

NÖETHER AND BELINFANTE CORRECTED TYPES OF CURRENTS FOR PERTURBATIONS IN THE EINSTEIN-GAUSS-BONNET GRAVITY

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Abstract

In the framework of an arbitrary D -dimensional metric theory, perturbations are considered on arbitrary backgrounds that are however solutions of the theory. Conserved currents for perturbations are presented following two known prescriptions: canonical Noether theorem and Belinfante symmetrization rule. Using generalized formulae, currents in the Einstein-Gauss-Bonnet (EGB) gravity for arbitrary types of perturbations on arbitrary curved backgrounds (not only vacuum) are constructed in an explicit covariant form. Special attention is paid to the energy-momentum tensors for perturbations which are an important part in the structure of the currents.

We use the derived expressions for two applied calculations: a) to present the energy density for weak flat gravitational waves in D -dimensional EGB gravity; b) to construct the mass flux for the Maeda-Dadhich-Molina 3D radiating black holes of a Kaluza-Klein type in 6D EGB gravity.

1 Introduction

As multidimensional theories become more and more popular, the study of the behaviour and properties of solutions in these theories has gained prominence. In particular, it is especially important to describe perturbations in such theories (see, for example, [1] - [6], and numerous references there in). It is very important to construct conserved quantities for such perturbations. Although several approaches have been developed, including our recent results (see [7] and references there in), these have been restricted, as a rule, to only constructing surface (*non-local*) expressions. The surface expressions are obtained after integration of so-called superpotentials. However, keeping in mind cosmological and astrophysical applications it is important to construct *local* conserved quantities (which usually are expressed by conserved currents for perturbations), and to connect them with non-local conserved quantities. In this approach, conservation laws are presented in a form where superpotentials are connected with correspondent conserved currents.

Arguably, at the present moment, among multidimensional generalizations of usual 4-dimensional general relativity (4D GR), a Gauss-Bonnet (GB) modification is the most popular. The action of the Einstein-Gauss-Bonnet (EGB) theory has a lower (quadratic in curvature) order of the action of the Lovelock gravity [8]. The latter is a generalization of GR, when an action includes higher order curvature terms preserving the diffeomorphism invariance and still leading to field equations containing no more than second order derivatives. On the other hand, independently, the GB term occurs in the effective lower energy action of superstring theory [9]. The EGB gravity has many new useful and interesting properties. Therefore in the framework of the EGB gravity numerous important topics and problems are intensively studied. They are multidimensional black hole solutions, black hole thermodynamics and conserved charges, AdS/CFT correspondence, wormhole solutions and their properties, cosmological dynamics, membrane paradigm, *etc.* It is impossible to give a full bibliography on studies related to Lovelock and EGB theories; for a review and further references one can recommend, e.g., [10], where many aforementioned aspects are discussed.

In [7], where superpotentials and correspondent charges in EGB gravity were constructed, we have used the following three approaches well known in 4D GR. The first approach, *canonical* (direct application of Noether theorem), starts from the Einstein pseudotensor [11] and the Freud superpotential [12]. The final and maximally generalized form in 4D GR is presented by Katz, Bičák and Lynden-Bell [13]. The second approach is based on the *Belinfante symmetrization* method [14], which firstly in 4D GR has been applied by Papapetrou [15] for symmetrization of the Einstein pseudotensor and for the correction of the Freud superpotential. Maximally generalized application of the Beinfante method in 4D GR

is presented in the works [16, 17]. The third approach is frequently called as *field-theoretical* (or *symmetrical*) one, where all perturbations (including metric ones) are presented as an united field configuration, which propagates in a background spacetime and is described by symmetrical (metric) energy-momentum tensor. For review of the above methods see [18].

To the best of our knowledge, unlike superpotentials, authors have not paid attention to constructing currents in modified theories. Here, at least in part, we try to close this gap. Following the proposals in [18], we present currents of a generalized form in an arbitrary metric theory in the *canonical* and *Belinfante symmetrization* approaches. Following this, the generalized formulae are used to construct the currents in EGB gravity. Thus, continuing the research begun in [19] and [7], we add superpotentials presented in [7] by corresponding currents. The symmetrical approach, due to its technical particularities, requires a separate investigation, therefore we do not consider it here.

The paper is organized as follows. In section 2, general definitions in an arbitrary D -dimensional metric theory are given, and general identities necessary for constructing conserved quantities are presented and discussed. In section 3, in an arbitrary metric theory conserved currents for arbitrary perturbations on arbitrary curved backgrounds that are however solutions of the theory, are presented in the framework of both the approaches. In section 4, the results of section 3 are used to construct explicit covariant expressions for the currents in EGB gravity. In section 5, the new expressions for the currents are used to construct energy density for weak flat gravitational waves and mass flux for radiating solutions in the EGB gravity. The concluding remarks are placed in the last section. In Appendix, we present the necessary, although somewhat cumbersome, expressions from EGB gravity.

2 Arbitrary D -dimensional metric theories. The main identities

2.1 Preliminaries

To present an arbitrary D -dimensional metric theory we consider the Lagrangian:

$$\hat{\mathcal{L}}_D = -\frac{1}{2\kappa_D}\hat{\mathcal{L}}_g(g_{\mu\nu}) + \hat{\mathcal{L}}_m(g_{\mu\nu}, \Phi). \quad (2.1)$$

One includes derivatives up to the second order of the metric $g_{\mu\nu}$ and Φ , where the last defines matter sources without concretization. Here and below “hat” means densities of the wight +1, for example, $\hat{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$, $\hat{\mathcal{L}} = \sqrt{-g}\mathcal{L}$, etc; $(, \alpha) \equiv \partial_\alpha$ means ordinary derivatives; Greek indexes enumerate D -dimensional spacetime coordinates. Variation of

(2.1) with respect to $g^{\mu\nu}$ leads to the gravitational equations:

$$\hat{\mathcal{G}}_{\mu\nu} = \kappa_D \hat{T}_{\mu\nu}. \quad (2.2)$$

Variation (2.1) with respect to Φ gives corresponding matter equations. In this section, we derive the identities applying both the Noether theorem only and the Noether theorem together with the Belinfante procedure to the gravitational part of the Lagrangian (2.1).

To examine perturbations, we need to consider the background D -dimensional spacetime. Let it belong the metric $\bar{g}_{\mu\nu}$, from which the background Christoffel symbols $\bar{\Gamma}_{\tau\rho}^\sigma$, covariant derivatives \bar{D}_α and the background Riemannian tensor $\bar{R}^\sigma{}_{\tau\rho\pi}$ are constructed; here and below “bar” means that a quantity is a background one. We also use the background Lagrangian defined as $\bar{\mathcal{L}}_D = \hat{\mathcal{L}}_D(\bar{g}_{\mu\nu}, \bar{\Phi})$ and corresponding background gravitational and matter equations. We assume that the background fields $\bar{g}_{\mu\nu}$ and $\bar{\Phi}$ satisfy the background equations, and, thus are known (fixed). Here, for our purposes, we incorporate the background metric $\bar{g}_{\mu\nu}$ into $\hat{\mathcal{L}}_g$ changing the ordinary derivatives ∂_α by the covariant \bar{D}_α ones in the usual way: $\partial_\alpha g_{\mu\nu} \equiv \bar{D}_\alpha g_{\mu\nu} - \bar{\Gamma}_{\alpha\rho}^\sigma g_{\mu\nu}|_\sigma^\rho$. The generalized notation $g_{\mu\nu}|_\beta^\alpha$ corresponds to the Lie derivative definition:

$$\mathcal{L}_\xi g_{\mu\nu} = -\xi^\alpha \bar{D}_\alpha g_{\mu\nu} + g_{\mu\nu}|_\beta^\alpha \bar{D}_\alpha \xi^\beta, \quad (2.3)$$

thus $g_{\mu\nu}|_\beta^\alpha = -2g_{\beta(\mu}\delta_{\nu)}^\alpha$; the choice of a sign corresponds to [20]. Then, using

$$\Delta_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha - \bar{\Gamma}_{\mu\nu}^\alpha = \frac{1}{2}g^{\alpha\rho} (\bar{D}_\mu g_{\rho\nu} + \bar{D}_\nu g_{\rho\mu} - \bar{D}_\rho g_{\mu\nu}), \quad (2.4)$$

$$R^\lambda{}_{\tau\rho\sigma} = \bar{D}_\rho \Delta_{\tau\sigma}^\lambda - \bar{D}_\sigma \Delta_{\tau\rho}^\lambda + \Delta_{\rho\eta}^\lambda \Delta_{\tau\sigma}^\eta - \Delta_{\eta\sigma}^\lambda \Delta_{\tau\rho}^\eta + \bar{R}^\lambda{}_{\tau\rho\sigma} = \delta R^\lambda{}_{\tau\rho\sigma} + \bar{R}^\lambda{}_{\tau\rho\sigma}, \quad (2.5)$$

we transform the pure metric Lagrangian $\hat{\mathcal{L}}_g$ into an explicitly covariant form: $\hat{\mathcal{L}}_g = \hat{\mathcal{L}}_c = \hat{\mathcal{L}}_c(g_{\mu\nu}; \bar{D}_\alpha g_{\mu\nu}; \bar{D}_{\beta\alpha} g_{\mu\nu})$, where $\bar{D}_{\beta\alpha} \equiv \bar{D}_\beta \bar{D}_\alpha$. Here and below δ means a difference between a dynamical and a background quantity. Thus for an arbitrary tensor density Q :

$$\delta Q = Q - \bar{Q} \quad (2.6)$$

that is a finite (exact, not infinitesimal) perturbation.

2.2 The Noether method and identities

A direct application of the *canonical* Noether procedure to $-\hat{\mathcal{L}}_c/2\kappa_D$, as a *scalar density*, gives the identity $-1/2\kappa_D (\mathcal{L}_\xi \hat{\mathcal{L}}_c + \partial_\alpha (\xi^\alpha \hat{\mathcal{L}}_c)) \equiv 0$, which is equivalent to

$$-\bar{D}_\alpha [\hat{u}_\sigma{}^\alpha \xi^\sigma + \hat{m}_\sigma{}^{\alpha\tau} \bar{D}_\tau \xi^\sigma + \hat{n}_\sigma{}^{\alpha\tau\beta} \bar{D}_\beta \bar{D}_\tau \xi^\sigma] \equiv \bar{D}_\alpha \hat{i}_C^\alpha \equiv \partial_\alpha \hat{i}_C^\alpha \equiv 0. \quad (2.7)$$

Here, the coefficients are defined in unique way (without ambiguities) by the Lagrangian:

$$\hat{u}_\sigma{}^\alpha = -\frac{1}{\kappa_D} [\hat{\mathcal{G}}_\sigma^\alpha + \kappa_D \hat{\mathcal{U}}_\sigma{}^\alpha + \kappa_D \hat{n}_\lambda{}^{\alpha\tau\beta} \bar{R}^\lambda{}_{\tau\beta\sigma}], \quad (2.8)$$

$$\hat{m}_\sigma^{\alpha\tau} = -\frac{1}{2\kappa_D} \left[\frac{\delta\hat{\mathcal{L}}_c}{\delta(\overline{D}_\alpha g_{\mu\nu})} g_{\mu\nu}|_\sigma^\tau - \frac{\partial\hat{\mathcal{L}}_c}{\partial(\overline{D}_{\tau\alpha} g_{\mu\nu})} \overline{D}_\sigma g_{\mu\nu} + \frac{\partial\hat{\mathcal{L}}_c}{\partial(\overline{D}_{\beta\alpha} g_{\mu\nu})} \overline{D}_\beta (g_{\mu\nu}|_\sigma^\tau) \right], \quad (2.9)$$

$$\hat{n}_\sigma^{\alpha\tau\beta} = -\frac{1}{4\kappa_D} \left[\frac{\partial\hat{\mathcal{L}}_c}{\partial(\overline{D}_\beta \overline{D}_\alpha g_{\mu\nu})} g_{\mu\nu}|_\sigma^\tau + \frac{\partial\hat{\mathcal{L}}_c}{\partial(\overline{D}_\tau \overline{D}_\alpha g_{\mu\nu})} g_{\mu\nu}|_\sigma^\beta \right]. \quad (2.10)$$

It can be seen that $\hat{n}_\sigma^{\alpha\tau\beta} = \hat{n}_\sigma^{\alpha\beta\tau}$. We use the notations

$$\hat{\mathcal{G}}_\sigma^\alpha \equiv \frac{1}{2} \frac{\delta\hat{\mathcal{L}}_c}{\delta g_{\mu\nu}} g_{\mu\nu}|_\sigma^\alpha \equiv -\frac{\delta\hat{\mathcal{L}}_c}{\delta g_{\mu\alpha}} g_{\mu\sigma} \equiv \frac{\delta\hat{\mathcal{L}}_c}{\delta g^{\mu\sigma}} g^{\mu\alpha}, \quad (2.11)$$

$$\hat{\mathcal{U}}_\sigma^\alpha \equiv -\frac{1}{2\kappa_D} \left(\frac{\partial\hat{\mathcal{L}}_c}{\partial(\overline{D}_{\beta\alpha} g_{\mu\nu})} \overline{D}_{\sigma\beta} g_{\mu\nu} + \frac{\delta\hat{\mathcal{L}}_c}{\delta(\overline{D}_\alpha g_{\mu\nu})} \overline{D}_\sigma g_{\mu\nu} - \delta_\sigma^\alpha \hat{\mathcal{L}}_c \right), \quad (2.12)$$

$$\frac{\delta\hat{\mathcal{L}}_c}{\delta(\overline{D}_\alpha g_{\mu\nu})} \equiv \frac{\partial\hat{\mathcal{L}}_c}{\partial(\overline{D}_\alpha g_{\mu\nu})} - \overline{D}_\beta \left(\frac{\partial\hat{\mathcal{L}}_c}{\partial(\overline{D}_{\alpha\beta} g_{\mu\nu})} \right). \quad (2.13)$$

As usual, $\delta\hat{\mathcal{L}}_c/\delta g_{\mu\nu}$ means Lagrangian derivatives, $\hat{\mathcal{G}}_\sigma^\alpha$ is exactly the symmetrical left hand side of (2.2), and $\hat{\mathcal{U}}_\sigma^\alpha$ is the generalized canonical energy-momentum related to the gravitational Lagrangian in (2.1).

The generalized current in (2.7) can be rewritten as

$$\hat{i}_C^\alpha = - \left[(\hat{u}_\sigma^\alpha + \hat{n}_\lambda^{\alpha\beta\gamma} \overline{R}^\lambda_{\beta\gamma\sigma}) \xi^\sigma + \hat{m}^{\rho\alpha\beta} \partial_{[\beta} \xi_{\rho]} + \hat{z}_C^\alpha \right] \quad (2.14)$$

where z -term is defined as

$$\hat{z}_C^\alpha(\xi) = \hat{m}^{\sigma\alpha\beta} \zeta_{\sigma\beta} + \hat{n}^{\rho\alpha\beta\gamma} \left(2\overline{D}_\gamma \zeta_{\beta\rho} - \overline{D}_\rho \zeta_{\beta\gamma} \right), \quad (2.15)$$

and $2\zeta_{\rho\sigma} = -\mathcal{L}_\xi \overline{g}_{\rho\sigma} = 2\overline{D}_{(\rho} \xi_{\sigma)}$. Thus, z -term disappears, if $\xi^\mu = \lambda^\mu$ is a Killing vector of the background spacetime. Then only the current (2.14) is determined by the energy-momentum ($u + n\overline{R}$)-term and the spin m -term.

Opening the identity (2.7) and, since ξ^σ , $\partial_\alpha \xi^\sigma$, $\partial_{\beta\alpha} \xi^\sigma$ and $\partial_{\gamma\beta\alpha} \xi^\sigma$ are arbitrary at every world point, equating independently to zero the coefficients at ξ^σ , $\overline{D}_\alpha \xi^\sigma$, $\overline{D}_{(\beta\alpha)} \xi^\sigma$ and $\overline{D}_{(\gamma\beta\alpha)} \xi^\sigma$ we get a system of covariant identities of the Belinfante-Rosenfeld type:

$$\overline{D}_\alpha \hat{u}_\sigma^\alpha + \frac{1}{2} \hat{m}_\lambda^{\alpha\rho} \overline{R}_{\sigma\rho\alpha}^\lambda + \frac{1}{3} \hat{n}_\lambda^{\alpha\rho\gamma} \overline{D}_\gamma \overline{R}_{\sigma\rho\alpha}^\lambda \equiv 0, \quad (2.16)$$

$$\hat{u}_\sigma^\alpha + \overline{D}_\lambda \hat{m}_\sigma^{\lambda\alpha} + \hat{n}_\lambda^{\tau\alpha\rho} \overline{R}_{\sigma\rho\tau}^\lambda + \frac{2}{3} \hat{n}_\sigma^{\lambda\tau\rho} \overline{R}_{\tau\rho\lambda}^\alpha \equiv 0, \quad (2.17)$$

$$\hat{m}_\sigma^{(\alpha\beta)} + \overline{D}_\lambda \hat{n}_\sigma^{\lambda(\alpha\beta)} \equiv 0, \quad (2.18)$$

$$\hat{n}_\sigma^{(\alpha\beta\gamma)} \equiv 0. \quad (2.19)$$

These are the generalization to arbitrary curved backgrounds of the expressions given by Mitzkevich [20]. The above identities are not independent: the first one (2.16) is a consequence of the other three.

As a rule, a divergence $\hat{\mathcal{L}}' = \hat{d}^{\nu}{}_{,\nu}$ in the Lagrangian is not essential for deriving field equations. However, in a definition of canonical conserved quantities it plays an important role. Let us illustrate this below. For the scalar density $\hat{\mathcal{L}}' = \hat{d}^{\nu}{}_{,\nu}$ one has the Noether identity

$$(\mathcal{L}_\xi \hat{d}^\alpha + \xi^\alpha \hat{d}^{\nu}{}_{,\nu})_{,\alpha} \equiv 0, \quad (2.20)$$

which has to be considered together with (2.7). An expression under divergence in (2.20) defines an additional contribution

$$\hat{i}'^\alpha = - \left[\hat{u}'_\sigma{}^\alpha \xi^\sigma + \hat{m}'_\sigma{}^{\alpha\tau} \overline{D}_\tau \xi^\sigma \right]; \quad (2.21)$$

$$\hat{u}'_\sigma{}^\alpha = 2\overline{D}_\beta(\delta_\sigma^{[\alpha} \hat{d}^{\beta]}), \quad \hat{m}'_\sigma{}^{\alpha\beta} = 2\delta_\sigma^{[\alpha} \hat{d}^{\beta]}, \quad \hat{n}'_\sigma{}^{\alpha\beta\gamma} = 0. \quad (2.22)$$

into the current (2.14). Note that the construction of the quantities (2.22) in the additional current (2.21) does not depend on the inner structure of \hat{d}^ν .

2.3 The Belinfante symmetrization

Using the Belinfante rule [14] generalized in [17] we define the Belinfante correction:

$$\hat{s}^{\alpha\beta\sigma} = -\hat{s}^{\beta\alpha\sigma} = -\hat{m}_\lambda{}^{\sigma[\alpha} \overline{g}^{\beta]\lambda} - \hat{m}_\lambda{}^{\alpha[\sigma} \overline{g}^{\beta]\lambda} + \hat{m}_\lambda{}^{\beta[\sigma} \overline{g}^{\alpha]\lambda}, \quad (2.23)$$

and modify (2.14). Thus, the Belinfante corrected current is

$$\hat{i}_B^\alpha = \hat{i}_C^\alpha + \overline{D}_\beta(\hat{s}^{\alpha\beta\sigma} \xi_\sigma) = - \left[(\hat{u}_\sigma{}^\alpha + \hat{n}_\lambda{}^{\alpha\beta\gamma} \overline{R}^\lambda{}_{\beta\gamma\sigma} - \overline{D}_\beta \hat{s}^{\alpha\beta}{}_\sigma) \xi^\sigma + \hat{z}_B^\alpha(\xi) \right]. \quad (2.24)$$

By definition, it does not contain the spin term (coefficient at $\partial_{[\beta} \xi_{\rho]}$). The new z -term

$$\hat{z}_B^\alpha(\xi) = \left(\overline{g}^{\lambda\sigma} \hat{m}_\lambda{}^{\beta\alpha} + 2\overline{g}^{\lambda[\sigma} \hat{m}_\lambda{}^{\alpha]\beta} \right) \zeta_{\sigma\beta} + \hat{n}_\rho{}^{\alpha\beta\gamma} \left(2\overline{D}_\gamma \zeta_\beta^\rho - \overline{D}^\rho \zeta_{\beta\gamma} \right) \quad (2.25)$$

disappears for Killing vectors of the background as well. Due to antisymmetry in (2.23), this current (2.24) is also identically conserved:

$$\partial_\alpha \hat{i}_B^\alpha \equiv \overline{D}_\alpha \hat{i}_B^\alpha \equiv 0. \quad (2.26)$$

It is important to note that the Belinfante procedure cancels the addition (2.21) induced by a divergence in Lagrangian.

Because the currents \hat{i}_C^α and \hat{i}_B^α satisfy the identities (2.7) and (2.26) they have to be expressed through correspondent antisymmetrical tensor densities (superpotentials) $\hat{i}_C^{\alpha\beta}$ and $\hat{i}_B^{\alpha\beta}$, for which $\partial_\alpha \partial_\beta \hat{i}_C^{\alpha\beta} \equiv \partial_\alpha \partial_\beta \hat{i}_B^{\alpha\beta} \equiv 0$. Indeed, following the standard prescription [18] and (2.17) - (2.19), one can construct these superpotentials satisfying

$$\hat{i}_C^\alpha \equiv \overline{D}_\beta \hat{i}_C^{\alpha\beta} \equiv \partial_\beta \hat{i}_C^{\alpha\beta}, \quad (2.27)$$

$$\hat{i}_B^\alpha \equiv \overline{D}_\beta \hat{i}_B^{\alpha\beta} \equiv \partial_\beta \hat{i}_B^{\alpha\beta}. \quad (2.28)$$

These demonstrate a principal form of a connection of the currents constructed here with the superpotentials in [7].

Here, the coefficients (2.8) - (2.10) are *uniquely* defined by the pure metric part of the Lagrangian (2.1). Consequently, Noether's and Noether-Belinfante's procedures give uniquely defined currents. The same claim is related, of course, to all the quantities constructed below for perturbations and based on the identities presented here.

3 Currents in arbitrary D -dimensional metric theories

In previous section, we have derived the identities and the identically conserved currents related to the *external* background spacetime which looks as an *auxiliary* structure. Here, we use these results to describe perturbations, which are determined when one (dynamical) solution of the theory is considered as a perturbed system with respect to another solution (background) of the same theory. Perturbations in such a scenario are exact (not infinitesimal or approximate), and the background spacetime acquires a *real* sense, and is not just an auxiliary structure. The same scheme, which can be named as bimetric, has been explored in [7]. Linear and higher order approximations simply follow once the exact form is presented.

3.1 Canonical Noether current

The expressions presented in subsection 2.2 are maximally adopted to construct Noether canonical conserved quantities in the framework of the bimetric formulation. Following the Katz-Bičák-Lynden-Bell (KBL) ideology [13] we construct the Lagrangian:

$$\hat{\mathcal{L}}_G = -\frac{1}{2\kappa_D} \left(\hat{\mathcal{L}}_g - \bar{\mathcal{L}}_g + \partial_\alpha \hat{d}^\alpha \right). \quad (3.1)$$

This Lagrangian, constructed for perturbations, has to be vanishing for vanishing perturbations. Thus usually \hat{d}^α is chosen to satisfy this requirement, i.e., to disappear for vanishing perturbations, see, e.g., [13, 21]. Applying the barred procedure to (2.14) and taking into account the divergence (using (2.21) and (2.22)), one obtains the current corresponding to (3.1): $\hat{I}_C^\alpha = \hat{i}_C^\alpha - \bar{\hat{i}}_C^\alpha + \hat{i}'^\alpha$. We then use the dynamical equations (2.2) in \hat{u}_σ^α . We change $\mathcal{G}_{\mu\nu}$ (as a part of \hat{u}_σ^α , see (2.8)) by the matter energy-momentum $T_{\mu\nu}$ at right hand side of (2.2). Next, we do the same combining $\bar{\hat{u}}_\sigma^\alpha$ and the barred equations (2.2). In the result one obtains that the identically conserved current \hat{I}_C^α related to (3.1) transforms into the current

$$\hat{\mathcal{I}}_C^\alpha(\xi) = {}_C\hat{\Theta}_\sigma{}^\alpha \xi^\sigma + {}_C\hat{\mathcal{M}}^{\sigma\alpha\beta} \partial_{[\sigma} \xi_{\beta]} + {}_C\hat{\mathcal{Z}}^\alpha(\xi) \quad (3.2)$$

for perturbations. Now, the conservation law:

$$\overline{D}_\alpha \hat{\mathcal{I}}_C^\alpha(\xi) = \partial_\alpha \hat{\mathcal{I}}_C^\alpha(\xi) = 0 \quad (3.3)$$

holds in place due to the field equations (not identically). The generalized canonical energy-momentum, spin and Z -term are

$${}_C \hat{\Theta}_\sigma^\alpha \equiv \delta \hat{T}_\sigma^\alpha + \delta \hat{\mathcal{U}}_\sigma^\alpha + \kappa_D^{-1} \overline{D}_\beta (\delta_\sigma^{[\alpha} \hat{d}^{\beta]}), \quad (3.4)$$

$${}_C \hat{\mathcal{M}}^{\sigma\alpha\beta} \equiv \delta \hat{m}_\rho^{\alpha\beta} \bar{g}^{\sigma\rho} - \kappa_D^{-1} \bar{g}^{\sigma[\alpha} \hat{d}^{\beta]}, \quad (3.5)$$

$${}_C \hat{\mathcal{Z}}^\alpha(\xi) \equiv -\delta \hat{z}_C^\alpha + \kappa_D^{-1} \zeta_\beta^{[\alpha} \hat{d}^{\beta]} \quad (3.6)$$

where perturbations are constructed following the general definition (2.6). To calculate perturbations one has to use quantities presented in (2.2), (2.9), (2.10), (2.12) and (2.15).

3.2 Belinfante symmetrized current

To construct the Belinfante corrected conserved currents for the perturbed system (3.1) we turn to subsection 2.3. We subtract the barred expression (2.24) from the original one (2.24): $\hat{I}_B^\alpha = \hat{i}_B^\alpha - \overline{\hat{i}}_B^\alpha$. Of course, the same is obtained after applying the Noether-Belinfante method directly to the Lagrangian in (3.1). Again, using the equations (2.2) and their barred version in \hat{u}_σ^α and $\overline{\hat{u}}_\sigma^\alpha$, the current \hat{I}_B^α related to (3.1) transforms into

$$\hat{\mathcal{I}}_B^\alpha(\xi) = {}_B \hat{\Theta}_\sigma^\alpha \xi^\sigma + {}_B \hat{\mathcal{Z}}^\alpha(\xi). \quad (3.7)$$

Thus, one has a conservation law

$$\overline{D}_\alpha \hat{\mathcal{I}}_B^\alpha(\xi) = \partial_\alpha \hat{\mathcal{I}}_B^\alpha(\xi) = 0 \quad (3.8)$$

for perturbations satisfying the field equations. As it has to be, the current (3.7) does not contain a spin term, unlike (3.2). The Belinfante corrected energy-momentum and Z -term are

$${}_B \hat{\Theta}_\sigma^\alpha \equiv \delta \hat{T}_\sigma^\alpha + \delta \hat{\mathcal{U}}_\sigma^\alpha + \overline{D}_\beta \delta \hat{s}^{\alpha\beta}{}_\sigma, \quad (3.9)$$

$${}_B \hat{\mathcal{Z}}^\alpha(\xi) \equiv -\delta \hat{z}_B^\alpha(\xi). \quad (3.10)$$

To calculate perturbations one has to use quantities presented in (2.2), (2.9), (2.10), (2.12), (2.23) and (2.25).

Note that, the energy-momenta (3.4) and (3.9) are separated into the two parts: matter and pure gravitational ones. However, this separation is conventional because the relation between these two parts can be changed easily by another combination with the field equations (2.2) that is quite permissible.

4 Currents in the Einstein-Gauss-Bonnet gravity

4.1 Preliminaries

In this section, we apply the theoretical results of the previous sections to derive the explicit structure of the currents in the EGB gravity of both the kinds (3.2) and (3.7). We search for the expressions in the most general form: they are not to be restricted by any concrete backgrounds or dynamic solutions. As a basis for calculation, we use expressions presented in the Appendix. Because Z-terms in (3.2) and (3.7) disappear for the Killing vectors of the background they are not so essential. Therefore, we do not give their explicit form, since these can be easily reconstructed using the auxiliary expressions from the Appendix.

The action of the Einstein D -dimensional theory with a bare cosmological term Λ_0 and a Gauss-Bonnet correction term (see, for example, [22]) is

$$\begin{aligned} S &= -\frac{1}{2\kappa_D} \int d^D x \hat{\mathcal{L}}_{EGB} + \int d^D x \hat{\mathcal{L}}_m \\ &= -\frac{1}{2\kappa_D} \int d^D x \sqrt{-g} [R - 2\Lambda_0 + \alpha(RR)_{GB}] + \int d^D x \hat{\mathcal{L}}_m, \end{aligned} \quad (4.1)$$

$$(RR)_{GB} \equiv R^2_{\alpha\beta\gamma\delta} - 4R^2_{\alpha\beta} + R^2, \quad (4.2)$$

where $\kappa_D = 2\Omega_{D-2}G_D > 0$ and $\alpha > 0$; G_D is the D -dimension Newton's constant. Below, the subscripts “ E ” is related to the pure Einstein part of the action (4.1), and the subscript “ $_{GB}$ ” is related to the Gauss-Bonnet part connected with α -coefficient. The field equations that follow from (4.1) have the form of (2.2) with

$$\begin{aligned} \hat{\mathcal{G}}^{\mu\nu} &= -\frac{\delta}{\delta g_{\mu\nu}} \hat{\mathcal{L}}_{EGB} = \sqrt{-g} \left\{ \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + g^{\mu\nu} \Lambda_0 \right) \right. \\ &\quad \left. + 2\alpha \left[RR^{\mu\nu} - 2R^\mu{}_\sigma{}^\nu{}_\rho R^{\sigma\rho} + R^\mu{}_{\sigma\rho\tau} R^{\nu\sigma\rho\tau} - 2R^\mu{}_\sigma R^{\sigma\nu} - \frac{1}{4} g^{\mu\nu} (RR)_{GB} \right] \right\}. \end{aligned} \quad (4.3)$$

4.2 Canonical prescription

Let us turn to the current (3.2). Its structure (3.4) - (3.6) essentially depends on the divergence in the Lagrangian. We choose the divergence induced by the Katz-Lifshits approach [23] (see discussions in [7, 24]). Thus, in (3.4) - (3.6) we consider

$$\hat{d}^\alpha = {}_{(E)}\hat{d}^\alpha + {}_{(GB)}\hat{d}^\alpha = 2\Delta_{\tau\beta}^{[\tau} \hat{g}^{\alpha]\beta} + 4\alpha \left(\hat{R}_\rho{}^{\beta\tau\alpha} - 2\hat{R}_\rho^{[\tau} g^{\alpha]\beta} - 2\delta_\rho^{[\tau} \hat{R}^{\alpha]\beta} + \delta_\rho^{[\tau} g^{\alpha]\beta} \hat{R} \right) \Delta_{\tau\beta}^\rho. \quad (4.4)$$

In D -dimensional GR, the Katz and Livshits superpotential [23] turns out *uniquely* the KBL superpotential [13]. In EGB gravity, their superpotential (essentially connected with (4.4) and the GB term (4.2)) naturally transfers into the KBL superpotential for $D = 4$. Thus,

although the GB term does not affect the derivation of the field equations for $D = 4$, it plays an important role (as a criterium) in definition of superpotentials of canonical type. The use of the term (4.2) in the Lagrangian even in *four dimensions* turns out important when the other ideas are elaborated. For example, in [25] Olea includes the GB term to regularize conserved quantities, in [26] Mišković and Olea show that the standard holographic regularization procedure of AdS gravity with counterterms is topological and, thus, can be presented by the addition of the GB term.

Now, not calculating Z -term, we construct (3.4) and (3.5). The matter part in (3.4) is defined by the sources in (2.2), however its concrete form will not be presented here. For present purpose, it is more interesting to focus on the gravitational part of (3.4) denoted below as $\hat{\mathcal{T}}_\sigma^\alpha$. Thus, we calculate (3.4) with the use of (A.5) and (4.4):

$$\begin{aligned}
\hat{\mathcal{C}}_\sigma^\alpha &= \delta \hat{T}_\sigma^\alpha + \hat{\mathcal{T}}_\sigma^\alpha = \sqrt{-g} T_\sigma^\alpha - \sqrt{-\bar{g}} \bar{T}_\sigma^\alpha + \frac{1}{2\kappa_D} \delta_\sigma^\alpha \left[\bar{R}_{\rho\tau} \hat{l}^{\rho\tau} - 2\Lambda_0 \delta \sqrt{-g} \right] \\
&+ \frac{\sqrt{-g}}{\kappa_D} \left[\left(\Delta_{\rho[\tau}^\alpha \Delta_{\pi]\sigma}^\pi + \Delta_{\rho[\sigma}^\alpha \Delta_{\pi]\tau}^\pi + \delta_\sigma^\alpha \Delta_{\beta[\tau}^\pi \Delta_{\pi]\rho}^\beta - \delta_\rho^\alpha \Delta_{\beta[\sigma}^\pi \Delta_{\pi]\tau}^\beta \right) g^{\tau\rho} \right] \\
&+ \frac{1}{\kappa_D} \bar{D}_\beta (\delta_\sigma^{[\alpha} (GB) \hat{d}^{\beta]}) + \frac{\alpha}{2\kappa_D} \delta_\sigma^\alpha \delta (\hat{R}R)_{GB} \\
&+ \frac{2\alpha\sqrt{-g}}{\kappa_D} \left[\left(R^{\alpha\beta\rho}{}_\tau - 4g^{\rho[\alpha} R_{\tau}^{\beta]} + Rg^{\rho[\alpha} \delta_{\tau}^{\beta]} \right) \bar{D}_\sigma \Delta_{\beta\rho}^\tau \right. \\
&+ \left. 2g^{\beta\mu} \left(\bar{D}_\beta R^{\alpha\nu} + 2\Delta_{\beta\rho}^{(\alpha} R^{\nu)\rho} \right) \Delta_{\sigma(\mu}^\tau g_{\nu)\tau} - g^{\rho(\alpha} \Delta_{\sigma\rho}^{\nu)} \partial_\nu R \right]. \tag{4.5}
\end{aligned}$$

The symbol δ without subscripts, once again, means a perturbation of a quantity with respect to a background (2.6); we also use the notation: $\hat{l}^{\rho\tau} = \delta \hat{g}^{\rho\tau} = \delta(\sqrt{-g} g^{\rho\tau})$ [17].

To calculate the spin term (3.5) we use (A.6) subtracting the barred (A.6) and taking into account (4.4). Thus

$$\begin{aligned}
{}_C \hat{\mathcal{M}}^{\sigma\alpha\beta} &= -\frac{\sqrt{-g}}{2\kappa_D} \left[\Delta_{\rho\tau}^\tau \left(2\bar{g}^{\sigma[\alpha} g^{\beta]\rho} + \bar{g}^{\sigma\rho} g^{\alpha\beta} \right) - \Delta_{\rho\tau}^\alpha \left(2\bar{g}^{\sigma[\tau} g^{\beta]\rho} + \bar{g}^{\sigma\rho} g^{\tau\beta} \right) \right] \\
&- \frac{1}{\kappa_D} \bar{g}^{\sigma[\alpha} (GB) \hat{d}^{\beta]} + \frac{2\alpha\sqrt{-g}}{\kappa_D} \left[R^{\alpha\tau\rho}{}_\lambda \Delta_{\tau\rho}^\beta - 2R^{\alpha(\tau\beta)}{}_\rho \Delta_{\tau\lambda}^\rho \right] \bar{g}^{\lambda\sigma} \\
&+ \frac{4\alpha\sqrt{-g}}{\kappa_D} \left[4g^{\rho[\alpha} R_{\tau}^{\beta]} \Delta_{\rho\lambda}^\tau + 2R_\lambda^{[\alpha} g^{\tau]\rho} \Delta_{\tau\rho}^\beta + 2g^{\alpha[\beta} R_{\rho}^{\tau]} \Delta_{\tau\lambda}^\rho + g^{\tau\beta} \left(R_\rho^\alpha \Delta_{\tau\lambda}^\rho - R_{(\tau}^\rho \Delta_{\lambda)\rho}^\alpha \right) \right. \\
&- \left. \left(g^{\rho\beta} \Delta_{\rho(\tau}^\tau + g^{\rho\tau} \Delta_{\rho(\tau}^\beta - g^{\tau\beta} \Delta_{\rho(\tau}^\rho \right) R_{\lambda)}^\alpha \right] \bar{g}^{\lambda\sigma} - \frac{4\alpha}{\kappa_D} \bar{g}^{\lambda\sigma} \bar{D}_{(\tau} \delta (\hat{g}^{\tau\beta} R_\lambda^\alpha) \\
&+ \frac{2\alpha\sqrt{-g}}{\kappa_D} R \left[\Delta_{\rho\lambda}^{[\alpha} g^{\rho]\beta} + \Delta_{\rho\lambda}^{(\alpha} g^{\beta)\rho} \right] \bar{g}^{\lambda\sigma} + \frac{2\alpha}{\kappa_D} \bar{g}^{\sigma(\alpha} \bar{D}_\rho \delta (\hat{g}^{\rho)\beta} R). \tag{4.6}
\end{aligned}$$

As expected, these expressions disappear for vanishing perturbations. The Einstein parts in (4.5) and (4.6) exactly coincide with the energy-momentum and the spin tensor presented in [13]. We do not present explicitly the terms with $(GB) \hat{d}^\alpha$ because this does not simplify the expression as a whole.

4.3 Prescription of the generalized Belinfante procedure

Here, we turn to the current (3.7). Its structure, see (3.9) and (3.10), unlike the canonical case, does not depend on a spin term and a divergence in the Lagrangian. As before, we do not consider Z -term. Constructing (3.9) we calculate explicitly the pure gravitational part only, it is denoted below as ${}_B\hat{\mathcal{T}}^{\sigma\alpha}$. We substitute (A.5) and (A.8) into (3.9), raise indices, use the field equations (2.2) with (4.3) and, as a result, obtain

$$\begin{aligned}
{}_B\hat{\Theta}^{\sigma\alpha} &= \delta\hat{T}_\rho^{(\alpha}\bar{g}^{\sigma)\rho} + {}_B\hat{\mathcal{T}}^{\sigma\alpha} = \sqrt{-g}T_\rho^{(\alpha}\bar{g}^{\sigma)\rho} - \sqrt{-\bar{g}}\bar{T}^{\alpha\sigma} \\
&+ \frac{1}{2\kappa_D} \left[\hat{l}^{\rho\tau}\bar{R}_{\rho\tau}\bar{g}^{\alpha\sigma} + 2\hat{l}^{\lambda[\alpha}\bar{R}_\lambda^{\sigma]} - 2\bar{g}^{\alpha\sigma}\Lambda_0\delta\sqrt{-g} \right] \\
&+ \frac{1}{2\kappa_D} \left[\left(\hat{l}^{\alpha\sigma}\bar{g}^{\rho\tau} - \bar{g}^{\alpha\sigma}\hat{l}^{\rho\tau} \right) \bar{D}_\tau\Delta_{\rho\lambda}^\lambda + 2 \left(\hat{l}^{\rho\tau}\bar{g}^{\lambda(\alpha} - \bar{g}^{\rho\tau}\hat{l}^{\lambda(\alpha)} \right) \bar{D}_\tau\Delta_{\lambda\rho}^{\sigma]} \right] \\
&+ \frac{1}{2\kappa_D} \left[\bar{g}^{\rho\tau} \left(\hat{g}^{\alpha\sigma}\Delta_{\rho\lambda}^\lambda\Delta_{\tau\eta}^\eta + 2\hat{g}^{\lambda\eta}\Delta_{\lambda\rho}^{(\alpha}\Delta_{\eta\tau}^{\sigma)} \right) + \hat{g}^{\lambda\eta}\bar{g}^{\alpha\sigma}\Delta_{\rho\lambda}^\tau\Delta_{\tau\eta}^\rho \right] \\
&+ \frac{1}{\kappa_D} \left[\bar{g}^{\rho\tau} \left(\Delta_{\tau\eta}^\lambda\Delta_{\lambda\rho}^{(\alpha}\hat{g}^{\sigma)\eta} - 2\Delta_{\tau\lambda}^\lambda\Delta_{\eta\rho}^{(\alpha}\hat{g}^{\sigma)\eta} \right) + \hat{g}^{\lambda\eta} \left(\Delta_{\rho\tau}^\tau\Delta_{\lambda\eta}^{(\alpha} - \Delta_{\lambda\eta}^\tau\Delta_{\rho\tau}^{(\alpha} - \Delta_{\lambda\rho}^\tau\Delta_{\eta\tau}^{(\alpha)} \right) \bar{g}^{\sigma)\rho} \right] \\
&+ \frac{2\alpha\sqrt{-g}}{\kappa_D}\bar{g}^{\lambda\sigma} \left[\left(R^{\alpha\beta\rho}{}_\tau - 4g^{\rho[\alpha}R_\tau^{\beta]} + Rg^{\rho[\alpha}\delta_\tau^{\beta]} \right) \bar{D}_\lambda\Delta_{\beta\rho}^\tau - g^{\rho(\alpha}\Delta_{\lambda\rho}^{\beta)}\bar{D}_\beta R \right. \\
&+ \left. 2g^{\beta\mu} \left(\bar{D}_\beta R^{\alpha\nu} + 2R^{\rho(\alpha}\Delta_{\beta\rho}^{\nu)} \right) \Delta_{\lambda(\mu}^\tau g_{\nu)\tau} \right] + \frac{\alpha}{2\kappa_D}\bar{g}^{\alpha\sigma}\delta(\hat{R}R)_{GB} + \bar{D}_\beta\delta_{(GB)}\hat{s}^{\alpha\beta\sigma} \\
&- \frac{\alpha\sqrt{-g}}{\kappa_D}\bar{g}^{\lambda[\alpha} \left[R^{\sigma]}{}_{\pi\rho\tau}R_\lambda{}^{\pi\rho\tau} - 2R^{\sigma]}{}_{\pi\lambda\rho}R^{\pi\rho} - 2R_\rho^{[\sigma]}R_\lambda^{\rho]} + R_\lambda^{[\sigma]}R^{\lambda]} \right]. \tag{4.7}
\end{aligned}$$

The Einstein part exactly coincides with the one presented in [17]. Recall that, even this part (symmetrized) is not symmetrical in general, see [17]. Here, we do not open the divergence of the GB-part $\delta_{(GB)}\hat{s}^{\alpha\beta\sigma}$ of the Belinfante correction, see (2.23), because this does not simplify the expression; for calculations it is more convenient to use the already known/calculated components obtained with using (A.8). The *symmetrical* matter part and the last line in (4.7) are the result of the secondary use of the field equations (2.2) with (4.3). Like (4.5), the energy-momentum (4.7) disappears for vanishing perturbations, note that the barred last line in (4.7) vanishes due to the antisymmetrization.

5 Applications

5.1 Weak flat gravitational waves

Here, we use formulae from previous sections to calculate energy density for weak flat gravitational wave in the EGB gravity. Such a gravitational wave propagates in D -dimensional flat spacetime and is described by the linearized vacuum equations (2.2) with (4.3). Due

to the requirement of the flat background the GB part in (4.3), being quadratic in curvature components, does not contribute to the linearized equations, and $\Lambda_0 = 0$. Thus, effectively, these are the linear Einstein equations in D dimensions without cosmological term. Assume that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and that the Lorentz coordinates are used. Then one has $\eta_{\mu\nu} = \text{diag}(-1, 1 \dots 1)$ and $\Delta_{**}^* \sim h_{**, *}$. Linear Einstein equations after applying the standard technique of the TT -gauge [27] have a form

$$R_{\mu\nu} = -\frac{1}{2}h_{\mu\nu, \alpha}{}^{, \alpha} = 0. \quad (5.1)$$

Assuming $h_{\mu\nu} = h_{\mu\nu}(t - x) = h_{\mu\nu}(x^0 - x^1)$ one obtains $h_{0\alpha} = h_{1\alpha} = h_{\alpha}^{\alpha} = 0$, and non-zero components are h_{kl} where the Latin indices from the middle of alphabet numerate: $k, l \dots = 2, 3 \dots, D - 1$.

Let us turn to the canonical prescription. To calculate the energy density one has to calculate the 0-component of the current (3.2) with the Killing vector $\xi^\alpha = \lambda^\alpha = (-1, \mathbf{0})$; $\lambda_\alpha = (1, \mathbf{0})$. Then, only the 00-component of the pure gravitational energy-momentum ${}_C\hat{\mathcal{T}}_0^0$ in (4.5) contributes (without spin term (4.6)); for the linearized wave we calculate ${}_C\hat{\mathcal{T}}_0^0$ up to the second order. In direct calculations we take into account: a) a flat background with zero Riemannian tensor and its contractions; b) zero linear parts of the Ricci tensor and curvature scalar due to (5.1); c) proportionality $\Delta_{**}^* \sim h_{**, *}$. In the end we show that, including the quadratic terms, the GB part of ${}_C\hat{\mathcal{T}}_0^0$ is equal to zero as a whole. Considering the Einstein part one obtains in quadratic approximation:

$${}_C\mathcal{T}_0^0 = -\frac{1}{4\kappa_D} \sum_{k,l=2}^{D-1} \dot{h}_{kl}^2 \quad (5.2)$$

where dot means differentiation with respect to $t = x^0$.

Now we turn to the Belinfante prescription to calculate the 0-component of the current (3.7). Keeping in mind the above assumptions and using the Killing vector λ^α we need to calculate only the 00-component of the pure gravitational energy momentum ${}_B\hat{\mathcal{T}}^{00}$ in (4.7). Again the GB part is equal to zero in quadratic approximation. Thus, ${}_B\hat{\mathcal{T}}^{00}$ is also defined by the Einstein part only:

$${}_B\mathcal{T}^{00} = \frac{1}{4\kappa_D} \sum_{k,l=2}^{D-1} \dot{h}_{kl}^2. \quad (5.3)$$

Contracting both (5.2) and (5.3) with the Killing vector λ^α one obtains the unique expression for the energy density of the flat weak gravitational waves:

$$\mathcal{I}_C^0 = \mathcal{I}_B^0 = \frac{1}{4\kappa_D} \sum_{k,l=2}^{D-1} \dot{h}_{kl}^2. \quad (5.4)$$

This is in a full correspondence with the standard results in 4D GR [27]. Because the equations (5.1), in fact, are the Einstein ones the energy density (5.4) is acceptable. Thus, the new energy-momentum expressions applied here to describe flat gravitational waves satisfy simple, but important and non-trivial, test. Indeed, the equations (5.1) are from the *start* the linearized EGB equations, not the Einstein equations; the currents (3.2) (with (4.5) and (4.6)) and (3.7) (with (4.7)) from the *start* have been constructed in the framework of the EGB gravity, not the Einstein gravity.

5.2 Radiative 3D black hole of the Kaluza-Klein type

In this subsection, we apply the new formulae to describe mass fluxes for interesting and important solutions obtained recently in the works [28, 29]. The main assumption is that a spacetime is to be locally homeomorphic to $\mathcal{M}^d \times \mathcal{K}^{D-d}$ with the metric $g_{\mu\nu} = \text{diag}(g_{AB}, r_0^2 \gamma_{ab})$, $A, B = 0, \dots, d-1$; $a, b = d, \dots, D-1$. Thus, g_{AB} is an arbitrary Lorentz metric on \mathcal{M}^d ; γ_{ab} is the unit metric on the $(D-d)$ -dimensional space of constant curvature \mathcal{K}^{D-d} with $k = 0, \pm 1$. Factor r_0 is a small scale of extra dimensions. Vacuum gravitational equations $\mathcal{G}^\mu{}_\nu = 0$ (see (4.3)) are decomposed into two separate systems $\mathcal{G}^A{}_B = 0$ and $\mathcal{G}^a{}_b = 0$. The first one is a tensorial equation on \mathcal{M}^d , whereas the second one is a constraint for it. However, to obtain more interesting solutions one has to consider a special case, when the expression $\mathcal{G}^A{}_B$ disappears *identically*. This is possible for $d \leq 4$ only. In this case, constants are chosen so as to suppress the coefficients in $\mathcal{G}^A{}_B$, which is possible when $D \geq d+2$, $k = -1$ and $\Lambda_0 < 0$. After taking into account all of the above, a single governing equation is $\mathcal{G}^a{}_b = 0$. In reality it is the unique scalar equation on \mathcal{M}^d because $\mathcal{G}^a{}_b \sim \delta_b^a$ and depends on g_{AB} only.

Here, we consider the case $D = 6$ and $d = 3$ presented in [29]. A suitable set of constraints for the constants is $r_0^2 = 12\alpha = -3/\Lambda_0$. Then, the unique scalar equation is

$${}_{(d)}R = 2\Lambda_0, \quad (5.5)$$

where subscript ‘ ${}_{(d)}$ ’ imply that a quantity is constructed with the use of g_{AB} only. This scalar equation is satisfied by both the static and the radiative metric [29]. Here, for constructing the mass fluxes, it is quite appropriate to use the new current expressions. We apply them to radiative solution $g_{AB}(v, r)$ of the Vaidya type [29]:

$$ds^2 = -fdv^2 + 2dvdr + r^2d\phi, \quad f \equiv r^2/l^2 + q(v)/r - \mu(v) \quad (5.6)$$

where $l^2 \equiv -3/\Lambda_0$. In this concrete case $\mu(v)$ and $q(v)$ depend on the advanced time v . Non-zero components corresponding to the solution (5.6), $d = 3$ sector, are as follows.

Metric components are $g_{00} = -f(v)$, $g_{01} = 1$, $g_{22} = r^2$; Christoffel symbols, components of Riemannian and Ricci tensors, curvature scalar, and components of the Einstein tensor are

$$\begin{aligned}\Gamma_{00}^1 &= (ff' - \dot{f})/2, \Gamma_{00}^0 = f'/2, \Gamma_{01}^1 = -f'/2, \Gamma_{12}^2 = 1/r, \Gamma_{22}^1 = -rf, \Gamma_{22}^0 = -r, \quad (5.7) \\ R^{0101} &= \frac{1}{2}f'', \quad R^{0212} = -\frac{f'}{2r^3}, \quad R^{1212} = -\frac{1}{2r^3}(ff' + \dot{f}); \\ R^{11} &= -\frac{1}{2r} [f(rf'' + f') + \dot{f}], \quad R^{01} = -\frac{1}{2r}(rf'' + f'), \quad R^{22} = -\frac{f'}{r^3}; \\ R &= -\frac{1}{r}(rf'' + 2f'), \quad (5.8)\end{aligned}$$

$$G_0^0 = G_1^1 = 1/l^2 - q/2r^3, \quad G_0^1 = (\dot{\mu}r - \dot{q})/2r^2, \quad G_2^2 = 1/l^2 + q/r^3, \quad (5.9)$$

where ‘prime’ and ‘dot’ mean $\partial/\partial r$ and $\partial/\partial v$. The scalar curvature of $D - d = 3$ sector is

$${}_{(D-d)}R = 6k/r_0^2 = 2\Lambda_0 = -1/2\alpha. \quad (5.10)$$

In fact, (5.6) - (5.9) together with (5.10) present 6D solution in EGB gravity. Whereas (5.6) - (5.9) without (5.10) can be considered as a solution to the Einstein 3D equations on \mathcal{M}^3 , which are not vacuum equations with redefined cosmological constant $\Lambda = -1/l^2$:

$${}_{(3)}R_{AB} - \frac{1}{2}g_{AB}{}_{(3)}R + g_{AB}\Lambda = \kappa_3 T_{AB}. \quad (5.11)$$

A natural treating in [29] is that T_{AB} , corresponding to (5.9), is created by extra dimensions.

Here, both the full 6D presentation in the framework of the EGB gravity and the 3D interpretation (5.11) are explored. We consider a cylinder $S := r = \text{const}$. The wall S can be thought as 5D timelike hypersurface in 6D spacetime, or as 2D timelike hypersurface in 3D spacetime; $\partial\Sigma$ is an intersection of S with a lightlike hypersurface $v = \text{const}$. To present a mass flux through $\partial\Sigma$ one has to calculate the component $\hat{\mathcal{T}}^1$ of the currents (3.2) or (3.7) and integrate it over $\partial\Sigma$. The total background metric in the 6D derivation can be chosen as $\bar{g}_{\mu\nu} = \bar{g}_{AB} \times r_0^2 \gamma_{ab}$ with the AdS₃ metric \bar{g}_{AB} presented by $\bar{f} \equiv r^2/l^2 + 1$ in the element of the type (5.6), see [24]. Whereas the background metric in the 3D derivation is chosen as the same AdS₃ metric \bar{g}_{AB} only. Background components are derived from (5.6) - (5.10) after applying the barred procedure. To calculate the *mass* flux we use the *timelike* background Killing vector $\xi^\alpha = \lambda^\alpha = (-1, \mathbf{0})$; $\lambda_\alpha = (\bar{f}, -1, \mathbf{0})$ where $\mathbf{0}$ includes all the rest space dimensions both for 6D and for 3D derivations.

Let us present results of calculations in the framework of the canonical prescription of subsection 3.1 with the formulae of subsection 4.2 in detail. Turn to the 6D derivation. Then using all the components (5.6) - (5.10) together with the barred ones, we substitute them into (4.5) and (4.6). Recall that $Z(\lambda)$ -term disappears, and note that, unlike subsection 5.1,

we need to calculate the spin term (4.6). After very prolonged and cumbersome calculations we obtain for (3.2): ${}_{(E)}\hat{\mathcal{I}}_C^1(\lambda) \equiv {}_{(GB)}\hat{\mathcal{I}}_C^1(\lambda) \equiv 0$ that gives $\hat{\mathcal{I}}_C^1(\lambda) \equiv 0$. Thus,

$$\dot{M} = \oint_{\partial\Sigma} dx^{D-2} \hat{\mathcal{I}}_C^1 \equiv 0. \quad (5.12)$$

At a first glance, the result (5.12) looks strange. However, it is in full correspondence with the results in [24] where we have just calculated the masses of $d = 3$ objects in $D = 6$ EGB gravity with the use of the superpotentials constructed in [7]. Let us demonstrate this correspondence. A general expression for the total mass has been obtained as a surface integral in $D = 6$ dimensions [24]:

$$M = \oint_{\partial\Sigma} dx^{D-2} \sqrt{-\bar{g}_D} \mathcal{I}^{01} = \oint_{r \rightarrow \infty} d\phi \sqrt{-\bar{g}_d} \mathcal{I}^{01} \oint_{r_0} dx^{D-d} \sqrt{-\bar{g}_{D-d}} = V_{r_0} \oint_{r \rightarrow \infty} d\phi \sqrt{-\bar{g}_d} \mathcal{I}^{01}. \quad (5.13)$$

Integration over $d = 3$ sector gives zero. In the canonical approach, it is so because $\mathcal{I}_C^{01} \equiv 0$ (in more details ${}_{(GB)}\hat{\mathcal{I}}_C^{01} \equiv -{}_{(E)}\hat{\mathcal{I}}_C^{01} \neq 0$). For the Belinfante corrected approach the integration over $d = 3$ sector gives zero due to the asymptotic behaviour of \mathcal{I}_B^{01} , in spite of $\mathcal{I}_B^{01} \neq 0$. The formula (5.13) shows that one needs to consider two possibilities: (i) when extra $D - d = 3$ dimensions are not compactified; (ii) when they are compactified by appropriate identifications.

In the case (i), one has to consider objects as 6 dimensional ones. Of course, in spite of $V_{r_0} \rightarrow \infty$, their masses in (5.13) have to be equated to zero. Next, because M is defined for arbitrary $\partial\Sigma$ one has $M_{\partial\Sigma_0} = M_{\partial\Sigma_1} = 0$ that determines null flux through $\partial\Sigma$. One finds just a correspondence of (5.12) with the corresponding conclusion in [24].

The case (ii), in our opinion, has a more physical sense. Now, V_{r_0} in (5.13) is finite. Then, because $\mathcal{I}^{01} \sim 1/\kappa_6$ one can set $\kappa_3 = \kappa_6/V_{r_0}$, and, really, in (5.13) one has $M \sim 1/\kappa_3$. This means that the 6D Einstein constant κ_6 is reduced to the 3-dimensional one κ_3 that is the standard Kaluza-Klein prescription. One has to reject 6D derivation and turn to 3D derivation with the Einstein presentation (5.11) and with the evident interpretation of κ_3 . In this case, null mass is quite unacceptable. Therefore one has to use ingredients of the Einstein theory only, and not the EGB one. Thus, applying superpotentials constructed in [7], we have used their Einstein parts only, changing κ_6 by κ_3 : ${}_{(E)}\hat{\mathcal{I}}^{01} = (\mu + 1 - q/r)/2\kappa_3$ that gives acceptable mass for the solution (5.6) on the AdS₃ background [24]:

$${}_{(E)}M = \oint_{\partial\Sigma} dx^{D-2} {}_{(E)}\hat{\mathcal{I}}_C^{01} = (\mu + 1)\pi/\kappa_3. \quad (5.14)$$

Exploring the expressions for currents presented here one has to use the Einstein interpretation (5.11) also. However now, unlike the superpotential application, we cannot use the reduced the Einstein part of the current of the 6D description because ${}_{(GB)}\hat{\mathcal{I}}_C^1 \equiv {}_{(E)}\hat{\mathcal{I}}_C^1 \equiv 0$.

Nevertheless, there is no a contradiction. Recall that in 6D picture we are based on the *vacuum* EGB equation (pure gravitational), whereas for the 3D description (5.11) constructing currents (see (3.2) and (3.4)) we must use the created matter at the right hand side of (5.11). Thus, in 3D derivation, the component $T^1_0 = (\dot{\mu}r - \dot{q})/2\kappa_3 r^2$ in (5.11) just determines ${}_{(E)}\hat{\mathcal{I}}^1_C = \sqrt{-g}T^1_0\lambda^0 = -(\dot{\mu} - \dot{q}/r)/2\kappa_3$ by a crucial way. This gives the flux:

$${}_{(E)}\dot{M} = \oint_{\partial\Sigma} dx^{D-2} {}_{(E)}\hat{\mathcal{I}}^1_C = -\dot{\mu}\pi/\kappa_3. \quad (5.15)$$

Differentiating mass (5.14) (obtained in the framework of the superpotential derivation) with respect to v one obtains: ${}_{(E)}\dot{M} = \dot{\mu}\pi/\kappa_3$ [24]. One can see a difference in a sign, however there is no contradiction. A simplified differentiation of M with respect to v gives, in fact, an absolute value of the flux. A check with using $\hat{\mathcal{I}}^1 = \partial_0\hat{\mathcal{I}}^{10}$ and antisymmetry $\hat{\mathcal{I}}^{10} = -\hat{\mathcal{I}}^{01}$ shows a correspondence in signs also.

The same conclusions follow when the Belinfante symmetrization method developed in subsections 3.2 and 4.3 is applied. Though, unlike the canonical approach, in 6D description: $\mathcal{I}^1_B \neq 0$; and, in 3D interpretation, ${}_{(E)}\mathcal{I}^1_B$ is not determined by the created energy-momentum in (5.11) only. However, due to the asymptotic behaviour, additional terms do not contribute into the final expressions after integration. Thus, once again, in the case (i), one obtains a zero mass flux (5.12), whereas, in the case (ii), one needs to use the 3D (5.11) interpretation and has the flux (5.15).

6 Concluding remarks

The modern development of multidimensional metric theories themselves, naturally, includes/induces a development of methods for constructing conserved quantities. In the present paper, in the framework of the D -dimensional EGB gravity we have presented the explicit covariant expressions for the conserved currents of perturbations of an arbitrary type on arbitrary curved backgrounds. The two methods, *canonical* and *Belinfante corrected*, have been applied. The main parts in the structure of the canonical and Belinfante corrected currents, which are the energy-momentum tensors (4.5) and (4.7), are the generalization of the Einstein pseudotensor [11] and of the Papapetrou pseudotensor [15], respectively.

Together with an evident academic interest, a construction of such currents can be very useful in applications. Indeed, many solutions of modified metric theories need to be examined in detail. It is necessary because frequently such solutions look quite exotic, and one has to understand the physical meaning they represent, how contradictive or non-contradictive they are, *etc.* Thus, by presenting rules for constructing conserved quantities including

conserved currents that are important physical characteristics of objects, we present the instrument for analyzing these objects.

Applications in section 5, should be seen as tests for the new expressions. Indeed, weak flat gravitational wave is the standard object with well studied properties; also properties of the 3D radiating black holes already have been studied by us in [24]. However, the direct calculation of the mass flux using the new current expressions can be viewed as an important independent result. In [28, 29], the matter presented by the energy-momentum at the right hand side of (5.11) is treated as being created by all the extra dimensions as a whole. Of course, such a derivation differs from the standard Kaluza-Klein picture where each of compactified extra dimensions determines its own charge. Nevertheless, as we show in subsection 5.2, the compactified dimensions are reduced in the standard Kaluza-Klein prescription. Also, we demonstrate that the created matter in (5.11) determines the *classically* defined mass (5.14) (see [24]) and mass flux (5.15) of the objects. Thus, keeping in mind the above comments, we support the claim of the authors of [28, 29] that their solutions present the objects of the Kaluza-Klein type.

It is important to compare our results with the results by Cai, Cao and Ohta [30]. The authors, in the framework of the Lovelock gravity of an arbitrary order, have constructed and analyzed a new solutions analogous to (5.6)-(5.10), only static. Using the Wald technique [31], they have proved that the objects corresponding to such solutions have zero entropy and, consequently, zero mass. This coincides with our conclusions. Indeed, [30] uses the EGB Lagrangian in all D dimensions. In our consideration, in the case (i), when objects are examined in all 6 dimensions, both mass of the objects [24] and their mass flux (even in radiative regime (5.12)) are equal to zero.

In the future, we intend to continue to construct conserved quantities in EGB gravity, and present conserved currents in the framework of the *symmetrical* approach (see Introduction). Also, we plan to use the new expressions, both for the superpotentials and for the currents in EGB gravity, to describe interesting solutions, say, 4D objects in D -dimensional EGB gravity [28].

Lastly, the possibility of a connection between AdS gravity and a conformal field theory (CFT) living on its boundary induces a considerable attention. A definition of conserved quantities and an existence of nonzero energy for asymptotically AdS vacuum spacetime could be useful to identify the AdS/CFT correspondence at the boundary. To define finite conserved quantities one requires a regularization procedure, the mechanism of which does not invoke the subtraction of background configurations. In this context, one of more popular approaches is the *boundary counterterm method*. It is developing more intensively

in the framework of EGB gravity, one can recommend, e.g., interesting works [32, 33] and numerous references there in. Unlike the boundary counterterm method, the prescriptions explored here (and in [7, 18, 24]) use the background spacetime in a crucial way. Nevertheless, it could be very useful to compare these approaches, and we plan to do this in future.

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A Auxiliary expressions in EGB gravity

In this appendix, we calculate the coefficients, which are derived following the definitions (2.8), (2.9) and (2.10), and correspond to the Lagrangian $\hat{\mathcal{L}}_{EGB}$ in (4.1). However, at the first it is useful to present the next derivatives:

$$\begin{aligned}
& \frac{-2\kappa_D}{\sqrt{-g}} \frac{\partial \hat{\mathcal{L}}_{EGB}}{\partial g_{\mu\nu}} = \frac{-2\kappa_D}{\sqrt{-g}} \left(\frac{\partial \hat{\mathcal{L}}_E}{\partial g_{\mu\nu}} + \frac{\partial \hat{\mathcal{L}}_{GB}}{\partial g_{\mu\nu}} \right) \\
& = \frac{1}{2} g^{\mu\nu} (R - 2\Lambda_0) + \left[2 \left(g^{\rho(\mu} \Delta_{[\sigma}^{\nu)} \Delta_{\rho] \tau}^\alpha + g^{\alpha\rho} \Delta_{\tau[\sigma}^{(\mu} \Delta_{\rho]\alpha}^{\nu)} + g^{\rho(\mu} \bar{D}_{[\sigma} \Delta_{\rho]\tau}^{\nu)} \right) g^{\tau\sigma} - R^{\mu\nu} \right] \\
& + \frac{\alpha g^{\mu\nu}}{2} \left(R_{\lambda\tau\rho\sigma} R^{\lambda\tau\rho\sigma} - 4R_{\rho\sigma} R^{\rho\sigma} + R^2 \right) \\
& + 2\alpha \left[2 \left(g^{\lambda(\mu} \Delta_{\alpha\sigma}^{\nu)} \Delta_{\rho\tau}^\alpha + g^{\alpha\lambda} \Delta_{\tau\sigma}^{(\mu} \Delta_{\rho\alpha}^{\nu)} + g^{\lambda(\mu} \bar{D}_{\sigma} \Delta_{\rho\tau}^{\nu)} \right) R_{\lambda}{}^{\tau\rho\sigma} - R^{\mu\tau\rho\sigma} R^{\nu}{}_{\tau\rho\sigma} \right] \\
& - 8\alpha \left[2 \left(g^{\rho(\mu} \Delta_{\alpha[\sigma}^{\nu)} \Delta_{\rho]\tau}^\alpha + g^{\alpha\rho} \Delta_{\tau[\sigma}^{(\mu} \Delta_{\rho]\alpha}^{\nu)} + g^{\rho(\mu} \bar{D}_{[\sigma} \Delta_{\rho]\tau}^{\nu)} \right) R^{\tau\sigma} - R^{\mu\rho} R_{\rho}^{\nu} \right] \\
& + 2\alpha R \left[2 \left(g^{\rho(\mu} \Delta_{\alpha[\sigma}^{\nu)} \Delta_{\rho]\tau}^\alpha + g^{\alpha\rho} \Delta_{\tau[\sigma}^{(\mu} \Delta_{\rho]\alpha}^{\nu)} + g^{\rho(\mu} \bar{D}_{[\sigma} \Delta_{\rho]\tau}^{\nu)} \right) g^{\tau\sigma} - R^{\mu\nu} \right]; \tag{A.1}
\end{aligned}$$

$$\begin{aligned}
& \frac{-2\kappa_D}{\sqrt{-g}} \frac{\partial \hat{\mathcal{L}}_{EGB}}{\partial (\bar{D}_{\alpha} g_{\mu\nu})} = \frac{-2\kappa_D}{\sqrt{-g}} \left(\frac{\partial \hat{\mathcal{L}}_E}{\partial (\bar{D}_{\alpha} g_{\mu\nu})} + \frac{\partial \hat{\mathcal{L}}_{GB}}{\partial (\bar{D}_{\alpha} g_{\mu\nu})} \right) \\
& = 2 \left[\Delta_{\sigma\rho}^{\alpha} g^{\sigma[\rho} g^{\mu]\nu} + g^{\alpha\sigma} \Delta_{\sigma\rho}^{(\mu} g^{\nu)\rho} - g^{\alpha(\mu} \Delta_{\sigma\rho}^{\nu)} g^{\sigma\rho} \right] \\
& + 4\alpha \left[2R^{\alpha\sigma\rho(\mu} \Delta_{\sigma\rho}^{\nu)} - \Delta_{\sigma\rho}^{\alpha} R^{\sigma\mu\nu\rho} \right] \\
& - 4\alpha \left[2R^{\alpha\sigma} \Delta_{\sigma\rho}^{(\mu} g^{\nu)\rho} - 2g^{\alpha(\mu} \Delta_{\sigma\rho}^{\nu)} R^{\sigma\rho} + 2g^{\alpha\sigma} \Delta_{\sigma\rho}^{(\mu} R^{\nu)\rho} - 2R^{\alpha(\mu} \Delta_{\sigma\rho}^{\nu)} g^{\sigma\rho} \right] \\
& + \Delta_{\sigma\rho}^{\alpha} R^{\sigma\rho} g^{\mu\nu} + \Delta_{\sigma\rho}^{\alpha} g^{\sigma\rho} R^{\mu\nu} - 2\Delta_{\sigma\rho}^{\alpha} R^{\sigma(\mu} g^{\nu)\rho} \\
& + 4\alpha R \left[\Delta_{\sigma\rho}^{\alpha} g^{\sigma[\rho} g^{\mu]\nu} + g^{\alpha\sigma} \Delta_{\sigma\rho}^{(\mu} g^{\nu)\rho} - g^{\alpha(\mu} \Delta_{\sigma\rho}^{\nu)} g^{\sigma\rho} \right]; \tag{A.2}
\end{aligned}$$

$$\begin{aligned}
& \frac{-2\kappa_D}{\sqrt{-g}} \frac{\partial \hat{\mathcal{L}}_{EGB}}{\partial (\bar{D}_{\beta\alpha} g_{\mu\nu})} = \frac{-2\kappa_D}{\sqrt{-g}} \left(\frac{\partial \hat{\mathcal{L}}_E}{\partial (\bar{D}_{\beta\alpha} g_{\mu\nu})} + \frac{\partial \hat{\mathcal{L}}_{GB}}{\partial (\bar{D}_{\beta\alpha} g_{\mu\nu})} \right) \\
& = \left[g^{\alpha(\mu} g^{\nu)\beta} - g^{\alpha\beta} g^{\mu\nu} \right] \\
& + 2\alpha \left[2R^{\alpha(\mu\nu)\beta} - 4R^{\alpha(\mu} g^{\nu)\beta} + 2g^{\mu\nu} R^{\alpha\beta} + 2g^{\alpha\beta} R^{\mu\nu} + R \left(g^{\alpha(\mu} g^{\nu)\beta} - g^{\alpha\beta} g^{\mu\nu} \right) \right]. \tag{A.3}
\end{aligned}$$

With using (A.1) - (A.3) we calculate the coefficients (2.8) - (2.10) for the EGB Lagrangian in (4.1). Because (2.8) is defined by (2.11), (2.12) and (2.10) we present these parts separately. Thus

$$\begin{aligned}\hat{\mathcal{G}}_\sigma^\alpha &= {}_{(E)}\hat{\mathcal{G}}_\sigma^\alpha + {}_{(GB)}\hat{\mathcal{G}}_\sigma^\alpha \\ &= \sqrt{-g} \left(R_\sigma^\alpha - \frac{1}{2} \delta_\sigma^\alpha R + \delta_\sigma^\alpha \Lambda_0 \right) \\ &+ \alpha \sqrt{-g} \left[2 \left(RR_\sigma^\alpha - 2R^\alpha{}_{\tau\sigma\rho} R^{\tau\rho} + R^\alpha{}_{\pi\rho\tau} R_\sigma{}^{\pi\rho\tau} - 2R_\rho^\alpha R^{\rho\sigma} \right) - \frac{1}{2} \delta_\sigma^\alpha (RR)_{GB} \right].\end{aligned}\tag{A.4}$$

$$\begin{aligned}\hat{\mathcal{U}}_\sigma^\alpha &= {}_{(E)}\hat{\mathcal{U}}_\sigma^\alpha + {}_{(GB)}\hat{\mathcal{U}}_\sigma^\alpha \\ &= -\frac{\sqrt{-g}}{2\kappa_D} \left[2g^{\rho[\tau} \bar{D}_\sigma \Delta_{\rho\tau}^\alpha] - \delta_\sigma^\alpha (R - 2\Lambda_0) \right] \\ &+ \frac{2\alpha\sqrt{-g}}{\kappa_D} \left[\left(R^{\alpha\beta\rho}{}_\tau - 4g^{\rho[\alpha} R_\tau^{\beta]} + Rg^{\rho[\alpha} \delta_\tau^{\beta]} \right) \bar{D}_\sigma \Delta_{\beta\rho}^\tau \right. \\ &+ \left. 2g^{\beta\mu} \left(\bar{D}_\beta R^{\alpha\nu} + 2R^{\rho(\alpha} \Delta_{\beta\rho}^{\nu)} \right) \Delta_{\sigma(\mu}^\tau g_{\nu)\tau} - g^{\rho(\alpha} \Delta_{\sigma\rho}^{\beta)} \bar{D}_\beta R \right] + \frac{\alpha\sqrt{-g}}{2\kappa_D} \delta_\sigma^\alpha (RR)_{GB}.\end{aligned}\tag{A.5}$$

$$\begin{aligned}\hat{m}_\sigma^{\alpha\beta} &= {}_{(E)}\hat{m}_\sigma^{\alpha\beta} + {}_{(GB)}\hat{m}_\sigma^{\alpha\beta} \\ &= -\frac{\sqrt{-g}}{2\kappa_D} \left[\delta_\sigma^\alpha \Delta_{\rho\tau}^\beta g^{\rho\tau} - 2\Delta_{\sigma\rho}^\alpha g^{\beta\rho} + \Delta_{\rho\sigma}^\rho g^{\alpha\beta} \right] \\ &+ \frac{2\alpha\sqrt{-g}}{\kappa_D} \left[R^{\alpha\tau\rho}{}_\sigma \Delta_{\tau\rho}^\beta - 2R^{\alpha(\tau\beta)}{}_\rho \Delta_{\tau\sigma}^\rho \right] \\ &+ \frac{4\alpha\sqrt{-g}}{\kappa_D} \left[4g^{\rho[\alpha} R_\tau^{\beta]} \Delta_{\rho\sigma}^\tau + 2R_\sigma^{[\alpha} g^{\tau]\rho} \Delta_{\tau\rho}^\beta + 2g^{\alpha[\beta} R_\rho^{\tau]} \Delta_{\tau\sigma}^\rho \right. \\ &- \left. g^{\tau\beta} \left(\bar{D}_{(\tau} R_{\sigma)}^\alpha + R_{(\tau}^\rho \Delta_{\sigma)\rho}^\alpha - R_\rho^\alpha \Delta_{\tau\sigma}^\rho \right) \right] \\ &- \frac{\alpha\sqrt{-g}}{\kappa_D} \left[\left(\delta_\sigma^\alpha \Delta_{\rho\tau}^\beta g^{\rho\tau} - 2\Delta_{\sigma\rho}^\alpha g^{\beta\rho} + \Delta_{\rho\sigma}^\rho g^{\alpha\beta} \right) R - 2\delta_\sigma^{(\alpha} g^{\tau)\beta} \partial_\tau R \right]\end{aligned}\tag{A.6}$$

$$\begin{aligned}\hat{n}_\sigma^{\lambda\alpha\beta} &= {}_{(E)}\hat{n}_\sigma^{\lambda\alpha\beta} + {}_{(GB)}\hat{n}_\sigma^{\lambda\alpha\beta} \\ &= \frac{\sqrt{-g}}{2\kappa_D} \left\{ g^{\alpha\beta} \delta_\sigma^\lambda - g^{\lambda(\alpha} \delta_\sigma^{\beta)} \right\} \\ &+ \frac{\alpha\sqrt{-g}}{\kappa_D} \left\{ -2R_\sigma^{(\alpha\beta)\lambda} - 4R_\sigma^\lambda g^{\alpha\beta} + 4R_\sigma^{(\alpha} g^{\beta)\lambda} + R \left(g^{\alpha\beta} \delta_\sigma^\lambda - g^{\lambda(\alpha} \delta_\sigma^{\beta)} \right) \right\}.\end{aligned}\tag{A.7}$$

It was checked directly that the coefficients \hat{u}_σ^α in (2.8) (calculated with the use of (A.4), (A.5) and (A.7)), and the coefficients (A.6) and (A.7) themselves satisfy exactly the identities (2.16) - (2.19).

At last, using (A.6) we calculate the Belinfante correction (2.23) for the EGB gravity:

$$\begin{aligned}\hat{s}^{\alpha\beta\sigma} &= {}_{(E)}\hat{s}^{\alpha\beta\sigma} + {}_{(GB)}\hat{s}^{\alpha\beta\sigma} \\ &= \frac{\sqrt{-g}}{\kappa_D} \left[\Delta_{\tau\rho}^{[\alpha} \bar{g}^{\beta]\sigma} g^{\tau\rho} + \Delta_{\lambda\rho}^\rho g^{\sigma[\alpha} \bar{g}^{\beta]\lambda} - \Delta_{\lambda\rho}^\sigma g^{\rho[\alpha} \bar{g}^{\beta]\lambda} - 2\Delta_{\lambda\rho}^{[\alpha} \bar{g}^{\beta]\lambda} g^{\rho\sigma} + \Delta_{\lambda\rho}^{[\alpha} g^{\beta]\rho} \bar{g}^{\sigma\lambda} \right]\end{aligned}$$

$$\begin{aligned}
& + \frac{2\alpha\sqrt{-g}}{\kappa_D} \left[\bar{g}^{\lambda[\alpha} \left(R^{\beta]\sigma\tau}{}_{\rho} - 2R^{\beta]\tau\sigma}{}_{\rho} \right) \Delta_{\lambda\tau}^{\rho} + \bar{g}^{\lambda[\alpha} \Delta_{\tau\rho}^{\beta]} R^{\sigma\tau\rho}{}_{\lambda} + \bar{g}^{\lambda[\alpha} R^{\beta]\tau\rho}{}_{\lambda} \Delta_{\tau\rho}^{\sigma} \right. \\
& + \left. \bar{g}^{\sigma\lambda} \left(R_{\lambda}{}^{\tau\rho[\alpha} \Delta_{\tau\rho}^{\beta]} - \frac{3}{2} R^{\alpha\beta\tau}{}_{\rho} \Delta_{\lambda\tau}^{\rho} \right) \right] \\
& + \frac{4\alpha\sqrt{-g}}{\kappa_D} \left[\left(R_{\lambda}^{\tau} g^{\rho(\sigma} \Delta_{\tau\rho}^{\alpha)} - g^{\tau\rho} R_{\lambda}^{(\sigma} \Delta_{\tau\rho}^{\alpha)} - g^{\sigma\alpha} R_{\rho}^{\tau} \Delta_{\tau\lambda}^{\rho} + \bar{D}_{(\tau} R_{\lambda)}^{(\sigma} g^{\alpha)\tau} + R_{(\tau}^{\rho} \Delta_{\lambda)\rho}^{(\sigma} g^{\alpha)\tau} \right) \bar{g}^{\beta\lambda} \right. \\
& - \left(R_{\lambda}^{\tau} g^{\rho(\sigma} \Delta_{\tau\rho}^{\beta)} - g^{\tau\rho} R_{\lambda}^{(\sigma} \Delta_{\tau\rho}^{\beta)} - g^{\sigma\beta} R_{\rho}^{\tau} \Delta_{\tau\lambda}^{\rho} + \bar{D}_{(\tau} R_{\lambda)}^{(\sigma} g^{\beta)\tau} + R_{(\tau}^{\rho} \Delta_{\lambda)\rho}^{(\sigma} g^{\beta)\tau} \right) \bar{g}^{\alpha\lambda} \\
& + \left. \left(2g^{\tau[\alpha} R_{\rho}^{\beta]} \Delta_{\tau\lambda}^{\rho} - R_{\lambda}^{\tau} g^{\rho[\alpha} \Delta_{\tau\rho}^{\beta]} + g^{\tau\rho} R_{\lambda}^{[\alpha} \Delta_{\tau\rho}^{\beta]} - \bar{D}_{(\tau} R_{\lambda)}^{[\alpha} g^{\beta]\tau} - R_{(\tau}^{\rho} \Delta_{\lambda)\rho}^{[\alpha} g^{\beta]\tau} \right) \bar{g}^{\sigma\lambda} \right] \\
& + \frac{2\alpha\sqrt{-g}}{\kappa_D} \left[\left(g^{\sigma[\alpha} \bar{g}^{\beta]\lambda} \Delta_{\lambda\rho}^{\rho} - \Delta_{\lambda\rho}^{(\sigma} g^{\alpha)\rho} \bar{g}^{\beta\lambda} + \Delta_{\lambda\rho}^{(\sigma} g^{\beta)\rho} \bar{g}^{\alpha\lambda} - \bar{g}^{\sigma[\alpha} \Delta_{\lambda\rho}^{\beta]} g^{\lambda\rho} - \bar{g}^{\sigma\lambda} g^{\rho[\alpha} \Delta_{\lambda\rho}^{\beta]} \right) R \right. \\
& + \left. \left(\bar{g}^{\lambda[\alpha} g^{\beta]\sigma} + \bar{g}^{\sigma[\alpha} g^{\beta]\lambda} \right) \partial_{\lambda} R \right] \tag{A.8}
\end{aligned}$$

Of course, the Einstein part exactly coincides with the one presented in [17].

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