An asymptotic theory of cloning of classical state families

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Abstract

Cloning, or approximate cloning, is one of basic operations in quantum information processing. In this paper, we deal with cloning of classical states, or probability distribution in asymptotic setting. We study the quality of the approximate (n, rn)-clone, with n being very large and r being constant.

The result turns out to be $\|\mathrm{N}(0, r\mathbf{1}) - \mathrm{N}(0, \mathbf{1})\|_1$, where $\mathrm{N}(\mu, \Sigma)$ is the Gaussian distribution with mean μ and covariance Σ. Notablly, this value does not depend on the the family of porbability distributions to be cloned.

The key of the argument is use of local asymptotic normality: If the curve $\theta \to P_\theta$ is sufficiently smooth in θ, then, the behavior of $P_{\theta'}^{\otimes n}$ where $\theta' - \theta = o\left(\sqrt{1/n}\right)$, is approximated by Gaussian shift. Using this, we reduce the general case to Gaussian shift model.

1 Introduction

Cloning, or approximate cloning, is one of basic operations in quantum information processing. It is related to optimal eavesdropping of quantum key distribution, and also to optimal estimation efficiency. The quality of the approximate clone, thus, has been studied extensively [8].

In this paper, we deal with cloning of classical states, or probability distributions in asymptotic setting. The study of (approximate) cloning of classical states had started even earlier than the proposal of no-cloning theorem, to give a measure of information contained in additional observations : they studied the quality of approximate $(n + r)$-copies made from n-copies ($(n, n + r)$-clone, hereafter), with n being very large and r being constant [2][5][6].

This paper explores another direction: we study the quality of the approximate (n, rn)-clone with n being very large and r being constant, since its extension to quantum system seems to be easier.

In the argument, we make full use of local asymptotic normality: If the curve $\theta \to P_\theta$ is sufficiently smooth, then, the family $\left\{P_{\theta+hn^{-1/2}}^{\otimes n}\right\}_{h\in\mathbb{R}^m}$ is approximated by Gaussian shift $\left\{\mathrm{N}\left(h, J_\theta^{-1}\right)\right\}_{h\in\mathbb{R}^m}$, where J_θ is the Fisher information matrix of $\{P_\theta\}_{\theta\in\Theta}$ at θ. Using this fact , we reduce the general case to the Gaussian shift model . More concretely, letting $D_{r,\Sigma}$ be the loss of optimal $(1,r)$-cloner of the Gaussian shift $\{\mathrm{N}(h,\Sigma)\}_{h\in\mathbb{R}^m}$, we show

$$\sup_{a\geq 0} \varliminf_{n\to\infty} \inf_{\Lambda} \sup_{\|\theta'-\theta\|\leq an^{-1/2}} \left\|\Lambda\left(P_{\theta'}^n\right) - P_{\theta'}^{rn}\right\|_1 \geq D_{r,J_\theta^{-1}}, \tag{1}$$

where Λ moves over all the Markov maps. In other words, the loss of the optimal asymptotic (n,nr)-cloner is asymptotically lower-bounded by $D_{r,J_\theta^{-1}}$, at each $\theta\in\Theta$. This loss turns out to be achievable: we construct a cloner $\Lambda_{\delta,\varepsilon}^{n,r}$ with

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty} \left\|\Lambda_{\delta,\varepsilon}^{n,r}\left(P_\theta^n\right) - P_\theta^{rn}\right\|_1 = D_{r,J_\theta^{-1}}. \tag{2}$$

Also, we find more explicit expression of $D_{r,\Sigma}$, which is

$$D_{r,\Sigma} = \left\|\mathrm{N}(0, r\mathbf{1}) - \mathrm{N}(0,\mathbf{1})\right\|_1.$$

It is notable that $D_{r,\Sigma}$ does not depend on Σ. This means that $D_{r,J_\theta^{-1}}$, the smallest asymptotic loss of (n,rn)-cloner, does not depend on the family $\{P_\theta\}_{\theta\in\Theta}$ to be cloned.

Since there is a (finite dimensional) quantum version of local asymptotic normality, this argument may be extended to finite dimensional quantum case.

The paper is organized as follows. First, we give the optimal approximate cloners for Gaussian shift families, and find some properties of them. Second, we state local asymptotic normality of smooth family of probability distributions, and its uniform version. Finally, we give asymptotic analysis of approximate (n,rn)-clone of smooth families. The paper is concluded by discussions.

2 Gaussian shift family

2.1 Reduction of cloning to amplification

The contents of the subsection is well-known, but added for the sake of completion.

Consider the Gaussian shift family $\{\mathrm{N}(h,\Sigma)\}_{h\in\mathbb{R}^m}$. Then, the problem of optimum approximate $(1,r)$-clone, or finding a map achieving

$$C_{r,\Sigma} := \inf_{\Lambda:\text{Markov}} \sup_{h\in\mathbb{R}^m} \left\|\Lambda\left(\mathrm{N}(h,\Sigma)\right) - \mathrm{N}(h,\Sigma)^{\otimes r}\right\|_1$$

is equivalent to finding the optimum r-amplifier, or a Markov map achieving

$$D_{r,\Sigma} := \inf_{\Lambda:\text{Markov}} \sup_{h\in\mathbb{R}^m} \left\|\Lambda\left(\mathrm{N}(h,\Sigma)\right) - \mathrm{N}\left(\sqrt{r}h,\Sigma\right)\right\|_1. \tag{3}$$

To see this, let $X_1, \cdots, X_r \sim \mathrm{N}(h, \Sigma)$, and

$$X_i' = \sum_{j=1}^{r} O_{i,j} X_j$$

where O is an orthogonal matrix with $O_{1,1} = O_{1,2} = \cdots = O_{1,r} = \frac{1}{\sqrt{r}}$. Then, $X_1' \sim \mathrm{N}(\sqrt{r}h, \Sigma)$ and $X_2', \cdots, X_r' \sim \mathrm{N}(0, \Sigma)$.

Therefore, if

$$\sup_{h \in \mathbb{R}^m} \left\| \Lambda_0 \left(\mathrm{N}(h, \Sigma) \right) - \mathrm{N}(h, \Sigma)^{\otimes r} \right\|_1 = C_{r,\Sigma} + \varepsilon,$$

then

$$\sup_{h \in \mathbb{R}^m} \left\| \Psi \circ \Lambda_0 \left(\mathrm{N}(h, \Sigma) \right) - \mathrm{N}\left(\sqrt{r}h, \Sigma \right) \right\|_1 \leq C_{r,\Sigma} + \varepsilon, ,$$

where Ψ is a Markov map corresponding to application of O followed by restriction to the first variable. Hence,

$$C_{r,\Sigma} \geq D_{r,\Sigma}.$$

On the other hand, let Ψ' be a Markov map corresponding to the map

$$X \to (X, X_2', \cdots, X_r') \,,\ X_2', \cdots, X_r' \sim \mathrm{N}(0, \Sigma)$$

followed by O^{-1}. If

$$\sup_{h \in \mathbb{R}^m} \left\| \Lambda_1 \left(\mathrm{N}(h, \Sigma) \right) - \mathrm{N}\left(\sqrt{r}h, \Sigma \right) \right\|_1 = D_{r,\Sigma} + \varepsilon,$$

then

$$\sup_{h \in \mathbb{R}^m} \left\| \Psi' \circ \Lambda_1 \left(\mathrm{N}(h, \Sigma) \right) - \mathrm{N}(h, \Sigma)^{\otimes r} \right\|_1 \leq D_{r,\Sigma} + \varepsilon.$$

Hence,

$$C_{r,\Sigma} \leq D_{r,\Sigma}.$$

After all, we have $C_{r,\Sigma} = D_{r,\Sigma}$.

2.2 Amplifier for Gaussian shift families

In this subsection, we find the optimum r-amplifier ($r \geq 1$) and its loss $D_{r,\Sigma} = C_{r,\Sigma}$ for the Gaussian shift family $\{\mathrm{N}(h, \Sigma)\}_{h \in \mathbb{R}^m}$.

Observe first that

$$\Psi_{\sqrt{r}} \left(\mathrm{N}(h, \Sigma) \right) = \mathrm{N}\left(\sqrt{r}h, r\Sigma \right), \ \Psi_{r^{-1/2}} \left(\mathrm{N}\left(\sqrt{r}h, r\Sigma \right) \right) = \mathrm{N}(h, \Sigma).$$

where Ψ_a describes the Markov map corresponding to scale change. Hence,

$$\begin{aligned} D_{r,\Sigma} &\leq \inf_{\Lambda} \sup_{h \in \mathbb{R}^m} \left\| \Lambda \circ \Psi_{\sqrt{r}} \left(\mathrm{N}(h, \Sigma) \right) - \mathrm{N}\left(\sqrt{r}h, \Sigma \right) \right\|_1 \\ &= \inf_{\Lambda} \sup_{h \in \mathbb{R}^m} \left\| \Lambda \left(\mathrm{N}\left(\sqrt{r}h, r\Sigma \right) \right) - \mathrm{N}\left(\sqrt{r}h, \Sigma \right) \right\|_1 \end{aligned}$$

and

$$\begin{aligned} D_{r,\Sigma} &= \inf_{\Lambda} \sup_{h\in\mathbb{R}^m} \left\| \Lambda \circ \Psi_{r^{-1/2}} \left(\mathrm{N}\left(\sqrt{r}h, r\Sigma\right)\right) - \mathrm{N}\left(\sqrt{r}h, \Sigma\right)\right\|_1 \\ &\geq \inf_{\Lambda} \sup_{h\in\mathbb{R}^m} \left\| \Lambda \left(\mathrm{N}\left(\sqrt{r}h, r\Sigma\right)\right) - \mathrm{N}\left(\sqrt{r}h, \Sigma\right)\right\|_1 . \end{aligned}$$

Thus,

$$D_{r,\Sigma} = \inf_{\Lambda} \sup_{h\in\mathbb{R}^m} \left\| \Lambda \left(\mathrm{N}\left(\sqrt{r}h, r\Sigma\right)\right) - \mathrm{N}\left(\sqrt{r}h, \Sigma\right)\right\|_1 , \tag{4}$$

and Λ^r_{amp} achieving (3) and Λ^r achieving (4) are, if exists, related by

$$\Lambda^r_{\mathrm{amp}} = \Lambda^r \circ \Psi_{\sqrt{r}}.$$

Now, we refer to Theorem 3 of [11]: applying to our case, it says that

$$\begin{aligned} D_{r,\Sigma} &= \sup_{f:\sup_x |f(x)|\leq 1} \left\{ \int f(y)\, p_{0,\Sigma}(y)\, \mathrm{d}y - \sup_x \int f\left(y+\sqrt{r}x\right) p_{0,r\Sigma}(y)\, \mathrm{d}y \right\} \\ &= \sup_{f:\sup_x |f(x)|\leq 1} \inf_x \left\{ \int f(y) \left\{ p_{0,\Sigma}(y) - p_{x,r\Sigma}(y)\right\} \mathrm{d}y \right\} \\ &= \sup_{f:\sup_x |f(x)|\leq 1} \inf_x \left\{ \int f(y) \left\{ p_{0,\mathbf{1}}(y) - p_{x,r\mathbf{1}}(y)\right\} \mathrm{d}y \right\}, \end{aligned} \tag{5}$$

where $p_{x,\Sigma}$ is probability density function of $\mathrm{N}(x,\Sigma)$.

The right most side of (5) is evaluated as follows. Observe

$$\begin{aligned} D_{r,\Sigma} &\leq \inf_x \left\| p_{0,\mathbf{1}} - p_{x,r\mathbf{1}},\right\|_1 \\ &= \left\| p_{0,\mathbf{1}} - p_{0,r\mathbf{1}}\right\|_1 . \end{aligned} \tag{6}$$

(The proof of (6) is in the appendix.) On the other hand, define $B_r := \{y\,; p_{\mathbf{1}}(y) \geq p_{r\mathbf{1}}(y)\}$, which is a ball centered at origin. Then,

$$\begin{aligned} D_{r,\Sigma} &\geq \inf_x \left\{ \int \left(2I_{B_r}(y) - 1\right) \left\{ p_{0,\mathbf{1}}(y) - p_{x,r\mathbf{1}}(y)\right\} \mathrm{d}y \right\} \\ &= \int \left(2I_{B_r}(y) - 1\right) p_{0,\mathbf{1}}(y)\, \mathrm{d}y - \sup_x \int \left(2I_{B_r}(y) - 1\right) p_{x,r\mathbf{1}}(y)\, \mathrm{d}y \\ &= \int \left(2I_{B_r}(y) - 1\right) p_{0,\mathbf{1}}(y)\, \mathrm{d}y - \int \left(2I_{B_r}(y) - 1\right) p_{0,r\mathbf{1}}(y)\, \mathrm{d}y \\ &= \left\| p_{0,\mathbf{1}} - p_{0,r\mathbf{1}}\right\|_1 . \end{aligned} \tag{7}$$

(N.B. in the case of $r<1$, $\sup_x \int \left(2I_{B_r}(y) - 1\right) p_{x,r\mathbf{1}}(y)\, \mathrm{d}y$ is achieved as $\|x\| \to \infty$.)

After all, we have, if $r \geq 1$,

$$D_{r,\Sigma} = \left\| p_{0,\mathbf{1}} - p_{0,r\mathbf{1}}\right\|_1 = \left\| \mathrm{N}(0,\mathbf{1}) - \mathrm{N}(0,r\mathbf{1})\right\|_1 . \tag{8}$$

Obviously, corresponding Λ^r is the identity map. Thus,

$$\Lambda^r_{\mathrm{amp}} = \Psi_{\sqrt{r}}. \tag{9}$$

2.3 Bounded shifts

Define

$$D_{r,\Sigma,a} := \inf_{\Lambda} \sup_{\|h\| \le a} \left\| \Lambda\left(\mathrm{N}\left(h,\Sigma\right)\right) - \mathrm{N}\left(h, \sqrt{r}\Sigma\right) \right\|_1 .$$

Then, if $a' \geq a$ and

$$\sup_{\|h\| \le a'} \left\| \Lambda\left(\mathrm{N}\left(h,\Sigma\right)\right) - \mathrm{N}\left(h, \sqrt{r}\Sigma\right) \right\|_1 = D_{r,\Sigma,a'} + \varepsilon,$$

then

$$\sup_{\|h\| \le a} \left\| \Lambda\left(\mathrm{N}\left(h,\Sigma\right)\right) - \mathrm{N}\left(h, \sqrt{r}\Sigma\right) \right\|_1 \leq D_{r,\Sigma,a'} + \varepsilon.$$

Since $\varepsilon > 0$ can be arbitrary, therefore,

$$D_{r,\Sigma,a} \leq D_{r,\Sigma,a'}.$$

Hence, since $D_{r,\Sigma,a} \leq 2$, $\lim_{a\to\infty} D_{r,\Sigma,a}$ exists.

Lemma 1

$$\lim_{a\to\infty} D_{r,\Sigma,a} = D_{r,\Sigma}.$$

Proof. Let us consider a decision problem taking values in $[-1,1]^{\mathbb{R}^m}$. Let ρ be a Markov kernel from $\mathbb{R}^m$ to $[-1,1]^{\mathbb{R}^m}$, and $F\left(h,\cdot\right) : \mathbb{R}^m \to [-1,1]$ be a lower continuous function. Also, we define $\mathcal{P}_a$ be the set of probability distributions over $\{x; \|x\| \leq a\}$ with finite support. Then, we define, for $\pi \in \mathcal{P}_a$,

$$R_\pi\left(\Sigma, F, \rho\right) := \int\int F\left(h,a\right) \rho\left(\mathrm{d}a, x\right) p_{h,r\Sigma}\left(x\right) \mathrm{d}x \mathrm{d}\pi\left(h\right).$$

Due to the randomization criteria (Theorem 1.10 of [9], Theorem 55.9 of [10]),

$$D_{r,\Sigma} = \sup_{\pi\in\mathcal{P}_\infty} \sup_F \left\{ \inf_\rho R_\pi\left(r\Sigma, F, \rho\right) - \inf_\rho R_\pi\left(\Sigma, F, \rho\right) \right\},$$

and

$$D_{r,\Sigma,a} = \sup_{\pi\in\mathcal{P}_a} \sup_F \left\{ \inf_\rho R_\pi\left(r\Sigma, F, \rho\right) - \inf_\rho R_\pi\left(\Sigma, F, \rho\right) \right\}.$$

Comparing the right hand sides of them,

$$D_{r,\Sigma} = \sup_{a\geq 0} D_{r,\Sigma,a} = \lim_{a\to\infty} D_{r,\Sigma,a}.$$

■

3 Smooth family

3.1 Settings and description of results

Consider a family of probability distributions $\{P_\theta; \theta \in \Theta\}$ over the measurable space $(\Omega, \mathcal{X})$, where Θ is an open region in $\mathbb{R}^m$, Ω is a Polish space (a separable completely metrizable topological space, e.g. $\mathbb{R}^k$, $\mathbb{Z}^k$, etc.) , and P_θ has density p_θ with respect to a measure μ. Define $P_\theta^n := P_\theta^{\otimes n}$, $p_\theta^n := p_\theta^{\otimes n}$, $\Omega^n := \Omega^{\times n}$, $\mathcal{X}^n := \mathcal{X}^{\otimes n}$, and

$$Z_{\theta,h}^n := \frac{p_{\theta+hn^{-1/2}}^n}{p_\theta^n}.$$

Also, E_θ and E_θ^n refers to expectation with respect to P_θ or P_θ^n, respectively. $W_{\theta,\kappa}$ $(\kappa = 1, \cdots, n)$ are the random variables with $W_{\theta,\kappa} \sim P_\theta$, and define $W_\theta^n := (W_{\theta,1}, \cdots, W_{\theta,n})$, which obeys P_θ^n.

Under this setting, we investigate the quality of (n, nr)-clone of $\{P_\theta; \theta \in \Theta\}$. More specifically, we show

$$\sup_{a\geq 0} \varliminf_{n\to\infty} \inf_{\Lambda^{n,r}:\mathrm{Markov}} \sup_{\|\theta'-\theta\|\leq an^{-1/2}} \left\|\Lambda^{n,r}\left(P_{\theta'}^n\right) - P_{\theta'}^{rn}\right\|_1 \geq D_{r,J_\theta^{-1},\infty} = D_{r,J_\theta^{-1}}, \tag{10}$$

which means the loss of the optimal asymptotic (n, nr)-cloner is lower bounded by $D_{r,J_\theta^{-1}}$, at each $\theta \in \Theta$. Also, we show this loss is achievable: we construct a cloner $\Lambda_{\delta,\varepsilon}^{n,r}$ with

$$\lim_{\delta\downarrow 0} \lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \left\|\Lambda_{\delta,\varepsilon}^{n,r}\left(P_\theta^n\right) - P_\theta^{rn}\right\|_1 = D_{r,J_\theta^{-1}}. \tag{11}$$

3.2 Local asymptotic normality and its uniform version

The map $\theta \to p_\theta$ is *differentiable in quadratic mean*, if

$$\lim_{h\to 0} \frac{1}{\|h\|^2} \int \left(\sqrt{p_{\theta+h}} - \sqrt{p_\theta} - \frac{h^T}{2}\ell_\theta\sqrt{p_\theta}\right)^2 \mathrm{d}\mu = 0, \ \forall\theta \in \Theta. \tag{12}$$

If the map $\theta \to \ell_\theta$ is continuous, we say $\theta \to p_\theta$ is *continuously differentiable in quadratic mean*.

We define, with $\omega^n \in \Omega^n$ and $\omega_\kappa \in \Omega$,

$$\ell_\theta^n\left(\omega^n\right) := \frac{1}{\sqrt{n}} \sum_{\kappa=1}^n \ell_\theta\left(\omega_\kappa\right),$$

$J_\theta := [\mathrm{E}_\theta \ell_{\theta,i}\ell_{\theta,j}]$, and

$$Z_{\theta,h}\left(x\right) := \exp\left(h^T x - \frac{1}{2}h^T J_\theta h\right).$$

The following Lemma is recasting of Remark 1 of [3] and Theorem 7.2 of [12].

Lemma 2 *Suppose Θ is an open region in $\mathbb{R}^m$ and $\theta \to p_\theta$ is continuously differentiable in quadratic mean. Then, $\mathrm{E}_\theta \ell_\theta = 0$, and, for any compact set $K \subset \Theta$ and $K' \subset \mathbb{R}^m$,*

$$\lim_{n\to\infty} \sup_{h\in K'} \sup_{\theta\in K} P_\theta^n \left\{ \left| \ln Z_{\theta,h}^n - \ln Z_{\theta,h}\left(\ell_\theta^n\right) \right| > \varepsilon \right\} = 0, \forall \varepsilon > 0.$$

The following Lemma is recasting of Remark 1 of Theorem 7.2 of [12].

Lemma 3 *Suppose Θ is an open region in $\mathbb{R}^m$ and $\theta \to p_\theta$ is differentiable in quadratic mean. Then, $\mathrm{E}_\theta \ell_\theta = 0$, and, for any compact set $K' \subset \mathbb{R}^m$,*

$$\lim_{n\to\infty} \sup_{h\in K'} P_\theta^n \left\{ \left| \ln Z_{\theta,h}^n - \ln Z_{\theta,h}\left(\ell_\theta^n\right) \right| > \varepsilon \right\} = 0, \forall \varepsilon > 0.$$

In addition, we assume the following conditions:

$$J_\theta \text{ is continuous in } \theta, \tag{13}$$

$$\inf_{\theta\in\Theta} \alpha_\theta > 0, \tag{14}$$

where α_θ is the minimum eigenvalue of J_θ, and

$$\sup_{\theta\in K} \mathrm{E}_\theta e^{h^T \ell_\theta} < \infty, \ \ \forall h \in \mathbb{R}^m, \text{ for any compact set } K \subset \Theta. \tag{15}$$

Observe that

$$\mathrm{E}_\theta \left(h^T \ell_\theta\right)^{2k} \le (2k)! \left\|h\right\|^{2k} \mathrm{E}_\theta \cosh\left(e^T \ell_\theta\right),$$
$$\mathrm{E}_\theta \left|h^T \ell_\theta\right|^{2k-1} \le \left\|h\right\|^{2k-1} \left\{1 + \mathrm{E}_\theta \left(e^T \ell_\theta\right)^{2k}\right\} \le \left\|h\right\|^{2k-1} \left\{1 + (2k)! \mathrm{E}_\theta \cosh\left(e^T \ell_\theta\right)\right\},$$

where $e = h/\left\|h\right\|$, implying

$$\sup_{\theta\in K} \mathrm{E}_\theta \left|h^T \ell_\theta\right|^k < \infty, \ \ \forall h \in \mathbb{R}^m, \text{ for any compact set } K \subset \Theta. \tag{16}$$

Also, one can show that, for any compact set $K \subset \Theta$ and $K' \subset \mathbb{R}^m$,

$$\sup_{n \ge n_{K,K'}} \mathrm{E}_\theta e^{h^T \ell_\theta^n} \le e^{h^T J_\theta h}, \forall \theta \in \Theta, \forall h \in K', \exists n_{K,K'} \tag{17}$$

The proof of (17) is as follows. Observe, since $\mathrm{E}_\theta \ell_\theta = 0$ due to Lemma 3,

$$\begin{aligned} \mathrm{E}_\theta^n e^{h^T \ell_\theta^n} &= \left(\mathrm{E}_\theta e^{-\frac{h^T}{\sqrt{n}} \ell_\theta} \right)^n \\ &= \left(1 + \frac{h^T J_\theta h}{2n} + f_{\mathrm{rem}}\left(\theta, h, n\right) \right)^n, \end{aligned} \tag{18}$$

where

$$
\begin{aligned}
&|f_{\mathrm{rem}}(\theta,h,n)| \\
&\le \sum_{k=3}^{\infty}\frac{1}{k!}\left(\frac{\|h\|}{\sqrt{n}}\right)^k \mathrm{E}_\theta\left|e^T\ell_\theta\right|^k \\
&\le \frac{1}{2}\sum_{k\ge 3,k:\mathrm{even}}^{\infty}\left(\frac{\|h\|}{\sqrt{n}}\right)^k \mathrm{E}_\theta\cosh e^T\ell_\theta \\
&+ \sum_{k\ge 3,k:\mathrm{odd}}^{\infty}\left\{\frac{1}{k!}\left(\frac{\|h\|}{\sqrt{n}}\right)^k + \frac{(k+1)!}{k!}\left(\frac{\|h\|}{\sqrt{n}}\right)^k \mathrm{E}_\theta\cosh e^T\ell_\theta\right\} \\
&\le \sum_{k\ge 3}^{\infty}(k+1)\left(\frac{\|h\|}{\sqrt{n}}\right)^k\left\{\mathrm{E}_\theta\cosh e^T\ell_\theta+1\right\} \\
&= \left(\frac{\|h\|}{\sqrt{n}}\right)^3\frac{4-5\|h\|/\sqrt{n}}{1-\|h\|/\sqrt{n}}\left(\mathrm{E}_\theta\cosh e^T\ell_\theta+1\right). \qquad (19)
\end{aligned}
$$

Therefore, for each compact set $K\subset\Theta$ and $K'\subset\mathbb{R}^m$, there is $n_{K,K'}$ such that

$$
\mathrm{E}_\theta^n e^{h^T\ell_\theta^n} \le \left(1+\frac{h^TJ_\theta h}{n}\right)^n \le e^{h^TJ_\theta h},\ \forall n\ge n_{K,K'}.
$$

Hence, we have (17).

Also, we use the following identity :

$$
\lim_{a\to\infty}\sup_{n\ge n_{K,K'}}\sup_{h\in K'}\sup_{\theta\in K}\mathrm{E}_\theta^n\left[e^{h^T\ell_\theta^n}:e^{h^T\ell_\theta^n}\ge a\right]=0, \qquad (20)
$$

which is proved as follows.

$$
\begin{aligned}
&\lim_{a\to\infty}\sup_{n\ge n_{K,K'}}\sup_{h\in K'}\sup_{\theta\in K}\mathrm{E}_\theta^n\left[e^{h^T\ell_\theta^n}:e^{h^T\ell_\theta^n}\ge a\right] \\
&\le \lim_{a\to\infty}\sup_{n\ge n_{K,K'}}\sup_{h\in K'}\sup_{\theta\in K}\sqrt{\mathrm{E}_\theta^n\left[e^{2h^T\ell_\theta^n}\right]P_\theta^n\left\{e^{h^T\ell_\theta^n}\ge a\right\}} \\
&\le \lim_{a\to\infty}\sup_{n\ge n_{K,K'}}\sup_{h\in K'}\sup_{\theta\in K}e^{2h^TJ_\theta h}\sqrt{P_\theta^n\left\{e^{h^T\ell_\theta^n}\ge a\right\}} \\
&\le \lim_{a\to\infty}\sup_{n\ge n_{K,K'}}\sup_{h\in K'}\sup_{\theta\in K}e^{2h^TJ_\theta h}\sqrt{\frac{1}{a}\mathrm{E}_\theta^n e^{h^T\ell_\theta^n}} \\
&\le \lim_{a\to\infty}\frac{1}{a}\sup_{h\in K'}\sup_{\theta\in K}e^{\frac{5}{2}h^TJ_\theta h}=0.
\end{aligned}
$$

Lemma 4 *Suppose random variables $X_{n,t}$, and $Y_{n,t}$, $n\ge 1$, $t\in T$, taking values in $\mathbb{R}^k$, satisfies*

$$
\lim_{n\to\infty}\sup_{t\in T}\Pr\left\{\|X_{n,t}-Y_{n,t}\|>\varepsilon\right\}=0, \qquad (21)
$$

Let f be a continuously differentiable function from $\mathbb{R}^k$ to $\mathbb{R}$ such that,

$$\sup_{x:f(x)\leq a} \|\nabla_x f(x)\| < \infty, \tag{22}$$

and

$$\lim_{a\to\infty} \lim_{n\to\infty} \sup_{t\in T} \mathrm{E}\left[f(X_{n,t}) : f(X_{n,t}) > a\right] < \infty, \tag{23}$$

$$\lim_{a\to\infty} \lim_{n\to\infty} \sup_{t\in T} \mathrm{E}\left[f(X_{n,t}) : f(Y_{n,t}) > a\right] < \infty. \tag{24}$$

Then,

$$\lim_{n\to\infty} \sup_{t\in T} |\mathrm{E}f(X_{n,t}) - \mathrm{E}f(Y_{n,t})| = 0$$

Proof. Define

$$f^a(x) := f(x) \wedge a.$$

Then,

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{t\in T} |\mathrm{E}f(X_{n,t}) - \mathrm{E}f(Y_{n,t})| \\
&\leq \lim_{n\to\infty} \sup_{t\in T} |\mathrm{E}f^a(X_{n,t}) - \mathrm{E}f^a(X_t)| \\
&+ \lim_{n\to\infty} \sup_{t\in T} |\mathrm{E}\left[f(X_{n,t}) : f(X_{n,t}) > a\right]| \\
&+ \lim_{n\to\infty} \sup_{t\in T} |\mathrm{E}\left[f(Y_{n,t}) : f(Y_{n,t}) > a\right]|.
\end{aligned} \tag{25}$$

The first term of the right hand side is evaluated as follows.

$$|f^a(X_{n,t}) - f^a(Y_{n,t})| \leq C \|X_{n,t} - Y_{n,t}\|, \forall t \in T$$

where

$$C \leq \sup_{x:f(x)\leq a} \|\nabla_x f(x)\| < \infty.$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \sup_{t\in T} |\mathrm{E}f^a(X_{n,t}) - \mathrm{E}f^a(Y_{n,t})| &\leq \varepsilon + a \times \lim_{n\to\infty} \sup_{t\in T} \Pr\{|f^a(X_{n,t}) - f^a(Y_{n,t})| > \varepsilon\} \\
&= \varepsilon + a \times \lim_{n\to\infty} \sup_{t\in T} \Pr\{C|X_{n,t} - Y_{n,t}| > \varepsilon\} \\
&= \varepsilon.
\end{aligned}$$

This can be made arbitrarily small, since $\varepsilon > 0$ is arbitrary.

The second and the third terms of the right hand side of (25) can be made arbitrarily small by taking a large. Hence, we have the assertion. ■

Lemma 5 *Suppose $\theta \to p_\theta$ is continuously differentiable in quadratic mean, and (15) holds. Then, for any compact set $K \subset \Theta$ and $K' \subset \mathbb{R}^m$,*

$$\lim_{n\to\infty} \sup_{h\in K'} \sup_{\theta\in K} \mathrm{E}_\theta^n \left| Z_{\theta,h}^n - Z_{\theta,h}(\ell_\theta^n) \right| = 0.$$

Proof. We apply Lemma 4, with $f(x) := e^x$, $t = (\theta, h)$, $X_{n,t} := \ln Z^n_{\theta,h}$ and $Y_{n,t} := \ln Z_{\theta,h}(\ell^n_\theta) = h^T \ell^n_\theta - \frac{1}{2} h^T J_\theta h$. Then, the premises (21) and (22) are obviously satisfied.

Due to (20), (23) is satisfied:

$$\lim_{n\to\infty} \sup_{h\in K'} \sup_{\theta\in K} \mathrm{E}^n_\theta \left[Z_{\theta,h}(\ell^n_\theta) : Z_{\theta,h}(\ell^n_\theta) \geq a \right] \to 0, a \to \infty.$$

(24) is proved as follows. Let $g_a(x)$ be a continuous function on $\mathbb{R}_+$ such that $g_a(x) = 1$ for $x \leq a-1$ and $g_a(x) = 0$ for $x \geq a$. Then,

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{h\in K'} \sup_{\theta\in K} \mathrm{E}^n_\theta \left[Z^n_{\theta,h} : Z^n_{\theta,h} \geq a \right] \\
&\leq \lim_{n\to\infty} \sup_{h\in K'} \sup_{\theta\in K} \left\{ 1 - \mathrm{E}^n_\theta \left[Z^n_{\theta,h} g_a\left(Z^n_{\theta,h} \right) \right] \right\} \\
&\leq \lim_{n\to\infty} \sup_{h\in K'} \sup_{\theta\in K} \left\{ 1 - \mathrm{E}^n_\theta \left[Z_{\theta,h}(\ell^n_\theta)\ g_a\left(Z_{\theta,h}(\ell^n_\theta) \right) \right] \right\} \\
&+ \sup_x \left\{ (x+\varepsilon) g_a(x+\varepsilon) - x g_a(x) \right\} \\
&+ a \lim_{n\to\infty} P^n_\theta \left\{ \left| Z^n_{\theta,h} - Z_{\theta,h}(\ell^n_\theta) \right| > \varepsilon \right\} \\
&\leq \lim_{n\to\infty} \sup_{h\in K'} \sup_{\theta\in K} \mathrm{E}^n_\theta \left[Z_{\theta,h}(\ell^n_\theta) : Z_{\theta,h}(\ell^n_\theta) \geq a-1 \right] \\
&+ \sup_x \left\{ (x+\varepsilon) g_a(x+\varepsilon) - x g_a(x) \right\} \\
&+ a \lim_{n\to\infty} P^n_\theta \left\{ \left| Z^n_{\theta,h} - Z_{\theta,h}(\ell^n_\theta) \right| > \varepsilon \right\} \\
&= \lim_{n\to\infty} \sup_{h\in K'} \sup_{\theta\in K} \mathrm{E}^n_\theta \left[Z_{\theta,h}(\ell^n_\theta) : Z_{\theta,h}(\ell^n_\theta) \geq a-1 \right] \\
&+ \sup_x \left\{ (x+\varepsilon) g_a(x+\varepsilon) - x g_a(x) \right\}.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and g_a is continuous,

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{h\in K'} \sup_{\theta\in K} \mathrm{E}^n_\theta \left[Z^n_{\theta,h} : Z^n_{\theta,h} \geq a \right] \\
&\leq \lim_{n\to\infty} \sup_{h\in K'} \sup_{\theta\in K} \mathrm{E}^n_\theta \left[Z_{\theta,h}(\ell^n_\theta) : Z_{\theta,h}(\ell^n_\theta) \geq a-1 \right] \\
&\to 0,\ a \to \infty.
\end{aligned}$$

So, we have the assertion. ■

Lemma 6 *Suppose $\theta \to p_\theta$ is differentiable in quadratic mean, and (15) holds. Then, for any compact set $K' \subset \mathbb{R}^m$,*

$$\lim_{n\to\infty} \sup_{h\in K'} \mathrm{E}^n_\theta \left| Z^n_{\theta,h} - Z_{\theta,h}(\ell^n_\theta) \right| = 0.$$

Proof. The proof is almost parallel with the one of Lemma 5, except that Lemma 3 is used instead of Lemma 2 and that $\sup_{\theta\in K}$ at each step is removed. ■

Below, we denote by $C(h, r)$ the closed m-dimensional hypercube which is centered at $h \in \mathbb{R}^m$, parallel to the coordinate axis, and of edge length $2r$. Also, $2^{-k}\mathbb{Z}^m$ is an element of $\mathbb{R}^m$ whose coordinates are integer multiple of 2^{-k}.

Lemma 7 *Let Θ_0 be a countable subset of Θ and c_n be a positive constant. Then, to every ordered correction $(i_1, i_2, \cdots, i_k)$ associate a Borel set $S_{(i_1,i_2,\cdots,i_k)}$ in $\mathbb{R}^m$ such that*

$$S_{(i_1,i_2,\cdots,i_k)} \cap S_{(j_1,j_2,\cdots,j_k)} = \emptyset, \quad (i_1, i_2, \cdots, i_k) \neq (j_1, j_2, \cdots, j_k), \tag{26}$$

$$\textit{Diameter of } S_{(i_1,i_2,\cdots,i_k)} < \sqrt{m}2^{-k+2} \quad (k \geq 1), \tag{27}$$

$$P_\theta^n \left\{ \ell_\theta^n \in \partial S_{(i_1,i_2,\cdots,i_k)} \right\} = 0, \ \forall \theta \in \Theta_0, \forall n \tag{28}$$

$$\bigcup_{j=1}^{N_n} S_j \supset [-c_n, c_n]^m, N_n := (2c_n + 1)^m, \tag{29}$$

$$\bigcup_{j=1}^{\infty} S_j = \mathbb{R}^m, \tag{30}$$

$$\bigcup_{j=1}^{5^m} S_{(i_1,\cdots,i_{k-1},j)} = S_{(i_1,\cdots,i_{k-1})}. \tag{31}$$

Proof. Since Θ_0 is a countable set, we can choose an r_0 with $c_n < r_0 < c_n + \frac{1}{2}$ and

$$P_\theta^n \left\{ \ell_\theta^n \in C(0, r_0) \right\} = 0, \forall \theta \in \Theta_0, \forall n. \tag{32}$$

Also, we can choose r_k with $2^{-k} < r_k < 2^{-k+1}$ and

$$P_\theta^n \left\{ \ell_\theta^n \in C(h, r_k) \right\} = 0, \forall \theta \in \Theta_0, \forall n, \tag{33}$$

for all $h \in 2^{-k+1}\mathbb{Z}^m$.

First, we compose S_1, S_2, $\cdots$. Define $h_j \in \mathbb{Z}^m$ so that $h_1, \cdots, h_{N_n} \in [-c_n, c_n]^m$, and that $\{h_j ; j = 1, 2, \cdots\} = \mathbb{Z}^m$. Then, recursively define, for $j = 1, \cdots, N_n$,

$$S_1 := C(0, r_0) \cap C(h_1, r_1), \ \ S_j := C(0, r_0) \cap \left\{ C(h_j, r_1) - \bigcup_{i=1}^{j-1} S_i \right\},$$

and, for $j \geq N_n + 1$,

$$S_j := C(h_j, r_1) - \bigcup_{i=1}^{j-1} S_i.$$

Since $\bigcup_{j=1}^{N_n} C\left(h_j, 2^{-1}\right) = C\left(0, c_n + \frac{1}{2}\right)$, we have $\bigcup_{j=1}^{N_n} C(h_j, r_1) \supset C(0, r_0)$. Also,

$$\bigcup_{j=1}^{N_n} S_j = C(0, r_0) \cap \left\{ \bigcup_{j=1}^{N_n} C(h_j, r_1) \right\}.$$

Therefore,

$$\bigcup_{j=1}^{N_n} S_j = C(0, r_0) \supset [-c_n, c_n]^m,$$

indicating (29). Similarly, we have

$$\bigcup_{j=1}^{\infty} S_j = C(0, r_0) \cup \bigcup_{j=N_n+1}^{\infty} S_j = C(0, r_0) \cup \bigcup_{j=N_n+1}^{\infty} C(h_j, r_1)$$
$$\supset [-c_n, c_n]^m \cup \bigcup_{j=N_n+1}^{\infty} C\left(h_j, \frac{1}{2}\right) = \mathbb{R}^m,$$

which is (30).

Next, we compose $S_{(i_1,\cdots,i_k)}$. For each $k \geq 2$, let $h_{i_1,\cdots,i_k}$ $(i_k = 1,\cdots, 5^m)$ be an element of $2^{-k+1}\mathbb{Z}^m$ with $h_{i_1,\cdots,i_k} \in C\left(h_{i_1,\cdots,i_{k-1}}, 2^{-k+2}\right)$. Then, we define, recursively,

$$S_{(i_1,\cdots,i_{k-1},1)} := S_{(i_1,\cdots,i_{k-1})} \cap C\left(h_{i_1,\cdots,i_{k-1},1}, r_k\right),$$
$$S_{(i_1,\cdots,i_k)} := S_{(i_1,\cdots,i_{k-1})} \cap \left\{ C\left(h_{i_1,\cdots,i_{k-1},i_k}, r_k\right) - \bigcup_{j=1}^{i_k-1} S_{(i_1,\cdots,i_{k-1},j)} \right\}.$$

Since

$$\bigcup_{j=1}^{5^m} S_{(i_1,\cdots,i_{k-1},j)} \supset \bigcup_{j=1}^{5^m} C\left(h_{i_1,\cdots,i_{k-1},j}, 2^{-k}\right)$$
$$= C\left(h_{i_1,\cdots,i_{k-1}}, 2^{-k+2} + 2^{-k}\right)$$

and

$$C\left(h_{i_1,\cdots,i_{k-1}}, 2^{-k+2} + 2^{-k}\right) \supset C\left(h_{i_1,\cdots,i_{k-1}}, r_{k-1}\right) \supset S_{(i_1,\cdots,i_{k-1})},$$

we have

$$\bigcup_{j=1}^{5^m} S_{(i_1,\cdots,i_{k-1},j)} \supset S_{(i_1,\cdots,i_{k-1})},$$

which implies (31).

(26) is trivial by composition. (27) is due to

$$\partial S_{(i_1,\cdots,i_{k-1},i_k)} \subset \partial S_{(i_1,\cdots,i_{k-1})} \cup \bigcup_{j=1}^{i_k} \partial C\left(h_{i_1,\cdots,i_{k-1},j}, r_k\right).$$

Hence, by (32) and (33), recursively we have (28). (27) is obvious from that $S_{(i_1,\cdots,i_k)}$ is a subset of $C(h_{i_1,\cdots,i_k}, r_k)$. ■

Lemma 8 *Suppose $\theta \to p_\theta$ is continuously differentiable in quadratic mean, and (15) holds. Also, let Θ_0 be a countable subset of Θ. Then, there are random variables η_θ and η_θ^n $(n \geq 1)$ over $([0,1], \mathcal{B}([0,1] \times \mathbb{R}^m), \nu)$, such that ν is Lebesgue measure,*

$$\mathcal{L}(\eta_\theta|\nu) = \mathrm{N}(0, J_\theta), \quad \mathcal{L}(\eta_\theta^n|\nu) = \mathcal{L}(\ell_\theta^n|P_\theta^n), \tag{34}$$
$$\lim_{n\to\infty} \sup_{\theta\in K\cap\Theta_0} \nu\{\|\eta_\theta^n - \eta_\theta\| \geq \varepsilon\} = 0. \tag{35}$$

Proof. Let $S_{(i_1,\cdots,i_k)}$ be as of Lemma 7, and for each k, and order

$$\{(i_1,\cdots,i_k)\,;i_1\in\mathbb{N},1\leq i_j\leq 5^m\}$$

lexicographically. For $\theta\in\Theta_0$, define intervals $\Delta_\theta^n(i_1,\cdots,i_k)$ of the form $[a,b)$ in $[0,1)$ such that the length of $\Delta_\theta^n(i_1,\cdots,i_k)$ is $P_\theta^n\left\{\ell_\theta^n\in S_{(i_1,\cdots,i_n)}\right\}$, and that, with $(j_1,\cdots,j_k)>(i_1,\cdots,i_k)$, the left end point of $\Delta_\theta^n(j_1,\cdots,j_k)$ lies to the right of $\Delta_\theta^n(i_1,\cdots,i_k)$. Then, we have

$$\bigcup_{i_1\in\mathbb{N},1\leq i_j\leq 5^m}\Delta_\theta^n(i_1,\cdots,i_k)=[0,1).$$

If $P_\theta^n\left\{\ell_\theta^n\in S_{(i_1,\cdots,i_n)}\right\}$ is non-zero for some n, by (28), its interior is non-empty. Thus we may take a point $x_{(i_1,\cdots,i_k)}$ in its interior. For $\varpi\in[0,1]$, define

$$\eta_\theta^{n,k}(\varpi):=x_{(i_1,\cdots,i_k)},\ \varpi\in\Delta_\theta^n(i_1,\cdots,i_k).$$

Then,

$$\left\|\eta_\theta^{n,k}(\varpi)-\eta_\theta^{n,k+k'}(\varpi)\right\|\leq\sqrt{m}2^{-k+2},\tag{36}$$

making the sequence $\left\{\eta_\theta^{n,k}(\varpi)\right\}_{k=1}^\infty$ Cauchy for each ϖ,n, and θ. Hence, $\eta_\theta^n(\varpi):=\lim_{k\to\infty}\eta_\theta^{n,k}(\varpi)$ exists.

Define the intervals $\Delta_\theta(i_1,\cdots,i_k)$ of the form $[a,b)$ in $[0,1)$ such that the length of $\Delta_\theta(i_1,\cdots,i_k)$ is $P_{\mathrm{N}(0,J_\theta)}\left(S_{(i_1,\cdots,i_n)}\right)$, and that, with $(j_1,\cdots,j_k)>(i_1,\cdots,i_k)$, the left end point of $\Delta_\theta(j_1,\cdots,j_k)$ lies to the right of $\Delta_\theta(i_1,\cdots,i_k)$. Also, one can define $\eta_\theta^k(\varpi)$ and $\eta_\theta(\varpi)$ in the parallel manner with $\eta_\theta^{n,k}(\varpi)$ and $\eta_\theta^n(\varpi)$.

Then, by (28) and the multi-dimensional Berry Esseen theorem (Corollary 11.1 of [1]), we have

$$\sup_{\theta\in K\cap\Theta_0}\left|\nu\left(\Delta_\theta^n(i_1,\cdots,i_k)\right)-\nu\left(\Delta_\theta(i_1,\cdots,i_k)\right)\right|\leq\frac{\beta}{\sqrt{n}},$$

where $\beta:=400m^{1/4}\sup_{\theta\in K}\mathrm{E}_\theta\left\|J_\theta^{-1}\ell_\theta\right\|^3$.Therefore,

$$\nu\left(\Delta_\theta^n(i_1,\cdots,i_k)\triangle\Delta_\theta(i_1,\cdots,i_k)\right)\leq\frac{2\beta5^{mk}i_1}{\sqrt{n}}.$$

Also, by Markov's inequality,

$$\sum_{j=N_n+1}^\infty\nu\left(\Delta_\theta^n(j)\right)\leq\frac{\sup_{\theta\in K}\operatorname{tr}J_\theta}{c_n^2},\ \sum_{j=N_n+1}^\infty\nu\left(\Delta_\theta(j)\right)\leq\frac{\sup_{\theta\in K}\operatorname{tr}J_\theta}{c_n^2}$$

Thus,

$$\begin{aligned}&\sup_{\theta\in K\cap\Theta_0}\sum_{i_1\in\mathbb{N},1\leq i_j\leq 5^m}\nu\left(\Delta_\theta^n(i_1,\cdots,i_k)\triangle\Delta_\theta(i_1,\cdots,i_k)\right)\\&\leq\frac{2\sup_{\theta\in K}\operatorname{tr}J_\theta}{c_n^2}+\frac{\beta5^{2mk}\left((2c_n+1)^m+1\right)^2}{\sqrt{n}}.\end{aligned}$$

Here, set

$$k = k_n := \frac{\ln n}{16m \ln 5}, \;\; c_n := n^{\frac{1}{16m}}.$$

Then,

$$\begin{aligned}
&\sup_{\theta \in K \cap \Theta_0} \sum_{i_1 \in \mathbb{N}, 1 \leq i_j \leq 5^m} \nu \left(\Delta_\theta^n \left(i_1, \cdots, i_{k_n} \right) \triangle \Delta_\theta \left(i_1, \cdots, i_{k_n} \right) \right) \\
&= O\left(n^{-\frac{1}{8m}} \right) + O\left(n^{-1/4} \right) \to 0, \, n \to \infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{n \to \infty} \sup_{\theta \in K \cap \Theta_0} \nu \left\{ \eta_\theta^{n, k_n} \left(\varpi \right) \neq \eta_\theta^{k_n} \left(\varpi \right) \right\} \\
&\leq \lim_{n \to \infty} \sup_{\theta \in K} \sum_{i_1 \in \mathbb{N}, 1 \leq i_j \leq 5^m} \nu \left(\Delta_\theta^n \left(i_1, \cdots, i_{k_n} \right) \triangle \Delta_\theta \left(i_1, \cdots, i_{k_n} \right) \right) = 0. \qquad (37)
\end{aligned}$$

Observe, due to (36)

$$\begin{aligned}
\left\| \eta_\theta^n \left(\varpi \right) - \eta_\theta^{n, k_n} \left(\varpi \right) \right\| &= \lim_{k' \to \infty} \left\| \eta_\theta^{n, k'} \left(\varpi \right) - \eta_\theta^{n, k_n} \left(\varpi \right) \right\| \leq \sqrt{m} 2^{-k_n + 2}, \\
\left\| \eta_\theta \left(\varpi \right) - \eta_\theta^{k_n} \left(\varpi \right) \right\| &= \lim_{k' \to \infty} \left\| \eta_\theta^{k'} \left(\varpi \right) - \eta_\theta^{k_n} \left(\varpi \right) \right\| \leq \sqrt{m} 2^{-k_n + 2}.
\end{aligned}$$

Therefore, due to

$$\begin{aligned}
&\left\| \eta_\theta^n \left(\varpi \right) - \eta_\theta \left(\varpi \right) \right\| \\
&\leq \left\| \eta_\theta^n \left(\varpi \right) - \eta_\theta^{n, k_n} \left(\varpi \right) \right\| + \left\| \eta_\theta^{n, k_n} \left(\varpi \right) - \eta_\theta^{k_n} \left(\varpi \right) \right\| + \left\| \eta_\theta^{k_n} \left(\varpi \right) - \eta_\theta \left(\varpi \right) \right\|,
\end{aligned}$$

and (37), we have

$$\begin{aligned}
&\sup_{\theta \in K \cap \Theta_0} \nu \left\{ \left\| \eta_\theta^n - \eta_\theta \right\| \geq \varepsilon \right\} \\
&\leq \sup_{\theta \in K \cap \Theta_0} \nu \left\{ \left\| \eta_\theta^{n, k_n} - \eta_\theta^{k_n} \right\| + 2\sqrt{m} 2^{-k_n + 2} \geq \varepsilon \right\} \\
&\to 0, \, n \to \infty,
\end{aligned}$$

which is (35).

To prove (34), observe that every open set in $\mathbb{R}^m$ can be expressed as a disjoint countable union of $S_{(i_1, \cdots, i_k)}$'s. Therefore, for any open set G, by Fatou's lemma,

$$\underline{\lim}_{k \to \infty} \nu \left\{ \eta_\theta^{n,k} \in G \right\} \geq P_\theta^n \left\{ \ell_\theta^n \in G \right\}.$$

Hence, by Portmanteau theorem (Lemma 2.2 of [12]), $\lim_{k \to \infty} \mathcal{L}\left(\eta_\theta^{n,k} | \nu \right) = \mathcal{L}\left(\ell_\theta^n | P_\theta^n \right)$. Since $\lim_{k \to \infty} \eta_\theta^{n,k} = \eta_\theta^n$ almost surely, we have the second identity of (34). The first identity is proved parallelly. ■

Lemma 9 *Suppose $\theta \to p_\theta$ is continuously differentiable in quadratic mean, and (15) holds. Also, let Θ_0 be a countable subset of Θ. Then, there are probability measure $\tilde{P}_\theta^n$ over a measurable space $(\Omega^n \times \Omega', \mathcal{X}^n \otimes \mathcal{X}')$, where $(\Omega', \mathcal{X}') := (\mathbb{R}^m \times [0,1], \mathcal{B}(\mathbb{R}^m \times [0,1]))$, $n \geq 1$, and random variables λ'^n, $n \geq 1$ over $\left(\Omega^n \times \Omega', \mathcal{X}^n \otimes \mathcal{X}', \tilde{P}_\theta^n\right)$, such that, $\tilde{P}_\theta^n$ is an extension of P_θ^n and*

$$\lambda'^n \sim \mathrm{N}(0, J_\theta), \tag{38}$$

$$\lim_{n\to\infty} \sup_{\theta \in K \cap \Theta_0} \tilde{P}_\theta^n \{\|\ell_\theta^n - \lambda'^n\| \geq \varepsilon\} = 0, \tag{39}$$

for any compact set $K \subset \Theta$.

Proof. The proof much draws upon the second proof of Lemma 2.2 of [9]. Define a kernel $K_\theta^n(x, \mathrm{d}y)$ from $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ to $([0,1], \mathcal{B}([0,1]))$ by the identity

$$\delta_{\eta_\theta^n(y)}(\mathrm{d}x)\,\nu(\mathrm{d}y) = R_\theta^n(\mathrm{d}x)\,K_\theta^n(x, \mathrm{d}y), \tag{40}$$

where δ_y is Dirac measure, ν is the Lebesgue measure, η_θ^n is as of Lemma 8, and $R_\theta^n = \mathcal{L}(\ell_\theta^n | P_\theta^n) = \mathcal{L}(\eta_\theta^n | \nu)$. (Since $[0,1]$ is Polish, such K_θ^n exists, see 342E of [4].) Define, with $\tilde{\omega}^n = (\omega^n, x, y) \in \Omega^n \times \Omega'$,

$$\begin{aligned}
\tilde{P}_\theta^n(\mathrm{d}\tilde{\omega}^n) &:= P_\theta^n(\mathrm{d}\omega^n)\,\delta_{\ell_\theta^n(\omega^n)}(\mathrm{d}x)\,K_\theta^n(x, \mathrm{d}y), \\
\lambda'^n(\tilde{\omega}^n) &:= \eta_\theta(y),
\end{aligned}$$

where η_θ is as of Lemma 8.

Since the restriction of $\tilde{P}_\theta^n$ on $([0,1], \mathcal{B}([0,1]))$ is ν,

$$\mathcal{L}\left(\lambda'^n(\tilde{\omega}^n) | \tilde{P}_\theta^n\right) = \mathcal{L}\left(\eta_\theta(y) | \tilde{P}_\theta^n\right) = \mathcal{L}(\eta_\theta(y) | \nu) = \mathrm{N}(0, J_\theta).$$

Hence, (38) is shown.

By abusing the notation, we denote the extension of $\ell_\theta^n : \Omega^n \to \mathbb{R}^m$ to $\Omega^n \times \Omega' \to \mathbb{R}^m$ also by ℓ_θ^n: in other words,

$$\ell_\theta^n(\omega^n, x, y) := \ell_\theta^n(\omega^n).$$

To verify (39), we show

$$\ell_\theta^n(\tilde{\omega}^n) = \eta_\theta^n(y), \quad \tilde{P}_\theta^n\text{-a.s..} \tag{41}$$

Observe that restriction of $\tilde{P}_\theta^n$ to $(\Omega', \mathcal{X}') = (\mathbb{R}^m \times [0,1], \mathcal{B}(\mathbb{R}^m \times [0,1]))$ is (40). Therefore, we have

$$\begin{aligned}
\tilde{P}_\theta^n(\{\ell_\theta^n(\tilde{\omega}^n) = x\}) &= \int_{\Omega^n} \int_{\mathbb{R}^m} I_{\left\{\ell_\theta^n(\tilde{\omega}^n) = x\right\}} P_\theta^n(\mathrm{d}\omega^n)\,\delta_{\ell_\theta^n(\tilde{\omega}^n)}(\mathrm{d}x) \\
&= \int_{\Omega^n} P_\theta^n(\mathrm{d}\omega^n) \int_{\mathbb{R}^m} I_{\left\{\ell_\theta^n(\tilde{\omega}^n) = x\right\}} \delta_{\ell_\theta^n(\tilde{\omega}^n)}(\mathrm{d}x) \\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}\tilde{P}_\theta^n\left(\{\eta_\theta^n(y)=x\}\right) &= \int_{\mathbb{R}^m}\int_{[0,1]} I_{\left\{\eta_\theta^n(y)=x\right\}}\delta_{\eta_\theta^n(y)}(\mathrm{d}x)\,\nu(\mathrm{d}y)\\ &= \int_{[0,1]}\nu(\mathrm{d}y)\int_{[0,1]} I_{\left\{\eta_\theta^n(y)=x\right\}}\delta_{\eta_\theta^n(y)}(\mathrm{d}x)\\ &= 1.\end{aligned}$$

Thus, (41) is shown.

By (41) and the definition of λ'^n,

$$\begin{aligned}\sup_{\theta\in K\cap\Theta_0}\tilde{P}_\theta^n\left\{\|\ell_\theta^n-\lambda'^{n,}\|\geq\varepsilon\right\} &= \sup_{\theta\in K\cap\Theta_0}\tilde{P}_\theta^n\left\{\|\eta_\theta^n-\eta_\theta\|\geq\varepsilon\right\}\\ &= \sup_{\theta\in K\cap\Theta_0}\nu\left\{\|\eta_\theta^n-\eta_\theta\|\geq\varepsilon\right\}\\ &\to 0,\ n\to\infty.\end{aligned}$$

■

Lemma 10 *Suppose $\theta\to p_\theta$ is differentiable in quadratic mean, and (15) holds. Then, there are probability measure $\tilde{P}_\theta^n$ over a measurable space $(\Omega^n\times\Omega',\mathcal{X}^n\otimes\mathcal{X}')$, where $(\Omega',\mathcal{X}'):=\left(\mathbb{R}^m\times[0,1],\mathcal{B}\left(\mathbb{R}^m\times[0,1]\right)\right)$, $n\geq 1$, and random variables λ'^n, $n\geq 1$ over $\left(\Omega^n\times\Omega',\mathcal{X}^n\otimes\mathcal{X}',\tilde{P}_\theta^n\right)$, such that, $\tilde{P}_\theta^n$ is an extension of P_θ^n and*

$$\lambda'^n\sim\mathrm{N}\left(0,J_\theta\right), \tag{42}$$

$$\lim_{n\to\infty}\tilde{P}_\theta^n\left\{\|\ell_\theta^n-\lambda'^n\|\geq\varepsilon\right\}=0, \tag{43}$$

for any compact set $K\subset\Theta$.

Proof. This is only the combination of Lemma 2.2 of [9] and the central limit theorem. ■

Theorem 11 *Suppose $\theta\to p_\theta$ is continuously differentiable in quadratic mean, and (15) holds. Also, let Θ_0 be a countable subset of Θ. Then, there are probability measures $\tilde{P}_\theta^n$ over measurable spaces $(\Omega^n\times\Omega',\mathcal{X}^n\otimes\mathcal{X}')$, where $(\Omega',\mathcal{X}'):=\left(\mathbb{R}^m\times[0,1],\mathcal{B}\left(\mathbb{R}^m\times[0,1]\right)\right)$, $n\geq 1$, and random variables λ_h^n, $n\geq 1$ over $\left(\Omega^n\times\Omega',\mathcal{X}^n\otimes\mathcal{X}',\tilde{P}_\theta^n\right)$, such that, $\tilde{P}_\theta^n$ is an extension of P_θ^n and*

$$\begin{aligned}&\lim_{n\to\infty}\sup_{h\in K'}\sup_{\theta\in K\cap\Theta_0}\left\|\tilde{P}_{\theta+hn^{-1/2}}^n-Q_{\theta,h}^n\right\|_1=0,\\ &\mathcal{L}\left(\lambda_h^n\right)=\mathrm{N}\left(h,J_\theta^{-1}\right),\\ &Q_{\theta,h}^n(A):=\mathrm{E}^{\lambda^n}R_\theta^n\left(A|\lambda_h^n\right).\end{aligned}$$

Here, K is an arbitrary compact set in Θ, K' is an arbitrary compact set in $\mathbb{R}^m$, and $R_\theta^n(\cdot|\lambda^n)$ is a measure on $(\Omega^n\times\Omega',\mathcal{X}^n\otimes\mathcal{X}')$, which may depend on θ, but is independent of h.

Proof. We use Lemma 4, with $f(x) = e^x$, $t = (\theta, h)$, $X_{n,t} := h^T \lambda'^n - \frac{1}{2} h^T J_\theta h$ and $Y_{n,t} := h^T \ell_\theta^n - \frac{1}{2} h^T J_\theta h$. Obviously, (21) and (22) are satisfied.

Due to (20), we have (23):

$$\lim_{n\to\infty} \sup_{h\in K'} \sup_{\theta\in K\cap\Theta_0} \mathrm{E}_\theta^n \left[Z_{\theta,h}\left(\ell_\theta^n\right) : Z_{\theta,h}\left(\ell_\theta^n\right) > a \right] \to 0, a \to \infty.$$

Due to (38) , we have

$$\begin{aligned} &\lim_{n\to\infty} \sup_{h\in K'} \sup_{\theta\in K\cap\Theta_0} \tilde{\mathrm{E}}_\theta^n \left[Z_{\theta,h}\left(\lambda'^n\right) : Z_{\theta,h}\left(\lambda'^n\right) > a \right] \\ &= \sup_{h\in K'} \sup_{\theta\in K\cap\Theta_0} \mathrm{E}^X \left[Z_{\theta,h}(X) : Z_{\theta,h}(X) > a \right] \to 0, \ a \to \infty, \end{aligned}$$

where $\mathcal{L}(X) = \mathrm{N}\left(0, J_\theta^{-1}\right)$. Thus (24). Therefore, we have

$$\lim_{n\to\infty} \sup_{h\in K'} \sup_{\theta\in K\cap\Theta_0} \tilde{\mathrm{E}}_\theta^n \left| Z_{\theta,h}\left(\ell_\theta^n\right) - Z_{\theta,h}\left(\lambda'^n\right) \right| = 0.$$

Therefore, combining Lemma 5,

$$\begin{aligned} &\sup_{h\in K'} \sup_{\theta\in K} \tilde{\mathrm{E}}_\theta^n \left| Z_{\theta,h}^n - Z_{\theta,h}\left(\lambda'^n\right) \right| \\ &\le \sup_{h\in K'} \sup_{\theta\in K} \mathrm{E}_\theta^n \left| Z_{\theta,h}^n - Z_{\theta,h}\left(\ell_\theta^n\right) \right| + \sup_{h\in K'} \sup_{\theta\in K} \tilde{\mathrm{E}}_\theta^n \left| Z_{\theta,h}\left(\ell_\theta^n\right) - Z_{\theta,h}\left(\lambda'^n\right) \right| \\ &\to 0. \end{aligned}$$

Let $\tilde{W}_\theta^n$ be the random variable with $\mathcal{L}\left(\tilde{W}_\theta^n\right) = \tilde{P}_\theta^n$. Then

$$\begin{aligned} &\tilde{\mathrm{E}}_\theta^n Z_{\theta,h}\left(\lambda'^n\right) I_A\left(\tilde{W}_\theta^n\right) \\ &= \int \tilde{\mathrm{E}}_\theta^n \left[Z_{\theta,h}\left(\lambda'^n\right) I_A\left(\tilde{W}_\theta^n\right) | \lambda'^n = x \right] \frac{e^{-\frac{1}{2} x^T J_\theta^{-1} x} \mathrm{d}x}{(2\pi)^{m/2} \left(\det J_\theta\right)^{1/2}} \\ &= \int \tilde{\mathrm{E}}_\theta^n \left[I_A\left(\tilde{W}_\theta^n\right) | \lambda'^n = x \right] \exp\left\{ -\frac{1}{2} x^T J_\theta^{-1} x + h^T x - \frac{1}{2} h^T J_\theta h \right\} \frac{\mathrm{d}x}{(2\pi)^{m/2} \left(\det J_\theta\right)^{1/2}} \\ &= \int \tilde{\mathrm{E}}_\theta^n \left[I_A\left(\tilde{W}_\theta^n\right) | \lambda'^n = J_\theta x \right] \exp\left\{ -\frac{1}{2} (x-h)^T J_\theta (x-h) \right\} \frac{\left(\det J_\theta\right)^{1/2}}{(2\pi)^{m/2}} \mathrm{d}x. \end{aligned}$$

Since $\Omega^n \times \Omega'$ is Polish, there is a nice version of $R_\theta^n(A|x) := \tilde{\mathrm{E}}_\theta^n \left[I_A\left(\tilde{W}_\theta^n\right) | \lambda'^n = J_\theta x \right]$ which is a probability measure in $\mathcal{X}^n \times \mathcal{X}'$ for every $\not{x} \in \mathbb{R}^m$ (see, for example, 342E of [4]). By definition,

$$\tilde{\mathrm{E}}_\theta Z_{\theta,h}\left(\lambda'^n\right) I_A\left(\tilde{W}_\theta^n\right) = \mathrm{E}^{\lambda_h^n} R_\theta^n\left(A|\lambda_h^n\right).$$

Therefore, we have the assertion:

$$\begin{aligned}
&\sup_{h\in K'}\sup_{\theta\in K}\left|\tilde{P}^n_{\theta+n^{-1/2}h}\left(A\right)-Q^n_{\theta,h}\left(A\right)\right| \\
&=\sup_{h\in K'}\sup_{\theta\in K}\left|\tilde{P}^n_{\theta+n^{-1/2}h}\left(A\right)-\mathrm{E}^{\lambda^n}R^n_\theta\left(A|\lambda^n\right)\right| \\
&=\sup_{h\in K'}\sup_{\theta\in K}\left|\tilde{\mathrm{E}}^n_\theta Z^n_{\theta,h}I_A\left(\tilde{W}^n_\theta\right)-\tilde{\mathrm{E}}^n_\theta Z_{\theta,h}\left(\lambda'^n\right)I_A\left(\tilde{W}^n_\theta\right)\right| \\
&\leq\sup_{h\in K'}\sup_{\theta\in K}\tilde{\mathrm{E}}^n_\theta\left|Z^n_{\theta,h}-Z_{\theta,h}\left(\lambda'^n\right)\right|\to 0,\ n\to\infty.
\end{aligned}$$

■

Theorem 12 *Suppose $\theta\to p_\theta$ is differentiable in quadratic mean, and (15) holds. Then, there are probability measures $\tilde{P}^n_\theta$ over a measurable spaces $(\Omega^n\times\Omega',\mathcal{X}^n\otimes\mathcal{X}')$, where $(\Omega',\mathcal{X}'):=(\mathbb{R}^m\times[0,1],\mathcal{B}(\mathbb{R}^m\times[0,1]))$, $n\geq 1$, and random variables λ^n_h, $n\geq 1$ over $\left(\Omega^n\times\Omega',\mathcal{X}^n\otimes\mathcal{X}',\tilde{P}^n_\theta\right)$, such that, $\tilde{P}^n_\theta$ is an extension of P^n_θ and*

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{h\in K'}\left\|\tilde{P}^n_{\theta+hn^{-1/2}}-Q^n_{\theta,h}\right\|_1=0, \\
&\lambda^n_h\sim\mathrm{N}\left(h,J^{-1}_\theta\right), \\
&Q^n_{\theta,h}\left(A\right):=\mathrm{E}^{\lambda^n}R^n_\theta\left(A|\lambda^n_h\right).
\end{aligned}$$

Here, K' is an arbitrary compact set in $\mathbb{R}^m$, and $R^n_\theta\left(\cdot|\lambda^n\right)$ is a measure on $(\Omega^n\times\Omega',\mathcal{X}^n\otimes\mathcal{X}')$, which may depend on θ, but is independent of h.

Proof. The proof is parallel with the one of Theorem 11, except that Lemma 10 is used instead of Lemma 9, and that $\sup_{\theta\in\Theta}$ at each step is removed. ■

3.3 Asymptotic cloner using the optimal amplifier for the Gaussian shift family

Hereafter, we assume the existence of a sequence $\left\{\hat{\theta}^n\right\}$ of estimate of θ, such that

$$\lim_{a\to\infty}\lim_{n\to\infty}P^n_\theta\left\{\sqrt{n}\left\|\hat{\theta}^n-\theta\right\|\geq a\right\}=0. \tag{44}$$

Without loss of generality, one can suppose that

$$\hat{\theta}^n\in n^{-1/2}\mathbb{Z}. \tag{45}$$

If (45) is not satisfied, we redefine $\hat{\theta}^n$ as the closest element of $n^{-1/2}\mathbb{Z}$ to $\hat{\theta}^n$. Obviously, newly defined $\hat{\theta}^n$ satisfies (44). Therefore, letting

$$\Theta_0:=\left\{k^{-1/2}\cdot l;k\in\mathbb{N},l\in\mathbb{Z}\right\},$$

we can suppose

$$\hat{\theta}^n\in\Theta_0$$

and the cardinality of Θ_0 is countable.

We consider the following procedure of (n, rn)-cloner $\Lambda^{n,r}_{\delta,\varepsilon}$. For the composition, we use the optimal r-amplifier $\Lambda^r_{\mathrm{amp}} = \Psi_{\sqrt{r}}$ of the Gaussian shift family $\left\{\mathrm{N}\left(h, J_\theta^{-1}\right)\right\}_{h\in\mathbb{R}^m}$. Also, define

$$L^{n,\varepsilon}_\theta\left(\omega^n\right) := J_\theta^{-1}\ell^n_\theta\left(\omega^n\right) + Y_\varepsilon,$$

where $\mathcal{L}\left(Y_\varepsilon\right) = \mathrm{N}\left(0, \varepsilon\mathbf{1}\right)$.

Then, for a given $\varepsilon > 0$ and $0 < \delta < 1$, we construct a cloner $\Lambda^{n,r}_{\delta,\varepsilon}$ as follows.

(I) Estimate θ using n_1-data, $(n_1 := \delta n)$ and let $n_2 := (1-\delta)\, n$.

(II) Apply $\Lambda^{\sqrt{r/1-\delta}}_{\mathrm{amp}}$ to $\mathcal{L}\left(L^{n_2,\varepsilon}_{\hat\theta^{n_1}}|P^{n_2}_\theta\right)$. Denote the resulting random variable by $\tilde X^n_{\hat\theta^{n_1}}$.

(III) Generate (ω^{rn}, ω') according to $R^{rn}_{\hat\theta^{n_1}}\left(\cdot|\tilde X^n_{\hat\theta^{n_1}}\right)$, and discard ω'.

The output probability distribution is

$$\Lambda^{n,r}_{\delta,\varepsilon}\left(P^n_\theta\right) = \mathrm{E}^{\hat\theta^{n_1}}\mathrm{E}^{\tilde X^n_{\hat\theta^{n_1}}} R^{rn}_{\hat\theta^{n_1}}\left(A\times\Omega'|\tilde X^n_{\hat\theta^{n_1}}\right).$$

We will show this is asymptotically optimal.

Lemma 13 *Suppose $\theta \to p_\theta$ is continuously differentiable in quadratic mean, and (15) holds. Moreover, suppose (13) is satisfied. Then, for any compact set $K' \subset \mathbb{R}^m$,*

$$\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\left\|\mathcal{L}\left(L^{n,\varepsilon}_{\theta_{n,h}}|P^n_{\theta_{n,h}}\right) - \mathrm{N}\left(0, J_\theta^{-1}\right)\right\|_1 = 0,$$

where $\theta_{n,h} := \theta + n^{-1/2}h$.

Proof. Define h_n so that

$$\left\|\mathcal{L}\left(L^{n,\varepsilon}_{\theta_{n,h_n}}|P^n_{\theta_{n,h_n}}\right) - \mathrm{N}\left(0, J_\theta^{-1}\right)\right\|_1 \geq \sup_{h\in K'}\left\|\mathcal{L}\left(L^{n,\varepsilon}_{\theta_{n,h}}|P^n_{\theta_{n,h}}\right) - \mathrm{N}\left(0, J_\theta^{-1}\right)\right\|_1 - \varepsilon'$$

holds, and let $\theta^n := \theta_{n,h_n}$. Then, $\lim_{n\to\infty}\theta^n = \theta$.

Denote by $\phi_{\theta'}$ the characteristic function of the distribution of $J^{-1}_{\theta'}\ell_{\theta'}\left(W_{\theta,\kappa}\right)$. Then, the density of $\mathcal{L}\left(L^{n,\varepsilon}_{\theta^n}|P^n_{\theta^n}\right)$ with respect to Lebesgue measure is

$$\frac{1}{2\pi}\int\left\{\phi_{\theta^n}\left(\frac{t}{\sqrt{n}}\right)\right\}^n e^{-\frac{1}{2}\varepsilon\|t\|^2}e^{-\sqrt{-1}t\cdot x}\mathrm{d}t.$$

Observe

$$\int\left|\left\{\phi_{\theta^n}\left(\frac{t}{\sqrt{n}}\right)\right\}^n e^{-\frac{1}{2}\varepsilon\|t\|^2}e^{-\sqrt{-1}t\cdot x}\right|\mathrm{d}t \leq \int e^{-\varepsilon\|t\|^2}\mathrm{d}t < \infty.$$

Hence, by Lebesgue's dominated convergence theorem, we have, with $f_{\rm rem}$ being as of (18),

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{2\pi}\int\left\{\phi_{\theta^n}\left(\frac{t}{\sqrt{n}}\right)\right\}^n e^{-\frac{1}{2}\varepsilon\|t\|^2}e^{-\sqrt{-1}t\cdot x}\mathrm{d}t\\
&=\frac{1}{2\pi}\int\lim_{n\to\infty}\left\{\phi_{\theta^n}\left(\frac{t}{\sqrt{n}}\right)\right\}^n e^{-\frac{1}{2}\varepsilon\|t\|^2}e^{-\sqrt{-1}t\cdot x}\mathrm{d}t,\\
&=\frac{1}{2\pi}\int\lim_{n\to\infty}\left\{1-\frac{1}{2n}\left(t^TJ_{\theta^n}^{-1}t\right)+f_{\rm rem}\left(\theta,\sqrt{-1}t,n\right)\right\}^n e^{-\frac{1}{2}\varepsilon\|t\|^2}e^{-\sqrt{-1}t\cdot x}\mathrm{d}t\\
&=\frac{1}{2\pi}\int\exp\left\{-\frac{1}{2}t^T\left(J_\theta^{-1}+\varepsilon^2\mathbf{1}\right)t\right\}e^{-\sqrt{-1}t\cdot x}\mathrm{d}t\ \text{ a.e.}
\end{aligned}$$

Here, in the third line, we used Lemma 3 to show that the first order term of the Taylor expansion ($=\mathrm{E}_\theta^n\ell_\theta^n$) vanishes. Also, to obtain the fourth line, we used the inequality (19).

Therefore, the density of $\mathcal{L}\left(L_{\theta^n}^{n,\varepsilon}|P_{\theta^n}^n\right)$ converges to the one of $\mathrm{N}\left(0,J_\theta^{-1}+\varepsilon\right)$, as $n\to\infty$. Therefore, By Schefe's lemma, we have

$$\lim_{n\to\infty}\left\|\mathcal{L}\left(L_{\theta^n}^{n,\varepsilon}|P_{\theta^n}^n\right)-\mathrm{N}\left(0,J_\theta^{-1}+\varepsilon\right)\right\|_1=0.$$

Therefore,

$$\begin{aligned}
&\lim_{\varepsilon\downarrow0}\lim_{n\to\infty}\left\|\mathcal{L}\left(L_{\theta^n}^{n,\varepsilon}|P_{\theta^n}^n\right)-\mathrm{N}\left(0,J_\theta^{-1}\right)\right\|_1\\
&\le\lim_{\varepsilon\downarrow0}\lim_{n\to\infty}\left\|\mathcal{L}\left(L_{\theta^n}^{n,\varepsilon}|P_{\theta^n}^n\right)-\mathrm{N}\left(0,J_\theta^{-1}+\varepsilon\right)\right\|_1+\lim_{\varepsilon\downarrow0}\left\|\mathrm{N}\left(0,J_\theta^{-1}\right)-\mathrm{N}\left(0,J_\theta^{-1}+\varepsilon\right)\right\|_1\\
&=0.
\end{aligned}$$

Therefore,

$$\lim_{\varepsilon\downarrow0}\lim_{n\to\infty}\sup_{h\in K'}\left\|\mathcal{L}\left(L_{\theta_{n,h}}^{n,\varepsilon}|P_{\theta_{n,h}}^n\right)-\mathrm{N}\left(0,J_\theta^{-1}\right)\right\|_1\le\varepsilon'.$$

Since $\varepsilon'>0$ is arbitrary, we have the assertion. ■

Lemma 14 *Suppose $\theta\to p_\theta$ is continuously differentiable in quadratic mean, and (15) holds. Moreover, we suppose (13) and (14) hold. Then, for any compact set $K'\in\mathbb{R}^m$, we have*

$$\lim_{\varepsilon\downarrow0}\lim_{n\to\infty}\sup_{h\in K'}\left\|\mathcal{L}\left(L_{\theta_{n,h}}^{n,\varepsilon}|P_\theta^n\right)-\mathrm{N}\left(-h,J_\theta^{-1}\right)\right\|_1=0,$$

where $\theta_{n,h}:=\theta+n^{-1/2}h$

Proof. Observe, for any measurable function f with $\sup_{x\in\mathbb{R}^m}|f(x)|\le1$,

$$\mathrm{E}^{Y_\varepsilon}\mathrm{E}_\theta^n\left[f\left(L_{\theta_{n,h}}^{n,\varepsilon}\right)\right]=\mathrm{E}^{Y_\varepsilon}\mathrm{E}_{\theta_{n,h}}^n\left[f\left(L_{\theta_{n,h}}^{n,\varepsilon}\right)Z_{\theta_{n,h},-h}^n\right],$$

Observe also, due to Lemma 5, with K_θ being a compact subset of Θ containing θ and K' being an arbitrary compact subset of $\mathbb{R}^m$,

$$\begin{aligned}
&\sup_{h\in K'}\left|\mathrm{E}^{Y_\varepsilon}\mathrm{E}^n_{\theta_{n,h}}\left[f\left(L^{n,\varepsilon}_{\theta_{n,h}}\right)\left(Z^n_{\theta_{n,h},-h}-Z_{\theta_{n,h},-h}\left(\ell^n_{\theta_{n,h}}\right)\right)\right]\right|\\
&\le \sup_{h\in K'}\mathrm{E}^n_{\theta_{n,h}}\left[\left|Z^n_{\theta_{n,h},-h}-Z_{\theta_{n,h},-h}\left(\ell^n_{\theta_{n,h}}\right)\right|\right]\\
&\le \sup_{h\in K'}\sup_{\theta'\in K_\theta}\mathrm{E}^n_{\theta'}\left[\left|Z^n_{\theta',-h}-Z_{\theta',-h}\left(\ell^n_{\theta'}\right)\right|\right]\\
&\to 0,\ n\to\infty. \qquad (46)
\end{aligned}$$

Therefore, we have to evaluate

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{h\in K'}\mathrm{E}^{Y_\varepsilon}\mathrm{E}^n_{\theta_{n,h}}\left[f\left(L^{n,\varepsilon}_{\theta_{n,h}}\right)Z_{\theta_{n,h},-h}\left(\ell^n_{\theta_{n,h}}\right)\right]\\
&=\lim_{n\to\infty}\sup_{h\in K'}\mathrm{E}^{Y_\varepsilon}\mathrm{E}^n_{\theta_{n,h}}\left[f\left(L^{n,\varepsilon}_{\theta_{n,h}}\right)Z_{\theta_{n,h},-h}\left(J_{\theta_{n,h}}L^{n,\varepsilon}_{\theta_{n,h}}\right)e^{h^TJ_{\theta_{n,h}}Y_\varepsilon}\right]\\
&=\lim_{n\to\infty}\sup_{h\in K'}\left\{E_1+E_2+E_3\right\},
\end{aligned}$$

where

$$\begin{aligned}
E_1&:=\mathrm{E}^{Y_\varepsilon}\mathrm{E}^n_{\theta_{n,h}}\left[I_{\left\{\left\|L^{n,\varepsilon}_{\theta_{n,h}}\right\|\le a\,,\,\|Y_\varepsilon\|\le\varepsilon^{1/4}\right\}}f\left(L^{n,\varepsilon}_{\theta_{n,h}}\right)Z_{\theta_{n,h},-h}\left(J_{\theta_{n,h}}L^{n,\varepsilon}_{\theta_{n,h}}\right)e^{h^TJ_{\theta_{n,h}}Y_\varepsilon}\right],\\
E_2&:=\mathrm{E}^{Y_\varepsilon}\mathrm{E}^n_{\theta_{n,h}}\left[I_{\left\{\,\|Y_\varepsilon\|>\varepsilon^{1/4}\right\}}f\left(L^{n,\varepsilon}_{\theta_{n,h}}\right)Z_{\theta_{n,h},-h}\left(J_{\theta_{n,h}}L^{n,\varepsilon}_{\theta_{n,h}}\right)e^{h^TJ_{\theta_{n,h}}Y_\varepsilon}\right],\\
E_3&:=\mathrm{E}^{Y_\varepsilon}\mathrm{E}^n_{\theta_{n,h}}\left[I_{\left\{\left\|L^{n,\varepsilon}_{\theta_{n,h}}\right\|>a\,,\,\|Y_\varepsilon\|\le\varepsilon^{1/4}\right\}}f\left(L^{n,\varepsilon}_{\theta_{n,h}}\right)Z_{\theta_{n,h},-h}\left(J_{\theta_{n,h}}L^{n,\varepsilon}_{\theta_{n,h}}\right)e^{h^TJ_{\theta_{n,h}}Y_\varepsilon}\right].
\end{aligned}$$

The first term of the right most side of E_1 is evaluated as follows.

$$E_1=\mathrm{E}^{Y_\varepsilon}\mathrm{E}^n_{\theta_{n,h}}\left[I_{\left\{\left\|L^{n,\varepsilon}_{\theta_{n,h}}\right\|\le a\right\}}f\left(L^{n,\varepsilon}_{\theta_{n,h}}\right)Z_{\theta_{n,h},-h}\left(J_{\theta_{n,h}}L^{n,\varepsilon}_{\theta_{n,h}}\right)\mathrm{E}\left[I_{\left\{|Y_\varepsilon|\le\varepsilon^{1/4}\right\}}e^{h^TJ_{\theta_{n,h}}Y_\varepsilon}\middle|L^{n,\varepsilon}_{\theta_{n,h}}\right]\right], \qquad (47)$$

whose second factor can be evaluated as

$$\begin{aligned}
&\lim_{\varepsilon\downarrow0}\lim_{n\to\infty}\sup_{h\in K'}\left|\mathrm{E}\left[I_{\left\{\|Y_\varepsilon\|\le\varepsilon^{1/2}\right\}}e^{h^TJ_{\theta_{n,h}}Y}\middle|L^{n,\varepsilon}_{\theta_{n,h}}\right]-1\right|\\
&\le\lim_{\varepsilon\downarrow0}\lim_{n\to\infty}\sup_{h\in K'}e^{\|h\|\left\|J_{\theta_{n,h}}\right\|\varepsilon^{1/4}}=0 \qquad (48)
\end{aligned}$$

To evaluate the first factor of E_1, or

$$E_{1,1}:=\mathrm{E}^{Y_\varepsilon}\mathrm{E}^n_{\theta_{n,h}}\left[I_{\left\{\left\|L^{n,\varepsilon}_{\theta_{n,h}}\right\|\le a\right\}}f\left(L^{n,\varepsilon}_{\theta_{n,h}}\right)Z_{\theta_{n,h},-h}\left(J_{\theta_{n,h}}L^{n,\varepsilon}_{\theta_{n,h}}\right)\right],$$

observe

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{h\in K'}\left|E_{1,1}-\mathrm{E}^{Y_\varepsilon}\mathrm{E}^n_{\theta_{n,h}}\left[I_{\left\{\left\|L^{n,\varepsilon}_{\theta_{n,h}}\right\|\le a\right\}}f\left(L^n_{\theta_{n,h}}\right)Z_{\theta,-h}\left(J_\theta L^{n,\varepsilon}_{\theta_{n,h}}\right)\right]\right|\\
&\le\lim_{n\to\infty}\sup_{h\in K'}\sup_{\|L\|\le a}\left|Z_{\theta_{n,h},-h}\left(J_{\theta_{n,h}}L\right)-Z_{\theta,-h}\left(J_\theta L\right)\right|\\
&=\lim_{n\to\infty}\sup_{h\in K'}\sup_{\|L\|\le a}\left|\exp\left\{-h^TJ_{\theta_{n,h}}L\right\}e^{\frac{1}{2}h^TJ_{\theta_{n,h}}h}-\exp\left\{-h^TJ_\theta L\right\}e^{\frac{1}{2}h^TJ_\theta h}\right|\\
&=0.
\end{aligned}$$

Therefore, letting X_h be a random variable with $\mathcal{L}\left(X_h\right)=\mathrm{N}\left(h,J_\theta^{-1}\right)$,

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\left|E_{1,1}-\mathrm{E}^{X_{-h}}\left[f\left(X_{-h}\right);\|X_{-h}\|\le a\right]\right|\\
&=\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\left|\begin{array}{c}\mathrm{E}^{Y_\varepsilon}\mathrm{E}^n_{\theta_{n,h}}\left[f\left(L^n_{\theta_{n,h}}\right)Z_{\theta,-h}\left(J_\theta L^{n,\varepsilon}_{\theta_{n,h}}\right):\left\|L^{n,\varepsilon}_{\theta_{n,h}}\right\|\le a\right]\\ -\mathrm{E}^{X_0}\left[f\left(X_0\right)Z_{\theta,-h}\left(J_\theta X_0\right):\|X_0\|\le a\right]\end{array}\right|\\
&\le\sup_{h\in K'}e^{\|h\|\|J_\theta\|a}e^{-\frac{1}{2}h^TJ_\theta h}\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\left\|\mathcal{L}\left(L^{n,\varepsilon}_{\theta_{n,h}}|P^n_{\theta_{n,h}}\right)-\mathrm{N}\left(0,J_\theta^{-1}\right)\right\|_1\\
&=0,
\end{aligned}\tag{49}$$

where the last identity is due to Lemma 13.

Therefore, by (48) and (49),

$$\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\left|E_1-\mathrm{E}\left[f\left(X_{-h}\right);\|X_{-h}\|\le a\right]\right|=0.\tag{50}$$

On the other hand, by (17), $E_2$the second term of the right most side of (46) is evaluated as

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}E_2\le\lim_{n\to\infty}\sup_{h\in K'}\mathrm{E}^{Y_\varepsilon}\mathrm{E}^n_{\theta_{n,h}}\left[Z_{\theta_{n,h},-h}\left(\ell^n_{\theta_{n,h}}\right):\|Y_\varepsilon\|>\varepsilon^{1/4}\right]\\
&=\lim_{\varepsilon\downarrow 0}\Pr\left\{\|Y_\varepsilon\|>\varepsilon^{1/4}\right\}\cdot\lim_{n\to\infty}\sup_{h\in K'}e^{h^TJ_{\theta_{n,h}}h}e^{-\frac{1}{2}h^TJ_{\theta_{n,h}}h}\\
&=0.
\end{aligned}\tag{51}$$

Also, E_3 is evaluated as, by (17),

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'} E_3 \\
&\le \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'} \mathrm{E}^{Y_\varepsilon}\mathrm{E}_{\theta_{n,h}}\left[Z_{\theta_{n,h},-h}\left(\ell^n_{\theta_{n,h}}\right) : \left\|L^{n,\varepsilon}_{\theta_{n,h}}\right\| > a\,,\ \|Y_\varepsilon\| \le \varepsilon^{1/4}\right] \\
&\le \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'} \mathrm{E}^n_{\theta_{n,h}}\left[I_{\left\{\left\|J^{-1}_{\theta_{n,h}}\ell^n_{\theta_{n,h}}\right\| > a-\varepsilon^{1/4}\right\}} Z_{\theta_{n,h},-h}\left(\ell^n_{\theta_{n,h}}\right)\right] \\
&\le \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sqrt{\sup_{h\in K'} P^n_{\theta_{n,h}}\left\{\left\|\ell^n_{\theta_{n,h}}\right\| > \alpha_{\theta_{n,h}}\left(a-\varepsilon^{1/4}\right)\right\}\mathrm{E}^n_{\theta_{n,h}}\left[Z_{\theta_{n,h},-h}\left(\ell^n_{\theta_{n,h}}\right)\right]^2} \\
&\le \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sqrt{\sup_{h\in K'} P^n_{\theta_{n,h}}\left\{\left\|\ell^{n_2}_{\theta_{n,h}}\right\| > \alpha_{\theta_{n,h}}\left(a-\varepsilon^{1/4}\right)\right\} e^{2h^T J_{\theta_{n,h}}h} e^{-h^T J_{\theta_{n,h}}h}} \\
&\le \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sqrt{\sup_{h\in K'} \frac{1}{\left\{\alpha_{\theta_{n,h}}\left(a-\varepsilon^{1/4}\right)\right\}^2}\mathrm{E}^n_{\theta_{n,h}}\left\|\ell^n_{\theta_{n,h}}\right\|^2 e^{h^T J_{\theta_{n,h}}h}} \\
&\le \sqrt{\frac{\operatorname{tr} J_\theta}{\alpha^2_\theta a^2}\sup_{h\in K'} e^{h^T J_\theta h}}, \qquad (52)
\end{aligned}$$

where α_θ is the minimal eigenvalue of J_θ.

After all, combining (46), (50), (51) and (52), we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\left|\mathrm{E}^{Y_\varepsilon}\mathrm{E}^n_\theta\left[f\left(L^{n,\varepsilon}_{\theta_{n,h}}\right)\right] - \mathrm{E}\left[f\left(X_{-h}\right)\right]\right| \\
&\le \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\left|\mathrm{E}^{Y_\varepsilon}\mathrm{E}^n_{\theta_{n,h}}\left[f\left(L^{n,\varepsilon}_{\theta_{n,h}}\right)Z_{\theta_{n,h},-h}\left(\ell^n_{\theta_{n,h}}\right)\right] - \mathrm{E}\left[f\left(X_{-h}\right)\right]\right| \\
&= \sqrt{\frac{\operatorname{tr} J_\theta}{\alpha^2_\theta a^2}\sup_{h\in K'} e^{h^T J_\theta h}} + \sup_{h\in K'}\Pr\left\{\|X_{-h}\| > a\right\}.
\end{aligned}$$

Since a is arbitrary, letting $a\to\infty$, we have

$$\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\left\|\mathcal{L}\left(L^{n,\varepsilon}_{\theta_{n,h}}|P^n_\theta\right) - \mathrm{N}\left(-h, J^{-1}_\theta\right)\right\|_1 = 0.$$

■

Theorem 15 *Suppose $\theta\to p_\theta$ is continuously differentiable in quadratic mean, and (15) holds. Moreover, we suppose (13), (14), and (44) hold. Then, we have (11).*

Proof. Observe, for any compact set $K' \subset \mathbb{R}^m$,

$$\begin{aligned}
&\left\| \Lambda^{n,r}_{\delta,\varepsilon}\left(P^n_\theta\right) - P^{rn}_\theta \right\|_1 . \\
&\leq \left\| \mathrm{E}^{\hat{\theta}^{n_1}} \mathrm{E}^{\tilde{X}^n_{\hat{\theta}^{n_1}}} R^{rn}_{\hat{\theta}^{n_1}}\left(\cdot|\tilde{X}^n_{\hat{\theta}^{n_1}}\right) - \tilde{P}^{rn}_\theta \right\|_1 \\
&\leq \mathrm{E}^{\hat{\theta}^{n_1}} \left\| \mathrm{E}^{\tilde{X}^n_{\hat{\theta}^{n_1}}} R^{rn}_{\hat{\theta}^{n_1}}\left(\cdot|\tilde{X}^n_{\hat{\theta}^{n_1}}\right) - \tilde{P}^{rn}_\theta \right\|_1 \\
&\leq \mathrm{E}^{\hat{\theta}^{n_1}} \left[\left\| \mathrm{E}^{\tilde{X}^n_{\hat{\theta}^{n_1}}} R^{rn}_{\hat{\theta}^{n_1}}\left(\cdot|\tilde{X}^n_{\hat{\theta}^{n_1}}\right) - \tilde{P}^{rn}_\theta \right\|_1 : \sqrt{n_2}\left(\hat{\theta}^{n_1} - \theta\right) \in K' \right] \\
&+ \mathrm{E}^{\hat{\theta}^{n_1}} \left[\left\| \mathrm{E}^{\tilde{X}^n_{\hat{\theta}^{n_1}}} R^{rn}_{\hat{\theta}^{n_1}}\left(\cdot|\tilde{X}^n_{\hat{\theta}^{n_1}}\right) - \tilde{P}^{rn}_\theta \right\|_1 : \sqrt{n_2}\left(\hat{\theta}^{n_1} - \theta\right) \notin K' \right] \qquad (53)
\end{aligned}$$

The second term of (53) is evaluated as

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \mathrm{E}^{\hat{\theta}^{n_1}} \left\{ \left\| \mathrm{E}^{\tilde{X}^n_{\hat{\theta}^{n_1}}} R^{rn}_{\hat{\theta}^{n_1}}\left(\cdot|\tilde{X}^n_{\hat{\theta}^{n_1}}\right) - \tilde{P}^{rn}_\theta \right\|_1 : \sqrt{rn}\left(\hat{\theta}^{n_1} - \theta\right) \notin K' \right\} \\
&\leq 2 \overline{\lim_{n\to\infty}} P^{n_1}_\theta \left\{ \sqrt{rn}\left(\hat{\theta}^{n_1} - \theta\right) \notin K' \right\},
\end{aligned}$$

whose left hand side becomes arbitrarily small as $K' \uparrow \mathbb{R}^m$, due to (44).

Next, we evaluate the first term of (53).

$$\begin{aligned}
&\mathrm{E}^{\hat{\theta}^{n_1}} \left[\left\| \mathrm{E}^{\tilde{X}^n_{\hat{\theta}^{n_1}}} R^{rn}_{\hat{\theta}^{n_1}}\left(\cdot|\tilde{X}^n_{\hat{\theta}^{n_1}}\right) - \tilde{P}^{rn}_\theta \right\|_1 : \sqrt{n_2}\left(\hat{\theta}^{n_1} - \theta\right) \in K' \right] \\
&\leq \mathrm{E}^{\hat{\theta}^{n_1}} \left[\left\| \mathrm{E}^{\tilde{X}^n_{\hat{\theta}^{n_1}}} R^{rn}_{\hat{\theta}^{n_1}}\left(\cdot|\tilde{X}^n_{\hat{\theta}^{n_1}}\right) - Q^{rn}_{\hat{\theta}^{n_1}, \sqrt{rn}\left(\theta - \hat{\theta}^{n_1}\right)} \right\|_1 : \sqrt{n_2}\left(\hat{\theta}^{n_1} - \theta\right) \in K' \right] \\
&+ \mathrm{E}^{\hat{\theta}^{n_1}} \left[\left\| Q^{rn}_{\hat{\theta}^{n_1}, \sqrt{rn}\left(\theta - \hat{\theta}^{n_1}\right)} - \tilde{P}^{rn}_\theta \right\|_1 : \sqrt{n_2}\left(\hat{\theta}^{n_1} - \theta\right) \in K' \right]. \qquad (54)
\end{aligned}$$

The first term of (54) is evaluated as follows. Let

$$h := \sqrt{n_2}\left(\hat{\theta}^{n_1} - \theta\right) = \sqrt{\frac{1-\delta}{r}} \sqrt{rn}\left(\hat{\theta}^{n_1} - \theta\right),$$

or equivalently,

$$\hat{\theta}^{n_1} = \theta_{n_2,h} = \theta_{rn,\tilde{h}},$$

where $\tilde{h} := \sqrt{r\left(1-\delta\right)^{-1}}h$. Then,

$$
\begin{aligned}
&\sup_{\tilde{A}} \left| \mathrm{E}^{\tilde{X}^n_{\hat{\theta}^{n_1}}} R^{rn}_{\hat{\theta}^{n_1}} \left(\tilde{A} | \tilde{X}^n_{\hat{\theta}^{n_1}} \right) - Q^{rn}_{\hat{\theta}^{n_1}, \sqrt{rn}\left(\theta - \hat{\theta}^{n_1}\right)} \left(\tilde{A} \right) \right| \\
&= \sup_{\tilde{A}} \left| \mathrm{E}^{\tilde{X}^n_{\hat{\theta}^{n_1}}} R^{rn}_{\hat{\theta}^{n_1}} \left(\tilde{A} | \tilde{X}^n_{\hat{\theta}^{n_1}} \right) - Q^{rn}_{\hat{\theta}^{n_1}, -\tilde{h}} \left(\tilde{A} \right) \right| \\
&= \sup_{\tilde{A}} \left| \mathrm{E}^{\tilde{X}^n_{\hat{\theta}^{n_1}}} R^{rn}_{\hat{\theta}^{n_1}} \left(\tilde{A} | \tilde{X}^n_{\hat{\theta}^{n_1}} \right) - \mathrm{E}^{\lambda^{rn}_{-\tilde{h}}} R^{rn}_{\hat{\theta}^{n_1}} \left(\tilde{A} | \lambda^{rn}_{-\tilde{h}} \right) \right| \\
&\leq \left\| \mathcal{L}\left(\tilde{X}^n_{\hat{\theta}^{n_1}} \right) - \mathrm{N}\left(-\tilde{h}, J_\theta^{-1} \right) \right\|_1 \\
&\leq \left\| \mathcal{L}\left(\tilde{X}^n_{\hat{\theta}^{n_1}} \right) - \Lambda^{\sqrt{r/(1-\delta)}}_{\mathrm{amp}} \left(\mathrm{N}\left(-h, J_\theta^{-1} \right) \right) \right\|_1 + \left\| \Lambda^{\sqrt{r/(1-\delta)}}_{\mathrm{amp}} \left(\mathrm{N}\left(-h, J_\theta^{-1} \right) \right) - \mathrm{N}\left(-\tilde{h}, J_\theta^{-1} \right) \right\|_1 \\
&= \left\| \Lambda^{\sqrt{r/(1-\delta)}}_{\mathrm{amp}} \left(\mathcal{L}\left(L^{n_2,\varepsilon}_{\hat{\theta}^{n_1}} | P^{n_2}_\theta \right) \right) - \Lambda^{\sqrt{r/(1-\delta)}}_{\mathrm{amp}} \left(\mathrm{N}\left(-h, J_\theta^{-1} \right) \right) \right\|_1 \\
&+ \left\| \Lambda^{\sqrt{r/(1-\delta)}}_{\mathrm{amp}} \left(\mathrm{N}\left(-h, J_\theta^{-1} \right) \right) - \mathrm{N}\left(-\sqrt{r/\left(1-\delta\right)}h, J_\theta^{-1} \right) \right\|_1 \\
&\leq \left\| \mathcal{L}\left(L^{n_2,\varepsilon}_{\hat{\theta}^{n_1}} | P^{n_2}_\theta \right) - \mathrm{N}\left(-h, J_\theta^{-1} \right) \right\|_1 + \sup_{h' \in \mathbb{R}^m} \left\| \Lambda^{\sqrt{r/(1-\delta)}}_{\mathrm{amp}} \left(\mathrm{N}\left(h', J_\theta^{-1} \right) \right) - \mathrm{N}\left(\sqrt{r/\left(1-\delta\right)}h', J_\theta^{-1} \right) \right\|_1 .
\end{aligned}
$$

Therefore, due to Lemma 14,

$$
\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \overline{\lim_{n \to \infty}} \, \mathrm{E}^{\hat{\theta}^{n_1}} \left[\sup_{\tilde{A}} \left| \mathrm{E}^{\tilde{X}^n_{\hat{\theta}^{n_1}}} R^{rn}_{\hat{\theta}^{n_1}} \left(\tilde{A} | \tilde{X}^n_{\hat{\theta}^{n_1}} \right) - Q^{rn}_{\hat{\theta}^{n_1}, \sqrt{rn}\left(\theta - \hat{\theta}^{n_1}\right)} \left(\tilde{A} \right) \right| ; \sqrt{n_2} \left(\hat{\theta}^{n_1} - \theta \right) \in K' \right] \\
&\leq \lim_{\varepsilon \downarrow 0} \overline{\lim_{n \to \infty}} \sup_{h \in K'} \left\| \mathcal{L}\left(L^{n_2,\varepsilon}_{\theta_{n_2,h}} | P^{n_2}_\theta \right) - \mathrm{N}\left(-h, J_\theta^{-1} \right) \right\|_1 \\
&+ \sup_{h \in K'} \sup_{h' \in \mathbb{R}^m} \left\| \Lambda^{\sqrt{r/(1-\delta)}}_{\mathrm{amp}} \left(\mathrm{N}\left(h', J_\theta^{-1} \right) \right) - \mathrm{N}\left(\sqrt{r/\left(1-\delta\right)}h', J_\theta^{-1} \right) \right\|_1 \\
&= \sup_{h \in \mathbb{R}^m} \left\| \Lambda^{\sqrt{r/(1-\delta)}}_{\mathrm{amp}} \left(\mathrm{N}\left(h, J_\theta^{-1} \right) \right) - \mathrm{N}\left(\sqrt{r/\left(1-\delta\right)}h, J_\theta^{-1} \right) \right\|_1 \\
&= D_{r/(1-\delta), J_\theta^{-1}} .
\end{aligned} \tag{55}
$$

The second term of (54) is evaluated as follows. Let K' be an arbitrary compact set in $\mathbb{R}^m$ and K_θ be an arbitrary compact set in Θ with $\theta \in K$. Then, due to (45) and Theorem 11,

$$
\begin{aligned}
&\mathrm{E}^{\hat{\theta}^{n_1}} \left\{ \left\| Q^{rn}_{\hat{\theta}^{n_1}, \sqrt{rn}\left(\theta - \hat{\theta}^{n_1}\right)} - \tilde{P}^{rn}_\theta \right\|_1 : \sqrt{rn} \left(\hat{\theta}^{n_1} - \theta \right) \in K' \right\} \\
&\leq \sup_{\theta' \in K_\theta} \sup_{h \in K'} \left\| Q^{rn}_{\theta', -h} - \tilde{P}^{rn}_{\theta' - h/\sqrt{rn}} \right\|_1 \to 0, \ n \to \infty .
\end{aligned} \tag{56}
$$

Therefore, combining (55) and (56), we have

$$
\lim_{\varepsilon \downarrow 0} \overline{\lim_{n \to \infty}} \, \mathrm{E}^{\hat{\theta}^{n_1}} \left[\left\| \mathrm{E}^{\tilde{X}^n_{\hat{\theta}^{n_1}}} R^{rn}_{\hat{\theta}^{n_1}} \left(\cdot | \tilde{X}^n_{\hat{\theta}^{n_1}} \right) - \tilde{P}^{rn}_\theta \right\|_1 : \sqrt{n_2} \left(\hat{\theta}^{n_1} - \theta \right) \in K' \right] \leq D_{r/(1-\delta), J_\theta^{-1}}
$$

After all, we have

$$\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\left\|\Lambda_{\delta,\varepsilon}^{n,r}\left(P_\theta^n\right)-P_\theta^{rn}\right\|\leq D_{r/(1-\delta),J_\theta^{-1}}.$$

Since the map $x\to D_{x,\Sigma}$ is continuous about x (see (8)), letting $\delta\to 0$, we have the asserted result. ■

3.4 Local minimax property

Based on (n,rn)-cloner $\Lambda^{n,r}$ for $\{P_\theta^n\}_{\theta\in\Theta}$, we compose an amplifier $\Lambda_{\mathrm{amp}}^{\theta,\sqrt{r},n,\varepsilon}$ for the Gaussian shift $\left\{\mathrm{N}\left(h,J_\theta^{-1}\right)\right\}_{h\in\mathbb{R}^m}$ as follows.

(I) Given X_h with $\mathcal{L}\left(X_h\right)=\mathrm{N}\left(h,J_\theta^{-1}\right)$, compose

$$Q_{\theta,h}^{\prime n}\left(A\right):=Q_{\theta,h}^n\left(A\times\Omega^\prime\right)=\mathrm{E}^{X_h}R_\theta^n\left(A\times\Omega^\prime|X_h\right).$$

(II) Apply $\Lambda^{n,r}$ to $Q_{\theta,h}^{\prime n}$. Denote by $W_h^{n,r}$ the random variable with $\mathcal{L}\left(W_h^{n,r}\right)=\Lambda^{n,r}\left(Q_{\theta,h}^{\prime n}\right)$.

(III) The output random variable is $\tilde{X}_h^{r,n,\varepsilon}:=L_\theta^{rn,\varepsilon}\left(W_h^{n,r}\right)$, where $L_\theta^{n,\varepsilon}\left(\omega^n\right):=J_\theta^{-1}\ell_\theta^n\left(\omega^n\right)+Y_\varepsilon$ and $\mathcal{L}\left(Y_\varepsilon\right)=\mathrm{N}\left(0,\varepsilon\right)$.

Lemma 16 *Suppose $\theta\to p_\theta$ is differentiable in quadratic mean, and (15) and (14) hold. Then, for any compact set $K^\prime\in\mathbb{R}^m$, we have*

$$\lim_{n\to\infty}\sup_{h\in K^\prime}\left\|\mathcal{L}\left(L_\theta^{n,\varepsilon}|P_{\theta_{n,h}}^n\right)-\mathrm{N}\left(h,J_\theta^{-1}\right)\right\|_1=0.$$

Proof. Observe, for any measurable function f with $\sup_{x\in\mathbb{R}^m}\left|f\left(x\right)\right|\leq 1$,

$$\mathrm{E}^{Y_\varepsilon}\mathrm{E}_{\theta_{n,h}}^n\left[f\left(L_\theta^{n,\varepsilon}\right)\right]=\mathrm{E}^{Y_\varepsilon}\mathrm{E}_\theta^n\left[f\left(L_\theta^{n,\varepsilon}\right)Z_{\theta,h}^n\right],$$

Observe also, due to Lemma 6, with $K^\prime$ being an arbitrary compact subset of $\mathbb{R}^m$,

$$\begin{aligned}&\sup_{h\in K^\prime}\left|\mathrm{E}^{Y_\varepsilon}\mathrm{E}_\theta^n\left[f\left(L_\theta^{n,\varepsilon}\right)\left(Z_{\theta,h}^n-Z_{\theta,h}\left(\ell_\theta^n\right)\right)\right]\right| \\ &\leq\sup_{h\in K^\prime}\mathrm{E}_{\theta_{n,h}}^n\left[\left|Z_{\theta,h}^n-Z_{\theta,h}\left(\ell_\theta^n\right)\right|\right] \\ &\to 0,\ n\to\infty.\end{aligned}\tag{57}$$

Therefore, we have to evaluate

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K^\prime}\mathrm{E}^{Y_\varepsilon}\mathrm{E}_\theta^n\left[f\left(L_\theta^{n,\varepsilon}\right)Z_{\theta,h}\left(\ell_\theta^n\right)\right] \\ &=\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K^\prime}\mathrm{E}^{Y_\varepsilon}\mathrm{E}_\theta^n\left[f\left(L_\theta^{n,\varepsilon}\right)Z_{\theta,h}\left(J_\theta L_\theta^{n,\varepsilon}\right)e^{-h^TJ_\theta Y_\varepsilon}\right] \\ &=\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K^\prime}\left(E_1+E_2+E_3\right),\end{aligned}$$

where

$$E_1 := \mathrm{E}^{Y_\varepsilon}\mathrm{E}_\theta^n\left[I_{\left\{\left\|L_\theta^{n,\varepsilon}\right\|\le a\,,\,\|Y_\varepsilon\|\le\varepsilon^{1/4}\right\}}f\left(L_\theta^{n,\varepsilon}\right)Z_{\theta,h}\,\left(J_\theta L_\theta^{n,\varepsilon}\right)e^{-h^TJ_\theta Y_\varepsilon}\right],$$
$$E_2 := \mathrm{E}^{Y_\varepsilon}\mathrm{E}_\theta^n\left[I_{\left\{\|Y_\varepsilon\|>\varepsilon^{1/4}\right\}}f\left(L_\theta^{n,\varepsilon}\right)Z_{\theta,h}\,\left(J_\theta L_\theta^{n,\varepsilon}\right)e^{-h^TJ_\theta Y_\varepsilon}\right],$$
$$E_3 := \mathrm{E}^{Y_\varepsilon}\mathrm{E}_\theta^n\left[I_{\left\{\left\|L_\theta^{n,\varepsilon}\right\|>a\,,\,\|Y_\varepsilon\|\le\varepsilon^{1/4}\right\}}f\left(L_\theta^{n,\varepsilon}\right)Z_{\theta,h}\,\left(J_\theta L_\theta^{n,\varepsilon}\right)e^{-h^TJ_\theta Y_\varepsilon}\right].$$

E_1 is evaluated as follows. Observe

$$E_1 = \mathrm{E}^{Y_\varepsilon}\mathrm{E}_{\theta_{n,h}}^n\left[I_{\left\{\left\|L_\theta^{n,\varepsilon}\right\|\le a\right\}}f\left(L_\theta^{n,\varepsilon}\right)Z_{\theta,h}\,\left(J_\theta L_\theta^{n,\varepsilon}\right)\mathrm{E}\left[I_{\left\{\|Y_\varepsilon\|\le\varepsilon^{1/4}\right\}}e^{-h^TJ_\theta Y_\varepsilon}\middle|\,L_\theta^{n,\varepsilon}\right]\right],$$

whose second factor can be evaluated as

$$\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\left|\mathrm{E}\left[I_{\left\{\|Y_\varepsilon\|\le\varepsilon^{1/4}\right\}}e^{-h^TJ_\theta Y}\middle|\,L_{\theta_{n,h}}^{n,\varepsilon}\right]-1\right|\le e^{\|h\|\|J_\theta\|\varepsilon^{1/4}} \tag{58}$$

To evaluate the first factor

$$E_{1,1} := \mathrm{E}^{Y_\varepsilon}\mathrm{E}_\theta^n\left[f\left(L_\theta^{n,\varepsilon}\right)Z_{\theta,h}\,\left(J_\theta L_\theta^{n,\varepsilon}\right):\left\|L_\theta^{n,\varepsilon}\right\|\le a\right],$$

observe, with $\mathcal{L}\left(X_h\right)=\mathrm{N}\left(h,J_\theta^{-1}\right)$,

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\left|E_{1,1}-\mathrm{E}\left[f\left(X_{-h}\right):\|X_{-h}\|\le a\right]\right|\\
&=\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\left|\begin{array}{c}\mathrm{E}^{Y_\varepsilon}\mathrm{E}_\theta^n\left[f\left(L_\theta^{n,\varepsilon}\right)Z_{\theta,h}\,\left(J_\theta L_\theta^{n,\varepsilon}\right):\left\|L_\theta^{n,\varepsilon}\right\|\le a\right]\\ -\mathrm{E}^{X_0}\left[f\left(X_0\right)Z_{\theta,h}\,\left(J_\theta X_0\right):\|X_0\|\le a\right]\end{array}\right|\\
&\le\sup_{h\in K'}e^{\|h\|\|J_\theta\|a-\frac{1}{2}h^TJ_\theta h}\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\left\|\mathcal{L}\left(L_\theta^{n,\varepsilon}|P_\theta^n\right)-\mathrm{N}\left(0,J_\theta^{-1}\right)\right\|_1\\
&=0,
\end{aligned} \tag{59}$$

where the last identity is due to Lemma 13.

Therefore, by (58) and (59),

$$\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\left|E_1-\mathrm{E}\left[f\left(X_{-h}\right):\|X_{-h}\|\le a\right]\right|=0. \tag{60}$$

On the other hand, by (17),

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}E_2\\
&\le\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\mathrm{E}^{Y_\varepsilon}\mathrm{E}_\theta^n\left[Z_{\theta,h}\,\left(\ell_\theta^n\right):\|Y_\varepsilon\|>\varepsilon^{1/4}\right]\\
&=\lim_{\varepsilon\downarrow 0}\Pr\left\{\|Y_\varepsilon\|>\varepsilon^{1/4}\right\}\cdot\sup_{h\in K'}e^{h^TJ_\theta h}e^{-\frac{1}{2}h^TJ_\theta h}\\
&=0,
\end{aligned} \tag{61}$$

and

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'} E_3 \\
&\le \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'} \mathrm{E}^{Y_\varepsilon}\mathrm{E}_\theta\left[Z_{\theta,h}\left(\ell_\theta^n\right) : \|L_\theta^{n,\varepsilon}\| > a\,,\ \|Y_\varepsilon\| \le \varepsilon^{1/2}\right] \\
&\le \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'} \mathrm{E}_\theta^n\left[I_{\left\{\|\ell_\theta^n\|>\alpha_\theta\left(a-\varepsilon^{1/4}\right)\right\}} Z_{\theta,h}\left(\ell_\theta^n\right)\right] \\
&\le \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'} \sqrt{P_\theta^n\left\{\|\ell_\theta^n\| > \alpha_\theta\left(a-\varepsilon^{1/4}\right)\right\}}\sqrt{\mathrm{E}_\theta^n\left[Z_{\theta,h}\left(\ell_\theta^n\right)\right]^2} \\
&\le \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty} \sqrt{\sup_{h\in K'} P_\theta^n\left\{\|\ell_\theta^n\| > \alpha_\theta\left(a-\varepsilon^{1/4}\right)\right\} \sup_{h\in K'}\left(\mathrm{E}_{\theta'}^n e^{-\frac{2h^T}{\sqrt{n}}\ell_\theta}\right)^n e^{-h^T J_\theta h}} \\
&\le \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty} \sqrt{\sup_{h\in K'} \frac{1}{\left\{\alpha_\theta\left(a-\varepsilon^{1/4}\right)\right\}^2}\mathrm{E}_\theta^n\left\|\ell_\theta^n\right\|^2 \sup_{h\in K'} e^{h^T J_\theta h}} \\
&= \sqrt{\sup_{h\in K'}\frac{\operatorname{tr} J_\theta}{\alpha_\theta^2 a^2}\sup_{h\in K'} e^{h^T J_\theta h}}. \qquad (62)
\end{aligned}
$$

After all, by (57), (60), (51), and (52), we have

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\left|\mathrm{E}^{Y_\varepsilon}\mathrm{E}_{\theta_{h_n}}^n\left[f\left(L_\theta^{n,\varepsilon}\right)\right] - \mathrm{E}\left[f\left(X_h\right)\right]\right| \\
&= \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\left|\mathrm{E}^{Y_\varepsilon}\mathrm{E}_\theta^n\left[f\left(L_\theta^{n,\varepsilon}\right) Z_{\theta,h}\left(\ell_\theta^n\right)\right] - \mathrm{E}\left[f\left(X_h\right)\right]\right| \\
&= \sqrt{\sup_{h\in K'}\frac{\operatorname{tr} J_\theta}{\alpha_\theta^2 a^2}\sup_{h\in K'} e^{h^T J_\theta h}} + \sup_{h\in K'}\Pr\left\{\|X_h\| > a\right\}.
\end{aligned}
$$

Since a is arbitrary, letting $a\to\infty$, we have

$$
\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\sup_{h\in K'}\left\|\mathcal{L}\left(L_\theta^{n,\varepsilon}|P_{\theta_{n,h}}^n\right) - \mathrm{N}\left(h, J_\theta^{-1}\right)\right\|_1 = 0.
$$

■

Theorem 17 *Suppose $\theta\to p_\theta$ is differentiable in quadratic mean, and (15) and (14) hold. Then, for any Markov map $\Lambda^{n,r}$ and for any $\theta\in\Theta$, we have (10).*

Proof. Observe

$$
\begin{aligned}
&\left\|\Lambda_{\mathrm{amp}}^{\theta,\sqrt{r},n,\varepsilon}\left(\mathrm{N}\left(h,J_\theta^{-1}\right)\right)-\mathrm{N}\left(\sqrt{r}h,J_\theta^{-1}\right)\right\|_1\\
&\le\left\|\Lambda_{\mathrm{amp}}^{\theta,\sqrt{r},n,\varepsilon}\left(\mathrm{N}\left(h,J_\theta^{-1}\right)\right)-\mathcal{L}\left(L_\theta^{rn,\varepsilon}|P_{\theta_{n,h}}^{rn}\right)\right\|_1+\left\|\mathcal{L}\left(L_\theta^{rn,\varepsilon}|P_{\theta_{n,h}}^{rn}\right)-\mathrm{N}\left(\sqrt{r}h,J_\theta^{-1}\right)\right\|_1\\
&=\left\|\mathcal{L}\left(L_\theta^{rn,\varepsilon}|\Lambda^{n,r}\left(Q'^n_{\theta,h}\right)\right)-\mathcal{L}\left(L_\theta^{rn,\varepsilon}|P_{\theta_{n,h}}^{rn}\right)\right\|_1+\left\|\mathcal{L}\left(L_\theta^{rn,\varepsilon}|P_{\theta_{n,h}}^{rn}\right)-\mathrm{N}\left(\sqrt{r}h,J_\theta^{-1}\right)\right\|_1\\
&\le\left\|\Lambda^{n,r}\left(Q'^n_{\theta,h}\right)-P_{\theta_{n,h}}^{rn}\right\|_1+\left\|\mathcal{L}\left(L_\theta^{rn,\varepsilon}|P_{\theta_{n,h}}^{rn}\right)-\mathrm{N}\left(\sqrt{r}h,J_\theta^{-1}\right)\right\|_1\\
&\le\left\|\Lambda^{n,r}\left(Q'^n_{\theta,h}\right)-\Lambda^{n,r}\left(P_{\theta_{n,h}}^{n}\right)\right\|_1+\left\|\Lambda^{n,r}\left(P_{\theta_{n,h}}^{n}\right)-P_{\theta_{n,h}}^{rn}\right\|_1\\
&+\left\|\mathcal{L}\left(L_\theta^{rn,\varepsilon}|P_{\theta_{n,h}}^{rn}\right)-\mathrm{N}\left(\sqrt{r}h,J_\theta^{-1}\right)\right\|_1\\
&\le\left\|Q'^n_{\theta,h}-P_{\theta_{n,h}}^{n}\right\|_1+\left\|\Lambda^{n,r}\left(P_{\theta_{n,h}}^{n}\right)-P_{\theta_{n,h}}^{rn}\right\|_1+\left\|\mathcal{L}\left(L_\theta^{rn,\varepsilon}|P_{\theta_{n,h}}^{rn}\right)-\mathrm{N}\left(\sqrt{r}h,J_\theta^{-1}\right)\right\|_1.
\end{aligned}
$$

By Theorem 12,the first term vanishes,

$$
\lim_{n\to\infty}\sup_{h\in K'}\left\|Q'^n_{\theta,h}-P_{\theta_{n,h}}^{n}\right\|_1=0.
$$

By Lemma 16, since $\theta_{n,h}=\theta_{rn,\sqrt{r}h}$, the third term vanishes,

$$
\lim_{\varepsilon\downarrow0}\lim_{n\to\infty}\sup_{h\in K'}\left\|\mathcal{L}\left(L_\theta^{rn,\varepsilon}|P_{\theta_{n,h}}^{rn}\right)-\mathrm{N}\left(\sqrt{r}h,J_\theta^{-1}\right)\right\|_1=0.
$$

After all, we have

$$
\lim_{\varepsilon\downarrow0}\varliminf_{n\to\infty}\sup_{h\in K'}\left\|\Lambda_{\mathrm{amp}}^{\theta,\sqrt{r},n,\varepsilon}\left(\mathrm{N}\left(h,J_\theta^{-1}\right)\right)-\mathrm{N}\left(\sqrt{r}h,J_\theta^{-1}\right)\right\|_1\le\varliminf_{n\to\infty}\sup_{h\in K'}\left\|\Lambda^{n,r}\left(P_{\theta_{n,h}}^{n}\right)-P_{\theta_{n,h}}^{rn}\right\|_1.
$$

Since

$$
\inf_{\Lambda}\sup_{h\in K'}\left\|\Lambda_{\mathrm{amp}}^{\theta,\sqrt{r}}\left(\mathrm{N}\left(h,J_\theta^{-1}\right)\right)-\mathrm{N}\left(\sqrt{r}h,J_\theta^{-1}\right)\right\|_1\le\varliminf_{n\to\infty}\sup_{h\in K'}\left\|\Lambda_{\mathrm{amp}}^{\theta,\sqrt{r},n,\varepsilon}\left(\mathrm{N}\left(h,J_\theta^{-1}\right)\right)-\mathrm{N}\left(\sqrt{r}h,J_\theta^{-1}\right)\right\|_1
$$

holds due to optimality of $\Lambda_{\mathrm{amp}}^{\theta,\sqrt{r}}$, we have

$$
\varliminf_{n\to\infty}\sup_{h\in K'}\left\|\Lambda^{n,r}\left(P_{\theta_{n,h}}^{n}\right)-P_{\theta_{n,h}}^{rn}\right\|_1\ge\inf_{\Lambda}\sup_{h\in K'}\left\|\Lambda_{\mathrm{amp}}^{\theta,\sqrt{r}}\left(\mathrm{N}\left(h,J_\theta^{-1}\right)\right)-\mathrm{N}\left(\sqrt{r}h,J_\theta^{-1}\right)\right\|_1.
$$

Here, letting $K'=\{x\,;\|x\|\le a\}$ and $a\to\infty$, we have (10). ■

4 Discussion

Using quantum LAN, we can produce similar results for finite dimensional quantum system.

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A Proof of (6)

In this appendix, we prove

$$\inf_x \|p_{0,\mathbf{1}} - p_{x,r\mathbf{1}}\|_1 = \|p_{0,\mathbf{1}}(y) - p_{0,r\mathbf{1}}(y)\|_1 ,$$

where $p_{x,\Sigma}$ is the density of $\mathrm{N}(x,\Sigma)$. Due to the symmetry, we can suppose $x=(t,0,\cdots,0), t\geq 0$. Define $B_{r,t}:=\{y; p_{0,\mathbf{1}}(y)\geq p_{x,r\mathbf{1}}(y)\}$. Observe

$$\begin{aligned} & y\in B_r \\ & \Leftrightarrow \frac{1}{(2\pi)^{m/2}} e^{-\frac{(y_1)^2+\sum_{\kappa=2}^m (y_\kappa)^2}{2}} \geq \frac{1}{(2\pi r)^{m/2}} e^{-\frac{(y_1-t)^2+\sum_{\kappa=2}^m (y_\kappa)^2}{2r}} \\ & \Leftrightarrow \left(y_1-\frac{t}{r-1}\right)^2+\sum_{\kappa=2}^m (y_\kappa)^2 \leq \frac{2m}{r-1}\log r+\left(\frac{t}{r-1}\right)^2. \end{aligned}$$

Hence, B_r is a ball.

For $z\in\mathbb{R}^{m-1}$ and $t\in\mathbb{R}$, define t_1 and t_2 by

$$\begin{gathered} p_{0,\mathbf{1}}((t_\kappa,z)) = p_{0,r\mathbf{1}}((t_\kappa-t,z)), \\ t_1\leq t_2. \end{gathered}$$

One can verify

$$p_{0,\mathbf{1}}((t_1,z))\geq p_{0,\mathbf{1}}((t_2,z))$$

as follows. In case of $0\leq t_1(t,z)\leq t_2(t,z)$, this holds because $p_{0,\mathbf{1}}((\cdot,z))$ is monotone decreasing on $\mathbb{R}_+$. In case of $t_1(t,z)\leq 0\leq t_2(t,z)$, observe $t_1(0,z)\leq t_1(t,z)\leq 0\leq t_2(0,z)\leq t_2(t,z)$. Hence,

$$p_{0,\mathbf{1}}((t_1(t,z),z))\geq p_{0,\mathbf{1}}((t_1(0,z),z))=p_{0,\mathbf{1}}((t_2(0,z),z))\geq p_{0,\mathbf{1}}((t_2(t,z),z)).$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|p_{0,\mathbf{1}}-p_{x,r\mathbf{1}}\right\|_1=\int\left\{p_{0,\mathbf{1}}((t_1,z))-p_{0,\mathbf{1}}((t_2,z))\right\}\mathrm{d}z\geq 0.$$

Therefore, the minimum is achieved $t=0$, and we have the asserted result.