The Aeolian harp sound from the Madelung model of quantum mechanics

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Abstract

According to Madelung, Bohm and Vigier, Wilhelm, Rosen and others, the original Schroedinger equation can be transformed into the hydrodynamical system of equations by using the so called Madelung ansatz. We derive in such quantum hydrodynamics, the non-relativistic and relativistic Strouhal number from the so called von Kármán's vortex street. The relativistic derivation of this formula follows from the addition formula for velocities. The Strouhal friction tones are generated also during the motion of cosmic rays in relic photon sea, during the motion of bolids in atmosphere, during the Saturn rings motion in the relic black-body sea, during the motion of bodies in superfluid helium and so on.

1 Introduction

It is well known from the experimental artillery physics and ballistics, that the moving ballistic projectile generates not only the Mach cone but also the sound. Similarly, the moving projectile of a gun, moving bolid in atmosphere, moving tactic and ballistic misails generate sound. The physical origin of this sound is not caused by the vibration of the surface of the projectile, or by the vibration of the Mach cone, or by the micro-structure of the Mach cone, but it is caused by the periodic generation of vortexes in the vicinity of the surface of the projectile during the air flowing around it. Such sound is generated also by the air flow around the cylinders, or strings. The system of strings generating the sound is named Aeolean's harp (Aeolus being God of winds in the Greek mythology) and the tones generated in a such a way are so called the Strouhal friction tones. If the diameter of the string, or cylinder immersed in the flow is D and the velocity of the flow is v then the frequency f of the sound is given by the Strouhal formula (Blokhintsev, 1981):

$$f = \kappa(Re)\frac{v}{D}, \tag{1}$$

where κ is the Strouhal number named after Vincent Strouhal, a Czech physicist who experimented in 1878 with wires experiencing vortex shedding and singing in the wind (Strouhal, 1878; White, 1999). The dimensionless symbol Re is the Reynolds number given by the formula $Re = vD/\nu$, where ν is the kinematic viscosity. The Strouhal number was late generalized to involve obertons, or, (Blokhintsev, 1981):

$$f = \kappa(Re)\frac{v}{D}n, \tag{2}$$

where n is the integer number of the oberton.

2 The Madelung model of quantum mechanics

According to Madelung (1926), Bohm and Vigier (1954), Wilhelm (1970), Rosen (1974) and others, the original Schrödinger equation can be transformed into the hydrodynamical system of equations by using the so called Madelung ansatz:

$$\Psi = \sqrt{n}e^{\frac{i}{\hbar}S}, \tag{3}$$

where n is interpreted as the density of particles and S is the classical action for $\hbar \to 0$. The mass density is defined by relation $\varrho = nm$ where m is mass of a particle.

It is well known that after insertion of the relation (3) into the original Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi, \tag{4}$$

where V is the potential energy, we get, after separating the real and imaginary parts, the following system of equations:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V = \frac{\hbar^2}{2m}\frac{\Delta\sqrt{n}}{\sqrt{n}} \tag{5}$$

$$\frac{\partial n}{\partial t} + \mathrm{div}(n\mathbf{v}) = 0 \tag{6}$$

with

$$\mathbf{v} = \frac{\nabla S}{m}. \tag{7}$$

Equation (5) is the Hamilton-Jacobi equation with the additional term

$$V_q = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{n}}{\sqrt{n}}, \tag{8}$$

which is called the quantum Bohm potential and equation (6) is the continuity equation.

After application of operator $\bigtriangledown$ on eq. (5), it can be cast into the Euler hydrodynamical equation of the form:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{m}\nabla(V + V_q). \tag{9}$$

It is evident that this equation is from the hydrodynamical point of view incomplete as a consequence of the missing term $-\varrho^{-1}\nabla p$ where p is hydrodynamical pressure. We use here this fact just as the crucial point for derivation of the nonlinear Schrödinger equation. We complete the eq. (9) by adding the pressure term and in such a way we get the total Euler equation in the form:

$$m\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla(V + V_q) - \nabla F, \tag{10}$$

where

$$\nabla F = \frac{1}{n}\nabla p. \tag{11}$$

The equation (10) can be obtained by the Madelung procedure from the following extended Schrödinger equation

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi + F\Psi \tag{12}$$

on the assumption that it is possible to determine F in term of the wave function. From the vector analysis follows that the necessary condition of the existence of F as the solution of the eq. (11) is rot grad F = 0, or,

$$\mathrm{rot}(n^{-1}\nabla p) = 0, \tag{13}$$

which enables to take the linear solution in the form

$$p = -bn = -b|\Psi|^2, \tag{14}$$

where b is some arbitrary constant. We do not consider the more general solution of eq. (13). Then, from eq. (11) i.e. grad $F = \mathbf{a}$ we have:

$$F = \int a_i\, dx_i = -b\int \frac{1}{n}\, dn = -b\ln|\Psi|^2, \tag{15}$$

where we have omitted the additive constant which plays no substantial role in the Schrödinger equation.

Now, we can write the generalized Schrödinger equation which corresponds to the complete Euler equation (10) in the following form:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi - b(\ln|\Psi|^2)\Psi. \tag{16}$$

The last equation is continuation of the de Broglie ideas on quantum mechanics (Broglie, 1960) and leads to interesting consequences as can be seen it many articlle on the nonlinear quantum mechanics. Namely, (Bialynicky-Birula, et al., 1976), (Pardy, 1993; 1994; 2001), (Gaehler, et al., 1981) and so on. Let us remark that only nonlinear equation (16) has the correct classsical limit for the particle of the infinite mass (Pardy, 2001).

3 The Quantum Pascal law

Let us firstly, consider quantum mechanics with the electron confined in the box with the infinite barriers at point 0 and l. Then, the energy levels of electron inside the box is (Sokolov et al. 1962)

$$E_n = \frac{\pi^2\hbar^2 n^2}{2ml^2} \tag{17}$$

and the corresponding wave function is

$$\psi_n = \sqrt{\frac{2}{l}}\sin\left(\pi n\frac{x}{l}\right). \tag{18}$$

The quantum pressure caused by the quantum mechanical motion of particle is obtained by the same operation as in the Casimir effect. Or,

$$F = -\frac{\partial E_n}{\partial l} = \frac{\pi^2\hbar^2 n^2}{2ml^3}. \tag{19}$$

The physical interpretation of eq. (19) is, that if some pressure p is at point 0, then the same pressure is instantaneously without retardation in point l,

In case that the thermal box is three-dimensional, we get (Sokolov et al., 1962)

$$E_{n_1,n_2,n_3} = \frac{\pi^2\hbar^2}{2m}\left[\left(\frac{n_1}{l_1}\right)^2 + \left(\frac{n_2}{l_2}\right)^2 + \left(\frac{n_3}{l_3}\right)\right] \tag{20}$$

and the corresponding wave function is

$$\psi_{n_1,n_2,n_3} = \sqrt{\frac{8}{l_1 l_2 l_3}}\sin\left(\pi n_1\frac{x}{l_1}\right)\sin\left(\pi n_2\frac{x}{l_2}\right)\sin\left(\pi n_3\frac{x}{l_3}\right). \tag{21}$$

The corresponding pressures are

$$p_{23} = -\frac{1}{l_2 l_3}\frac{\partial E_{n_1,n_2,n_3}}{\partial l_1} \tag{22}$$

$$p_{13} = -\frac{1}{l_1 l_3}\frac{\partial E_{n_1,n_2,n_3}}{\partial l_2} \tag{23}$$

$$p_{12} = -\frac{1}{l_1 l_2}\frac{\partial E_{n_1,n_2,n_3}}{\partial l_3}. \tag{24}$$

The physical interpretation of eqs. (22–24) is, that if some pressure p is at wall l_{ij}, then the same pressure is instantaneously without retardation at opposite wall of l_{ij}. (the partial quantum Pascal law).

The wave function af the electron in a box abc in the coordinate system xyz in the momentum representation is as follows (Grashin, 1974):

$$\psi(\mathbf{p}) = \int_0^a dx \int_0^b dy \int_0^c dz \frac{\exp -(\mathbf{pr}/\hbar)}{\sqrt{\pi^3 \hbar^3 abc}}$$

$$\sin\left(\pi\frac{x}{a}\right)\sin\left(\pi\frac{y}{b}\right)\sin\left(\pi\frac{z}{c}\right). \tag{25}$$

Let us only remark that the quantum pressure derived here is the perfect proof that the wave function in quantum mechanics is the mathematical form of physical reality - form of matter - or, the wave function is in such a way the objective form of matter, where matter is continuum independent on observer. The similar quantum model is the Casimir effect, which is based on the quantum field theory, where quantum field is form of matter independent on the observer and not only the mathematical construct. So, the Madelung hydrodynamical model of quantum mechanics is the model of medium which can be connsidered as the physical medium involving sound waves.

4 The von Kármán vortex street

The von Kármán vortex street is named after the engineer and fluid dynamicist Theodore von Kármán (1963; 1994). It is produced for instance by wind interacting with the suspended telephone, or, by the power lines, or, by a car antenna at certain speeds of a car.

The potential flow of the ideal liquid can be described by the complex function

$$w(z) = \varphi(x, y) + i\psi(x, y), \tag{26}$$

with $z = x + iy$ (Kočin et al., 1963). It is supposed that the function is analytical, which means that the Cauchy-Riemann conditions are fulfilled:

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}; \quad \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}. \tag{27}$$

The corresponding velocities of the two-dimensional liquid fluid are as follows:

$$v_x = \frac{\partial \varphi}{\partial x}; \quad v_y = \frac{\partial \varphi}{\partial y}. \tag{28}$$

Then, derivation of w gives:

$$\frac{dw}{dz} = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \varphi}{\partial x} - i\frac{\partial \varphi}{\partial y} = v_x - iv_y. \tag{29}$$

The vortex potential was derived in the theory of complex function theory of the fluid dynamics as

$$w(z) = \frac{\Gamma}{2\pi i} \ln \frac{(z - z_k)}{l}, \tag{30}$$

where Γ is so called circulation of liquid and l is the arbitrary constant with the dimensionality of length. Let us suppose that the center of the vortexes is at points $z_0, \pm z_1, \pm z_2, \pm z_3, ...$ with $x_k = lk, k = 0, \pm 1, \pm 2, \pm 3, ...$ and $y_k = H/2$, where H is the arbitrary parameter.

It may be easy to see that the complex potential of the system of vortexes is (Kočin et al. 1963):

$$w(z) = \frac{\Gamma}{2\pi i} \left\{ \ln \frac{(z - z_0)\pi}{l} + \sum_{k=1}^{\infty} \left[\ln \frac{(z - z_k)}{-lk} + \ln \frac{(z - z_{-k})}{lk} \right] \right\} + const, \tag{31}$$

where we have multiplied $z - z_0$ by π/l and $z - z_k$ by $1/(-kl)$, which leads to the change of additional constant in the complex potential and not to the change of physics following from the complex potential. After some mathematical manipulations, we get the last formula in the following form:

$$w(z) = \frac{\Gamma}{2\pi i} \ln \left\{ \frac{(z - z_0)\pi}{l} \prod_{k=1}^{\infty} \frac{(z - z_k)}{-lk} \frac{(z - z_{-k})}{lk} \right\}. \tag{32}$$

Since we have used $z_k = z_0 - lk, z_{-k} = z_0 + lk$, then

$$w(z) = \frac{\Gamma}{2\pi i} \ln \left\{ \frac{(z - z_0)\pi}{l} \prod_{k=1}^{\infty} \left[1 - \left(\frac{(z - z_0)^2}{lk} \right) \right] \right\}. \tag{33}$$

Now, using the formula

$$\sin \pi x = \pi x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2} \right), \tag{34}$$

we get

$$w = \frac{\Gamma}{2\pi i} \ln \sin \frac{\pi}{l}(z - z_0). \tag{35}$$

The corresponding complex velocity is as follows:

$$v_x - iv_y = \frac{\Gamma}{2li} \cot \frac{\pi}{l}(z - z_0). \tag{36}$$

In case of two vortexes with circulation Γ_1, Γ_2, we get the complex velocities in the form:

$$v_x - iv_y = \frac{\Gamma_1}{2li} \cot \frac{\pi}{i}(z - z_1) + \frac{\Gamma_2}{2li} \cot \frac{\pi}{l}(z - z_2) = -\frac{dw}{dz}. \tag{37}$$

It is possible to see that (Kočin et al., 1963)

$$v_{1x} - iv_{1y} = \frac{\Gamma_2}{2li} \cot \frac{\pi}{l}(z_1 - z_2) \tag{38}$$

and

$$v_{2x} - iv_{2y} = -\frac{\Gamma_1}{2li} \cot \frac{\pi}{l}(z_1 - z_2). \tag{39}$$

We have from equal complex velocities:

$$v_{1x} - iv_{1y} = v_{2x} - iv_{2y}, \tag{40}$$

the following evident relation

$$\Gamma_1 = -\Gamma_2. \tag{41}$$

In case that y-velocities of vortexes are zero, or, $v_{1y} = v_{2y} = 0$, then

$$z_1 - z_2 = b + Hi, \tag{42}$$

where b, H are some constants.

Now, let us use the formula:

$$\cot \frac{\pi}{l}(b + Hi) = \frac{\sin \frac{2\pi b}{l}}{\cosh \frac{2\pi H}{l} - \cos \frac{2\pi b}{l}} - i \frac{\sinh \frac{2\pi H}{l}}{\cosh \frac{2\pi H}{l} - \cos \frac{2\pi b}{l}}, \tag{43}$$

Then, it follows from the last equation that

$$v_{1,2x} = \frac{\Gamma}{2l} \frac{\sinh \frac{2\pi H}{l}}{\cosh \frac{2\pi H}{l} - \cos \frac{2\pi b}{l}}, \tag{44a}$$

$$v_{1,2y} = -\frac{\Gamma}{2l} \frac{\sin \frac{2\pi b}{l}}{\cosh \frac{2\pi H}{l} - \cos \frac{2\pi b}{l}}. \tag{44b}$$

We have from $v_{1y} = v_{2y} = 0$, that

$$\sin \frac{2\pi b}{l} = 0, \tag{45}$$

or, $b = 0$, $b = l/2$.

The situation with $b = 0$ is called the symmetrical configuration which is non-stable (Kočin et al., 1963) and the situation with $b = l/2$ which is the chess stable configuration. We have two velocities:

$$v_{1x} = \frac{\Gamma}{2l}\coth\left(\frac{\pi H}{l}\right); \quad (b = 0), \tag{46}$$

$$v_{2x} = \frac{\Gamma}{2l}\tanh\left(\frac{\pi H}{l}\right); \quad (b = l/2). \tag{47}$$

5 The derivation of the Strouhal number from the vortex street

The period forming by the vortex street, where the relative velocities is $v - u$, is (Blokhintsev, 1981):

$$T = \frac{l}{v - u} \tag{48}$$

and the frequency f is

$$f = \frac{v - u}{l} = \left(1 - \frac{u}{v}\right)\frac{D}{l}.\frac{v}{D} \tag{49}$$

It means in the last formula that the non-relativistic Strouhal number κ is

$$\kappa = \left(1 - \frac{u}{v}\right)\frac{D}{l}. \tag{50}$$

6 The relativistic Strouhal number

The rigorous derivation of the relativistic Strouhal number follows from the relativistic hydrodynamics (Landau et al. 1987), together with the derivation of the relativistic von Kármán's vortex theory. However, we here suppose that the relativistic Strouhal nummber follows immediately from the non-relativistic formula by the operation of the relativistic generalization.

The Strouhal formula contains quantity D with the dimensionality of length, and velocities v and u. According to special theory of relativity, length is not contracted when the cylinder or string is placed perpendicular to the direction of motion, and it means that it is not contracted if it is placed perpendicular to the air flow in the considered experiment. On the other hand, the special relativity addition theorem is necessary to apply for velocities v and u. In other words the relativistic formula is as follows (with $v \oplus u$ being the relativistic addition) :

$$v \oplus u = \frac{v + u}{1 + \frac{uv}{c^2}}. \tag{51}$$

Using the formula (25) for non-relativistic frequency generated by the vortexes, we get after some algebraic operations, the relativistic Strouhal number in the form:

$$\kappa = \frac{\left(1 - \frac{u}{v}\right)\frac{D}{l}}{\left(1 - \frac{uv}{c^2}\right)}. \tag{52}$$

Let us remark, that if we consider the Strouhal effect in the inertial system moving with velocity V with regard to the laboratory system, then it is necessary still transform the last formula according the relativistic Doppler formula.

7 Discussion

We have considered the aerodynamic and hydrodynamical situations where the Strouhal friction tones are generated. The non-relativistic and relativistic Strouhal number was derived from so called von Kármán's vortex street. The relativistic derivation of this formula followed from the relativistic addition formula for velocities.

The physical phenomenon called the Strouhal friction tones can be extended to the cosmic rays moving in the relic photon sea, motion of bolids in atmosphere and the ionospheric generation of sound escorting the aurora borealis/australis. In case of cosmic rays we consider the moving bunch of cosmical particles with its effective diameter D and not the individual particles. The detection of the generated sound is possible by special microphones. Motion of bodies in the solar wind produces also the von Kármán vortex street leading to the Strouhal friction tones.

The moving bunch of protons with effective diameter D in the accelerated tube in LHC of CERN generates also the Strouhal friction tones, because the tube is the black-body with the photon sea (Pardy, 2013a; ibid., 2013b), which enables the formation of the von Kármán photon vortex street.

The Strouhal friction tones are also generated by submarine during the formation of the von Kármán phonon vortex street.

The cylinder of the diameter D immersed in the flame perpendicularly to the flame flow generates also the Strouhal friction tones.

The Saturn rings $R_i, i = 1, 2, ...$ are composed from massive objects with diameters $D_{i1}, D_{i2}, D_{i3}, ...$, moving in the relic photon see (Pardy, 2013a; ibid., 2013b) and producing the Strouhal friction tones. In such a way, the Saturn rings form the Saturn Aeolian harp in our planetary system. It is not excluded that the Strouhal friction tones of the Saturn rings can be detected by the special microphones of the Bell laboratories. At the samme time we can say that every strategic bombarder is visible because it produces von Kármán's vortex street which are detectable by the special radars.

The von Kármán vortexes are generated also in the Earth atmosphere , or in the Jupiter atmosphere, or, in other planetary atmospheres. It is not excluded that in a cosmological space, the von Kármán vortexes are generated during formation of galaxies, however, if and only if the hydrodynamical limit of cosmological matter is possible.

Let us remark in conclusion that the von Kármán vortexes and friction tones are also generated by the motion of bodies in liquid helium, which can be easily verified by the experiments in the low temperature laboratories.

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