j]\|_{tr} \le \epsilon$ for self-adjoint matrices $H_1, H_2, \dots, H_k$ with $\|H_i\|_{op} \le 1$, then there exist pairwise commuting self-adjoint matrices $A_1, \dots, A_k$ such that $\|U_j - A_j\|_{tr} \le \delta(\epsilon, k)$ and $\|A_i\| \le 1$.*

Proof. The same as for Theorem 2. $\square$

7 Normal matrices

Observe that Theorem 4 implies the existence of commuting normal matrices close to almost commuting ones. Observe also, that Theorem 3 implies the existence of a normal matrix N close to an $\|\cdot\|_{tr}$-almost normal matrix M. Could it be done in a way that $\|N\|_{op} \le \|M\|_{op}$? In the section we give the affirmative answer to this question (Corollary 3)

Theorem 5. *Let U and H be unitary and positive matrices, correspondingly. Let $\|H\|_{op} \le 1$. Then there exists unitary and positive matrices V, A such that $[V, A] = [H, A] = 0$, $\|V - U\|_{tr} \le 30\|[U, H]\|_{tr}^{1/9}$ and $\|H - A\|_{tr} \le 30\|[U, H]\|_{tr}^{1/9}$, $\|A\|_{op} \le 1$.*

Proof. Let $H = diag(h_1, \dots, h_n)$. Make partition $\mathcal{I}$ and $\tilde{\mathcal{I}}$ as in the proof of Theorem 3. Construct A as A_1 in Theorem 3 and V as U_2 in Theorem1. $\square$

Lemma 7. *Let A, B be positive commuting matrices. Then $\|A - B\|_{tr} \le \sqrt{\|A^2 - B^2\|_{tr}}$.*

Proof. Without loss of generality we may assume that $A = diag(a_1, a_2, \dots, a_n)$ and $B = diag(b_1, b_2, \dots, b_n)$. Now,

$$\|A - B\|_{tr}^2 = \frac{1}{n}\sum_{j=1}^{n}(a_j - b_j)^2 \underset{(\mathbf{a})}{\le} \frac{1}{n}\sum_{j=1}^{n}|a_j^2 - b_j^2| =$$

$$\frac{1}{n}\sum_{j=1}^{n}\sqrt{(a_j^2 - b_j^2)^2} \underset{(\mathbf{b})}{\le} \sqrt{\frac{1}{n}\sum_{j=1}^{n}(a_j^2 - b_j^2)^2} = \|A^2 - B^2\|_{tr}$$

The inequality **(a)** is due to $(a - b)^2 \le |a^2 - b^2|$ for $a, b \ge 0$; the inequality **(b)** is due to concavity of $\sqrt{\cdot}$. $\square$

Theorem 5 with Lemma 7 implies

Corollary 3. *Let M be a matrix with $\|MM^* - M^*M\|_{tr} \le \epsilon$ and $\|M\|_{op} \le 1$. Then there exists a normal matrix N such that $\|M - N\|_{tr} \le 36\epsilon^{1/18}$ and $\|N\|_{op} \le 1$.*

Proof. Let $M = UH$ with unitary U and positive H. We have $\|UH^2 - H^2U\|_{tr} \le \epsilon$. So, by Theorem 5 we can find positive A and unitary V such that $\|H^2 - A\|_{tr} \le 30\epsilon^{1/9}$, $\|U - V\|_{tr} \le 30\epsilon^{1/9}$ and $[H^2, A] = [V, A] = 0$. By Lemma 7 we have $\|H - A^{1/2}\| \le 6\epsilon^{1/18}$ and $N = VA^{1/2}$ satisfies the Corollary. $\square$

8 Concluding remarks

We see that the normalized Hilbert-Schmidt norm is more friendly for almost-near questions for the commutator. We think that it is interesting to consider other relations. For example, if almost solutions of

$$U^k = V^{-1}UV$$

are near solutions?

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References

[1] J. J. Bastian and K. J. Harrison, Subnormal weighted shifts and asymptotic properties of normal operators, *Proc. Amer. Math. Soc.* **42** (1974), 475-479.

[2] Man-Duen Choi, Almost Commuting Matrices Need not be Nearly Commuting, *Proc. of AMS* **102**, N.3, 529-533, (1988)

[3] W. Fulton, Eigenvalues, invariant factors, highest weights, and Schubert calculus, *Bull. of AMS* **37** N.3, 209-249, (2000)

[4] Hastings, M. B. Making almost commuting matrices commute, *Comm. Math. Phys.* **291** (2009), no. 2, 321–345.

[5] Lin, Huaxin. Almost commuting selfadjoint matrices and applications, Operator algebras and their applications (Waterloo, ON, 1994/1995), 193–233, Fields Inst. Commun., 13, Amer. Math. Soc., Providence, RI, 1997

[6] D. Vioculescu, Asymptotically commuting finite rank unitaries without commuting approximants, *Acta Sci. Math. (Szeged)* **45** (1983), 429-431.

[7] Luxemburg, W. A. J.; Taylor, R. F. Almost commuting matrices are near commuting matrices, *Nederl. Akad. Wetensch. Proc. Ser. A 73=Indag. Math.* **32** (1970) 96–98.

[8] John von Neumann, Approximative properties of matrices of high finite order, *Port. math* **3** (1942), 1-62, also may be found in Collected works.

[9] C. Pearcy and A. Shields, Almost commuting matrices, *J. Funct. Anal.* **33** (1979), 332-338.

[10] V. Pestov, Hyperlinear and sofic groups: a brief guide, *The Bulletin of Symbolic Logic* **14** (2008), pp. 449-480

[11] Szarek, Stanislaw J. On almost commuting Hermitian operators, *Rocky Mountain J. Math.* **20** (1990), no. 2, 581–589.

[12] H. Weyl, Das asymtotische Verteilungsgesetz der Eigenwerte lineare partieller Differentialgleichungen, *Math. Ann.* **71** (1912), 441-479.