Scattering of Spin 1/2 Particles by the Coulomb Field

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Abstract

Using the Dirac equation for electron in the Coulomb field, we find the Green function by the iteration method and then we deduce the transition matrix. The corresponding differential cross-section is then derived. The article is written with the mathematical simplicity and the Schwinger pedagogical clarity.

1 Introduction

In a Coulomb scattering in the non-relativistic case the quantum particle scattering is governed by the same Rutherford formula as the classical one. In classical picture each particle is scattered by the field, so the above mentioned agreement for the field without interaction is impossible. In the relativistic region classical and quantum cross sections are different and a part of the difference may be attributed to the fact that quantum particle can fly through the field without being deflected.

The classical cross section in small angle region has terms which have different signs for attractive and repulsive potentials. Their analog in quantum case contains the Planck constant $\hbar$ in the denominator of the fine-structure constant $\alpha = e^2/(\hbar c)$. This suggests that unless αZ is made sufficiently large, the reproduction of classical terms from quantum ones is impossible. For this reason the theoretical investigation of the relationship between the classical and quantum scattering is of interest (Nikishov, 2009).

2 The source theory

Source theory, being the new quantum theory of fields (Schwinger, 1969; 1970; 1973; 1989; Dittrich, 1978), is the theoretical construction that uses quantum-mechanical particle language. Initially it was constructed for a description of the particle physics situations occurring in high-energy physics experiments. However, it was found that the original formulation simplifies the calculations in the electrodynamics and gravity where the interactions are mediated by the photon or graviton, respectively.

The basic formula in the source theory is the vacuum to vacuum amplitude (Schwinger et al., 1976):

$$\langle 0_+|0_-\rangle = e^{\frac{i}{\hbar}W(S)}, \tag{1}$$

where the minus and plus signs on the vacuum symbol are causal labels, referring to any time before and after the space-time region where sources are manipulated. The exponential form is introduced with regard to the existence of the physically independent experimental arrangements, which has a simple consequence that the associated probability amplitudes multiply and corresponding W expressions add (Schwinger, 1969; 1970; 1973; 1989).

3 Scattering of spin 1/2 particles by Coulomb field

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Exploiting the experience with the spin 0 particles, we investigate here the scattering of spin 1/2 particles in the static Coulomb field. The quantum motion of the spin 1/2 particles is described by the Dirac equation and their interaction with electromagnetic field is introduced by the gauge invariance of it which is equivalent with replacing the partial derivatives by the covariant ones, or,

$$\partial_\mu \quad \longrightarrow \quad \partial_\mu - ieqA_\mu \stackrel{d}{=} D_\mu, \tag{2}$$

where q is co-called charge matrix (Schwinger, 1970).

Then, the Dirac equation involving the interaction with the electromagnetic field is

$$\left(\gamma\frac{1}{i}D+m\right)\psi=0 \tag{3}$$

and the appropriate equation for the Green function is the following one:

$$\left(\gamma\frac{1}{i}\partial+m\right)G_+^A(x,x')=\delta(x-x')+eq\gamma A(x)G_+^A(x,x'), \tag{4}$$

which can be easily converted into the integral equation with the formal solution:

$$G_+^A(x,x')=G_+(x-x')+\int(d\xi)G_+(x-\xi)eqA(\xi)G_+^A(\xi,x'). \tag{5}$$

This equation can be solved by iteration. Such manipulation is facilitated by adopting a matrix notation in which the coordinates x and x' join the discrete spinor and charge indices as continuous row and column labels. In such notation eq. (5) is transformed into the form:

$$G_+^A=G_++G_+eq\gamma AG_+^A \tag{6}$$

and it can be written as the formal matrix solution

$$G_+^A=(1-G_+eq\gamma A)^{-1}G_+, \tag{7}$$

which enables the expansion of the form

$$G_+^A=\sum_{\alpha=0}(G_+eq\gamma A)^\alpha G_+=$$

$$G_++G_+eq\gamma AG_++G_+eq\gamma AG_+eq\gamma AG_++.... \tag{8}$$

Also here the information about scattering must be involved in the action which for the non-interacting particles is

$$W(\eta)=\frac{1}{2}\int(dx)(dx')\eta(x)\gamma^0G_+(x-x')\eta(x') \tag{9}$$

and for interacting particles is

$$W_A(\eta)=\frac{1}{2}\int(dx)(dx')\eta(x)\gamma^0G_+^A(x-x')\eta(x'). \tag{10}$$

To get information about scattering we must insert expansion (8) into eq. (10). Using

$$\psi(x) = \int (dx')G_+(x-x')\eta(x') \tag{11}$$

we get

$$W_{21} = \frac{1}{2}\int (dx)\psi(x)\gamma^0 eq\gamma A(x)\psi(x) \tag{12}$$

$$W_{22} = \frac{1}{2}\int (dx)(dx')\psi(x)\gamma^0 eq\gamma A(x)G_+(x-x')eq\gamma A(x')\psi(x') \tag{13}$$

$$W_{23} = \frac{1}{2}\int (dx)(dx')(dx'')\psi(x)\gamma^0 eq\gamma A(x)G_+(x-x')eq\gamma A(x') \quad \times$$

$$G_+(x'-x'')eq\gamma A(x'')\psi(x''). \tag{14}$$

The analogy with the spin 0 situation leads to (Schwinger, 1970):

$$iW_{21} = i\int (dx)\psi_1(x)(-eq)A^0(\mathbf{x})\psi_2(x) \quad =$$

$$i\sum i\eta^*_{p_1\sigma_1 q} 2m\left(d\omega_{p_1} d\omega_{p_2}\right)^{1/2}(-eq)\left(u^*_{p_1\sigma_1} u_{p_2\sigma_2}\right) \quad \times$$

$$\left[\int (dx)e^{i(p_2-p_1)x}A^0(\mathbf{x})\right] i\eta_{p_2\sigma_2 q}, \tag{15}$$

where we choose the Coulomb field with the components

$$A^0(\mathbf{x}) = \frac{Ze}{4\pi}\frac{1}{|\mathbf{x}|},\ \mathbf{A}(\mathbf{x}) = 0. \tag{16}$$

It corresponds to the electrodynamic four-vector J^μ with the components

$$J^0(\mathbf{x}) = Ze\delta(\mathbf{x}),\ \mathbf{J}(\mathbf{x}) = 0. \tag{17}$$

Then, from eq. (15) we can deduce the transition matrix (Schwinger, 1970)

$$\langle 1_{p_1\sigma_1 q}|T|1_{p_2\sigma_2 q}\rangle \quad =$$

$$2m\left(d\omega_{p_1} d\omega_{p_2}\right)^{1/2}\left[-\frac{Ze^2 q}{(\mathbf{p_1}-\mathbf{p_2})^2}\right]\left(u^*_{p_1\sigma_1} u_{p_2\sigma_2}\right). \tag{18}$$

The transition probability per unit time is (Schwinger, 1970)

$$(d\omega_{p_1} d\omega_{p_2})\, 2\pi\delta(p_1^0 - p_2^0) \left[\frac{2mZe^2}{(\mathbf{p_1} - \mathbf{p_2})^2}\right] |u^*_{p_1\sigma_1} u_{p_2\sigma_2}|^2. \tag{19}$$

When helicity states are used, we have:

$$\left(u^*_{p_1\sigma_1} u_{p_2\sigma_2}\right) = \left(\frac{p^0 + m}{2m} + \frac{p^0 - m}{2m}\sigma_1\sigma_2\right) \left(v^*_{\sigma_1} v_{\sigma_2}\right), \tag{20}$$

or,

$$\left(u^*_{p_1\sigma_1} u_{p_2\sigma_2}\right) = \frac{p^0}{m} \cos\frac{1}{2}\Theta; \quad \sigma_1 = \sigma_2 \tag{21}$$

$$\left(u^*_{p_1\sigma_1} u_{p_2\sigma_2}\right) = \sigma_1 \sin\frac{1}{2}\Theta; \quad \sigma_1 = -\sigma_2, \tag{22}$$

where the factor σ_1 in eq. (22) denotes the algebraic signs $\pm$. For either choice of σ_2 the summation over σ_1 gives the differential cross section of the form (Schwinger, 1970):

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} Z^2\alpha^2 \left[\frac{p^0}{(p^0)^2 - m^2}\right] \quad \times$$

$$\left(\cos^2\frac{1}{2}\Theta + \frac{m^2}{(p^0)^2}\sin^2\frac{1}{2}\Theta\right) \frac{1}{\sin^4\frac{1}{2}\Theta}. \tag{23}$$

Let us stil remark that from eq. (20) it follows that the electron retains its helicity at high energy, while the spin remains inert in space al low energy.

3.1 The correction to the scattering process

In analogy with spin 0 scattering process the same question arises, namely, what is the interpretation of terms $W_{22}, W_{23}, ...$, if W_{21} describes scattering by the fixed charge? In term W_{22}

$$W_{22} = \frac{1}{2} e^2 \int (dx)(dx')\psi(x) A^0(\mathbf{x}) G_+(x - x')\gamma^0 A^0(\mathbf{x}')\psi(x') \tag{24}$$

only field $\psi(x)$ refers to propagation of particles and therefore W_{22} also describes an electron scattering process and all the other $W_{2\alpha}$. In other words, the expansion in powers of the static potential A^0 represents the successive approximation to the complete treatment of the motion of the particle in the Coulomb field of the point source.

The considered term W_{22} leads to the modification of the transition matrix $\delta\langle 1_{p_1\sigma_1 q}|T|1_{p_2\sigma_2 q}\rangle$ which automatically leads to the modification of the differential cross section. It was shown that the correction coming from W_{22} belongs to transitions in which the helicity does not change (Schwinger, 1970).

The resulting differential cross section for unpolarized particles is (Schwinger, 1970):

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} Z^2 \alpha^2 \left[\frac{p^0}{(p^0)^2 - m^2} \right]^2 \frac{1}{\sin^4 \frac{1}{2}\Theta} \left[\cos^2 \frac{1}{2}\Theta + \left(\frac{m}{p^0} \right)^2 \sin^2 \frac{1}{2}\Theta \quad - \right.$$
$$\left. \pi Z \alpha q \left(1 - \frac{m^2}{(p)^2} \right)^{1/2} \sin \frac{1}{2}\Theta \left(1 - \sin \frac{1}{2}\Theta \right) \right]. \tag{25}$$

In this expression it can be recognized the existence of mechanism which is common for spin 0 and 1/2 particles. The term with $\sin^2(\Theta/2)$ is specific to spin 1/2. Further and more detailed discussion about this term is involved in the Schwinger book (Schwinger, 1970).

4 Discussion

The classical cross section for scattering in relativistic region is different for attractive and repulsive potentials and does not agree exactly with quantum cross section for scalar particle. It seems that this disagreement cannot be totally ascribed to the fault of classical approach. This suggests that quantum corrections to the first Born approximation should have such a structure that for sufficiently large αZ they give classical terms which are independent of αZ It is not excluded that with small probability a high-energy particle can rush through Coulomb field without deflection (Nikishov, 2009).

The used method of source theory can be evidently generalized to the Yukava potential and compare with the other methods (Goldberger et al., 1964).

$$V(r) = -g^2 \frac{e^{-\alpha m r}}{r}, \tag{26}$$

where g is a magnitude scaling constant, i.e. is the amplitude of potential, m is the mass of the particle, r is the radial distance to the particle, and α is another scaling constant.

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