Enhancement of Geometric Phase by Frustration of Decoherence: A Parrondo like Effect

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Geometric phase plays an important role in evolution of pure or mixed quantum states. However, when a system undergoes decoherence the development of geometric phase may be inhibited. Here, we show that when a quantum system interacts with two competing environments there can be enhancement of geometric phase. This effect is akin to Parronodo like effect on the geometric phase which results from quantum frustration of decoherence. Our result suggests that the mechanism of two competing decoherence can be useful in fault tolerant holonomic quantum computation.

Introduction.- Geometric phase (GP) is a consequence of the holonomy of the path traced by a quantum system in its Hibert space, thereby highlighting its connection to the intrinsic curvature of the space [1]. Even though its classical foundation was laid by Pancharatnam [2], in dealing with questions related to the characterization of interference of classical light in distinct states of polarization, its quantum counterpart was discovered much later by Berry [3] for cyclic adiabatic evolution. This was subsequently generalized to non-adiabatic [4] and non-cyclic evolutions [5]. Later, a quantum kinematic approach was provided for the GP [6] and a generalized gauge potential for the most general quantum evolution was introduced [7]. The concept of geometric phase is not limited to pure state quantum evolution, but do appear for mixed states [8–10] also. Experimentally measurable geometric phase for mixed states under unitary evolution was first introduced in [9] and then generalized to nonunitary evolutions [10]. Since geometric phase depends on the evolution path and not on the detailed dynamics, thereby suggesting an inherent fault tolerance [11], it can be a useful resource for quantum computation. Using NMR [12] and atom interferometry [13] pure and mixed state geometric phases have been realized experimentally. There have been various other proposals to observe GP in a coupled two-mode Bose-Einstein condensate [14], Bose-Einstein Josephson junction [15] and a superconducting nanostructure [16], in all of which it is imperative to consider the effect of the ambient environment on the system of interest [17]. In recent times, there have been attempts to connect GP with quantum correlations, in particular entanglement [18], in a variety of quantum systems.

The above reasons bring to focus the need to have an understanding of the impact of the environment on the study and practical implementation of GP. In fact, the effect of measurement on the GP was first investigated in [19] and it was shown that in the limit of continuous observation the GP can be suppressed. For mixed states it was shown that the Uhlmann phase also decreases under isotropic decoherence [20]. Open quantum systems is the systematic study of the influence of the environment, alternatively called the reservoir or bath, on the evolution of the system of interest. The basic idea is that one follows the evolution of the system of interest by tracing out the environmental degrees of freedom, resulting in a non-unitary evolution. Decoherence and dissipation are a natural consequence of this. Open quantum systems can be broadly classified into two categories, one that involves decoherence without dissipation [21, 22] and the other where dissipation occurs along with decoherence [22, 23]. Experiments with trapped atoms have been performed where both pure decoherence as well as dissipative type of evolution have been generated by coupling the atomic system to appropriate engineered reservoirs [24]. A practical implementation of GP would involve, for example, a qubit interacting with its environment, resulting in its inhibition. This calls for the need to have settings where the inhibition of GP, due to the ubiquitous environment, could be arrested. Quantum frustration of decoherence (QFD), as demonstrated in this letter, would be a potential candidate for achieving this.

QFD [25] is the term ascribed to the general phenomena when a quantum system coupled to two independent environments by canonically conjugate operators results in an enhancement of quantum fluctuations, that is, decoherence gets suppressed. The reason for this is attributable to the non-commuting nature of the conjugate coupling operators that prevents the selection of an appropriate pointer basis to which the quantum system could settle down to. It has been studied in various guises, such as an extension of the dissipative two-level system problem [25], where the two non-commuting spin operators of the central spin system were coupled to independent harmonic oscillator baths, or a harmonic oscillator, modelling a large spin impurity in a ferromagnet, coupled to two independent oscillator baths via its position and momentum operators [26]. In each case, irrespective of the system of interest or the coupling operators, QFD was observed. Another scenario where this has been put to use is in quantum error correction [27].

These considerations were extended to the case of spin baths [28], present, for example, in the case of quantum dots, with similar results. These motivate us to study GP in the presence of QFD. Interestingly, this could be also thought of as an example of Parrondo's paradox involving two games which when played individually lead to a loosing expectation, but when played in an alternative order produce a winning expectation [29, 30]. The underlying reason behind the surprising aspect of Parrondo's game is the breaking of an inherent symmetry in the problem. This feature is also shared by quantum frustration models where the symmetry in the decay channel, were only one bath present, is broken by the presence of coupling to two independent baths by non-commuting operators. In this letter, we take up a simple model of a frustrated open quantum system and explicitly show the enhancement of GP. This highlights the role of quantum frustrated decoherence leading to Parrondo like effect on the geometric phase.

Model.- We study the influence of QFD on GP by taking up a simple model involving a central spin, or a qubit which would be our system of interest, interacting with two independent spin baths via two non-commuting spin operators

$$\begin{aligned} H &= H_S + H_{SR} \\ &= \omega\frac{\sigma_z}{2} + \alpha_1\frac{\sigma_x}{2}\otimes\Sigma_{k=1}^{N} I_x^k + \alpha_2\frac{\sigma_y}{2}\otimes\Sigma_{l=1}^{N} J_y^l, \quad (1) \end{aligned}$$

where H_S is the system (single qubit) Hamiltonian and H_{SR} is the system-reservoir interaction Hamiltonian. Here σ_i, $i = x, y, z$ are the three Pauli matrices for the central spin, I_x^k and J_y^l are the bath spin operators. Also, α_1, α_2 are the two spin-bath coupling constants and ω comes from the basic system Hamiltonian, representing the initial magnetic field. The bath dynamics itself is not considered. This serves two purposes; it allows for an analytical treatment of the model and at the same time captures its essence, since in solid state spin systems with dominant spin-environment interactions, such as quantum dots where such a model could be envisaged, the internal bath dynamics composed of nuclear spins would be very slow compared to the central electronic spin [31].

Assume an uncorrelated system-reservoir initial state with the central spin in

$$\begin{aligned} \rho_S(0) &= \cos^2\left(\frac{\theta}{2}\right)|\downarrow\rangle\langle\downarrow| + \sin^2\left(\frac{\theta}{2}\right)|\uparrow\rangle\langle\uparrow| \\ &+ \frac{i}{2}\sin(\theta)e^{i\phi}\left[|\uparrow\rangle\langle\downarrow| - e^{-i2\phi}|\downarrow\rangle\langle\uparrow|\right]. \quad (2) \end{aligned}$$

Eq. (2) is the most general single qubit density matrix where $\theta \in \{0, \pi\}$ and $\phi \in \{0, 2\pi\}$ are the polar and azimuthal angles, respectively. The full form of the initial density matrix with an unpolarized initial bath state is $\rho_{SR}(0) = \frac{1}{2^{2N}}\rho_S(0)\otimes\mathcal{I}_{2^N}\otimes\mathcal{I}_{2^N}$, where N is the total number of spins present in each bath. Under the interaction Hamiltonian, the total state evolves as $\rho_{SR}(0) \to \rho_{SR}(t) = \exp[-i(H_S + H_{SR})t]\rho_{SR}(0)\exp[i(H_S + H_{SR})t]$. After interaction, the reduced state of the spin is given by $\rho_S(t) = \mathrm{Tr}_R[\rho_{SR}(t)]$. The Bloch vector representation of a spin-$\frac{1}{2}$ particle, which is the central spin here, is

$$\rho_S(t) = \frac{1}{2}\begin{bmatrix} 1+\langle\sigma_z(t)\rangle & \langle\sigma_x(t)\rangle - i\langle\sigma_y(t)\rangle \\ \langle\sigma_x(t)\rangle + i\langle\sigma_y(t)\rangle & 1-\langle\sigma_z(t)\rangle \end{bmatrix}, \quad (3)$$

where $\langle\sigma_i(t)\rangle = \sum_{m_1,m_2=-N/2}^{N/2}\zeta_{m_1}\zeta_{m_2}\mathrm{Tr}\left(\rho_{m_1,m_2}(0)\sigma_i(t)\right)$ and m_1, m_2 label the eigenvalues of bath spin operators and range from $-\frac{N}{2}$ to $\frac{N}{2}$. The average polarizations of the central spin come out to be

$$\begin{aligned} \langle\sigma_z(t)\rangle &= \frac{-1}{2^{2N+1}}\sum_{m_1,m_2=-N/2}^{N/2}\zeta_{m_1}\zeta_{m_2}\Big(\cos(\Gamma_{m_1m_2}t)\cos(\theta) \\ &+ \frac{\sin(\Gamma_{m_1m_2}t)}{\Gamma_{m_1m_2}}\sin(\theta)\left[m_1\alpha_1\cos(\phi) - m_2\alpha_2\sin(\phi)\right] \\ &+ \omega\left(m_1\alpha_1\sin(\theta)\sin(\phi) + m_2\alpha_2\sin(\theta)\cos(\phi)\right. \\ &+ \left.\omega\cos(\theta)\right)\frac{(1-\cos(\Gamma_{m_1m_2}t))}{\Gamma_{m_1m_2}^2}\Big), \quad (4) \end{aligned}$$

$$\begin{aligned} \langle\sigma_x(t)\rangle &= \frac{-1}{2^{2N+1}}\sum_{m_1,m_2=-N/2}^{N/2}\zeta_{m_1}\zeta_{m_2}\Big(\cos(\Gamma_{m_1m_2}t) \\ &\sin(\theta)\sin(\phi) - \frac{\sin(\Gamma_{m_1m_2}t)}{\Gamma_{m_1m_2}} \\ &\times \left[\omega\sin(\theta)\cos(\phi) - m_2\alpha_2\cos(\theta)\right] \\ &+ m_1\alpha_1\left(m_1\alpha_1\sin(\theta)\sin(\phi) + m_2\alpha_2\sin(\theta)\cos(\phi)\right. \\ &+ \left.\omega\cos(\theta)\right)\frac{(1-\cos(\Gamma_{m_1m_2}t))}{\Gamma_{m_1m_2}^2}\Big), \quad (5) \end{aligned}$$

and

$$\begin{aligned} \langle\sigma_y(t)\rangle &= \frac{-1}{2^{2N+1}}\sum_{m_1,m_2=-N/2}^{N/2}\zeta_{m_1}\zeta_{m_2}\Big(\cos(\Gamma_{m_1m_2}t) \\ &\sin(\theta)\cos(\phi) - \frac{\sin(\Gamma_{m_1m_2}t)}{\Gamma_{m_1m_2}} \\ &\times \left[-\omega\sin(\theta)\sin(\phi) + m_1\alpha_1\cos(\theta)\right] \\ &+ m_2\alpha_2\left(m_1\alpha_1\sin(\theta)\sin(\phi) + m_2\alpha_2\sin(\theta)\cos(\phi)\right. \\ &+ \left.\omega\cos(\theta)\right)\frac{(1-\cos(\Gamma_{m_1m_2}t))}{\Gamma_{m_1m_2}^2}\Big). \quad (6) \end{aligned}$$

Here $\zeta_m = \frac{N!}{(N/2-m)!(N/2+m)!}$ and $\Gamma_{m_1m_2} = \sqrt{\omega^2 + \alpha_1^2m_1^2 + \alpha_2^2m_2^2}$.

The vector $v(t) = \mathrm{Tr}\Big[\rho_S(t)\sigma(0)\Big]$ is called the Bloch vector of the system. For pure states $|v(t)| = 1$ while for mixed states, $|v(t)| < 1$, that is, the Bloch vector penetrates into the Bloch sphere.

GP of frustrated spin system: Explicit solution and analysis.- A general mixed state density matrix $\rho(t) =$

$\sum_k \lambda(k)|\phi_k(t)\rangle\langle\phi_k(t)|$ is subject to purification, by the introduction of an ancilla, as

$$|\Psi(t)\rangle = \sum_k \sqrt{\lambda(k)}|\phi_k(t)\rangle \otimes |a_k\rangle; t \in [0,\tau], \qquad (7)$$

where $\lambda(k)$, $|\phi_k(t)\rangle$ are the eigenvalues and eigenvectors of the reduced density matrix $\rho(t)$ under consideration, respectively, and $|a_k\rangle$ represent the ancilla. The Pancharatnam relative phase, $\alpha(t) = arg(\langle\Psi(0)|\Psi(t)\rangle)$ reduces to the GP when the parallel transport condition, $\langle\phi_k(t)|d/dt|\phi_k(t)\rangle = 0, k = 1...P$ corresponding to the P eigenstates, is satisfied. The GP for the mixed state, $\rho_S(t)$ [Eq. (3)], satisfying the parallel transport conditions assumes the form

$$\gamma_g(\tau) = arg\Big[\sum_k \sqrt{\lambda_k(\tau)\lambda_k(0)}\langle\phi_k(0)|\phi_k(\tau)\rangle \times e^{-\int_0^\tau \langle\phi_k(t')|\dot{\phi}_k(t')\rangle dt'}\Big] \qquad (8)$$

where $\lambda_k(\tau)$ are the eigenvalues and $\phi_k(\tau)$ are the corresponding eigenvectors of the reduced density matrix $\rho_S(\tau)$ [Eq. (3)].

The Eq. (8) can be shown to be

$$\begin{aligned}\gamma_g(\tau) &= \arg\Big[\Big\{\frac{1}{2}\Big(1+\sqrt{A^2(\tau)+4R^2(\tau)}\Big)\Big\}^{\frac{1}{2}} \\ &\times \Big\{\cos\Big(\frac{\theta_0}{2}\Big)\sin\Big(\frac{\theta_\tau}{2}\Big) \\ &+ e^{i(\chi(\tau)-\chi(0))}\sin\Big(\frac{\theta_0}{2}\Big)\cos\Big(\frac{\theta_\tau}{2}\Big)\Big\} \\ &\times e^{-i\int_0^\tau dt(\dot{\chi}(t))\cos^2\left(\frac{\theta_t}{2}\right)}\Big]. \end{aligned} \qquad (9)$$

Here $A = \langle\sigma_z(t)\rangle$, $R = \frac{1}{2}\sqrt{\langle\sigma_x(t)\rangle)^2 + \langle\sigma_y(t)\rangle)^2}$ and $\tan(\chi(t)) = \frac{\langle\sigma_y(t)\rangle}{\langle\sigma_x(t)\rangle}$. Also, $\sin\left(\frac{\theta_t}{2}\right) = \frac{2R}{\sqrt{4R^2+(\epsilon_+-A)^2}}$, and $\cos\left(\frac{\theta_0}{2}\right) = \sqrt{\frac{1+\langle\sigma_z(0)\rangle}{2}}$, $\epsilon_+ = \sqrt{A^2+4R^2}$. The GP in the presence of two competing decoherence processes [Eq. 9] can also be expressed as

$$\begin{aligned}\gamma_g(\tau) &= \tan^{-1}\left[\frac{\sin\frac{\theta_0}{2}\cos\frac{\theta_\tau}{2}\sin\delta\chi(t)}{\cos\left(\frac{\theta_0}{2}\right)\sin\frac{\theta_\tau}{2}+\sin\frac{\theta_0}{2}\cos\frac{\theta_\tau}{2}\cos\delta\chi(t)}\right] \\ &- \int_0^\tau dt\ \ \dot{\chi}(t)\cos^2\frac{\theta_t}{2}, \end{aligned} \qquad (10)$$

where $\delta\chi(t) = (\chi(t) - \chi(0))$. It can be easily seen from Eq. (10) that if we remove the influence of the environment, we obtain for $\tau = \frac{2\pi}{\omega}$, $\gamma_g = -\pi(1-\cos(\theta_0))$, as expected, which is the standard result for the unitary evolution of an initial pure state. If we take the angle $\theta_0 = \pi$, that is, the South Pole of the Bloch sphere of the spin of interest, then Eq. (10) simplifies to

$$\gamma_g(\tau) = \frac{1}{2}\int_0^\tau dt\ \ \dot{\chi}(t)\,(1-\cos\theta_t)\,. \qquad (11)$$

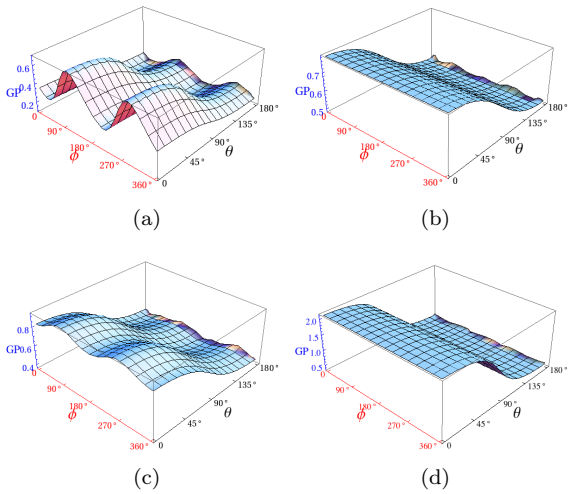

FIG. 1: (Color online) GP $[\gamma_g(\tau)]$ with respect to θ and ϕ (a) when $\alpha_1 = 1$ and $\alpha_2 = 0$, that is, for the case of a single bath, (b) when $\alpha_1 = \alpha_2 = \frac{1}{\sqrt{2}}$, (c) when $\alpha_1 = \frac{\sqrt{3}}{2}$ and $\alpha_2 = \frac{1}{2}$ (d) when $\alpha_1 = \alpha_2 = \frac{1}{4}$. Here $\omega = 2$, and time $t = 50$. A comparison between (a), (b), (c), and (d) reiterates the point that the decay of GP gets frustrated when both the baths are acting and one of the best strategies is seen to be the case where $\alpha_1 = \alpha_2 = \frac{1}{4}$.

It can be noticed that a contribution to the GP, in Eq. (11), coming from the argument of the exponential, resembles the solid-angle expression for GP in the usual demonstrations.

In Fig. 1, GP with respect to θ and ϕ for different values of coupling constants α_1 and α_2 for an evolution time $t = 50$ and $\omega = 2$ is depicted. A comparison between Figs. 1(a),1(b), 1(c) and 1(d) where $\alpha_1 = 1$ and $\alpha_2 = 0$, $\alpha_1 = \alpha_2 = 1/\sqrt{2}$, $\alpha_1 = \sqrt{3}/2$ and $\alpha_2 = 1/2$, and $\alpha_1 = \alpha_2 = 1/4$, respectively, brings out the point that the decay of GP gets frustrated when both the baths are acting and one of the best strategy seems to be the case of $\alpha_1 = \alpha_2 = \frac{1}{4}$. In Fig.2, GP for $\alpha_1 = \alpha_2 = 1/4$ when $t = 50$ and $t = 200$ is shown and we can note that the optimum value of α_1 and α_2 for maximum frustration of GP varies with time. In Fig. 3, a comparison is made of GP for different coupling constants with respect to θ and ϕ for time $t = 50$. For the QFD regime, that is, when $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ we observe that the value of GP is higher as compared to the case where QFD is not applicable, that is, for the case of single coupling constant. These observations bring out the inherent robustness of GP against decay of quantum fluctuations in the presence of QFD.

Analogy with Parrondo games.- The effect of frustration on GP could be thought of as a Parrondo'd game: each game on its own is "a single qubit interacting with its bath; one with σ_x, and with another σ_y"; this would result in decoherence and dissipation leading to inhibi-

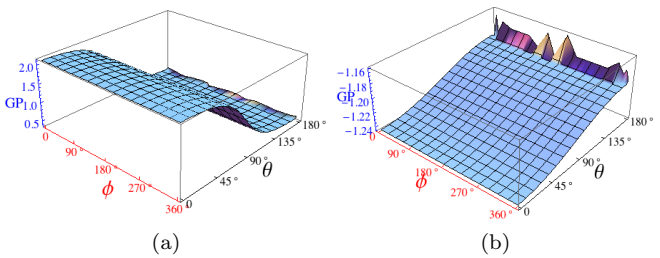

FIG. 2: (Color online) GP for $\alpha_1 = \alpha_2 = 1/4$ with respect to θ and ϕ for (a) time $t = 50$ and (b) time $t = 200$, respectively. Here $\omega = 2$. Among other sets of α_1, α_2, the figure corresponding to the case (a) is found to be optimum in resisting the depletion of GP. Therefore, for different time different α_1 and α_2 will help to enhance the GP.

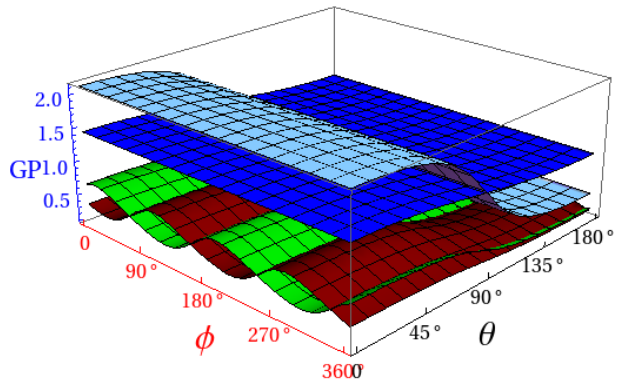

FIG. 3: (Color online) GP for different α_1, α_2 with respect to θ and ϕ for time $t = 50$ and $\omega = 2$. The red curve corresponds to $\alpha_1 = 1$, $\alpha_2 = 0$; the green curve corresponds to $\alpha_1 = 0$, $\alpha_2 = 1$; the light blue curve to $\alpha_1 = \alpha_2 = \frac{1}{4}$; while the dark blue curve corresponds to $\alpha_1 = \alpha_2 = \frac{1}{2}$. It is clearly evident from the plots that the decay of GP gets frustrated due to the presence of both the couplings α_1 and α_2.

tion of GP. This would be the situation where each player looses his game. However, when the two games are played in a synchronized fashion; corresponding, here, to the case of "the qubit interacting with two independent baths via non-commuting operators with coupling strengths α_1 and α_2", then the decoherence and dissipation can get frustrated leading to improvement in GP over some range of parameters. Though, we have presented the Parrondo like effect for GP for our model system, we expect this to be a generic feature of a quantum system interacting with two competing environments.

Conclusions.- To conclude, we have, by analyzing a simple model of QFD, illustrated the enhancement of geometric phase in the presence of two competing environments. The model being simple allows for an explicit evaluation, but is generic in the sense that it captures the essence of frustration on GP for other models as well. Here we consider a qubit interacting with two independent baths via non-commuting operators, for e.g., σ_x, σ_y. Naively, one would expect that due to interaction with two baths, the decoherence effect would increase leading to inhibition of geometric phase. However, in contrast to this, it is found that decoherence gets suppressed: thus providing a typical framework for the Parrondo kind of game. Parrondo's games take place when a symmetry in the original problem gets broken. In this case the broken symmetry would be the interaction of the qubit with the two independent baths via two non-commuting operators. Here a purely dephasing scheme would not work as that would require the system and interaction Hamiltonians to commute [21]. But it is the non-commutativity of operators in the interaction Hamiltonian that leads to the Parrondo like effect for the geometric phase. This suggests that for quantum frustration of decoherence to be effective, we need both decoherence as well as dissipation. We hope that the effect found here can be used in fault tolerant quantum computation. This may also find wide applications in enhancement of geometric phases in other systems under competing decoherence.

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