

# Log-concavity of Lucas sequences of first kind

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## Abstract

In these notes we address the study of the log-concave operator acting on Lucas sequences of first kind. We will find for which initial values a generic Lucas sequence is log-concave, and using this we show when the same sequence is infinite log-concave. The main result will help to fix the log-concavity of some well known recurrent sequences like Fibonacci and Mersenne numbers. Some possible generalization for a complete classification of the log-concave operator applied to general linear recurrent sequences is proposed.

## 1 Introduction

Log-concave sequences arise in many areas of algebra, combinatorics, and geometry as detailed by the survey article of Brenti [1]. During the years there have been some studies on the log-operator  $\mathcal{L}$  acting on recurrent sequences such as the work of Asai [2] on Bell numbers, Bóna [3] on sequences counting permutations, Liu [4] on combinatorial sequences and McNamara[5] with his work on Pascal's triangle. Lucas sequences were first introduced in 1874 by the French mathematician Edouard Lucas, an extensive reference is the book of Koshy [6]. By definition let  $P, Q$  two integer numbers such that  $P^2 - 4Q \geq 0$ , then the Lucas sequence of first kind  $U_n(P, Q)$  is the recurrent sequence defined by  $U_0 = 0, U_1 = 1, U_2 = p, U_n = PU_{n-1} - QU_{n-2}$ . As special case for some  $P, Q$  the Lucas sequence associated becomes a well known sequence, for example  $L(1, -1, n) = F_n$  where  $F_n$  is the Fibonacci sequence. In these notes we study the log-operator on these sequence to address the general problem to find which  $P, Q$  integer the corresponding Lucas sequence  $U_n(P, Q)$  is log-concave or  $\infty$ -log concave . In section one we will introduce some basic definition and some basic results on log-operator acting on recurrent sequences. Section two will show a general result on how to solve the log-concavity problem on a generic Lucas sequence of first kind. Last section will propose a generalization of the methods used on Lucas sequence to generic linear recurrent sequences.

## 2 Basic definition

We now remark some definitions of the log-operator . We refer to the notation to McNamara [5]. Let us start with

**Definition 1.** Let  $(a_n)_{n \in \mathbb{N}}$  a real sequence we define the log-operator as a function  $\mathcal{L} : \mathbb{R} \rightarrow \mathbb{R}$  such that  $b_n = \mathcal{L}(a_n) = a_n^2 - a_{n-1}a_{n+1}$ . If  $b_n \geq 0$  for all  $n \in \mathbb{N}$  then the sequence  $(a_n)_{n \in \mathbb{N}}$  is said to be log-concave.

Considering that log-concavity can deals with negative indexes, by convention we will extend a sequence  $(a_n)_{n \in \mathbb{N}}$  to a sequence  $(a_n)_{n \in \mathbb{Z}}$  where by definition  $a_n = 0$  if  $n < 0$ . In the same way if the sequence is finite so  $n \leq m, m \in \mathbb{N}$  then all other indexes  $n > m$  will be zero.

**Definition 2.** A real sequence  $(a_n)_{n \in \mathbb{N}}$  is said to be  $i$ -fold log-concave for  $i \in \mathbb{N}, i \geq 1$  if  $\mathcal{L}^i(a_n)$  is a nonnegative sequence. Where  $\mathcal{L}^i(a_n)$  is the log-operator applied to a sequence  $(a_n)_{n \in \mathbb{N}}$   $i$ -times so  $\mathcal{L}^i = \mathcal{L} \circ \mathcal{L} \circ \dots \circ \mathcal{L}$ .

Using McNamara [5] notation:

**Definition 3.** a real sequence  $(a_n)_{n \in \mathbb{N}}$  is said to be  $\infty$ -log concave if  $\mathcal{L}^i(a_n)$  is a nonnegative sequence for all  $i \in \mathbb{N}, i \geq 1$ .

So log-concavity in the ordinary sense is 1-fold log-concavity. To study log-concavity on Lucas sequences, we need some preliminary results, like the following:

**Lemma 4.** let  $(a_n)_{n \in \mathbb{N}}$  a sequence where  $a_n = k$  for all  $n \in \mathbb{N}$  and  $k$  is a real number, then  $(a_n)_{n \in \mathbb{N}}$  is  $\infty$ -log concave .

*Proof.* It is easy to check that

$$b_n = \mathcal{L}(a_n) = \mathcal{L}(a_n) = k^2 - (k \cdot k) = 0$$

for all  $n \in \mathbb{N}$ . It is also clear that the all zeros sequence  $b_n$  is invariant by the log-operator that is  $\mathcal{L}(b_n) = b_n$ . Being  $b_n \geq 0$  that means that also  $\mathcal{L}(b_n) \geq 0$  so the all zeros sequence is  $\infty$ -log concave .  $\square$

In the same way it is also easy to check that

**Lemma 5.** let  $(a_n)_{n \in \mathbb{N}}$  a sequence where for all  $n \in \mathbb{N}$   $a_n = kb^n$  where  $k, b \in \mathbb{R}, k \neq 0, b \neq 0$  then  $(a_n)_{n \in \mathbb{N}}$  is  $\infty$ -log concave .

*Proof.* By direct check

$$\mathcal{L}(a_n) = (a_n)^2 - a_{n-1}a_{n+1} = k^2b^{2n} - k^2b^{n-1+n+1} = k^2b^{2n} - k^2b^{2n} = 0$$

for all  $n \in \mathbb{N}$ . So  $a_n$  is 1-fold log-concave and the result sequence is the all zeros sequence than considering lemma 4 then the sequence  $a_n$  is also  $\infty$ -log concave .  $\square$

In next section we will detail our analysis of the log-operator to the Lucas sequence.

### 3 Log-operator and Lucas sequences

In these section we address the study of log-concavity, of a Lucas sequence of first kind Let start with the Lucas sequence definition:

**Definition 6.** Let  $(P, Q) \in \mathbb{Z} \times \mathbb{Z}$  two non-zero integer such that  $P^2 - 4Q \geq 0$  and let  $n \in \mathbb{N}$  an index. A Lucas sequence  $U_n(P, Q)$  of first kind is a recurrent sequence defined as follows:

$$\begin{aligned} U_0 &= 0 \\ U_1 &= 1 \\ U_n &= PU_{n-1} - QU_{n-2}. \end{aligned}$$

Choosing the correct  $P, Q$  it is possible to obtain some well known sequences for example:

- If  $P = 1, Q = -1$  then the Lucas sequence  $U_n(1, -1) = F_n$  where  $F_n$  is the Fibonacci sequence.
- If  $P = 2, Q = -1$  then the Lucas sequence  $U_n(2, -1)$  is the sequence of Pell numbers.
- If  $P = 1, Q = -2$  then the Lucas sequence  $U_n(1, 2)$  is the sequence of Jacobsthal numbers.
- If  $P = 3, Q = 2$  then the Lucas sequence  $U_n(3, 2)$  is the sequence of Mersenne numbers.

The main result of this section will prove for which initial  $P, Q$  the resulting Lucas sequence is  $\infty$ -log concave . Let us start by showing that in general if we choose a generic couple  $P, Q$  the Lucas sequence  $U_n(P, Q)$  is not 1-fold log-concave .

We use the following proposition

**Proposition 7.** *The Fibonacci sequence  $F_n$  is not 1-fold log-concave .*

*Proof.* Considering the log-operator applied to  $F_n$  we have

$$b_n = \mathcal{L}(F_n) = F_n^2 - F_{n-1}F_{n+1};$$

now by the Cassini's identity

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \tag{1}$$

we obtain

$$F_n^2 - F_{n-1}F_{n+1} = (-1) \cdot (-1)^n = (-1)^{n+1}.$$

So

$$\mathcal{L}(F_n) = (-1)^{n+1}.$$

thus  $F(n)$  is not 1-fold log-concave . If we applied the  $\mathcal{L}$  operator to the sequence  $b_n$  and we calculate  $\mathcal{L}^2(F(n)) = \mathcal{L}(\mathcal{L}(b_n))$  we obtain

$$\mathcal{L}^2(F(n)) = ((-1)^{n+1})^2 - (-1)^{n+2} \cdot (-1)^n = ((-1)^{n+1})^2 - (-1)^{2n+2} = 1 - 1 = 0$$

so after applying the log-operator more than once we obtain a sequence that is log-concave.  $\square$

We will now fix for what initial parameter  $P, Q$  the generate Lucas sequence  $U_n(P, Q)$  is a 1-fold log-concave Lucas sequence, and in these cases where for what  $P, Q$  the Lucas sequence becomes  $\infty$ -log concave . Instead of trying to apply directly the log-operator to the generic expression of the Lucas sequence  $U_n(P, Q)$ , we will use a more treatable expression for  $U_n(P, Q)$ . To do this, we first need to recall [7] that:

*Remark 8.* let  $U_n(P, Q)$  a Lucas sequence of first kind, than the characteristic equation of the recurrence relation is

$$x^2 - Px + Q = 0 \quad (2)$$

that has discriminant  $D = P^2 - 4Q$ . If the discriminant is positive so  $D \geq 0$  then the roots of the characteristic equation are

$$a = \frac{P + \sqrt{D}}{2}, \quad b = \frac{P - \sqrt{D}}{2} \quad (3)$$

and so if  $D \geq 0$  it is possible to rewrite  $U_n(P, Q)$  in the following way

$$U_n(P, Q) = \frac{a^n - b^n}{a - b} = \frac{a^n - b^n}{\sqrt{D}}. \quad (4)$$

Armed with this expression for Lucas sequence, we will divide our study in two main cases let us start with the simpler one.

**Proposition 9.** *Let  $U_n(P, Q)$  a Lucas sequence where  $P, Q$  are two integer and the discriminant  $D$  of the characteristic equation associated with  $U_n(P, Q)$  is zero then the Lucas sequence associated is 1-fold log-concave .*

*Proof.* It is easy to see that if  $D = 0$  then  $P^2 - 4Q = 0$  and so there exists an integer  $S$  such that  $P = 2S$  and  $Q = S^2$ . Using this fact the Lucas sequence associated can be rewritten in the form

$$U_n = nS^{n-1}. \quad (5)$$

So now, applying the  $\mathcal{L}$  operator, we see that

$$\begin{aligned} \mathcal{L}(U_n) &= \\ \mathcal{L}(nS^{n-1}) &= (nS^{n-1})^2 - [(n-1)S^{n-2} \cdot (n+1)S^n] \\ &= (n^2)S^{2n-2} - (n^2 - 1)S^{n-2+n} \\ &= (n^2)S^{2n-2} - (n^2 - 1)S^{2n-2} \\ &= (n^2 - n^2 + 1)S^{2n-2} \\ &= (S^{n-1})^2 \end{aligned}$$

and so  $\mathcal{L}(U_n) \geq 0$  for all  $S \in \mathbb{Z}$ . This prove that  $U_n$  is 1-fold log-concave .  $\square$

From proposition 9 we have also the following corollary

**Corollary 10.** *Let  $U_n(P, Q)$  a Lucas sequence where  $P, Q$  are two integer and there exist an  $S \in \mathbb{Z}$  such that  $P = 2S$  and  $Q = S^2$  then the Lucas sequence associated is  $\infty$ -log concave .*

*Proof.* We have seen that under the hypothesis  $\mathcal{L}(U_n) = (S^{n-1})^2 = (S^2)^{n-1}$ . By changing the index we have that the original sequence become a sequence of the form  $b_k = S^k$  where  $k \in \mathbb{Z}, k = 2n - 2, k \geq -2$ . Considering that for negative indexes  $b_k = 0$  we have that by lemma 5 the sequence  $b_k$  is  $\infty$ -log concave and so  $U_n$ .  $\square$

Let now consider the general case

If  $D = P^2 - 4Q > 0$  by remark 8 it is possible to rewrite  $U_n(P, Q)$  in the following way

$$U_n(P, Q) = \frac{a^n - b^n}{a - b} = \frac{a^n - b^n}{\sqrt{D}} \quad (6)$$

where

$$a = \frac{P + \sqrt{D}}{2}, \quad b = \frac{P - \sqrt{D}}{2} \quad (7)$$

we notice that, using direct calculation we have

$$\begin{aligned} \mathcal{L}(U_n) &= \\ \mathcal{L}\left(\frac{a^n - b^n}{\sqrt{D}}\right) &= \left(\frac{a^n - b^n}{\sqrt{D}}\right)^2 - \left[\frac{a^{n-1} - b^{n-1}}{\sqrt{D}} \cdot \frac{a^{n+1} - b^{n+1}}{\sqrt{D}}\right] \\ &= \left(\frac{a^{2n} - 2a^n b^n + b^{2n}}{D}\right) - \left(\frac{a^{n-1+n+1} - a^{n-1}b^{n+1} - a^{n+1}b^{n-1} + b^{n+1+n-1}}{D}\right) \\ &= \frac{a^{2n} - 2a^n b^n + b^{2n} - a^{2n} + a^{n-1}b^{n+1} + a^{n+1}b^{n-1} - b^{2n}}{D} \\ &= \frac{a^{n+1}b^{n-1} - 2a^n b^n + a^{n-1}b^{n+1}}{D} \\ &= \frac{a^{n-1}b^{n-1}(a^2 - 2ab + b^2)}{D} \\ &= \frac{(ab)^{n-1}(a^2 - 2ab + b^2)}{D} \\ &= \frac{(ab)^{n-1}(a - b)^2}{D} \end{aligned}$$

now then by definition

$$ab = \frac{P + \sqrt{D}}{2} \cdot \frac{P - \sqrt{D}}{2} = \frac{1}{4}(P^2 - D) = \frac{1}{4}(P^2 - P^2 + 4Q) = Q \quad (8)$$

and

$$a - b = \frac{P + \sqrt{D}}{2} - \frac{P - \sqrt{D}}{2} = \frac{2P}{2} = P. \quad (9)$$

So finally we have

$$\mathcal{L}(U_n) = \frac{Q^{n-1}P^2}{D} \quad (10)$$

So  $\mathcal{L}(U_n) \geq 0$  if  $Q \geq 0$ . Combining this with the assumption that  $P^2 - 4Q \geq 0$  we have that  $U_n(P, Q)$  is 1-fold log-concave if

$$\begin{cases} Q \geq 0 \\ P^2 - 4Q > 0 \end{cases}$$

that gives the following set of solutions  $Q \geq 0 \wedge P > 2\sqrt{Q}$  or  $Q \geq 0 \wedge P < -2\sqrt{Q}$ .

We can summarize the result in the following

**Theorem 11.** *Let  $P, Q$  two integer such that  $Q \geq 0 \wedge P > 2\sqrt{Q}$  or  $Q \geq 0 \wedge P < -2\sqrt{Q}$ , then the associated Lucas sequence  $U_n(P, Q)$  is 1-fold log-concave .*

using theorem 11 and the lemma 5, it is easy to check that

**Corollary 12.** *Let  $P, Q$  two integer such that  $Q \geq 0 \wedge P > 2\sqrt{Q}$  or  $Q \geq 0 \wedge P < -2\sqrt{Q}$ . Then the Lucas sequence  $U_n(P, Q)$  is  $\infty$ -log concave .*

*Proof.* Under the hypothesis we have that

$$b_n = \mathcal{L}(U_n) = \frac{Q^{n-1}P^2}{D}$$

that is a sequence of the form  $k_n a^n$  and by lemma 5  $U_n(P, Q)$  is  $\infty$ -log concave .  $\square$

At the end using the corollary 12 we can check that:

- $U_n(1, -1)$  is the Fibonacci sequence that is not 1-fold log-concave and so neither  $\infty$ -log concave .
- $U_n(2, -1)$  is the sequence of Pell numbers that is not 1-fold log-concave and so neither  $\infty$ -log concave .
- $U_n(1, -2)$  is the sequence of Jacobsthal numbers that is not 1-fold log-concave and so neither  $\infty$ -log concave .
- $U_n(3, 2)$  is the sequence of Mersenne numbers that is  $\infty$ -log concave .

## 4 Conclusion

In these notes we have studied the log-operator applied to a generic Lucas sequence of first kind  $U_n$ . We have shown that for initial parameter  $Q \geq 0, P \geq 2Q$  or  $Q \geq 0, P \leq -2Q$ , the associate Lucas sequence of first kind is  $\infty$ -log concave . As result we find that Fibonacci, Pell and Jacobsthal sequences are not  $\infty$ -log concave but the Mersenne numbers sequence is  $\infty$ -log concave . There is a natural question that arise from these results. As shown the key fact, that a sequence is recurrent, allow the sequence to be expressed in a more treatable way before applying the log-operator . It would be interesting giving a generic linear recurrent sequence that satisfy a generic characteristics equation of order  $k$ , to find sufficient condition on the coefficient of the equation to be sure that the sequence is 1-fold log-concave and after this fix which conditions leads to a  $\infty$ -log concave sequence. Formalizing a little, giving a

recurrent sequence define as  $a_n = k_1 a_{n-1} + k_2 a_{n-2} + \dots + k_m a_{n-m}$  that has a characteristic equation  $a_n - k_1 a_{n-1} - k_2 a_{n-2} - \dots - k_m a_{n-m} = 0$  is there is a sufficient condition on the  $k_1, k_2, \dots, k_m$  integer coefficient such that  $a_n$  is 1-fold log-concave and  $\infty$ -log concave . This question would be subject of further study.

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