Three–dimensional compact manifolds and the Poincare conjecture

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Abstract: The aim of the work is to prove the following main theorem.

Theorem. *Let M^3 be a three–dimensional, connected, simple connected, compact, closed, smooth manifold and S^3 be the three–dimensional sphere. Then the manifolds M^3 and S^3 are diffeomorphic.*

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0 Introduction

We can fix some Riemannian metric g on a manifold M^n of dimension n which defines the length of arc of a piecewise smooth curve and the continuous function ρ (x; y) of the distance between two points x, $y \in M^n$. The topology defined by the function of distance (metric) ρ is the same as the topology of the manifold M^n, **[5]**.

We should mention that it will suffice to prove that M^3 and S^3 are homeomorphic since the existence of a homeomorphism between M^n and S^n (n=dimM^n, $n \leq 6$, $n \neq 4$) implies the existence of a diffeomorphism between them. If n=7 then there exist such 28 smooth manifolds that every one from them is homeomorphic to S^7 but any two from them are not diffeomorphic.

The proof of the main theorem is based on some notions from **[1]**, **[2]** and that will be considered step by step in the following sections. Some results can be useful in the case when M^3 is not simply connected or can be generalized for manifolds of dimension n>3.

In section 1, using a smooth triangulation and a Riemannian metric we see that every compact, connected, closed manifold M^n of dimension n can be represented as a union of a n–dimensional cell C^n and a connected union K^{n-1} of some finite number of (n–1) — simplexes of the triangulation. A sufficiently small closed neighborhood of K^{n-1} we call a *geometric black hole.* In dimension 3 we have $M^3 = C^3 \cup K^2$.

In section 2, we get some technical results permitting to retract 2–dimensional and 1–dimensional simplexes from K^2 having boundaries *i. e.* to obtain a decomposition $M^3 = \widetilde{C}^3 \cup \widetilde{K}^2$ where $\widetilde{K}^2$ contains less simplexes than K^2 does.

In section 3, we consider the proof of the main theorem consisting of the realization of several algorithms. The number of 2–dimensional simplexes of the complex K^2 becomes less every step and finally we have a decomposition $M^3 = C^3 \cup K^1$ where K^1 is a connected and simply connected union of some 1–dimensional simplexes *i. e.* K^1 is a tree. Using the section 2 we can retract complex K^1 to a point x_0 therefore a decomposition $M^3 = C^3 \cup \{x_0\}$ is obtained and M^3 is homeomorphic to sphere.

1. On extension of coordinate neighborhood

1°. Let M^n be a connected, compact, closed and smooth manifold of dimension n and C^n be a cell (coordinate neighborhood) on M^n. A standard simplex Δ^n of dimension n is the set of points $x=(x_1, x_2, ..., x_n)\in \boldsymbol{R}^n$ defined by conditions

$$0\le x_i\le 1,\ i=\overline{1,\ n},\ x_1+x_2+...+x_n\le 1.$$

We consider the interval of a straight line connected the center of some face of Δ^n and the vertex which is opposite to this face. It is clear that the center of Δ^n belongs to the interval. We can decompose Δ^n as a set of intervals which are parallel to that mentioned above. If the center of Δ^n is connected by intervals with points of some face of Δ^n then a subsimplex of Δ^n is obtained. All the faces of Δ^n considered, Δ^n is seen as a set of all such subsimplexes. Let $U(\Delta^n)$ be some open neighborhood of Δ^n in $\boldsymbol{R}^n$. A diffeomorphism φ : $U(\Delta^n)\to M^n(\delta^n = \varphi(\Delta^n))$ is called a singular n–simplex on the manifold M^n. *Faces*, *edges*, *the center*, *vertexes* of the simplex δ^n are defined as the images of those of Δ^n with respect to φ.

The manifold M^n is triangulable, **[6]**. It means that for any $l,\ \ 0\le l\le n$ such a finite set Φ^l of diffeomorphisms φ : $\Delta^l \to M^n$ is defined that

a) M^n is a disjunct union of images $\varphi(Int\Delta^l),\ \ \varphi\in\Phi^l$;
b) if $\varphi\in\Phi^l$ then $\varphi\circ\varepsilon_i\in\Phi^{l-1}$ for every i where ε_i: $\Delta^{k-1}\to\Delta^k$ is the linear mapping transferring the vertexes $v_0,...,v_{k-1}$ of the simplex Δ^{k-1} in the vertexes $v_0,...,\hat{v}_i,...v_k$ of the simplex Δ^k.

2°. Let δ_0^n be some simplex of the fixed triangulation of the manifold M^n. We paint the inner part $Int\delta_0^n$ of the simplex δ_0^n white and the boundary $\partial\delta_0^n$ of δ_0^n black. There exist coordinates on $Int\delta_0^n$ given by diffeomorphism φ_0. A subsimplex $\delta_{01}^{n-1}\subset\delta_0^n$ is defined by a black face $\delta_{01}^{n-1}\subset\delta_0^n$ and the center c_0 of

δ_0^n. We connect c_0 with the center d_0 of the face δ_{01}^{n-1} and decompose the subsimplex δ_{01}^n as a set of intervals which are parallel to the interval c_0d_0. The face δ_{01}^{n-1} is a face of some simplex δ_1^n that has not been painted. We draw an interval between d_0 and the vertex v_1 of the subsimplex δ_1^n which is opposite to the face δ_{01}^{n-1} then we decompose δ_1^n as a set of intervals which are parallel to the interval d_0v_1. The set $\delta_{01}^n \cup \delta_1^n$ is a union of such broken lines every one from which consists of two intervals where the endpoint of the first interval coincides with the beginning of the second interval (in the face δ_{01}^{n-1}) the first interval belongs to δ_{01}^n and the second interval belongs to δ_1^n. We construct a homeomorphism (extension) $\varphi_{01}^1 : Int\delta_{01}^n \to Int(\delta_{01}^n \cup \delta_1^n)$. Let us consider a point $x \in Int\delta_{01}^n$ and let x belong to a broken line consisting of two intervals the first interval is of a length of s_1 and the second interval is of a length of s_2 and let x be at a distance of s from the beginning of the first interval. Then we suppose that $\varphi_{01}^1(x)$ belongs to the same broken line at a distance of $\frac{s_1+s_2}{s_1}\cdot s$ from the beginning of the first interval. It is clear that φ_{01}^1 is a homeomorphism giving coordinates on $Int(\delta_{01}^n \cup \delta_1^n)$. We paint points of $Int(\delta_{01}^n \cup \delta_1^n)$ white. Assuming the coordinates of points of white initial faces of subsimplex δ_{01}^n to be fixed we obtain correctly introduced coordinates on $Int(\delta_0^n \cup \delta_1^n)$. The set $\sigma_1 = \delta_0^n \cup \delta_1^n$ is called a *canonical polyhedron*. We paint faces of the boundary $\partial\sigma_1$ black.

We describe the contents of the successive step of the algorithm of extension of coordinate neighborhood. Let us have a canonical polyhedron σ_{k-1} with white inner points (they have introduced *white coordinates*) and the black boundary $\partial\sigma_{k-1}$. We look for such an n–simplex in σ_{k-1}, let it be δ_0^n that has such a black face, let it be δ_{01}^{n-1} that is simultaneously a face of some n–simplex, let it be δ_1^n, inner points of which are not painted. Then we apply the procedure described above to the pair δ_0^n, δ_1^n. As a result we have a polyhedron σ_k with one simplex more than σ_{k-1} has. Points of $Int\sigma_k$ are painted in white and the boundary $\partial\sigma_k$ is painted in black. The process is finished in the case when all the black faces of the last polyhedron border on the set of white points (the cell) from two sides.

After that all the points of the manifold M^n are painted in black or white, otherwise we would have that $M^n = M_0^n \cup M_1^n$ (the points of M_0^n would be painted and those of M_1^n would be not) with M_0^n and M_1^n being unconnected, which would contradict of connectivity of M^n.

Thus, we have proved the following

Theorem 1. *Let M^n be a connected, compact, closed, smooth manifold of dimension n. Then $M^n = C^n \cup K^{n-1}$, $C^n \cap K^{n-1} = \varnothing$, where C^n is an n–dimensional cell and K^{n-1} is a union of some finite number of (n–1)–simplexes of the triangulation.*

3°. We consider the initial simplex δ_0^n of the triangulation and its center c_0. Drawing intervals between the point c_0 and points of all the faces of δ_0^n we obtain a decomposition of δ_0^n as a set of the intervals. In **2**° the homeomorphism ψ : $Int\delta_0^n \to C^n$ was constructed and ψ evidently maps every interval above on a piecewise smooth broken line γ in C^n. We denote $\widetilde{M}^n = M^n \setminus \{c_0\}$. $\widetilde{M}^n$ is a connected and simply connected manifold if M^n is that. Let $I=[0;1]$, we define a homotopy $F: \widetilde{M}^n \times I \to \widetilde{M}^n : (x; t) \mapsto y=F(x;t)$ in the following way

a) $F(z; t)=z$ for every point $z \in K^{n-1}$;

b) if a point x belongs to the broken line γ in C^n and the distance between x and its limit point $z \in K^{n-1}$ is $s(x)$ then $y=F(x; t)$ is on the same broken line γ at a distance of $(1-t)s(x)$ from the point z.

It is clear that $F(x;0)=x$, $F(x;1)=z$ and we have obtained the following

Theorem 2. *The spaces* $\widetilde{M}^n$ and K^{n-1} *are homotopy–equivalent, in particular, the groups of singular homologies $H_k\left(\widetilde{M}^n\right)$ and $H_k\left(K^{n-1}\right)$ are isomorphic for every k.*

Corollary 2.1. *The space K^{n-1} is connected and if M^n is simply connected then K^{n-1} is simply connected too.*

Remark. *The white coordinates are extended from the simplex δ_0^n in the simplex δ_1^n through the face δ_{01}^{n-1} hence $Int\delta_{01}^{n-1}$ has also the white coordinates. On the other hand there exist two linear structures (intervals, the center etc) on δ_{01}^n induced from δ_0^n and δ_1^n respectively. Further, we set that the linear structure of δ_{01}^{n-1} is the structure induced from δ_0^n.*

2. On the complex K^2

For a three–dimensional, connected, compact, closed, smooth manifold M^3 we consider a decomposition $M^3 = C^3 \cup K^2$ obtained in theorem 1.

We call simplexes of dimension 3, 2, 1 by tetrahedrons, triangles, edges (intervals) respectively.

1°. **Definition 1.** a) *A triangle from the complex K^2 is called a f–triangle (free) if it has at least one free edge i. e. such an edge that it is not an edge of any other triangle from K^2.*

b) *A triangle from the complex K^2 is called a m–triangle if it has such an edge that is an edge of more than two triangle from K^2. By definition, a m–triangle can not be a f–triangle.*

c) *A triangle from the complex K^2 is called a s–triangle (standard) if every its edge is an edge of only one other triangle from K^2.*

Let us have a *f*–triangle $\bar{\delta}^2 \in K^2$ with some free edge $\bar{\delta}^1$. We consider such a polyhedron σ which σ is a set of all the tetrahedrons with $\bar{\delta}^1$ as their edge. Among them we have exactly two tetrahedrons, let they be $\bar{\delta}_1^3$ and $\bar{\delta}_l^3$ with $\bar{\delta}^2$ as their face. We call the *output* of $\bar{\delta}_1^3$ the face $\bar{\delta}_1^2$ with $\bar{\delta}^1$ as its edge. Inner points of the triangle $\bar{\delta}_1^2$ are white because the edge $\bar{\delta}^1$ is free. The face $\bar{\delta}_1^2$ is a face of another tetrahedron $\bar{\delta}_2^3$ that has only one another face $\bar{\delta}_2^2$ with the edge $\bar{\delta}^1$, moreover, all inner points of the triangle $\bar{\delta}_2^2$ are white. The faces $\bar{\delta}_1^2$ and $\bar{\delta}_2^2$ are called respectively the *input* and *output* (*conversions*) of the tetrahedron $\bar{\delta}_2^3$. The face $\bar{\delta}_2^2$ is called the input of some tetrahedron $\bar{\delta}_3^3$ *etc*. Taking a finite number of steps we come to the tetrahedron $\bar{\delta}_l^3$ with an input $\bar{\delta}_{l-1}^2$ with $\bar{\delta}^1$ as its edge and all inner points of the triangle $\bar{\delta}_{l-1}^2$ are white. Thus, we obtain $\sigma = \bigcup_{i=1}^{l} \bar{\delta}_i^3$ (minimal possible meaning is l=3). We have to note that all inner points of the faces of conversions $\bar{\delta}_1^2$, ..., $\bar{\delta}_{l-1}^2$ in the tetrahedrons of the polyhedron σ are white. It is clear that $(Int\sigma)/\bar{\delta}^2$ is a cell.

2°. We consider the closed cube Cu^3 in the three – dimensional coordinate space $\mathbf{R^3}$ having the vertexes $A(1;\ 1;\ 1)$, $B(1;\ -1;\ 1)$, $C(-1;\ 1;\ 1)$, $D(-1;\ -1;\ 1)$, $A_1(1;\ 1;\ -1)$, $B_1(1;\ -1;\ -1)$, $C_1(-1;\ 1;\ -1)$, $D_1(-1;\ -1;\ -1)$. Let Rc be the intersection of Cu^3 with the semiplane $\alpha = \{ M(x;y;z) \in \mathbf{R^3} \mid z=0; x \ge 0 \}$ and τ be the intersection of α with the square ABB_1A_1. It is easy to construct such a homeomorphism $\varphi_1 : (Cu^3) \setminus Rc \to (Cu^3) \setminus \tau$ that $\varphi_1 = id$ on $(\partial Cu^3) \setminus \tau$.

Proposition 3. *We can redistribute coordinates of white points of the polyhedron σ and introduce white coordinates of points from $Int\,\bar{\delta}^2 \cup \bar{\delta}^1$ (construct the corresponding homemorphism φ_σ) in such way that the following conditions are fulfilled*

a) *all the points of Int σ are painted in white i.e. have white coordinates,*

b) *white coordinates of points of boundary faces of the polyhedron σ are not changed.*

Proof. There exists a homeomorphism $\varphi_2 : \sigma \to Cu^3$, $\varphi_2\left(\bar{\delta}^2\right) = Rc$. Then $\varphi_\sigma = \varphi_2^{-1} o \varphi_1 o \varphi_2$ is a required homeomorphism.

QED.

Remark. *Further, we set that the linear structure (intervals, the center etc.) of* δ^2 *is the structure induced from* $\overline{\delta}_1^3$ *where* $\overline{\delta}_1^3$ *was equipped with white coordinates earlier than* $\overline{\delta}_l^3$.

3°. Definition 2. *An edge* $\overline{\delta}^1 = x_0x_1$ *is called semi-isolated if it is not an edge of any triangle from* K^2. *A semi-isolated edge* $\overline{\delta}^1$ *is called isolated if one of the endpoints of the interval* $\overline{\delta}^1$ *(let it be* x_1*) is free i.e. it is not an endpoint of any edge from* K^2.

An isolated edge $\overline{\delta}^1$ can appear as a result of painting white some neighboring *f*-triangles containing $\overline{\delta}^1$. We consider a polyhedrons σ where σ is the set of all tetrahedrons with x_1 as their vertex. It is clear that all the points of $Int\sigma$ are white with the exception of black points of $\overline{\delta}^1 \setminus \{x_0\}$.

Proposition 4. *We can redistribute coordinates of white points of the polyhedron* σ *and introduce white coordinates of points from* $Int\overline{\delta}^1 \cup \{x_1\}$ *(construct the corresponding homeomorphism) in such a way that the following condition are fulfilled)*

a) *all the points of* $Int\sigma$ *are painted in white i. e. have white coordinates,*

b) *white coordinates of points of boundary faces of the polyhedron* σ *are not changed.*

Proof. It is clear that $(Int\sigma)\setminus\overline{\delta}^1$ is a cell. There exists a homeomorphism $\varphi_2 : \sigma \to Cu^3$, $\varphi_2\left(\overline{\delta}^1\right) = \delta$, where Cu^2 was defined in 2^0 and $\delta = \left\{ (x;0;0) \mid x \in [0;1] \right\}$ is a closed interval in Cu^2. It is easy to construst such a homeomorphism $\varphi_1 : (Cu^3)\setminus\delta \to (Cu^3)\setminus\{E\}$ that $\varphi_1 = id$ on $(\partial Cu^3)\setminus\{E\}$ where $E \in \delta$, *E (1; 0; 0)*. Then $\varphi_\sigma = \varphi_2^{-1} o \varphi_1 o \varphi_1$ is a required homeomorphism.

QED.

4°. We assume that in the process of painting *f*-triangles white by the proposition 3 all the triangles from K^2 are white *i.e.* that we have a representation $M^3 = C^3 \cup K^1$, $C^3 \cap K^1 = \emptyset$, where C^3 is a three–dimensional cell and K^1 is a connected union of finite number of black edges of the triangulation. Since the process of painting *f*–triangles white does not influence simple connectedness of a space that is been obtained after every step then K^1 is a tree if the complex K^2 is simply connected. Painting isolated edges of K^1 in white by the proposition 4 as a result we have unique black point x_0. Thus, we obtain a representation $M^3 = C^3 \cup B(x_0;\ \varepsilon)$, where $B(x_0;\ \varepsilon)$ is an open geodesic ball with the center in

x_0 and of radius ε. The manifold M^3 is homeomorphic to sphere S^3 by the following lemma 5.

Lemma 5 [5]. *If a topological manifold M^n is a union of two n–dimensional cells then M^n is homeomorphic to sphere S^n.*

3. Proof of the main theorem

The proof has a combinatorial nature and assumes the realization of a number of algorithms. We consider that step by step. The initial complex K^2 is assumed to be connected, simply connected and without free triangles.

1°. We call a sequence of tetrahedrons (triangles, edges) a *simple chain (s – chain)* if every such a simplex participates in the sequence only one time and if every subsequent tetrahedron (triangle, edge) has a common face (edge, vertex) with the previous one. The number of elements of a *s*–chain is called the *length* of the s–chain.

Let δ_0^2 be a triangle from the complex K^2 with $\delta_0^1 = x_0x_1$ as its edge. The edge δ_0^1 can also be an edge of some *m*–triangles other than δ_0^2.

Lemma 6. *We can rebuild the complex K^2 in such a way that as a result we have got a black triangle δ_0^2 with the free edge $\delta_0^1 = x_0x_1$. A new rebuilt complex K^2 is connected and simple connected.*

Proof. We consider the *s*–chain of tetrahedrons with δ_0^1 as their edge the first of which has the *upper part* of δ_0^2 as its face and the last of which has the *lower part* of δ_0^2 as its face. In this *s*–chain we can find a tetrahedron, let it be $\bar{\delta}_1^3$, which is the first from the *s*–chain to have a black *m*–triangle $\bar{\delta}_1^2$ with the edge δ_0^1 as its face. The face $\bar{\delta}_1^2$ is the common face of $\bar{\delta}_1^3$ and $\bar{\delta}_2^3$. Thus, we obtain a *s*-chain $\bar{\delta}_1^3, \dots, \bar{\delta}_{l_1}^3$ of tetrahedrons (some of them have *m*-triangles as their faces) $\bar{\delta}_{l_1}^3$ has the lower part of δ_0^2 as its face.

We consider the graph *G* connecting by intervals the centers of all the tetrahedrons of the triangulation via the centers of all the white faces. There exists the maximal tree *L* connecting by intervals all the centers of the tetrahedrons of the triangulation via centers of some white faces. The tree *L* defines the maximal cell C^3. Really, if we consider a maximal tree *L* and some tetrahedron δ^3 then we can extend white coordinates from δ^3 on the maximal cell C^3 along the tree *L* as it was shown in section 1. We assume that the centers of all the tetrahedrons of the *s*-chain from the first to the $\bar{\delta}_1^3$ are connected by the broken line via the centers of their common white faces and the broken line is a part of *L*.

We cancel the white painting of points of $\bar{\delta}_2^3$ and paint the tetrahedron $\bar{\delta}_2^3$ black. The repainting of $\bar{\delta}_2^3$ black brings to a gap of L on three subtrees L_1, L_2, L_3 or less where the center of $\bar{\delta}_1^3$ belongs to L_1.

Further, we repaint inner parts of $\bar{\delta}_1^2$ and $\bar{\delta}_2^3$ white (extend white coordinates from $\bar{\delta}_1^3$ through the face $\bar{\delta}_1^2$ as it was shown in section **1)** and connect the centers of $\bar{\delta}_1^3$, $\bar{\delta}_1^2$, $\bar{\delta}_2^3$ by intervals. Those centers belong to the subtree L_1. Other faces of $\bar{\delta}_2^3$ are black and they are simultaneously faces of other tetrahedrons.

We consider the following cases.

a) $L_1 = L$. The black faces of $\bar{\delta}_2^3$ remain black.

b) We have got two subtrees L_1 and L_2 where L_2 defines a cell called a *dead end*. We repaint the closure of the dead end black. Then we are looking for a black face of $\bar{\delta}_2^3$ which is simultaneously a face of other tetrahedron with the center from L_2. We extend white coordinates from $\bar{\delta}_2^3$ through this face along the tree L_2 as it was shown in section **1** and repaint inner points of this face and inner points of the dead end white. Then we connect by intervals the center of this face with the centers of $\bar{\delta}_2^3$ and other tetrahedron obtaining a new maximal tree L defining a new maximal cell C^3. Two other faces of $\bar{\delta}_2^3$ remain black.

c) We have got three subtrees L_1, L_2, L_3 where L_2 and L_3 define two cells called dead ends. We repaint the closure of each the dead end black. Then we are looking for a black face of $\bar{\delta}_2^3$ which is simultaneously a face of other tetrahedron with the center from L_1. This face remains black. We extend white coordinates from $\bar{\delta}_2^3$ through two other black faces of $\bar{\delta}_2^3$ along the trees L_2 and L_3 as it was shown in section **1** and repaint inner points of this faces and inner points of the dead ends white. Then we connect by intervals the centers of this faces with the centers of $\bar{\delta}_2^3$ and two other tetrahedrons obtaining a new maximal tree L defining a new maximal cell C^3.

Further, we apply the process above to the tetrahedrons $\bar{\delta}_2^3$, $\bar{\delta}_3^3$ *etc*. All the centers of the tetrahedrons $\bar{\delta}_1^3$, $\bar{\delta}_2^3$, $\bar{\delta}_3^3$;… are connected by broken line which is a part of the subtree L_1 at every step. As a result we have got a black triangle δ_0^2 with the free edge $\delta_0^1 = x_0x_1$.

QED.

Remark. *We have obtained the s-chain of the tetrahedrons (with white inner part) having* δ_0^1 *as their edge and all the centers of the tetrahedrons are connected by the broken line via the centers of their common white faces. The broken line can be considered as a part of a subtree* L_1 *of some maximal tree L i.e.*

this broken line can be extended to L. Really, we can apply the method considered in the proof of the lemma 6 to the s-chain above sequentially repainting the tetrahedrons from the second ($\bar{\delta}_1^3$) to the last ($\bar{\delta}_{l_1}^3$) *to get the broken line in the end.*

2°. We choose a small ball with the center in x_0 which is diffeomorphic to a small ball in $\mathbf{R}^3$ and call a *trace* of a simplex with a vertex or an endpoint in the point x_0 its intersection with the sphere which is the surface of the ball where the sphere is supposed to be transversal to all the triangles with the vertex x_0 . Such a sphere exists because of the smoothness of the triangulation. All other vertexes of the triangles do not belong to the ball. The ball can be choosed in such a way that every edge with the endpoint x_0 has only one point of the intersection with the sphere and every triangle with the vertex x_0 is intersected with the sphere by only one segment of a curve. There exists one to one correspondence between the set of simplexes having a vertex (endpoint) x_0 and the set of their traces therefore all steps of the algorithm below can be illustrated on the sphere.

We continue a consideration of the edge $\delta_0^1 = x_0x_1$ and a set $Bt(x_0)$ of black triangles with x_0 as their vertex. We extract *pyramids* from the set $Bt(x_0)$. The trace of the *surface of a pyramid* formed by some black triangles is a *maximal loop* on the sphere *i.e.* a curve that divides the sphere into two parts. Any *exterior* white point of the sphere close to the loop can be connected with the trace of δ_0^1 (a black point) by a white curve and any *interior* white point with respect of the loop cannot. Such loops can be connected among themselves by segments of black curves. See Fig.1 as a possible picture of such traces.

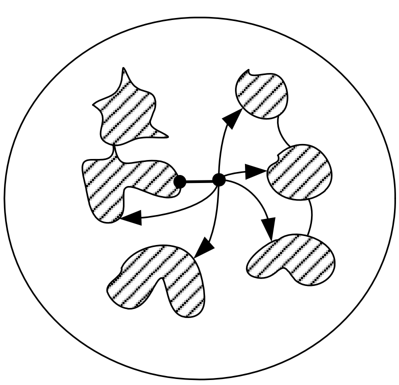
Fig. 1

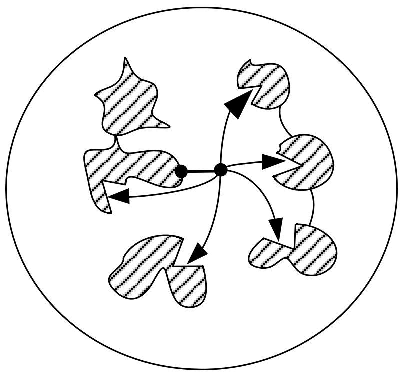
Fig. 2

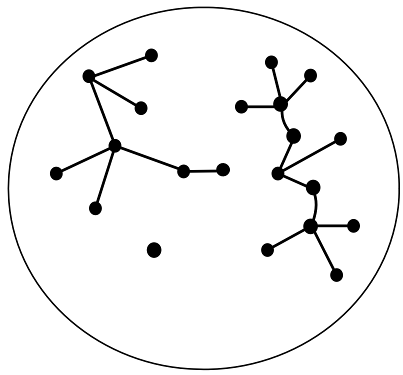
Fig. 3

Further, we consider one of the pyramids and any s-chain (with the white inner part) of tetrahedrons having x_0 as their vertex the first of which has δ_0^1 as its edge and last of which (the first in the s-chain) has a black triangle from the surface of the pyramid as its face. All the centers of the tetrahedrons of any such a s-chain are supposed belonging to a subtree L_1 of some maximal tree L. Really, we can apply the method considered in the proof of the lemma 6 to any such a s-chain above sequentially repainting tetrahedrons from the second to the last to get a broken line in the end. The subtree L_1 is the union of all such broken lines and the initial broken line considered in the remark in **1°**, **3.** The subtree L_1 can be extended to L and L_1 cannot get a gap by the procedure below. In the set of all possible similar *s*-chains we look for a *s*-chain of the minimal length. In the last tetrahedron δ_l^3 of the *s*–chain we consider the subtetrahedron $\delta_{l_0}^3$ with the center of δ_l^3 as its vertex and the mentioned above black triangle as its face. The latter belongs to tetrahedron $\delta_{l_1}^3$. The inner points of $\delta_{l_1}^3$ are simultaneously inner points of the pyramid. By definition, x_0x_1 cannot be an edge of such a tetrahedron $\delta_{l_1}^3$. Canceling white painting of those inner points and painting the tetrahedron $\delta_{l_1}^3$ black we extend white coordinates from $\delta_{l_0}^3$ into $\delta_{l_1}^3$ through their common face as it was described in section **1** and paint those inner points white again. A new one more length *s*–chain has been obtained (see Fig.2). If we obtain a gap of the maximal tree L then we eliminate it by the procedure described in lemma 6 using introduced above the subtree L_1. Further, we iterate the algorithm above and so on. It is clear that we cannot get a new black triangle having δ_0^1 as its edge by the procedure above. In the end any tetrahedron with a vertex in x_0 can be considered as an element of some *s*-chain with the white inner part connecting this tetrahedron with the edge δ_0^1 *i.e.* all the loops on the sphere are annihilated and we have got a number of trees on the sphere (see Fig.3). Any endpoint of a tree is simultaneously the trace of a free edge of some *f*-triangle and we can paint the *f*-triangle white by proposition 3. As it has been noted in the proof of this proposition the painting of boundary points of a polyhedron containing a black *f*-triangle is not changed. Sequentially painting all those *f*-triangles white we retract all the trees on the sphere to a number of black points which are traces of some semi-isolated (isolated) edges. As a result we have got a situation when the set $Bt(x_0)$ becomes empty.

Really, othervise if we have only one *f*-triangle δ_0^2 in $Bt(x_0)$ and δ_1^2 is an other triangle in $Bt(x_0)$ then we can construct some *s*-chain δ_1^2, δ_2^2, … , δ_n^2 of triangles from $Bt(x_0)$ where δ_n^2 has some a common edge $x_0\ x_l$ with a previous triangle from this *s*-chain *i.e.* we have got a pyramid. There exists a tetrahedron containing inner white points of the pyramid which can not be connected by any *s*-chain (with white inner part) of tetrahedrons having x_0 as their vertex with the edge δ_0^1 *i.e.* the contradiction to the situation above has been obtained.

It is obvious that the set of all the white points is a three–dimensional cell at every step. It is clear that the last rebuilt complex K^2 is connected and simple connected because of a homotopy–equivalence.

Remark. *Further, a structure consisting of a semi–isolated edge and a black subcomplex joined to it is called a «black flower» growing from the point x_0. Let bf_1 and bf_2 be any two black flowers connected by a system of semi-isolated edges. The simple connectivity of K^2 implies that if we paint the semi-isolated edge of bf_1*

white then the black subcomplex obtained can not be connected with bf_2 by a black curve.

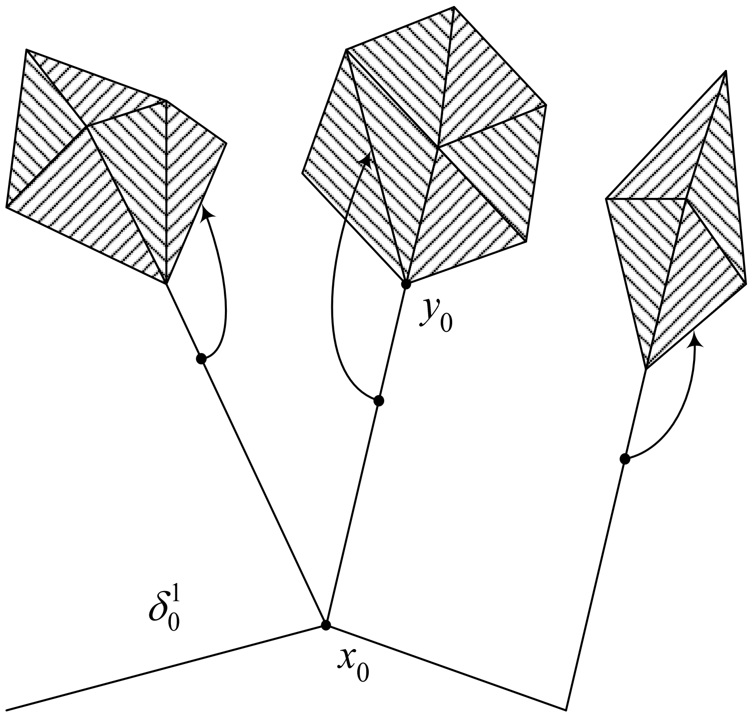

Fig. 4

3°. We consider a black flower consisting of a semi-isolated edge $\overline{\delta}^1$ with the endpoints x_0 , y_0 and a black two-dimensional subcomplex of some black triangles having y_0 as their vertex. Further, we apply the procedure considered in **2°** to the point y_0 and the edge $\overline{\delta}^1$. The simple connectivity of K^2 implies that we cannot get a black loop in K^2 having a semi-isolated edge as its part therefore the annihilation of black triangles of $Bt(y_0)$ cannot bring to an appearance of a black triangle in $Bt(x_0)$ and $Bt(x_0)$ remains empty. Similarly, if we have a s-chain of semi-isolated edges $\delta_1^1,\ldots,\delta_k^1= x_k\, y_k$ then the process of the annihilation of black triangles in $Bt(y_k)$ cannot bring to an appearance of a black triangle having a generic point with δ_i^1 $(i<k)$. Really, otherwise such a black triangle gives an opportunity to connect the endpoints x_k and y_k of δ_k^1 by a black curve which is different from δ_k^1. As a result a black loop with the semi-isolated edge δ_k^1 has been obtained and the loop is not contractible that is a contradiction to the simple connectivity of K^2. Thus, a number of the black isolated and semi-isolated edges is increased and the sets $Bt(x_0)$, $Bt(y_0)$, … remain empty.

It follows that a number of black triangles becomes less at every step. Finally, at some step of our algorithm the set of black triangles must be exhausted *i.e.* we come to **4°**, **2**.

Remark. *The obtained complex can be imagined as a «tree with flowers» growing in the endpoints of the branches of the tree. An iteration of the algorithm can be interpreted as a sequential transformation of those flowers into branches to get a tree in the end.*

The main theorem is completely proved.

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