

Improved decoding of affine–variety codes

Chiara Marcolla (chiara.marcolla@unitn.it)

Department of Mathematics, University of Trento, Italy

Emmanuela Orsini (orsini@posso.dm.unipi.it)

Department of Mathematics, University of Pisa, Italy

Massimiliano Sala (maxsalacodes@gmail.com)

Department of Mathematics, University of Trento, Italy

Abstract

General error locator polynomials are polynomials able to decode any correctable syndrome for a given linear code. Such polynomials are known to exist for all cyclic codes and for a large class of linear codes. We provide some decoding techniques for affine-variety codes using some multidimensional extensions of general error locator polynomials. We prove the existence of such polynomials for any correctable affine-variety code and hence for any linear code. We propose two main different approaches, that depend on the underlying geometry. We compute some interesting cases, including Hermitian codes. To prove our coding theory results, we develop a theory for special classes of zero-dimensional ideals, that can be considered generalizations of stratified ideals. Our improvement with respect to stratified ideals is twofold: we generalize from one variable to many variables and we introduce points with multiplicities.

Keywords: Affine–variety code, AG codes, coding theory, commutative algebra, decoding, general error locator polynomial, Gröbner basis, Hermitian codes, linear code, zero-dimensional ideal.

1 Introduction

Affine-variety codes were introduced by Fitzgerald and Lax in [FL98] and provide a way to represent any linear code as an evaluation code for a suitable polynomial ideal. Unsurprisingly, this rather general description does not provide immediately any efficient decoding algorithm. The lack of an efficient decoding algorithm is one of the main drawbacks of this nice approach, which has unfortunately still not received the attention it deserves, with few exceptions ([Gei03], [SDG06]). Some Gröbner basis techniques have been proposed

in [FL98] to decode these codes, which may be efficient depending on the underlying algebraic structure.

General error locator polynomials are polynomials introduced by us in [OS05] to decode cyclic codes. Their roots, when specialized to a syndrome, give the error locations. They can be used to decode any linear code, if it possesses them. Giorgetti and Sala in [GS06,GS09] have found a large family of linear codes possessing such polynomials. When the general error locator polynomial admits a sparse representation, the decoding for the code is very fast. Experimental evidence (and theoretical proofs for special cases) suggests their sparsity in many interesting cases ([MOS06], [OS07], [CL10], [LCCC10], [LCJC10]).

We report several other approaches on decoding linear and cyclic codes with Gröbner bases: [BP09], [ABF09], [GR09].

In this paper we generalize our formerly proposed locator polynomials to cover also the multi-dimensional case and hence the affine-variety case. By adapting the Gröbner techniques in [FL98], [OS05], [GS09], we can prove their existence for any affine-variety code. Excluding this introduction, this paper contains the following sections.

- In Section 2 we recall definitions and properties for affine-variety codes, stratified ideals (a special class of zero-dimensional ideals), general error locator polynomials and the Hasse derivative.
- In Section 3 we summarize the decoding proposed in [FL98] and we propose several alternatives, discussing their merits and drawbacks, especially taking into consideration the underlying geometric situation. In particular, we introduce the notion of “ghost points”, which are points added to the variety to play the role of non-valid error locations. This way we can define a first generalization of general error locator polynomials to the multivariate case (Definition 3.9), which provides a first decoding strategy. We also introduce evaluator polynomials (Definition 3.10) that permits a second strategy. While the existence of evaluator polynomials can be proved directly using the theory of stratified ideals, unfortunately in this section we lack the theoretical background to prove the existence of these multivariate locators.
- In Section 4 we extend the results in [OS05] for stratified ideals to cover also the “multi-dimensional case”, that is precisely the theoretical background that we need for any multivariate generalization of general locators. Unexpectedly, there is no obvious “natural” way to extend the core notion of stratified ideals. We present three generalizations in Definition 4.4, 4.5. We discuss their implications and provide some preliminary results.

Given a zero-dimensional ideal we can consider an order on its elimination ideals with a decreasing number of variables. By choosing two consecu-

tive elimination ideals I and I' , we have a natural projection from $\mathcal{V}(I)$ to $\mathcal{V}(I')$. At this stage, we highlight the role of two natural numbers: the maximum degree of some special polynomials in suitable Gröbner basis of I' and the maximum number of extensions to $\mathcal{V}(I)$ for points in $\mathcal{V}(I')$. It is convenient to present these values as functions, respectively, η (Definition 4.3) and ζ (13). Section 4 ends with the statement of Proposition 4.13, which is the main result claimed in this section (but not proved here). Proposition 4.13 is, in some sense, the multivariate analogue of Proposition 2.5 on stratified ideals, and shows that for our ideals η and ζ coincides in this setting.

- Section 5 is devoted to the long proof of Proposition 4.13. This proposition describes some features of the Gröbner basis of (the elimination ideals of) a zero-dimensional radical ideal J . The proof is constructive and relies on iterated applications of some versions of the Buchberger-Möller algorithm. To be more precise, we can start from the vanishing ideal of a single point. For any monomial ordering it is trivial to determine its Gröbner basis. In particular, ζ and η coincide. By adding more points, the shape of the Gröbner basis becomes more complex, but we can follow what happens to the leading terms, if we are only interested in the variable involved in the projection $\mathcal{V}(I) \rightarrow \mathcal{V}(I')$. When we have added enough points, we will get again J , since J is radical. To apply the Buchberger-Möller algorithm, we need to add the points one by one. The difficult part is choosing the point in such a way that η and ζ grow exactly by the same amount.
- Unfortunately, our result in the multidimensional case, Proposition 4.13, is not as strong as our result in the one-variable case, Proposition 2.5. In Section 6, it does allow us to prove the existence of our first generalization of locators in Theorem 6.1, but we show that better locators can be found, as in Definition 6.2. We discuss with examples a new decoding strategy by applying these locators, but again for the moment we are unable to prove their existence, since they use multiplicities. This will be done in the next section.
- In Section 7 we develop the theory for generalizing stratified ideals to the multivariate case with multiplicities. As usual, we are interested in suitable Gröbner bases of elimination ideals of some zero-dimensional ideals. First, we introduce the notion of *stuffed ideals* (Definition 7.1), which basically means that the roots of some polynomials in these Gröbner bases have the “expected” multiplicity. We give a constructive method (“stuffing”) to obtain stuffed ideals from special classes of ideals (in particular, radical ideals will do). Our main results here are Theorem 7.4, that ensures that the desired shape of our Gröbner bases is unchanged under stuffing, and Theorem 7.6, that ensures the existence of our sought-after locators (in our Gröbner bases).

- In Section 8 we compute some examples from different families of affine-variety codes. In particular, we formally determine the shape for multi-variate locator polynomials in the Hermitian case, for any $q \geq 2$ and $t = 2$ (Theorem 8.3), both in our weaker version and in our stronger version.
- In Section 9 we provide further comments and draw some conclusions.

2 Preliminaries

In this section we fix some notation and recall some known results.

We denote by \mathbb{F}_q the field with q elements, where q is a power of a prime, and by $n \geq 1$ a natural number. Let $(\mathbb{F}_q)^n$ be the vector space of dimension n over \mathbb{F}_q . Any vector subspace $C \subset (\mathbb{F}_q)^n$ is a linear code (over \mathbb{F}_q).

From now on, we denote by \mathbb{K} any (not necessarily finite) field and by $\overline{\mathbb{K}}$ its algebraic closure.

For any ideal I in a polynomial ring $\mathbb{K}[X]$, $X = \{x_1, \dots, x_m\}$, we denote by $\mathcal{V}(I)$ its variety in $\overline{\mathbb{K}}$. For any $Z \subset \overline{\mathbb{K}}^m$ we denote by $\mathcal{I}(Z)$ the vanishing ideal of Z , $\mathcal{I}(Z) \subset \mathbb{K}[X]$.

For any $f \in \mathbb{K}[X]$ and any term order $>$ on $\mathbb{K}[X]$, we denote by $\mathbf{T}(f)$ the leading term of f with respect to $>$. We assume the reader familiar with the standard theory of Gröbner bases, see e.g. [Mor05], [Morar], [Mor09b]. When we have fixed the polynomial ring and the term order, we write $\text{GB}(I)$ for the (unique) reduced Gröbner basis of I .

We briefly recall the notion of “block order”, since it is less frequently met in literature. Let $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_r\}$ be two variable sets. Let $<_X$ and $<_Y$ be two orders, on the monomials of X and on the monomials of Y , respectively. We can define an order $< = (<_X, <_Y)$ on the monomials of $X \cup Y$ (*block order*) as follows:

$$\begin{aligned}
 & x_1^{i_1} \dots x_m^{i_m} y_1^{j_1} \dots y_r^{j_r} < x_1^{a_1} \dots x_m^{a_m} y_1^{b_1} \dots y_r^{b_r} \text{ if} \\
 & \text{either } y_1^{j_1} \dots y_r^{j_r} <_Y y_1^{b_1} \dots y_r^{b_r} \\
 & \text{or } y_1^{j_1} \dots y_r^{j_r} = y_1^{b_1} \dots y_r^{b_r} \text{ and } x_1^{i_1} \dots x_m^{i_m} <_X x_1^{a_1} \dots x_m^{a_m}
 \end{aligned}$$

The definition of a block order for more variable sets is a direct generalization.

If $f \in \mathbb{K}[X, y]$, with $y > X$, we can write f as

$$f = a_u y^u + \dots + a_1 y + a_0, \quad \text{with } a_i \in \mathbb{K}[X] \text{ and } a_u \neq 0.$$

Then we say that a_u is the *leading polynomial* of f .

2.1 Affine-variety codes

Let $m \geq 1$ and $I \subseteq \mathbb{F}_q[X] = \mathbb{F}_q[x_1, \dots, x_m]$ be an ideal such that

$$\{x_1^q - x_1, x_2^q - x_2, \dots, x_m^q - x_m\} \subset I.$$

Let $\mathcal{V}(I) = \{P_1, P_2, \dots, P_n\}$. Since I is a zero-dimensional radical ideal, we have an isomorphism of \mathbb{F}_q vector spaces (an evaluation):

$$\begin{aligned} \phi : R = \mathbb{F}_q[x_1, \dots, x_m]/I &\longrightarrow (\mathbb{F}_q)^n \\ f &\longmapsto (f(P_1), \dots, f(P_n)). \end{aligned}$$

Let $L \subseteq R$ be an \mathbb{F}_q vector subspace of R of dimension r .

Definition 2.1 ([FL98]). *The **affine-variety code** $C(I, L)$ is the image $\phi(L)$ and the **affine-variety code** $C^\perp(I, L)$ is its dual code.*

If b_1, \dots, b_r is a linear basis for L over \mathbb{F}_q , then the matrix

$$\begin{pmatrix} b_1(P_1) & b_1(P_2) & \dots & b_1(P_n) \\ \vdots & \vdots & \dots & \vdots \\ b_r(P_1) & b_r(P_2) & \dots & b_r(P_n) \end{pmatrix}$$

is a generator matrix for $C(I, L)$ and a parity-check matrix for $C^\perp(I, L)$.

Theorem 2.2 ([FL98]). *Every linear code may be represented as an affine-variety code.*

From now on, q, m, n, I and L are understood to be defined as above.

For any $1 \leq i \leq m$, $\hat{\pi}_i$ denotes the natural projection $\hat{\pi}_i : (\mathbb{F}_q)^m \rightarrow \mathbb{F}_q$, such that $\hat{\pi}_i(\bar{x}_1, \dots, \bar{x}_m) = \bar{x}_i$.

2.2 Stratified ideals

In this subsection we summarize some definitions and results from [GS09].

Let $J \subset \mathbb{K}[\mathcal{S}, \mathcal{A}, \mathcal{T}]$ be a zero-dimensional radical ideal, with variables $\mathcal{S} = \{s_1, \dots, s_N\}$, $\mathcal{A} = \{a_1, \dots, a_L\}$, $\mathcal{T} = \{t_1, \dots, t_K\}$. We fix a term ordering $<$ on $\mathbb{K}[\mathcal{S}, \mathcal{A}, \mathcal{T}]$, with $\mathcal{S} < \mathcal{A} < \mathcal{T}$, such that the \mathcal{A} variables are ordered by $a_L < a_{L-1} < \dots < a_1$.

Let us define the elimination ideals $J_{\mathcal{S}} = J \cap \mathbb{K}[\mathcal{S}]$, $J_{\mathcal{S}, a_L} = J \cap \mathbb{K}[\mathcal{S}, a_L]$, \dots , $J_{\mathcal{S}, a_L, \dots, a_1} = J \cap \mathbb{K}[\mathcal{S}, a_L, \dots, a_1] = J \cap \mathbb{K}[\mathcal{S}, \mathcal{A}]$.

We want to view $\mathcal{V}(J_{\mathcal{S}})$ as a disjoint union of some sets. The way we define these sets is linked to the fact that any point P in $\mathcal{V}(J_{\mathcal{S}})$ can be extended to at least one point in $\mathcal{V}(J_{\mathcal{S}, a_L})$. But the number of all possible extensions of P

in $\mathcal{V}(J_{\mathcal{S}, \mathbf{a}_L})$ is finite, since the ideal is zero-dimensional, so we can partition $\mathcal{V}(J_{\mathcal{S}})$ in sets such that all points in the same set share the same number of extensions. We denote by $\lambda(L)$ the maximum number of extensions in $\mathcal{V}(J_{\mathcal{S}, \mathbf{a}_L})$ for any $P \in \mathcal{V}(J_{\mathcal{S}})$. The same principle applies when we consider the variety of another elimination ideal, e.g. $\mathcal{V}(J_{\mathcal{S}, \mathbf{a}_L, \dots, \mathbf{a}_h})$. We can partition it into subsets such that all points in the same subset share the same number of extensions in $\mathcal{V}(J_{\mathcal{S}, \mathbf{a}_L, \dots, \mathbf{a}_h, \mathbf{a}_{h-1}})$. The maximum number of extensions is denoted by $\lambda(h-1)$.

We write our partitioning in a formal way, as follows:

$$\begin{aligned} \mathcal{V}(J_{\mathcal{S}}) &= \sqcup_{l=1}^{\lambda(L)} \Sigma_l^L, \text{ with} \\ \Sigma_l^L &= \{(\bar{s}_1, \dots, \bar{s}_N) \in \mathcal{V}(J_{\mathcal{S}}) \mid \exists \text{ exactly } l \text{ distinct values } \bar{a}_L^{(1)}, \dots, \bar{a}_L^{(l)} \\ &\quad \text{s.t. } (\bar{s}_1, \dots, \bar{s}_N, \bar{a}_L^{(\ell)}) \in \mathcal{V}(J_{\mathcal{S}, \mathbf{a}_L}), 1 \leq \ell \leq l\}; \\ \mathcal{V}(J_{\mathcal{S}, \mathbf{a}_L, \dots, \mathbf{a}_h}) &= \sqcup_{l=1}^{\lambda(h-1)} \Sigma_l^{h-1}, \quad 2 \leq h \leq L, \text{ with} \end{aligned}$$

$$\begin{aligned} \Sigma_l^{h-1} &= \{(\bar{s}_1, \dots, \bar{s}_N, \bar{a}_L, \dots, \bar{a}_h) \in \mathcal{V}(J_{\mathcal{S}, \mathbf{a}_L, \dots, \mathbf{a}_h}) \mid \exists \text{ exactly } l \text{ distinct values} \\ &\quad \bar{a}_{h-1}^{(1)}, \dots, \bar{a}_{h-1}^{(l)} \text{ s.t. } (\bar{s}_1, \dots, \bar{s}_N, \bar{a}_L, \dots, \bar{a}_h, \bar{a}_{h-1}^{(\ell)}) \in \mathcal{V}(J_{\mathcal{S}, \mathbf{a}_L, \dots, \mathbf{a}_{h-1}}), 1 \leq \ell \leq l\}. \end{aligned}$$

For an arbitrary zero-dimensional ideal J , nothing can be said about $\lambda(h)$, except that $\lambda(h) \geq 1$ for any $1 \leq h \leq L$.

Definition 2.3 ([GS09]). *With the above notation, let J be a zero-dimensional radical ideal. We say that J is **stratified**, with respect to the \mathcal{A} variables, if:*

- (a) $\lambda(h) = h$, $1 \leq h \leq L$, and
- (b) $\sum_l^h \neq \emptyset$, $1 \leq h \leq L$, $1 \leq l \leq h$.

To explain conditions (a) and (b) in the above definition, let us consider $h = L$ and think of the projection

$$\pi : \mathcal{V}(J_{\mathcal{S}, \mathbf{a}_L}) \rightarrow \mathcal{V}(J_{\mathcal{S}}). \quad (1)$$

In this case, (a) in Definition 2.3 is equivalent to saying that any point in $\mathcal{V}(J_{\mathcal{S}})$ has at most L pre-images in $\mathcal{V}(J_{\mathcal{S}, \mathbf{a}_L})$ via π , and that there is at least one point with (exactly) L pre-images. On the other hand, (b) implies that, if for a point $P \in \mathcal{V}(J_{\mathcal{S}})$ we have $|\pi^{-1}(P)| = m \geq 2$, then there is at least another point $Q \in \mathcal{V}(J_{\mathcal{S}})$ such that $|\pi^{-1}(Q)| = m - 1$.

Example 2.4. Let $\mathcal{S} = \{s_1\}$, $\mathcal{A} = \{a_1, a_2, a_3\}$ ($L = 3$) and $\mathcal{T} = \{t_1\}$ such that $\mathcal{S} < \mathcal{A} < \mathcal{T}$ and $a_3 < a_2 < a_1$. Let us consider $\mathcal{J} = \mathcal{I}(Z) \subset \mathbb{C}[s_1, a_3, a_2, a_1, t_1]$ with $Z = \{(1, 2, 1, 0, 0), (1, 2, 2, 0, 0), (1, 4, 0, 0, 0), (1, 6, 0, 0, 0), (2, 5, 0, 0, 0), (3, 1, 0, 0, 0), (3, 3, 0, 0, 0), (5, 2, 0, 0, 0)\}$. Then:

$$\begin{aligned} \mathcal{V}(J_{\mathcal{S}}) &= \{1, 2, 3, 5\} \\ \mathcal{V}(J_{\mathcal{S}, \mathbf{a}_3}) &= \{(1, 2), (1, 4), (1, 6), (2, 5), (3, 1), (3, 3), (5, 2)\} \\ \mathcal{V}(J_{\mathcal{S}, \mathbf{a}_3, \mathbf{a}_2}) &= \{(1, 2, 1), (1, 2, 2)(1, 4, 0), (1, 6, 0), (2, 5, 0), (3, 1, 0), (3, 3, 0), (5, 2, 0)\} \\ \mathcal{V}(J_{\mathcal{S}, \mathbf{a}_3, \mathbf{a}_2, \mathbf{a}_1}) &= \{(1, 2, 1, 0), (1, 2, 2, 0)(1, 4, 0, 0), (1, 6, 0, 0), (2, 5, 0, 0), (3, 1, 0, 0), (3, 3, 0, 0), (5, 2, 0, 0)\} \end{aligned}$$

Let us consider the projection $\pi : \mathcal{V}(\mathcal{J}_{\mathcal{S}, \mathbf{a}_3}) \rightarrow \mathcal{V}(\mathcal{J}_{\mathcal{S}})$. Then:

$$|\pi^{-1}(\{5\})| = 1, \quad |\pi^{-1}(\{2\})| = 1, \quad |\pi^{-1}(\{3\})| = 2, \quad |\pi^{-1}(\{1\})| = 3,$$

so $\sum_1^3 = \{2, 5\}$, $\sum_2^3 = \{3\}$, $\sum_3^3 = \{1\}$ and $\sum_i^3 = \emptyset$, $i > 3$. This means that $\lambda(L) = \lambda(3) = 3$ and \sum_l^3 is not empty, for $l = 1, 2, 3$. Thus the conditions of Definition 2.3 are satisfied for $h = L = 3$ (see Fig. 1). In the same way, it is easy to verify said conditions also for $h = 1, 2$, and hence the ideal \mathcal{J} is stratified with respect to the \mathcal{A} variables.

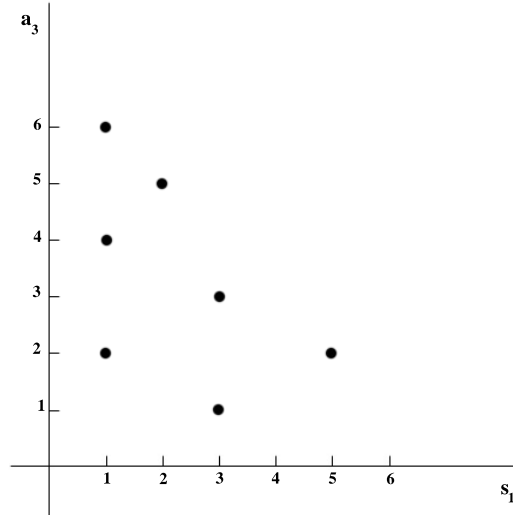


Figure 1. A variety in a stratified case

With the above notation, an immediate consequence of Theorem 3.6 in [GS06] (Theorem 32 in [GS09]) is the following proposition.

Proposition 2.5. *Let $<$ be any lexicographic term order with $\mathcal{S} < \mathcal{A} < \mathcal{T}$ and $\mathbf{a}_L < \mathbf{a}_{L-1} < \dots < \mathbf{a}_1$. Let J be a stratified ideal with respect to the \mathcal{A} variables. Let $G = \text{GB}(J)$. Then G contains one and only one polynomial g such that:*

$$g \in \mathbb{K}[\mathcal{S}, \mathbf{a}_L], \quad \mathbf{T}(g) = \mathbf{a}_L^L.$$

2.3 Root multiplicities and Hasse derivative

Definition 2.6. *Let $g = \sum_i a_i x^i \in \mathbb{K}[x]$. Then the **n-th Hasse derivative** of g is $\varphi^{(n)}(g)$ and the **n-th formal derivative** of g is $g^{(n)}$, where*

$$\varphi^{(n)}(g) = \sum_i \binom{i}{n} a_i x^{i-n} \text{ and } g^{(n)} = n! \sum_i \binom{i}{n} a_i x^{i-n}.$$

We can note that $g^{(n)} = n! \varphi^{(n)}(g)$. In a field with characteristic p , it is more convenient to use the Hasse derivative, because $n! = 0$ for all $n \geq p$.

Note that $\varphi^{(2)}(g) \neq \varphi^{(1)}(\varphi^{(1)}(g))$.

Definition 2.7. Let $g \in \mathbb{K}[x]$, $g \neq 0$, $P \in \mathbb{K}$ and $g(P) = 0$. The **multiplicity** of P as a root of g is the largest integer $r \geq 1$ such that

$$\varphi^{(k)}(g)(P) = \varphi^{(k)}(g)|_{x=P} = 0, \quad \text{for } 0 \leq k \leq r-1.$$

The following theorem is well-known, see e.g. [LN97].

Theorem 2.8. Let $g, f \in \mathbb{K}[x]$ and let g be irreducible. Then

$$g^r | f \iff g | \varphi^{(k)}(f) \text{ for } 0 \leq k \leq r-1.$$

As a consequence of the previous theorem when $g = (x-P)$ for any $P \in \mathbb{K}$, we have

$$(x-P)^r | f \iff \varphi^{(k)}(f)|_{x=P} = 0 \text{ for } 0 \leq k \leq r-1.$$

2.4 General error locator polynomials

Let C be an $[n, k, d]$ linear code over \mathbb{F}_q with correction capability $t \geq 1$. Choose any parity-check matrix with entries in an appropriate extension field \mathbb{F}_{q^M} of \mathbb{F}_q , $M \geq 1$. Its syndromes lie in $(\mathbb{F}_{q^M})^{n-k}$ and form a vector space of dimension $r = n - k$ over \mathbb{F}_q . Let α be a primitive n -th root of unity in \mathbb{F}_{q^M} .

Definition 2.9. Let \mathcal{L} be a polynomial in $\mathbb{F}_q[S, x]$, where $S = (s_1, \dots, s_r)$. Then \mathcal{L} is a **general error locator polynomial** of C if

- (1) $\mathcal{L}(S, x) = x^t + a_{t-1}x^{t-1} + \dots + a_0$, with $a_j \in \mathbb{F}_q[S]$, $0 \leq j \leq t-1$, that is, \mathcal{L} is a monic polynomial with degree t with respect to the variable x and its coefficients are in $\mathbb{F}_q[S]$;
- (2) given a syndrome $\mathbf{s} = (\bar{s}_1, \dots, \bar{s}_r) \in (\mathbb{F}_{q^M})^r$, corresponding to an error vector of weight $\mu \leq t$ and error positions $\{k_1, \dots, k_\mu\}$, if we evaluate the S variables at \mathbf{s} , then the roots of $\mathcal{L}(\mathbf{s}, x)$ are exactly $\{\alpha^{k_1}, \dots, \alpha^{k_\mu}, 0\}$, where the multiplicity of 0 is $t - \mu$.

Given any (correctable) linear code C , the existence of a general error locator polynomial is not known. In [OS05] the authors prove its existence for any cyclic code and recently in [GS06, GS09, Gio06] its existence has been proved for a large class of linear codes.

We can extend Definition 2.9 to the case when there are also erasures.

Definition 2.10. Let \mathcal{L} be a polynomial in $\mathbb{F}_q[S, W, x]$, $S = (s_1, \dots, s_r)$ and $W = (w_1, \dots, w_\nu)$, where ν is the number of occurred erasures. Let $2\tau + \nu < d$. Then \mathcal{L} is a **general error locator polynomial of type ν** of C if

- (1) $\mathcal{L}(S, W, x) = x^\tau + a_{\tau-1}x^{\tau-1} + \dots + a_0$, with $a_j \in \mathbb{F}_q[S, W]$, for any $0 \leq j \leq \tau-1$, that is, \mathcal{L} has degree τ w.r.t. x and coefficients in $\mathbb{F}_q[S, W]$;
- (2) for any syndrome $\mathbf{s} = (\bar{s}_1, \dots, \bar{s}_r)$ and any erasure location vector $\mathbf{w} = (\bar{w}_1, \dots, \bar{w}_\nu)$, corresponding to an error vector of weight $\mu \leq \tau$ and error locations $\{k_1, \dots, k_\mu\}$, if we evaluate the S variables at \mathbf{s} and the W variables at \mathbf{w} , then the roots of $\mathcal{L}(\mathbf{s}, \mathbf{w}, x)$ are $\{\alpha^{k_1}, \dots, \alpha^{k_\mu}, 0\}$, where the multiplicity of 0 is $\tau - \mu$.

For the benefit of readers unfamiliar with simultaneous correction of errors and erasures, we sketch how it works. When some (say ν) symbols are not recognised by the receiver, the decoder treats them as *erasures*. The decoder knows the positions of these erasures i_1, \dots, i_ν , which means in our notation that the decoder knows the erasure locations grouped for convenience in the *erasure location vector* $\mathbf{w} = (\bar{w}_1, \dots, \bar{w}_\nu) = (\alpha^{i_1}, \dots, \alpha^{i_\nu})$. A standard result in coding theory is that it is possible to correct simultaneously ν erasures and τ errors, provided that $2\tau + \nu < d$.

To be consistent with our notation, we may refer to a polynomial in Definition 2.9 also as a *general error locator polynomial of type 0*.

For a code C , the possession of a polynomial of each type $0 \leq \nu < d$ might be a stronger condition than the possession of a polynomial of type 0, but in [OS05] the authors prove that any cyclic code admits a polynomial of any type ν , for $0 \leq \nu < d$. In [GS09] the existence of general error locator polynomials (of any type) for a large class of linear codes was proved, but it is still unknown whether such a result holds for general linear codes.

3 Decoding the affine variety code with the Cooper philosophy

3.1 The approach by Fitzgerald and Lax

In [FL98] a decoding technique was proposed following what is known as the “Cooper philosophy”. Although this terminology has been established only recently ([MO09]), this decoding approach has a quite wide literature, e.g. [Coo90], [Coo93], [CM02], [Coo91], [CRHT94a]. We describe this technique for affine-variety codes, as follows (see Subsection 2.1). Let $C^\perp(I, L)$ be an affine-variety code with dimension $n - r$ and let $I = \langle g_1, \dots, g_\gamma \rangle$. Let L be linearly generated by b_1, \dots, b_r . Then we can denote by $J_{\mathcal{FL}}^{C,t}$ the ideal (\mathcal{FL} is for “FitzgeraldLax”)

$$J_{\mathcal{FL}}^{C,t} \subset \mathbb{F}_q[s_1, \dots, s_r, x_{t,1}, \dots, x_{t,m}, \dots, x_{1,1}, \dots, x_{1,m}, e_1, \dots, e_t] = \mathbb{F}_q[S, X_t, \dots, X_1, E]$$

where¹

$$J_{\mathcal{FL}}^{C,t} = \left\langle \left\{ \sum_{j=1}^t e_j b_\rho(x_{j,1}, \dots, x_{j,m}) - s_\rho \right\}_{1 \leq \rho \leq r}, \left\{ e_j^{q-1} - 1 \right\}_{1 \leq j \leq t}, \left\{ g_h(x_{j,1}, \dots, x_{j,m}) \right\}_{\substack{1 \leq h \leq \gamma, \\ 1 \leq j \leq t}} \right\rangle. \quad (2)$$

Let $<_S$ be any term ordering on the variables s_1, \dots, s_r and $<_{\text{lex}}$ be the lexicographic ordering on the variables X_t, \dots, X_1 , such that

$$x_{t,1} <_{\text{lex}} \dots <_{\text{lex}} x_{t,m} <_{\text{lex}} \dots <_{\text{lex}} x_{1,1} <_{\text{lex}} \dots <_{\text{lex}} x_{1,m}.$$

Let $<_E$ be any term ordering on the variables e_1, \dots, e_t .

Then let $<$ be the block order $(<_S, <_{\text{lex}}, <_E)$. We denote by $\mathcal{G}_{\mathcal{FL}}^{C,t}$ a Gröbner basis of $J_{\mathcal{FL}}^{C,t}$ with respect to $<$. In [FL98] we can find a method describing how to find the error locations and values, by applying elimination theory to the polynomials in $\mathcal{G}_{\mathcal{FL}}^{C,t}$.

Example 3.1. Let $C = C^\perp(I, L)$ be the Hermitian code from the curve $y^2 + y = x^3$ over \mathbb{F}_4 and with defining monomials $\{1, x, y, x^2, xy\}$. The eight points of the variety defined by I are

$$P_1 = (0, 0), P_2 = (0, 1), P_3 = (1, \alpha), P_4 = (1, \alpha^2), P_5 = (\alpha, \alpha), P_6 = (\alpha, \alpha^2), P_7 = (\alpha^2, \alpha), P_8 = (\alpha^2, \alpha^2),$$

where α is any primitive element of \mathbb{F}_4 . It is well-known that C corrects up to $t = 2$ errors. The ideal $J_{\mathcal{FL}}^{C,2} \subset \mathbb{F}_4[s_1, \dots, s_5, x_2, y_2, x_1, y_1, e_1, e_2]$ is

$$\begin{aligned} J_{\mathcal{FL}}^{C,2} = \langle & \{x_1^4 - x_1, x_2^4 - x_2, y_1^4 - y_1, y_2^4 - y_2, e_1^3 - 1, e_2^3 - 1, y_1^2 + y_1 - x_1^3, \\ & y_2^2 + y_2 - x_2^3, e_1 + e_2 - s_1, e_1 x_1 + e_2 x_2 - s_2, e_1 y_1 + e_2 y_2 - s_3, \\ & e_1 x_1^2 + e_2 x_2^2 - s_4, e_1 x_1 y_1 + e_2 x_2 y_2 - s_5\} \rangle. \end{aligned}$$

Typically the Gröbner basis of $J_{\mathcal{FL}}^{C,t}$ that has been obtained using the block order $<$ contains a large number of polynomials and most are not useful for decoding purposes. We would have to choose a polynomial in $\mathbb{F}_q[S, x_{t,1}]$ that, once specialized in the received syndrome, could be used to find the first coordinates of all the errors. It is important to observe that in this situation we do not know which polynomial is the right one, because after the specialization we can obtain a polynomial which vanishes identically.

3.2 Rationale for our decoding ideals

The approach presented in the previous section shares the same problem with other similar approaches ([CRHT94b],[LY97],[CM02]). In the portion of the Gröbner basis corresponding to the elimination ideal $I_{S,x_{t,1}}$, one should choose a polynomial g in $\mathbb{F}_q[S, x_{t,1}] \setminus \mathbb{F}_q[S]$, specialize it to the received syndrome, and then find its $x_{t,1}$ -roots. The problem is that it is not possible to

¹ To speed up the basis computation we can add $\{x_{j,\ell}^q - x_{j,\ell}\}_{\substack{1 \leq j \leq t, \\ 1 \leq \ell \leq m}}$ to the ideal.

know in advance which polynomial has to be chosen, and there might be hundreds of “candidate” polynomials. Let us call ideal $J_{\mathcal{FL}}^{C,t}$ the “Cooper ideal for affine-variety codes” (the convenience for this historically inaccurate name will be clear in a moment) and the “Cooper variety” its variety.

The same problem is present in the ideal for decoding cyclic codes presented in [CRHT94b], which we will call the “Cooper ideal for cyclic codes” (although again its formal definition was first presented in [CRHT94b]), where a huge number of polynomials can be found as soon as the code parameters are not trivial. In this case an improvement was proposed in [CM02]. Instead of specializing the whole polynomial, one can specialize only its leading polynomial, since it does not vanish identically if and only if the whole polynomial does not vanish (by the Gianni-Kalkbrener theorem [Gia89], [Kal89]). We could adopt exactly the same strategy for the “Cooper ideal for affine-variety codes” and thus get a significant improvement on the algorithm proposed in [FL98]. This improvement would reduce the cost of the specialization, but would still require an evaluation (in the worst case) for any candidate polynomial. In Section 7 of [CM02] a more refined strategy has been investigated, that is, the vanishing conditions coming from the leading polynomials were grouped and a decision tree was formed. In the example proposed there, this resulted in a drastic reduction of the computations required to identify the right candidate. Unfortunately, this strategy has not been deeply investigated in the general case, but we believe that it is obvious how this could be done also for the Cooper ideal for affine-variety codes, obtaining thus another improvement.

In [LY97] it was noted that the Cooper variety for cyclic codes contains also points that do not correspond to valid syndrome-error location pairs and thus are useless. In [OS05] the authors enlarge the Cooper ideal in order to remove exactly the non-valid pairs, which we call “spurious solutions”. The new ideal turns out to be stratified (although the notion of stratified ideal is established later in [GS09]) and hence to contain the general error locator polynomial, thanks to deep properties of some Gröbner bases of stratified ideals, which is the *only* polynomial that needs to be specialized. We are now going to explain how this improvement can be obtained also for the Cooper variety for affine-variety codes.

We define several modified versions of the Cooper ideal for decoding affine-variety codes. We summarize what we are going to do:

- In Subsection 3.3 we define a decoding ideal (8) that is able to correct any correctable error, even not knowing in advance the number of errors.
- However, in Subsection 3.4 we show why this decoding ideal does not necessarily contain locator polynomials that play the same role of generator error locator polynomials for cyclic codes. Still, these weak forms of locators (Definition 3.9) can be used to decode.
- In Section 4 we develop the commutative algebra necessary to show the

existence of weak locators, with Section 5 devoted to the long proof of the main result, and then in Section 6 we will finally be able to define a set of multi-dimensional general error locator polynomials (see Definition 6.2). We define a suitable ideal containing this set as we show in Theorem 7.6.

3.3 Decoding with ghost points

Note that Fitzgerald and Lax consider the possible error locations as t points in $\mathcal{V}(I)$, that we call $P_{\sigma_1}, \dots, P_{\sigma_t}$, but they denote their components dropping the reference to σ , that is, $P_{\sigma_l} = (x_{l,1}, \dots, x_{l,m})$ for $1 \leq l \leq t$. We adhere to this notation from now on.

We observe that in the Cooper ideal (2) there is not any constraint on point pairs. But we want that all error locations are distinct. We have to force this, i.e. any two locations $P_{\sigma_j} = (x_{j,1}, \dots, x_{j,m})$ and $P_{\sigma_k} = (x_{k,1}, \dots, x_{k,m})$ must differ in at least one component. So we add this condition:

$$\prod_{1 \leq \iota \leq m} ((x_{j,\iota} - x_{k,\iota})^{q-1} - 1) = 0 \quad \text{for } 1 \leq j < k \leq t.$$

In fact, if $\alpha \in \mathbb{F}_q$, then $\alpha \neq 0 \iff \alpha^{q-1} = 1$. Therefore, the product $\prod_{1 \leq \iota \leq m} ((x_{j,\iota} - x_{k,\iota})^{q-1} - 1)$ is zero if and only if at least for one ι we have $(x_{j,\iota} - x_{k,\iota})^{q-1} = 1$, i.e. $x_{j,\iota} \neq x_{k,\iota}$ and thus $P_{\sigma_j} \neq P_{\sigma_k}$. Our ideal becomes

$$\begin{aligned} \widehat{J}_{\mathcal{FL}}^{C,t} = \left\langle \left\{ \sum_{j=1}^t e_j b_\rho(x_{j,1}, \dots, x_{j,m}) - s_\rho \right\}_{1 \leq \rho \leq r}, \{e_j^{q-1} - 1\}_{1 \leq j \leq t}, \right. \\ \left. \{g_h(x_{j,1}, \dots, x_{j,m})\}_{\substack{1 \leq h \leq \gamma, \\ 1 \leq j \leq t}}, \right. \\ \left. \left\{ \prod_{1 \leq \iota \leq m} ((x_{j,\iota} - x_{k,\iota})^{q-1} - 1) \right\}_{1 \leq j < k \leq t} \right\rangle. \end{aligned} \quad (3)$$

Remark 3.2. Ideal $\widehat{J}_{\mathcal{FL}}^{C,t}$ can be used to correct and it will work better than $J_{\mathcal{FL}}^{C,t}$, since its variety does not contain spurious solutions. However, we cannot expect that $\widehat{J}_{\mathcal{FL}}^{C,t}$ contains polynomials with a role similar to that of the generic error locator in the cyclic case, because $\widehat{J}_{\mathcal{FL}}^{C,t}$ still depends on the knowledge of the error number.

In the following we modify (3) to allow for different-weight syndromes.

(a) First, we note that in $\widehat{J}_{\mathcal{FL}}^{C,t}$ the following condition is verified

$$e_j^{q-1} = 1 \text{ with } j = 1, \dots, t.$$

This is equivalent to saying that exactly t errors occurred, which are $e_1, \dots, e_t \in \mathbb{F}_q^*$. We must allow for some e_j with $j = 1, \dots, t$ to be equal to zero. We would obtain a new ideal where the conditions $e_j^{q-1} = 1$ are replaced with $e_j^q = e_j$ for any $j = 1, \dots, t$.

- (b) We recall the changes made to the Cooper ideal in [OS05] for cyclic codes. We consider the error vector

$$e = (\underbrace{0, \dots, 0}_{k_1-1}, \underbrace{e_1}_{\uparrow k_1}, 0, \dots, 0, \underbrace{e_l}_{\uparrow k_l}, 0, \dots, 0, \underbrace{e_\mu}_{\uparrow k_\mu}, \underbrace{0, \dots, 0}_{n-1-k_\mu}) \quad \text{with } \mu \leq t,$$

where k_1, \dots, k_μ are the error positions and e_1, \dots, e_μ are the error values. We consider the j -th syndrome and we obtain the following equation

$$\sum_{l=1}^{\mu} e_l (\alpha^{i_j})^{k_l} = s_j. \quad (4)$$

(For the n -th root codes in [GS06, GS09] the formulas are slightly more complicated). To arrive at the desired equation

$$\sum_{l=1}^t e_l (\alpha^{i_j})^{k_l} = s_j \quad (5)$$

we have to add the “virtual error position” k defined as $\alpha^k = 0 \ \forall \alpha \in \mathbb{F}$. Using the location $z_l = \alpha^{k_l}$ (and so the “virtual error location” is $\alpha^k = 0$), equation (5) becomes

$$s_j = \sum_{l=1}^{\mu} e_l (z_l)^{i_j} + \sum_{l=\mu+1}^t e_l (\alpha^k)^{k_l} = \sum_{l=1}^{\mu} e_l (z_l)^{i_j} + \sum_{l=\mu+1}^t e_l (0)^{k_l} = \sum_{l=1}^t e_l (z_l)^{i_j}.$$

We can rephrase what we did by saying that we are using 0 as a *ghost error location*, meaning that if we find ν zero roots in the error location polynomial, then $\mu = t - \nu$ (ν error locations are ghost locations and so they do not correspond to actual errors).

- (c) Let us come back to the affine-variety case. The error vector is

$$e = (\underbrace{0, \dots, 0}_{\sigma_1-1}, \underbrace{e_1}_{\uparrow P_{\sigma_1}}, 0, \dots, 0, \underbrace{e_l}_{\uparrow P_{\sigma_l}}, 0, \dots, 0, \underbrace{e_\mu}_{\uparrow P_{\sigma_\mu}}, \underbrace{0, \dots, 0}_{n-1-\sigma_\mu}).$$

The valid error locations are the points $P_{\sigma_l} = (x_{l,1}, \dots, x_{l,m})$, $1 \leq l \leq \mu$. The equation corresponding to (4) is

$$s_\rho = \sum_{l=1}^{\mu} e_l b_\rho(P_{\sigma_l}) = \sum_{l=1}^{\mu} e_l b_\rho(x_{l,1}, \dots, x_{l,m}). \quad (6)$$

We want a sum like (5), something like $s_\rho = \sum_{l=1}^t e_l b_\rho(P_{\sigma_l})$. In order to do that, we would need $\sum_{l=\mu+1}^t e_l b_\rho(P_{\sigma_l}) = 0$, for some convenient *ghost*

points $\{P_{\sigma_l}\}_{\mu+1 \leq l \leq t}$. Actually, we can use just one ghost point, that we call P_0 . But it must *not* lie on the variety, otherwise it could be confused with valid locations. In particular, we cannot hope to use always the ghost point $P_0 = (x_{0,1}, \dots, x_{0,m}) = (0, \dots, 0)$, since $(0, \dots, 0)$ could be a point on the variety. For example, the Hermitian curve $\chi : x^{q+1} = y^q + y$ contains $(0, 0)$ for any q .

Let P_0 be a ghost point. Not only we need to choose P_0 outside the variety, but we must also force $e_j = 0$ for the error values in P_0 , since we cannot hope that $b_\rho(P_0) = 0$ for each ρ . With these assumptions, we obtain

$$\begin{aligned} s_\rho &= \sum_{l=1}^{\mu} e_l b_\rho(x_{l,1}, \dots, x_{l,m}) + \sum_{l=\mu+1}^t e_l b_\rho(P_0) \\ &= \sum_{l=1}^{\mu} e_l b_\rho(x_{l,1}, \dots, x_{l,m}) + \sum_{l=\mu+1}^t 0 b_\rho(P_0) \\ &= \sum_{l=1}^{\mu} e_l b_\rho(x_{l,1}, \dots, x_{l,m}). \end{aligned} \tag{7}$$

- (d) For us a ghost point is any point $P_0 \in (\mathbb{F}_q)^m \setminus \mathcal{V}(I)$. Depending on the variety, there can be clever ways to choose P_0 .

Definition 3.3. Let $P_0 = (\bar{x}_{0,1}, \dots, \bar{x}_{0,m}) \in (\mathbb{F}_q)^m \setminus \mathcal{V}(I)$. We say that P_0 is an **optimal ghost point** if there is a $1 \leq j \leq m$ such that the hyperplane $x_j = \bar{x}_{0,j}$ does not intersect the variety. We call j the **ghost component**.

In other words, for any optimal ghost point there is at least a component not shared with any variety point. See Figure 2 for an example.

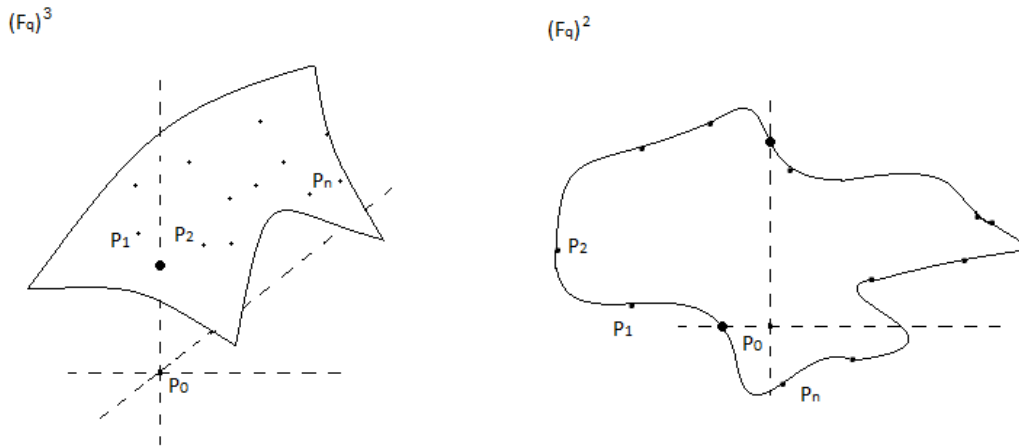


Figure 2. In the first picture we have a optimal ghost point with two ghost components. In the second, a non-optimal ghost point.

Remark 3.4. The advantage of using optimal ghost points is that it is enough to look at any ghost component in order to discard non-valid

locations.

If a curve is smooth and maximal (e.g., an Hermitian curve), it will probably intersect any hyperplane and so no optimal ghost point will exist in this case.

- (e) We are ready to define a new ideal, summarising the above argument. We start from equations (7):

$$\left\{ \sum_{j=1}^t e_j b_\rho(x_{j,1}, \dots, x_{j,m}) - s_\rho \right\}_{1 \leq \rho \leq r}$$

We choose a ghost point $P_0 = (x_{0,1}, \dots, x_{0,m}) \notin \mathcal{V}(I)$. We need to find a generator set for the radical ideal I' vanishing on $\mathcal{V}(I) \sqcup \{P_0\}$. The easiest way of doing this is to start from any Gröbner basis G of I and to use the Buchberger-Möller algorithm (see [MB82, Mor09a]) to compute the Gröbner basis G' of I' . We will summarize the Buchberger-Möller algorithm in Theorem 5.1. Let $G' = \{g'_h\}_{1 \leq h \leq \gamma'}$. We can insert in our new ideal the following polynomials

$$\{g'_h(x_{j,1}, \dots, x_{j,m})\}_{\substack{1 \leq h \leq \gamma' \\ 1 \leq j \leq t}}$$

In our new system we put $\{e_j^q = e_j\}$, because there can be zero values (corresponding to ghost locations). We enforce $(x_{j,1}, \dots, x_{j,m}) \neq P_0$ for all j corresponding to actual error locations. In order to do that, when $e_j \neq 0$ we must have at least one component of P_{σ_j} different from that of P_0 , that is, $e_j \prod_{1 \leq \iota \leq m} ((x_{j,\iota} - x_{0,\iota})^{q-1} - 1) = 0$. So we can add

$$\left\{ e_j \prod_{1 \leq \iota \leq m} ((x_{j,\iota} - x_{0,\iota})^{q-1} - 1) \right\}_{1 \leq j \leq t}.$$

On the other hand, when $e_j = 0$ we want $(x_{j,1}, \dots, x_{j,m}) = P_0$. To enforce it, we add

$$\{(e_j^{q-1} - 1)(x_{j,\iota} - x_{0,\iota})\}_{\substack{1 \leq j \leq t, \\ 1 \leq \iota \leq m}}.$$

Finally, if two points correspond to valid locations then they must be distinct. However, if at least one is a ghost point, then this requirement does not hold:

$$\left\{ e_j e_k \prod_{1 \leq \iota \leq m} ((x_{j,\iota} - x_{k,\iota})^{q-1} - 1) \right\}_{1 \leq j < k \leq t}.$$

We denote by $J_*^{C,t}$ the ideal in $\mathbb{F}_q[s_1, \dots, s_r, X_t, \dots, X_1, e_1, \dots, e_t]$, with $X_1 = \{x_{1,1}, \dots, x_{1,m}\}, \dots, X_t = \{x_{t,1}, \dots, x_{t,m}\}$ s.t.

$$J_*^{C,t} = \left\langle \left\{ \sum_{j=1}^t e_j b_\rho(x_{j,1}, \dots, x_{j,m}) - s_\rho \right\}_{1 \leq \rho \leq r}, \left\{ e_j^q - e_j \right\}_{1 \leq j \leq t}, \right. \\ \left. \left\{ g'_h(x_{j,1}, \dots, x_{j,m}) \right\}_{\substack{1 \leq h \leq \gamma', \\ 1 \leq j \leq t}}, \left\{ (e_j^{q-1} - 1)(x_{j,\iota} - x_{0,\iota}) \right\}_{\substack{1 \leq j \leq t, \\ 1 \leq \iota \leq m}}, \right. \\ \left. \left\{ e_j \prod_{1 \leq \iota \leq m} ((x_{j,\iota} - x_{0,\iota})^{q-1} - 1) \right\}_{1 \leq j \leq t}, \right. \\ \left. \left\{ e_j e_k \prod_{1 \leq \iota \leq m} ((x_{j,\iota} - x_{k,\iota})^{q-1} - 1) \right\}_{1 \leq j < k \leq t} \right\rangle. \quad (8)$$

Since $I' = \langle \{g'_h\}_{1 \leq h \leq H} \rangle$ contains the field equations, we may add them to reduce the computation of the Gröbner basis of $J_*^{C,t}$.

3.4 Weak locator polynomials

We would like to define some locator polynomials for affine-variety codes that play the same role as those in Definition 2.9. We would expect to find them in our ideal (8). These locators might look like

$$\mathcal{L}_i(S, x_1, \dots, x_i) = x_i^t + a_{t-1}x_i^{t-1} + \dots + a_0, \quad (9)$$

with $a_j \in \mathbb{F}_q[S, x_1, \dots, x_{i-1}]$, $0 \leq j \leq t-1$, that is, \mathcal{L}_i is a monic polynomial with degree t with respect to the variable x_i and its coefficients are in $\mathbb{F}_q[S, x_1, \dots, x_{i-1}]$. We would also want the following property.

Given a syndrome $\mathbf{s} = (\bar{s}_1, \dots, \bar{s}_r) \in (\mathbb{F}_q)^r$, corresponding to an error vector of weight $\mu \leq t$ and μ error locations $(\bar{x}_{1,1}, \dots, \bar{x}_{1,m}), \dots, (\bar{x}_{\mu,1}, \dots, \bar{x}_{\mu,m})$, if we evaluate the S variables at \mathbf{s} and the variables x_1, \dots, x_{i-1} at $\bar{x}_{j,1}, \dots, \bar{x}_{j,i-1}$ for any $1 \leq j \leq \mu$, then the roots of $\mathcal{L}_i(\mathbf{s}, \bar{x}_{j,1}, \dots, \bar{x}_{j,i-1}, x_i)$ are either $\{\bar{x}_{1,i}, \dots, \bar{x}_{t,i}\}$, when $\mu = t$, or $\{\bar{x}_{1,i}, \dots, \bar{x}_{\mu,i}, \bar{x}_{0,i}\}$, when $\mu \leq t-1$. Apart from the actual location components and possibly the ghost component, polynomial \mathcal{L}_i should not have other solutions.

To show that a polynomial of this kind does not necessarily exist in $J_*^{C,t}$, we consider the following examples.

Example 3.5. Let us consider an MDS code $C = C^\perp(I, L)$ $[5, 1, 5]$ from the plane curve $\{y^5 - y^4 + y^3 - y^2 + y - x = 0\} \cap \{x - 1 = 0\}$ over \mathbb{F}_7 and with

$$L = \{y - 3, y^2 - 1, y^3 + 3, y^4 - 1\}, \mathcal{V}(I) = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5)\}.$$

It is easy to see that C can correct up to $t = 2$ errors. Let us consider the lex term-ordering with $s_1 < s_2 < s_3 < s_4 < s_5 < x_2 < y_2 < x_1 < y_1 < e_2 < e_1$ in

$\mathbb{F}_7[s_1, s_2, s_3, s_4, s_5, x_2, y_2, x_1, y_1, e_1, e_2]$. Ideal $J_*^{C,t}$ is generated by

$$\begin{aligned} &\langle e_1^7 - e_1, e_2^7 - e_2, x_1 - 1, x_2 - 1, y_1^6 - y_1^5 + y_1^4 - y_1^3 + y_1^2 - y_1, y_2^6 - y_2^5 + y_2^4 - y_2^3 + y_2^2 - y_2, \\ &e_1(-y_1^4 + y_1^3 + y_1^2 - 2y_1 + 2) + e_2(-y_2^4 + y_2^3 + y_2^2 - 2y_2 + 2) - s_1, e_2((x_2 - 1)^6 - 1)(y_2^6 - 1), \\ &e_1(3y_1^4 - 2y_1^3 + 3y_1^2 + 3y_1) + e_2(3y_2^4 - 2y_2^3 + 3y_2^2 + 3y_2) - s_2, e_1(3y_1^4 - y_1^2 - 2) + e_2(3y_2^4 - y_2^2 - 2) - s_3, \\ &e_1(-y_1^4 + 2y_1^3 - y_1^2 - 3y_1 + 3) + e_2(-y_2^4 + 2y_2^3 - y_2^2 - 3y_2 + 3) - s_4, e_1((x_1 - 1)^6 - 1)(y_1^6 - 1), \\ &e_1e_2((x_1 - x_2)^6 - 1)((y_1 - y_2)^6 - 1), (e_2^6 - 1)(x_2 - 1), (e_2^6 - 1)y_2, (e_1^6 - 1)(x_1 - 1), (e_1^6 - 1)y_1 \rangle, \end{aligned}$$

where the ghost point is $P_0 = (1, 0)$. The reduced Gröbner basis G with respect to $s_1 < s_2 < s_3 < s_4 < s_5 < x_2 < y_2 < x_1 < y_1 < e_2 < e_1$ has 27 elements and the *new locators* are $\mathcal{L}_1(s_1, \dots, s_5, x_2) = \mathcal{L}_x$ and $\mathcal{L}_2(s_1, \dots, s_5, x_2, y_2) = \mathcal{L}_{xy}$ (see Appendix for polynomials a and b):

$$\mathcal{L}_x = \mathbf{x} - 1 \quad \text{and} \quad \mathcal{L}_{xy} = \mathbf{y}^2 + \mathbf{y}a + b.$$

We can note that \mathcal{L}_x does not play any role, because all x 's are equal to 1. So to apply the decoding we evaluate only \mathcal{L}_{xy} at $\bar{\mathbf{s}}$ and we expect to obtain the (second) components of error locations. We show it in two cases:

- We suppose that two errors occur at the points $P_1 = (1, 1)$ and $P_2 = (1, 2)$, both with error values 1, so the syndrome vector corresponding to the error vector $(1, 1, 0, 0, 0)$ is $\bar{\mathbf{s}} = (2, 1, 0, 0)$.
In order to find the error positions we evaluate \mathcal{L}_{xy} in $\bar{\mathbf{s}}$. We obtain two different solutions $\mathcal{L}_{xy}(\bar{\mathbf{s}}, y) = y^2 - 3y + 2 = (y - 2)(y - 1)$, that identify the two error locations.
- We consider $\bar{\mathbf{s}} = (0, 4, 4, 0, 1)$ corresponding to $(0, 0, 0, 4, 0)$, so only one error occurs in the point $(1, 3)$. Evaluating \mathcal{L}_{xy} at $\bar{\mathbf{s}}$ we obtain $\mathcal{L}_{xy}(\bar{\mathbf{s}}, y) = y^2 - 3y = y(y - 3)$. Also in this case we obtain a correct solutions (0 is the ghost component). So the above choice of \mathcal{L}_x and \mathcal{L}_{xy} seems correct.

Now we consider the above code but with a different ghost point. Also in the following example, we take an optimal ghost point.

Example 3.6. Let us consider the same MDS code $C = C^\perp(I, L)$ as in Example 3.5. In this example we choose the (optimal) ghost point $P_0 = (0, 0)$. The ideal $J_*^{C,t}$ is generated by

$$\begin{aligned} &\langle e_1^7 - e_1, e_2^7 - e_2, x_1y_1 - y_1, x_2y_2 - y_2, x_1^2 - x_1, x_2^2 - x_2, y_1^6 - y_1^5 + y_1^4 - y_1^3 + y_1^2 - y_1, \\ &y_2^6 - y_2^5 + y_2^4 - y_2^3 + y_2^2 - y_2, e_1(-y_1^4 + y_1^3 + y_1^2 - 2y_1 + 2) + e_2(-y_2^4 + y_2^3 + y_2^2 - 2y_2 + 2) - s_1, \\ &e_1(3y_1^4 - 2y_1^3 + 3y_1^2 + 3y_1) + e_2(3y_2^4 - 2y_2^3 + 3y_2^2 + 3y_2) - s_2, e_1(3y_1^4 - y_1^2 - 2) + e_2(3y_2^4 - y_2^2 - 2) - s_3, \\ &e_1(-y_1^4 + 2y_1^3 - y_1^2 - 3y_1 + 3) + e_2(-y_2^4 + 2y_2^3 - y_2^2 - 3y_2 + 3) - s_4, e_1(x_1^6 - 1)(y_1^6 - 1), \\ &e_2(x_2^6 - 1)(y_2^6 - 1), e_1e_2((x_1 - x_2)^6 - 1)((y_1 - y_2)^6 - 1), (e_2^6 - 1)x_2, (e_2^6 - 1)y_2, (e_1^6 - 1)x_1, (e_1^6 - 1)y_1 \rangle, \end{aligned}$$

The reduced Gröbner basis G with respect to $s_1 < s_2 < s_3 < s_4 < s_5 < x_2 < y_2 < x_1 < y_1 < e_2 < e_1$ has 27 elements and the *new locators* are $\mathcal{L}_1(\mathcal{S}, x_2) = \mathcal{L}_x$ and $\mathcal{L}_2(\mathcal{S}, x_2, y_2) = \mathcal{L}_{xy}$, where $\mathcal{S} = \{s_1, \dots, s_5\}$ (see Appendix for c and d):

$$\mathcal{L}_x = \mathbf{x}^2 - \mathbf{x} \quad \text{and} \quad \mathcal{L}_{xy} = \mathbf{y}^2 + \mathbf{y}c + d. \quad (10)$$

Also in this case \mathcal{L}_x does not depend on any syndrome, so to apply the decoding we just specialize $\mathcal{L}_{xy}(\bar{s}, \bar{x}, y)$. We would like that the solutions of $\mathcal{L}_{xy}(\bar{s}, \bar{x}, y) = 0$ are exactly the second components of error locations, but this is not always the case. Let us consider the same errors as in Example 3.5:

- We suppose that two errors occur at the points $P_1 = (1, 1)$ and $P_2 = (1, 2)$, with both error values 1, so the syndrome vector corresponding to the error vector $(1, 1, 0, 0, 0)$ is $\bar{s} = (2, 1, 0, 0)$. In order to find the error positions we evaluate \mathcal{L}_{xy} in \bar{s} . We obtain three different solutions

$$\begin{aligned}\mathcal{L}_{xy}(\bar{s}, 1, y) &= y^2 - 3y + 2 = (y - 1)(y - 2), \\ \mathcal{L}_{xy}(\bar{s}, 0, y) &= y^2 - 3y - 3 = (y + 2)^2.\end{aligned}$$

In this case, we are lucky, because $(0, 5)$ is not a point coordinate and so we can discard $y = 5$ finding the two error locations.

- We consider $\bar{s} = (0, 4, 4, 0, 1)$ corresponding to $(0, 0, 0, 4, 0)$, so only one error occurs in the point $(1, 3)$. Evaluating \mathcal{L}_{xy} in (\bar{s}) we obtain

$$\begin{aligned}\mathcal{L}_{xy}(\bar{s}, 1, y) &= y^2 - y + 1 = (y - 3)(y + 2), \\ \mathcal{L}_{xy}(\bar{s}, 0, y) &= y^2 - y = y(y - 1).\end{aligned}$$

In this case we have four possible solutions $(1, 3)$, $(1, 5)$, $(0, 0)$ and $(0, 1)$. but only three are acceptable, which are $(1, 3)$, $(1, 5)$ and $(0, 0)$. To individuate those corresponding to the syndrome vector \bar{s} , we must compute the two syndromes and we will see that $(1, 3)$ and $(0, 0)$, are correct. In this case, the above choice of \mathcal{L}_x and \mathcal{L}_{xy} is unfortunate.

One might think that the unpleasant behaviour of (10) is due to the degenerate geometric situation. Unfortunately, this is not entirely true, as next example shows (we end this long example with a horizontal line).

Example 3.7. Let us consider the Hermitian code $C = C^\perp(I, L)$ from the curve $y^2 + y = x^3$ over \mathbb{F}_4 and with defining monomials $\{1, x, y, x^2, xy\}$, as in Example 3.1. It is well-known that C can correct up to $t = 2$ errors. Let us consider the lex term-ordering with $s_1 < \dots < x_2 < y_2 < x_1 < y_1 < e_2 < e_1$ in $\mathbb{F}_4[s_1, s_2, s_3, s_4, s_5, x_2, y_2, x_1, y_1, e_1, e_2]$. Ideal $J_{\mathcal{FL}}^{C,t}$ is

$$\begin{aligned}\langle &x_1^4 - x_1, y_1^4 - y_1, x_2^4 - x_2, y_2^4 - y_2, e_1^3 - 1, e_2^3 - 1, y_1^2 + y_1 - x_1^3, y_2^2 + y_2 - x_2^3, \\ &e_1 + e_2 - s_1, \quad e_1 x_1 + e_2 x_2 - s_2, \quad e_1 y_1 + e_2 y_2 - s_3, \quad e_1 x_1^2 + e_2 x_2^2 - s_4, \\ &e_1 x_1 y_1 + e_2 x_2 y_2 - s_5 \rangle,\end{aligned}$$

and the reduced Gröbner basis G (with respect to $<$) has 53 elements. The authors of [FL98] report 119 polynomials because they do not use lex but a block order, which is faster to compute but which usually possesses larger Gröbner bases. In $G \cap (\mathbb{F}_4[S, x_2] \setminus \mathbb{F}_4[S])$ there are 5 polynomials of degree 2 in x_2 and these are our candidate polynomials:

$$\begin{aligned}
g_5 &= \mathbf{x}_2^2 s_5 + \mathbf{x}_2 (s_5 s_4 s_2^2 + s_4^2 s_3^2 s_2 s_1^2 + s_4^2 s_2 s_1 + s_4 s_3^2 s_1 + s_4 s_3 s_2^2 s_1^2 + s_4 s_3 s_1^2 + s_4 s_1^3 + s_3^2 s_2^2 s_1^3 + s_2^2 s_1^2) + \\
&\quad s_5^2 s_3 + s_5 s_4^2 s_3^2 s_2 + s_5 s_4^2 s_2 + s_4^2 s_3^2 s_2^2 s_1 + s_4^2 s_3^2 s_1 + s_4^2 s_3^2 s_2^2 s_1^2 + s_4^2 s_3^2 s_2^2 + s_4^2 s_3^2 s_2^2 s_1 + \\
&\quad s_4 s_3^2 s_2 + s_4 s_2 s_1^2 + s_3^2 s_2^2 s_1 + s_3 s_2^2 s_1^3 + s_3 s_2^2 + s_2^2 s_1 \\
g_4 &= \mathbf{x}_2^2 s_4 + \mathbf{x}_2 (s_4^2 s_2^2 + s_2^2 s_1 + s_1 + s_4^2 s_3^2 s_1^3) + s_4^2 s_2^2 + s_4^2 s_3^2 s_1^2 + s_4^2 s_1^2 + s_4 s_3 s_2^2 s_1^3 + s_4 s_3 s_2^2 + s_2 s_1^3 + s_2 \\
g_3 &= \mathbf{x}_2^2 s_3 + \mathbf{x}_2 (s_4^2 s_3 s_1 + s_4 s_3 s_2^2 s_1^3 + s_4 s_3 s_2^2 + s_3 s_2 s_1^2) + s_5^2 s_3^2 + s_5 s_3^2 s_2 + s_4^2 s_3^2 s_2 s_1 + s_4^2 s_3^2 s_2 s_1^2 + s_4 s_3^2 s_1^3 + \\
&\quad s_4 s_3^2 s_1 + s_3^2 s_2^2 s_1^2 + s_3^2 s_2^2 s_1^3 + s_3^2 s_2^2 \\
g_2 &= \mathbf{x}_2^2 s_2 + \mathbf{x}_2 (s_4^2 s_2 s_1 + s_4 s_1^3 + s_4 + s_2^2 s_1^2) + s_4^2 s_2^2 + s_4 s_3^2 s_2 s_1^3 + s_4 s_3^2 s_2 + s_4 s_2 s_1^2 + s_3 s_2^2 s_1^3 + s_3 s_2^2 + s_2^2 s_1 \\
g_1 &= \mathbf{x}_2^2 (s_1) + \mathbf{x}_2 (s_4^2 s_1^2 + s_2 s_1^3) + s_4^2 s_2 s_1 + s_4 s_1^3 + s_2^2 s_1^2
\end{aligned}$$

Of course, there are other similar polynomials in $J_{\mathcal{FL}}^{C,t} \cap (\mathbb{F}_4[S, x_2] \setminus \mathbb{F}_4[S])$ and they may be found for example by computing Gröbner bases with respect to other orderings. It is immediate that the leading polynomials are just $\{s_1, \dots, s_5\}$. Suppose that we receive a syndrome $\mathbf{s} = (\bar{s}_1, \dots, \bar{s}_5)$. If it is zero, then no errors occurred. Otherwise, we might follow the most obvious way to correct, that is, we might substitute \mathbf{s} in all g_i 's, until we find one which does not vanish identically. The improvement introduced by Caboara and Mora translates here in checking only the leading polynomials, i.e. checking which of the syndrome components \bar{s}_i is non-zero. Since clearly at least one is non-zero, with a negligible computational effort we are able to determine the right candidate.

Let us now follow our proposal. Ideal $J_*^{C,t}$ is generated by

$$\begin{aligned}
&\{x_1^4 - x_1, y_1^4 - y_1, x_2^4 - x_2, y_2^4 - y_2, e_1^4 - e_1, e_2^4 - e_2, y_1^2 x_1 + y_1^2 + y_1 x_1 + y_1 + x_1^3 + \\
&x_1, y_2^2 x_2 + y_2^2 + y_2 x_2 + y_2 + x_2^3 + x_2, y_1^3 + y_1 x_1^3 + y_1 + x_1^3, y_2^3 + y_2 x_2^3 + y_2 + x_2^3, e_1 + \\
&e_2 - s_1, e_1 x_1 + e_2 x_2 - s_2, e_1 y_1 + e_2 y_2 - s_3, e_1 x_1^2 + e_2 x_2^2 - s_4, e_1 x_1 y_1 + e_2 x_2 y_2 - s_5, \\
&e_1((x_1 - 1)^3 - 1)((y_1 - 1)^3 - 1), e_2((x_2 - 1)^3 - 1)((y_2 - 1)^3 - 1), (e_1^3 - 1)(x_1 - 1), \\
&(e_1^3 - 1)(y_1 - 1), (e_2^3 - 1)(x_2 - 1), (e_2^3 - 1)(y_2 - 1), e_1 e_2((x_1 - x_2)^3 - 1)((y_1 - y_2)^3 - 1)\}.
\end{aligned}$$

where the ghost point is $(1, 1)$ (note that $1^3 \neq 1^2 + 1$).

The reduced Gröbner basis G with respect to $s_1 < s_2 < s_3 < s_4 < s_5 < x_2 < y_2 < x_1 < y_1 < e_2 < e_1$ has 32 elements and the *new locators* are $\mathcal{L}_1(s_1, \dots, s_5, x_2) = \mathcal{L}_x$ and $\mathcal{L}_2(s_1, \dots, s_5, x_2, y_2) = \mathcal{L}_{xy}$, that are the polynomials of degree two in, respectively, x_2 and y_2 :

$$\begin{aligned}
\mathcal{L}_x &= \mathbf{x}^2 + \mathbf{x}(s_1^2 s_2 s_4^3 + s_4^3 + s_1 s_2^2 s_4^2 + s_1^2 s_2^2 s_4^2 + s_1 s_4^2 + s_2^2 s_4 + s_1 s_2 s_4 + s_2^2 + s_1^2 s_2 + s_1^3) + \\
&\quad s_3 s_5^2 + s_2 s_3 s_5 + s_1 s_2^2 s_4^3 + s_1^2 s_2^2 s_4^3 + s_2 s_3^2 s_4^2 + s_1 s_2 s_3^2 s_4^2 + s_1^2 s_2 s_3^2 s_4^2 + s_1 s_2^2 s_4^2 + s_1^2 s_2^2 s_4^2 + \\
&\quad s_2 s_4^2 + s_1^2 s_3^2 s_4 + s_1^2 s_3^2 s_4 + s_1 s_3 s_4 + s_1^2 s_2^2 s_4 + s_1^2 s_2^2 s_4 + s_1^2 s_4 + s_1^2 s_2^2 s_3^2 + s_2^2 s_3^2 + s_1 s_2^2 s_3^2 + \\
&\quad s_1^2 s_3^2 + s_3^2 + s_1^2 s_2^2 s_3^2 + s_1^2 s_2^2 s_3 + s_2^2 s_3 + s_1^2 s_3^2 + s_2^2 + s_1 s_2^2 + s_1^3 + 1 \\
\mathcal{L}_{xy} &= \mathbf{y}^2 + \mathbf{y}(s_3^3 + s_1 s_3^2 + s_1^2 s_3^2 s_3 + s_1^2 s_3 + s_1^3) + \mathbf{x}(s_2^2 s_3 s_4^3 + s_1 s_2^2 s_4^3 + s_1^2 s_2 s_3 s_4^2 + s_1^2 s_3^2 s_4 + s_2^2 s_4 + \\
&\quad s_1 s_3 s_4 + s_1^2 s_2^2 s_4) + s_5^3 + s_2 s_3^2 s_4^2 s_5 + s_3 s_4 s_5 + s_2^2 s_5 + s_3^2 s_4^3 + s_1 s_2^2 s_3^2 s_4^3 + s_2^2 s_4^3 + s_1^2 s_2^2 s_3^2 s_4^3 + \\
&\quad s_1^2 s_2 s_3^2 s_4 + s_1^2 s_2 s_3 s_4 + s_1 s_2 s_4 + s_2^2 s_3^2 + s_3^2 + s_1 s_2^2 s_3^2 + s_1 s_2^2 + s_1^2 s_3^2 s_3 + s_1^2 s_3 + s_1^2 s_2^2 + s_1^3 + 1
\end{aligned}$$

We can apply the decoding in this way: we specialize $\mathcal{L}_x(s, x)$ to $\bar{\mathbf{s}}$ for any received syndrome. If the syndrome corresponds to two errors, then we expect

that the roots of $\mathcal{L}_x(\bar{\mathbf{s}}, x)$ are the first components of error locations and the roots of $\mathcal{L}_{xy}(\bar{\mathbf{s}}, \bar{\mathbf{x}}, y)$ are exactly the second components of error locations. But it is not always true, we show it in three cases:

- We suppose that two errors occur at the points $P_6 = (\alpha, \alpha + 1)$ and $P_7 = (\alpha + 1, \alpha)$, with both error values 1, so the syndrome vector corresponding to the error vector $(0, 0, 0, 0, 0, 1, 1, 0)$ is $\bar{\mathbf{s}} = (0, 1, 1, 1, 0)$. In order to find the error positions we evaluate \mathcal{L}_x in $\bar{\mathbf{s}}$ and we obtain the correct values of x , in fact:

$$\mathcal{L}_x(\bar{\mathbf{s}}, x) = x^2 + x + 1 = (x - \alpha)(x - (\alpha + 1)).$$

Now we have to evaluate \mathcal{L}_{xy} in $(\bar{\mathbf{s}}, \bar{\mathbf{x}})$. We obtain four different solutions

$$\begin{aligned}\mathcal{L}_{xy}(\bar{\mathbf{s}}, \alpha, y) &= y^2 + y + 1 = (y - \alpha)(y - (\alpha + 1)) \\ \mathcal{L}_{xy}(\bar{\mathbf{s}}, \alpha + 1, y) &= y^2 + y + 1 = (y - \alpha)(y - (\alpha + 1)).\end{aligned}$$

But this is a *problem* for us, because all these solutions are curve points: $(\alpha, \alpha), (\alpha, \alpha + 1), (\alpha + 1, \alpha), (\alpha + 1, \alpha + 1)$. Only two are the correct locations. To individuate those corresponding to the syndrome vector $\bar{\mathbf{s}}$, we must compute the two syndromes and we will see that $(\alpha + 1, \alpha), (\alpha, \alpha + 1)$ are correct. This method of try-and-see works nice because the code is small, but soon it becomes unfeasible. So the above choice of \mathcal{L}_x and \mathcal{L}_{xy} is unfortunate.

- We suppose that the syndrome is $(\alpha + 1, 0, \alpha, 0, 0)$, corresponding to the error vector $(1, \alpha, 0, 0, 0, 0, 0, 0)$. So two errors have occurred and their values are 1 and α in the point, respectively, $P_1 = (0, 0)$ and $P_2 = (0, 1)$. In order to find the error locations we evaluate \mathcal{L}_x in $\bar{\mathbf{s}}$ and we obtain $\mathcal{L}_x(\bar{\mathbf{s}}, x) = x^2 + x = x(x - 1)$, then we evaluate \mathcal{L}_{xy} in $(\bar{\mathbf{s}}, 0)$ and $(\bar{\mathbf{s}}, 1)$ and we get $\mathcal{L}_{xy}(\bar{\mathbf{s}}, 0, y) = \mathcal{L}_{xy}(\bar{\mathbf{s}}, 1, y) = y^2 + y = y(y - 1)$. The equations

$$\mathcal{L}_x(\bar{\mathbf{s}}, x) = \mathcal{L}_{xy}(\bar{\mathbf{s}}, 1, y) = \mathcal{L}_{xy}(\bar{\mathbf{s}}, 0, y) = 0 \tag{11}$$

have four possible solutions: $(0, 0), (0, 1), (1, 0)$ and $(1, 1)$. Since the points $(1, 0)$ and $(1, 1)$ do not lie on the Hermitian curve, then only one solution couple is admissible: $\{(0, 0), (0, 1)\}$. This situation is better than the above case, because we can immediately understand what the correct solutions of system (11) are. This happens by chance and in any case the solutions of equation $\mathcal{L}_x(\bar{\mathbf{s}}) = 0$ are not what we want.

- Finally we consider $\bar{\mathbf{s}} = (\alpha + 1, \alpha + 1, 1, \alpha + 1, 1)$ corresponding to $(0, 0, \alpha + 1, 0, 0, 0, 0, 0)$, so only one error occurs. Evaluating \mathcal{L}_x and \mathcal{L}_{xy} , respectively, in $\bar{\mathbf{s}}$ and $(\bar{\mathbf{s}}, \bar{\mathbf{x}})$, we obtain

$$\begin{cases} \mathcal{L}_x(\bar{s}, x) = x^2 + 1 \\ \mathcal{L}_{xy}(\bar{s}, 1, y) = y^2 + (\alpha + 1)y + \alpha = (y - 1)(y - \alpha). \end{cases} \quad (12)$$

In this case we are extremely lucky because the two polynomials \mathcal{L}_x and \mathcal{L}_{xy} answer correctly: the solutions of system (12) are $(1, 1)$, which is the ghost point, and $(1, \alpha)$, which is the error location.

Remark 3.8. Since, in Example 3.7, the curve equation has all coefficients in \mathbb{F}_2 , the ideal $J_{\mathcal{FL}}^{C,t}$ actually lies in $\mathbb{F}_2[s_1, s_2, s_3, s_4, s_5, x_2, y_2, x_1, y_1, e_1, e_2]$. This is a special case of a more general fact: for any affine variety-code and any decoding ideal that we are considering in the whole paper, all polynomials defining these ideals have no coefficient different from $\{1, -1\}$, except possibly for the polynomials defining I . Therefore, if it is possible to have a basis for the ideal I with all coefficients in a smaller field, then any of its Gröbner bases will have elements with the same coefficient field, which means that the basis computation will be much faster.

Since polynomials like \mathcal{L}_x and \mathcal{L}_{xy} in Example 3.7 contain the right solutions (together with unwelcome parasite solutions), they deserve a definition. See Section 2.1 for our notation.

Definition 3.9. Let $C = C^\perp(I, L)$ be an affine-variety code. Let $1 \leq i \leq m$. Let $P_0 = (\bar{x}_{0,1}, \dots, \bar{x}_{0,m}) \in (\mathbb{F}_q)^m \setminus \mathcal{V}(I)$ be a ghost point. Let

$$t_i = \min \{t, |\{\hat{\pi}_i(P) \mid P \in \mathcal{V}(I) \cup P_0\}|\},$$

and let \mathcal{P}_i be a polynomial in $\mathbb{F}_q[S, x_1, \dots, x_i]$, where $S = \{s_1, \dots, s_r\}$. Then $\{\mathcal{P}_i\}_{1 \leq i \leq m}$ is a set of **weak multi-dimensional general error locator polynomials** of C if for any i

- $\mathcal{P}_i(S, x_1, \dots, x_i) = x_i^{t_i} + a_{t_i-1}x_i^{t_i-1} + \dots + a_0$, with $a_j \in \mathbb{F}_q[S, x_1, \dots, x_{i-1}]$, $0 \leq j \leq t_i - 1$, that is, \mathcal{P}_i is a monic polynomial with degree t_i with respect to the variable x_i and its coefficients are in $\mathbb{F}_q[S, x_1, \dots, x_{i-1}]$;
- given a syndrome $\bar{s} = (\bar{s}_1, \dots, \bar{s}_r) \in (\mathbb{F}_q)^r$, corresponding to an error vector of weight $\mu \leq t$, μ error locations $(\bar{x}_{1,1}, \dots, \bar{x}_{1,m}), \dots, (\bar{x}_{\mu,1}, \dots, \bar{x}_{\mu,m})$. If we evaluate the S variables at \bar{s} and the variables (x_1, \dots, x_{i-1}) at the truncated vectors $\bar{\mathbf{x}}^j = (\bar{x}_{j,1}, \dots, \bar{x}_{j,i-1})$ for $0 \leq j \leq \mu$, then the roots of $\mathcal{P}_i(\bar{s}, \bar{\mathbf{x}}^j, x_i)$ contain:

- either $\{\bar{x}_{h,i} \mid \bar{\mathbf{x}}^h = \bar{\mathbf{x}}^j, 0 \leq h \leq \mu\}$ (when $\mu < t$),
- or $\{\bar{x}_{h,i} \mid \bar{\mathbf{x}}^h = \bar{\mathbf{x}}^j, 1 \leq h \leq \mu\}$ (when $\mu = t$),

plus possibly some parasite solutions.

Note that the difference between $\{\bar{x}_{h,i} \mid \bar{\mathbf{x}}^h = \bar{\mathbf{x}}^j, 0 \leq h \leq \mu\}$ and $\{\bar{x}_{h,i} \mid \bar{\mathbf{x}}^h = \bar{\mathbf{x}}^j, 1 \leq h \leq \mu\}$ is that the latter set does not consider the ghost point.

Now we consider an alternative strategy to calculate the error locations, using the weak multi-dimensional general error locator polynomials and some other polynomials in ideal $J_*^{C,t}$.

Since it is convenient to know in advance the error number and the error values, we provide the following definition for a general correctable linear code. Let C be an $[n, k, d]$ linear code over \mathbb{F}_q with correction capability $t \geq 1$. Choose any parity-check matrix with entries in an appropriate extension field \mathbb{F}_{q^M} of \mathbb{F}_q , $M \geq 1$. Its syndromes lie in $(\mathbb{F}_{q^M})^{n-k}$ and form a vector space of dimension $r = n - k$ over \mathbb{F}_q .

Definition 3.10. *Let $\mathcal{E} \in \mathbb{F}_q[S, e]$, where $S = \{s_1, \dots, s_r\}$. Then \mathcal{E} is a **general error evaluator polynomial** of C if*

- $\mathcal{E}(S, e) = a_t e^t + a_{t-1} e^{t-1} + \dots + a_0$, with $a_j \in \mathbb{F}_q[S]$, $0 \leq j \leq t$, that is, \mathcal{E} is a polynomial with degree t with respect to the variable e and its coefficients are in $\mathbb{F}_q[S]$;
- Given a syndrome $\bar{s} = (\bar{s}_1, \dots, \bar{s}_r) \in (\mathbb{F}_{q^M})^r$ corresponding to an error vector of weight $\mu \leq t$ and with $\bar{e}_1, \dots, \bar{e}_\mu$ as error values, we evaluate the S variables at \bar{s} , then the roots of \mathcal{E} are the error values $\bar{e}_1, \dots, \bar{e}_\mu$ plus 0 with multiplicity $t - \mu$.

The importance of \mathcal{E} lies in fact that the error number is μ if and only if

$$e^{t-\mu} | \mathcal{E}(\bar{s}) \quad \text{and} \quad e^{(t-\mu+1)} \nmid \mathcal{E}(\bar{s}).$$

The ideal $J_*^{C,t} \cap \mathbb{K}[S, e_1, \dots, e_t]$ is easily seen to be stratified, as follows. There is a bijective correspondence between correctable syndromes and correctable errors (i.e., errors of weight $\tau \leq t$) and so if we fix $1 \leq l \leq t$ and $1 \leq s \leq t - l$ we can always find l error values e_1, \dots, e_l that have s extensions at level e_{l+1} . So we can apply Proposition 2.5 and obtain the existence of \mathcal{E} :

Theorem 3.11. *For any affine-variety code $C = C^\perp(I, L)$, the general error evaluator polynomial exists.*

Proof. We apply Proposition 2.5 to the stratified ideal $J_*^{C,t} \cap \mathbb{K}[S, e_1, \dots, e_t]$. It is enough to take g with $\mathbf{T}(g) = \mathbf{a}_L^t$ with $\mathcal{A} = \{e_1, \dots, e_t\}$ and $\mathcal{S} = S$. \square

Using \mathcal{E} , we know not only τ , but also the τ error values. In order to exploit this information, we can consider a straightforward generalisation of weak multi-dimensional general error locator polynomials (see Definition 3.9) where the locators are actually $\mathcal{P}_i^e \in \mathbb{F}_q[S, e, x_1, \dots, x_{i-1}]$. We do not give a long definition for these, since we think it is obvious.

We consider again Example 3.7 to show two alternative strategies.

Example 3.12. Let us consider the Hermitian code $C = C^\perp(I, L)$ from the curve $y^2 + y = x^3$ over \mathbb{F}_4 and with defining monomials $\{1, x, y, x^2, xy\}$, as

in the Example 3.7. The reduced Gröbner basis G of $J_*^{C,t}$ with respect to lex with $s_1 < s_2 < s_3 < s_4 < s_5 < e_2 < e_1 < x_2 < y_2 < x_1 < y_1$ has 33 elements and the general error evaluator polynomial \mathcal{E} is

$$\begin{aligned} \mathcal{E} = & \mathbf{e}^2 + \mathbf{e}s_1 + s_4^3s_3^2 + s_4^3s_3s_1 + s_4^3s_2^3s_1^2 + s_4^3s_1^2 + s_4^2s_3^2s_2^2s_1^2 + s_4^2s_3s_2^2s_1^3 + s_4s_3^2s_2s_1 + \\ & s_4s_3s_2s_1^2 + s_4s_2s_1^3 + s_4s_2 + s_3^2s_2^3s_1^3 + s_3^2 + s_3s_2^2s_1 + s_3s_1 + s_2^3s_1^2 \end{aligned}$$

In G there are also these polynomials:

$$\mathcal{P}_x^e = \mathbf{x}^2 + \mathbf{x}s_4s_2^2 + \mathbf{e}a_x + b_x \text{ and } g_x = \mathbf{x}_1 + \mathbf{x}_2 + c_x,$$

where $a_x, b_x, c_x \in \mathbb{F}_4[s_1, s_2, s_3, s_4, s_5]$ (see Appendix for the full polynomials). Now we change the lex ordering to $s_1 < \dots < s_5 < e_2 < e_1 < y_2 < x_2 < y_1 < x_1$. In the new Gröbner basis we have other two polynomials \mathcal{P}_y^e and g_y .

$$\mathcal{P}_y^e = \mathbf{y}^2 + \mathbf{y}(s_4s_3s_2 + s_2^3 + 1) + \mathbf{e}a_y + b_y \text{ and } g_y = \mathbf{y}_1 + \mathbf{y}_2 + c_y,$$

where $a_y, b_y, c_y \in \mathbb{F}_4[s_1, s_2, s_3, s_4, s_5]$ (see Appendix for the full polynomials). We can decode as follows. First we evaluate $\mathcal{E}(\bar{\mathbf{s}})$ and we find two error values $\mathbf{e}_1, \mathbf{e}_2$ (when $\tau = 1$, one is zero).

- If the syndrome corresponds to two errors, then the roots of $\mathcal{P}_x^e(\mathbf{s}, \mathbf{e}_2, x)$ are the first components of error locations,
- else if $\bar{\mathbf{s}}$ corresponds to one error, we specialize $g_x(s, e, x_1, x_2)$ in $(\mathbf{s}, \mathbf{e}_2, 1)$, where 1 is the ghost component, and again the root of $g_x(\mathbf{s}, \mathbf{e}_2, \mathbf{1}, x_2)$ is the first component of the error location.

Similarly we use \mathcal{P}_y^e and g_y to find the second location components. Let us explain in detail the above-mentioned decoding with the help of the three cases of Example 3.7.

- $\bar{\mathbf{s}} = (0, 1, 1, 1, 0)$ is the syndrome vector corresponding to the error vector $(0, 0, 0, 0, 0, 1, 1, 0)$. Evaluating \mathcal{E} in $\bar{\mathbf{s}}$ we obtain: $\mathcal{E}(\bar{\mathbf{s}}) = e^2 + 1$, so two errors have occurred and their values is 1. In order to find the error positions we evaluate \mathcal{P}_x^e and \mathcal{P}_y^e in $(\mathbf{s}, 1)$ and we obtain

$$\mathcal{P}_x^e(\bar{\mathbf{s}}, 1) = x^2 + x + 1 = (x - \alpha)(x - (\alpha + 1))$$

$$\mathcal{P}_y^e(\bar{\mathbf{s}}, 1) = y^2 + y + 1 = (y - \alpha)(y - (\alpha + 1)).$$

The system $\mathcal{P}_x^e(\bar{\mathbf{s}}, 1) = \mathcal{P}_y^e(\bar{\mathbf{s}}, 1) = 0$ have four possible solutions: (α, α) , $(\alpha + 1, \alpha + 1)$, $(\alpha + 1, \alpha)$ and $(\alpha, \alpha + 1)$. But only two solution pairs are admissible: $\{(\alpha + 1, \alpha), (\alpha, \alpha + 1)\}$ and $\{(\alpha, \alpha), (\alpha + 1, \alpha + 1)\}$, since both α and $\alpha + 1$ must appear as first components (and as second components). We are in the same ambiguous situation as in Example 3.7.

- Now we consider the syndrome $\bar{\mathbf{s}} = (\alpha + 1, 0, \alpha, 0, 0)$, corresponding to $(1, \alpha, 0, 0, 0, 0, 0, 0)$. Evaluating \mathcal{E} in $\bar{\mathbf{s}}$ we obtain $\mathcal{E}(\bar{\mathbf{s}}) = (e - 1)(e - \alpha)$, so two errors have occurred and their values are 1 and α . In order to find

the error positions we evaluate \mathcal{P}_x^e and \mathcal{P}_y^e in $(\mathbf{s}, 1)$ (or in (\mathbf{s}, α))

$$\mathcal{P}_x^e(\bar{\mathbf{s}}, 1) = f_x(\bar{\mathbf{s}}, \alpha) = x^2 \text{ and } \mathcal{P}_y^e(\bar{\mathbf{s}}, 1) = f_y(\bar{\mathbf{s}}, \alpha) = y^2 + y = y(y - 1).$$

The solutions of the system $f_x(\bar{\mathbf{s}}, 1) = f_y(\bar{\mathbf{s}}, 1) = 0$ are $\{(0, 0), (1, \alpha)\}$, in this case we find the correct error positions. Note that this case is an ambiguous situation in Example 3.7, while here it is not.

- Vector $\bar{\mathbf{s}} = (\alpha + 1, \alpha + 1, 1, \alpha + 1, 1)$ is the syndrome corresponding to $(0, 0, \alpha + 1, 0, 0, 0, 0, 0)$. We evaluate \mathcal{E} and we get $\mathcal{E}(\bar{\mathbf{s}}) = e^2 + (\alpha + 1)e$. So only one error occurred and its value is $\alpha + 1$. We evaluate g_x and g_y in $(\bar{\mathbf{s}}, \alpha + 1, 1)$, where 1 is the first ghost component, and we have

$$g_x(\bar{\mathbf{s}}, \alpha + 1, 1) = x_2 + 1 \text{ and } g_y(\bar{\mathbf{s}}, \alpha + 1, 1) = y_2 + \alpha.$$

Therefore the error location is $(1, \alpha)$.

Now we consider another type of decoding, using \mathcal{E} and taking polynomials from $\hat{J}_{\mathcal{FL}}^{C,t}$ as in (3). First, we evaluate $\mathcal{E}(\bar{\mathbf{s}})$ to know the number of errors. We do not need their values. Instead, we compute the Gröbner basis of ideal $\hat{J}_{\mathcal{FL}}^{C,\tau}$, with $1 \leq \tau \leq t$ and we collect polynomials in $\mathbb{F}_q[S, x]$ and $\mathbb{F}_q[S, y]$. For example, if two errors occur we use $s_1 < s_2 < s_3 < s_4 < s_5 < x_2 < y_2 < x_1 < y_1 < e_2 < e_1$ and $\dots s_5 < y_2 < \dots$ to get, for $\tau = 2$,

$$f_{2,x} = \mathbf{x}^2 + \mathbf{x}(s_4^2 s_1 + s_4 s_2^2 s_1^3 + s_4 s_2^2 + s_2 s_1^2) + s_5^2 s_3 + s_5 s_3 s_2 + s_4^2 s_3^3 s_2 + s_4^2 s_3^2 s_2 s_1 + s_4^2 s_3 s_2 s_1^2 + s_4^2 s_2 + s_4 s_3^3 s_1^2 + s_4 s_3^2 s_1^3 + s_4 s_3 s_1 + s_4 s_1^2 + s_3^3 s_2^2 s_1 + s_3^2 s_2^2 s_1^2 + s_3 s_2^2 s_1^3 + s_3 s_2^2 + s_2^2 s_1$$

$$f_{2,y} = \mathbf{y}^2 + \mathbf{y}(s_4^3 + s_4 s_3 s_2 s_1^3 + s_4 s_3 s_2 + s_4 s_2 s_1 + s_3^2 s_1 + s_3 s_1^2 + s_2^2 s_1^3 + s_1^3 + 1) + s_5^3 + s_5 s_4^2 s_3^2 s_2 + s_5 s_3^3 s_2^2 + s_5 s_2^2 + s_4^3 s_3 s_2^2 s_1^2 + s_4^3 s_3 s_1^2 + s_4^3 s_2^2 + s_4^2 + s_4^2 s_3^3 s_2^2 s_1^2 + s_4^2 s_3^2 s_2^2 + s_4^2 s_3 s_2^2 s_1 + s_4 s_3^3 s_2 s_1 + s_4 s_3 s_2 s_1^3 + s_4 s_3 s_2 + s_3^3 + s_3 s_3^2 s_1^2 + s_2^2 s_1^3 + s_2^2 + 1$$

and for $\tau = 1$

$$f_{1,x} = \mathbf{x} + s_2 s_1^2 \text{ and } f_{1,y} = \mathbf{y} + s_3 s_1^2.$$

The decoding with $\{f_{2,x}, f_{2,y}, f_{1,x}, f_{1,y}\}$ is obvious.

These polynomials are not the ideal polynomials yet, because again we may find parasite solutions (except with $\tau = 1$).

In the previous examples we have used some polynomials as weak multidimensional general error locator polynomials, as for example \mathcal{L}_x and \mathcal{L}_{xy} in Example 3.7. It is not obvious that such polynomials exist for any (affine-variety) code. To prove this, we need to analyse in depth the structure of the zero-dimensional ideal $J_*^{C,t}$. This ideal turns out to belong to several interesting classes of zero-dimensional ideals, defined as generalizations of stratified ideals. These ideal classes are rigorously studied in Section 4, where it is claimed in full generality that the sought-after polynomials can be found in a suitable Gröbner basis. Section 5 is devoted to the proof of this claim. In Section 6 we will come back to the coding setting.

4 Results on some zero-dimensional ideals

Our aim in this section is to describe the structure of the reduced Gröbner basis for some special classes of zero-dimensional ideals which are generalizations of stratified ideals. We suggest that the reader have a look at [OS05], since now we are generalizing our argument in [OS05].

First we provide a generalization of the material in Section 2. In this section $J \subset \mathbb{K}[\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_1, \mathcal{T}]$ is a zero-dimensional ideal, with $\mathcal{S} = \{s_1, \dots, s_N\}$, $\mathcal{A}_j = \{a_{j,1}, \dots, a_{j,m}\}$, $j = 1, \dots, L$, $\mathcal{T} = \{t_1, \dots, t_K\}$. We fix a block order $<$ on $\mathbb{K}[\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_1, \mathcal{T}]$, with $\mathcal{S} < \mathcal{A}_L < \dots < \mathcal{A}_1 < \mathcal{T}$, such that within \mathbb{A}_j we use lex with $a_{j,1} < a_{j,2} < \dots < a_{j,m}$ (for any j). Let \mathbb{A} and $\mathbb{A}_{j,i}$ denote the affine spaces $\mathbb{A} = \mathbb{K}^{N+mL+K}$ and $\mathbb{A}_{j,i} = \mathbb{K}^{N+m(L-j)+i}$.

With the usual notation for the elimination ideals, we want to partition $\mathcal{V}(J_S)$ according to the number of extensions in $\mathcal{V}(J_{S, \mathbf{a}_{L,1}})$, similarly to what done in Section 2.2 in the one-variable case, that is, when $m = 1$. The additional complication here is that the \mathbf{a} variables are not L any more, but rather they are collected into L blocks, each block having m variables. Since we order the \mathbf{a} variables first according to their block (block \mathcal{A}_L is the least) and then within the block from the least to the greatest, their first index denotes the block and their second index denotes their position within the block itself. So, the least \mathbf{a} variable is $a_{L,1}$ and the greatest is $a_{1,m}$.

The members of the partition of $\mathcal{V}(J_S)$ will be called $\{\Sigma_l^{L,1}\}$ (similarly to the previously defined Σ_l^L). The maximum number of extensions will be called $\eta(L, 1)$ (compare with $\lambda(L)$).

Remark 4.1. It is essential to count the number of extensions in $\mathcal{V}(J_{S, \mathbf{a}_{L,1}})$ discarding their multiplicities. In the definition of a stratified ideal we required radicality, so in that case multiplicities did not arise. However, in our following multidimensional generalisations of results and definitions from Sec. 2.2-2.4, we must drop radicality and so we have to be very careful when handling multiplicities.

In the general case, if we consider block j and variable $a_{j,i}$, we partition $\mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i}})$ into subsets $\{\Sigma_l^{j,i+1}\}$ according to the number of extensions to $\mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i}, \mathbf{a}_{j,i+1}})$, that is, adding the next variable $a_{j,i+1}$. The maximum number of extensions will be called $\eta(j, i+1)$. We meet a special case when we consider the last variable in a block (i.e., $i = m$), since in that case we extend from $\mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,m}})$ to $\mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j-1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,m}, \mathbf{a}_{j-1,1}})$. However, no confusion will arise if we follow our convention of naming the partition members according to the *added* variable, so they are called $\{\Sigma_l^{j-1,1}\}$ in this case, even if their union is $V = \mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j-1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,m}})$. Coherently, $\eta(j-1, 1)$ denotes the maximum number of extensions for points in V .

A formal description of the above discussion goes as follows, where l, j and

m are integers such that $l \geq 1$, $1 \leq j \leq L$ and $1 \leq i \leq m$:

$$\begin{aligned}
\Sigma_l^{L,1} &= \{(\bar{s}_1, \dots, \bar{s}_N) \in \mathcal{V}(J_S) \mid \exists \text{ exactly } l \text{ distinct values } \bar{a}_{L,1}^{(1)}, \dots, \bar{a}_{L,1}^{(l)} \\
&\quad \text{s.t. } (\bar{s}_1, \dots, \bar{s}_N, \bar{a}_{L,1}^{(\ell)}) \in \mathcal{V}(J_{S, \mathbf{a}_{L,1}}) \text{ with } 1 \leq \ell \leq l\}, \\
\Sigma_l^{j,1} &= \{(\bar{s}_1, \dots, \bar{s}_N, \bar{a}_{L,1}, \dots, \bar{a}_{L,m}, \dots, \bar{a}_{j+1,1}, \dots, \bar{a}_{j+1,m}) \in \mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}}) \mid \\
&\quad \exists \text{ exactly } l \text{ distinct values } \bar{a}_{j,1}^{(1)}, \dots, \bar{a}_{j,1}^{(l)} \text{ s.t. for any } 1 \leq \ell \leq l \\
&\quad (\bar{s}_1, \dots, \bar{s}_N, \bar{a}_{L,1}, \dots, \bar{a}_{L,m}, \dots, \bar{a}_{j+1,1}, \dots, \bar{a}_{j+1,m}, \bar{a}_{j,1}^{(\ell)}) \in \mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}})\} \\
&\quad j = 1, \dots, L-2, \\
\Sigma_l^{j,i} &= \{(\bar{s}_1, \dots, \bar{s}_N, \bar{a}_{L,1}, \dots, \bar{a}_{L,m}, \dots, \bar{a}_{j+1,1}, \dots, \bar{a}_{j+1,m}, \bar{a}_{j,1}, \dots, \bar{a}_{j,i-1}) \text{ in} \\
&\quad \mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}}) \mid \exists \text{ exactly } l \text{ distinct values } \bar{a}_{j,i}^{(1)}, \dots, \bar{a}_{j,i}^{(l)} \text{ s.t.} \\
&\quad (\bar{s}_1, \dots, \bar{s}_N, \bar{a}_{L,1}, \dots, \bar{a}_{L,m}, \dots, \bar{a}_{j+1,1}, \dots, \bar{a}_{j+1,m}, \bar{a}_{j,1}, \dots, \bar{a}_{j,i-1}, \bar{a}_{j,i}^{(\ell)}) \text{ is in} \\
&\quad \mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i}}) \mid 1 \leq \ell \leq l\}, \quad i = 2, \dots, m, j = 1, \dots, L-1.
\end{aligned}$$

The maximum number of extensions at any level, which is $\eta(j, i)$, plays an important role for our approach and therefore deserves a precise definition. Before defining it, we need an elementary result.

Fact 4.2. *Given J , there is a set of natural numbers $\{\eta(j, i)\}_{1 \leq j \leq L, 1 \leq i \leq m}$, such that*

- i) $\mathcal{V}(J_S) = \sqcup_{l=1}^{\eta(L,1)} \Sigma_l^{L,1}$;
- ii) $\mathcal{V}(J_{S, \mathbf{a}_{L,1}, \dots, \mathbf{a}_{L,i}}) = \sqcup_{l=1}^{\eta(L,i+1)} \Sigma_l^{L,i+1}$, $i = 1, \dots, m-1$;
- iii) $\mathcal{V}(J_{S, \mathcal{A}_L}) = \sqcup_{l=1}^{\eta(L-1,1)} \Sigma_l^{L-1,1}$;
- iv) $\mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}}) = \sqcup_{l=1}^{\eta(j,1)} \Sigma_l^{j,1}$, $j = 1, \dots, L-2$;
- v) $\mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i}}) = \sqcup_{l=1}^{\eta(j,i+1)} \Sigma_l^{j,i+1}$, $i = 1, \dots, m-1, j = 1, \dots, L-1$;
- vi) $\Sigma_{\eta(j,i)}^{j,i} \neq \emptyset$, $\forall i = 1, \dots, m, \forall j = 1, \dots, L$.

Proof. Since I is zero-dimensional ideal, $\mathcal{V}(I)$ is finite and so any variety projection $V = \mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}})$ has a finite number of points. Obviously V is the union of the corresponding $\Sigma_l^{j,i}$, which means that there can be only a finite number of non-empty $\Sigma_l^{j,i}$ and so we use the notation $\eta(j, i)$ to denote the largest l such that $\Sigma_l^{j,i}$ is non-empty. \square

Definition 4.3. *The level function of J (with respect to the $\mathcal{A}_L, \dots, \mathcal{A}_1$ variables) is the function $\eta : \{1 \dots L\} \times \{1 \dots m\} \rightarrow \mathbb{N}$ satisfying Fact 4.2.*

We want now to generalize our previous definition of stratified ideals (Definition 2.3) to the multivariate case, but dropping radicality (see Remark 4.1). It turns out that there are two ways of doing it: we have a weaker notion in the next definition and two stronger notions in the subsequent definition.

Definition 4.4. Let J be a zero-dimensional ideal with the above notation. We say that J is a **weakly stratified ideal** if

$$\Sigma_l^{j,i} \neq \emptyset \quad \text{for } 1 \leq l \leq \eta(j,i), 1 \leq i \leq m, 1 \leq j \leq L.$$

Being weakly stratified means that when considering the elimination ideal at level (j,i) (block j and variable $\mathbf{a}_{j,i}$) if there is a variety point with $l \geq 2$ extensions then there is another point with $l - 1$ extensions.

The following definition of multi-stratified ideal is given at variable-block level, rather than at a single-variable level. It contains two conditions: there is at least one point with exactly j extensions and there are no “gaps” in the number of extensions (for any integer $1 \leq l \leq j$ there is at least one point with l extensions). So it is exactly the multidimensional analogue of the definition of stratified ideals, except that we drop the radicality. Unfortunately, this straightforward generalization does not guarantee the existence of polynomials playing the role of “ideal” locators, and so in the same definition we provide an even stronger notion “strongly multi-stratified ideal”.

Definition 4.5. Let J be a zero-dimensional ideal with the above notation. Let us consider the natural projections

$$\begin{aligned} \pi_L &: \mathcal{V}(J_{\mathcal{S}, \mathcal{A}_L}) \longrightarrow \mathcal{V}(J_{\mathcal{S}}) \\ \pi_j &: \mathcal{V}(J_{\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathcal{A}_j}) \longrightarrow \mathcal{V}(J_{\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}}), \quad j = 1, \dots, L-1 \\ \rho_j &: \mathcal{V}(J_{\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathcal{A}_j}) \longrightarrow \mathcal{V}(J_{\mathcal{A}_j}), \quad j = 1, \dots, L \end{aligned}$$

Ideal J is a **multi-stratified ideal** (in the $\mathcal{A}_L, \dots, \mathcal{A}_1$ variables) if

- 1) for any $1 \leq j \leq L-1$ and for any $P \in \mathcal{V}(J_{\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}})$ we have that $|\pi_j^{-1}(\{P\})| \leq j$.

Moreover, for any $\bar{s} \in \mathcal{V}(J_{\mathcal{S}})$ we have that $|\pi_L^{-1}(\{\bar{s}\})| \leq L$;

- 2) for any $1 \leq j \leq L-1$ there is $Q \in \mathcal{V}(J_{\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}})$ s.t. $|\pi_j^{-1}(\{Q\})| = j$.

Moreover, there is $\bar{s} \in \mathcal{V}(J_{\mathcal{S}})$ s.t. $|\pi_L^{-1}(\{\bar{s}\})| = L$.

For any $1 \leq j \leq L$, let $Z_j = \rho_j(\mathcal{V}(J))$. We say that ideal J is a **strongly multi-stratified ideal** (in the $\mathcal{A}_L, \dots, \mathcal{A}_1$ variables) if 1) holds and

- 3) for any $1 \leq j \leq L-1$, for any $T \subset Z_j$ s.t. $1 \leq |T| \leq j$ there is a $Q \in \mathcal{V}(J_{\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}})$ s.t. $\rho_j(\pi_j^{-1}(\{Q\})) = T$.

Moreover, for any $T \subset Z_j$ s.t. $1 \leq |T| \leq L$ there is an $\bar{s} \in \mathcal{V}(J_{\mathcal{S}})$ s.t. $\rho_j(\pi_L^{-1}(\{\bar{s}\})) = T$.

Again, in the previous definition, we do not count multiplicities.

Remark 4.6. For any zero-dimensional ideal J with the above notation, let $Z = Z_1$. Once $\rho_{j'}(\mathcal{V}(J)) = \rho_j(\mathcal{V}(J))$ for any $1 \leq j, j' \leq L$, we obviously have $\rho_j(\pi_L^{-1}(\{\bar{s}\})) \subset Z$. Assuming this, 1) and 3) could be replaced by saying that there is a bijection between the sets of $\rho_j(\pi_L^{-1}(\{Q\}))$ and all (non-empty)

subsets of Z with up to j elements (and a similar condition at level L).

We note the following obvious fact.

Fact 4.7. *Let $m \geq 1$. If J is a strongly multi-stratified ideal then J is a multi-stratified ideal.*

Let $m = 1$. If J is a multi-stratified ideal then J is a weakly stratified ideal. If J is radical, then J is a multi-stratified ideal if and only if J is a stratified ideal.

The next two examples clarify (in the case $m = 1$) the notions of multi-stratified ideals and of weakly stratified ideals.

Example 4.8. Let $\mathcal{S} = \{s_1\}$, $\mathcal{A}_1 = \{a_{1,1}\}$, $\mathcal{A}_2 = \{a_{2,1}\}$, so that $m = 1$, and $\mathcal{T} = \{t_1\}$. Let $J = \mathcal{I}(Z) \subset \mathbb{C}[s_1, a_{2,1}, a_{1,1}, t_1]$ with $Z = \{(0, 0, 0, 0), (0, 1, 1, 0), (0, 2, 2, 0)\}$. The order $<$ is $s_1 < a_{2,1} < a_{1,1} < t_1$ and the varieties are

$$\mathcal{V}(J_{\mathcal{S}}) = \{0\}, \quad \mathcal{V}(J_{\mathcal{S}, a_{2,1}}) = \{(0, 0), (0, 1), (0, 2)\}, \quad \mathcal{V}(J_{\mathcal{S}, a_{2,1}, a_{1,1}}) = \{(0, 0, 0), (0, 1, 1), (0, 2, 2)\}.$$

Let us consider the projection $\pi_2 : \mathcal{V}(\mathcal{J}_{\mathcal{S}, a_{2,1}}) \rightarrow \mathcal{V}(\mathcal{J}_{\mathcal{S}})$. Then $|\pi_2^{-1}(\{0\})| = 3$. We have $\sum_3^{2,1} = \{0\}$ and $\sum_1^{2,1} = \emptyset$, $\sum_2^{2,1} = \emptyset$. So $\eta(2, 1) = 3$ and J is not a weakly stratified ideal (neither a stratified ideal).

Example 4.9. Let $\mathcal{S} = \{s_1\}$, $\mathcal{A}_1 = \{a_{1,1}\}$, $\mathcal{A}_2 = \{a_{2,1}\}$, $\mathcal{A}_3 = \{a_{3,1}\}$, $\mathcal{T} = \{t_1\}$ so that $m = 1$. Let $J = \mathcal{I}(Z) \subset \mathbb{C}[s_1, a_{3,1}, a_{2,1}, a_{1,1}, t_1]$ with $Z = \{(0, 1, 0, 0, 0), (0, 2, 1, 1, 2), (2, 2, 2, 0, 0)\}$. The order $<$ is $s_1 < a_{3,1} < a_{2,1} < a_{1,1} < t_1$ and the varieties are

$$\begin{aligned} \mathcal{V}(J_{\mathcal{S}}) &= \{0, 2\}, & \mathcal{V}(J_{\mathcal{S}, a_{3,1}}) &= \{(0, 1), (0, 2), (2, 2)\}, \\ \mathcal{V}(J_{\mathcal{S}, a_{3,1}, a_{2,1}}) &= \{(0, 1, 0), (0, 2, 1), (2, 2, 2)\}, \\ \mathcal{V}(J_{\mathcal{S}, a_{3,1}, a_{2,1}, a_{1,1}}) &= \{(0, 1, 0, 0), (0, 2, 1, 1), (2, 2, 2, 0)\}. \end{aligned}$$

Let us consider the projection $\pi_3 : \mathcal{V}(\mathcal{J}_{\mathcal{S}, a_{3,1}}) \rightarrow \mathcal{V}(\mathcal{J}_{\mathcal{S}})$. Then $|\pi_3^{-1}(\{0\})| = 2$ and $|\pi_3^{-1}(\{2\})| = 1$, so $\sum_2^{3,1} = \{0\}$, $\sum_1^{3,1} = \{2\}$ and $\eta(3, 1) = 2$, but $\sum_3^{3,1} = \emptyset$. Similarly, $\eta(2, 1) = \eta(1, 1) = 1$. So J is a weakly stratified ideal that is not multi-stratified (and not stratified).

However, if $m \geq 2$, a weakly stratified ideal is not necessarily a multi-stratified ideal and, viceversa, a multi-stratified ideal is not necessarily a weakly stratified ideal, as shown in the following example.

Example 4.10. Let $\mathcal{S} = \{s_1, s_2, s_3\}$, $\mathcal{A}_1 = \{a_{1,1}, a_{1,2}\}$, $\mathcal{A}_2 = \{a_{2,1}, a_{2,2}\}$, $\mathcal{A}_3 = \{a_{3,1}, a_{3,2}\}$, $\mathcal{T} = \{t_1\}$ so that $m = 2$.

Let $J = \mathcal{I}(Z) \subset \mathbb{C}[s_1, s_2, s_3, a_{3,1}, a_{3,2}, a_{2,1}, a_{2,2}, a_{1,1}, a_{1,2}, t_1, t_2]$, with $Z = \{(0, 0, 1, 1, 1, 1, 1, 3, 1, 1, 1), (0, 0, 1, 1, 2, 1, 3, 1, 2, 1, 1), (0, 0, 1, 1, 3, 0, 0, 2, 1, 1, 2), (1, 1, 2, 2, 1, 2, 1, 0, 0, 0, 1), (1, 1, 2, 0, 1, 1, 1, 0, 1, 2, 1), (1, 1, 2, 0, 1, 1, 0, 1, 0, 0, 1), (2, 3, 0, 3, 3, 1, 0, 1, 1, 1, 2)\}$. The order $<$ is $s_1 < s_2 < s_3 < a_{3,1} < a_{3,2} < a_{2,1} < a_{2,2} < a_{1,1} < a_{1,2} < t_1$

and the varieties are

$$\begin{aligned}\mathcal{V}(J_S) &= \{(0, 0, 1), (1, 1, 2), (2, 3, 0)\}, \\ \mathcal{V}(J_{S, \mathcal{A}_3}) &= \{(0, 0, 1, 1, 1), (0, 0, 1, 1, 2), (0, 0, 1, 1, 3), (1, 1, 2, 2, 1), (1, 1, 2, 0, 1), (2, 3, 0, 3, 3)\}, \\ \mathcal{V}(J_{S, \mathcal{A}_3, \mathcal{A}_2}) &= \{(0, 0, 1, 1, 1, 1), (0, 0, 1, 1, 2, 1, 2), (0, 0, 1, 1, 3, 0, 0), (1, 1, 2, 2, 1, 2, 1), (1, 1, 2, 0, 1, 1, 1), \\ &\quad (1, 1, 2, 0, 1, 1, 0), (2, 3, 0, 3, 3, 1, 0)\}, \\ \mathcal{V}(J_{S, \mathcal{A}_3, \mathcal{A}_2, \mathcal{A}_1}) &= \{(0, 0, 1, 1, 1, 1, 1, 3, 1), (0, 0, 1, 1, 2, 1, 2, 1, 2), (0, 0, 1, 1, 3, 0, 0, 2, 1), (1, 1, 2, 2, 1, 2, 1, 0, 0), \\ &\quad (1, 1, 2, 0, 1, 1, 1, 0, 1), (1, 1, 2, 0, 1, 1, 0, 1, 0), (2, 3, 0, 3, 3, 1, 0, 1, 1)\}.\end{aligned}$$

Let us consider the projection $\pi_3 : \mathcal{V}(J_{S, \mathcal{A}_3}) \rightarrow \mathcal{V}(J_S)$.

Then $|\pi_3^{-1}(\{(0, 0, 1)\})| = 3$, $|\pi_3^{-1}(\{(1, 1, 2)\})| = 2$ and $|\pi_3^{-1}(\{(2, 3, 0)\})| = 1$.

Similarly, if we consider $\pi_2 : \mathcal{V}(J_{S, \mathcal{A}_3, \mathcal{A}_2}) \rightarrow \mathcal{V}(J_{S, \mathcal{A}_3})$, then $|\pi_2^{-1}(\{(1, 1, 2, 0, 1)\})|$ is equal to 2 and for other $P \in \mathcal{V}(J_{S, \mathcal{A}_3})$ we have that $|\pi_2^{-1}(\{P\})| = 1$.

Finally, if we consider $\pi_1 : \mathcal{V}(J_{S, \mathcal{A}_3, \mathcal{A}_2, \mathcal{A}_1}) \rightarrow \mathcal{V}(J_{S, \mathcal{A}_3, \mathcal{A}_2})$, then for any $P \in \mathcal{V}(J_{S, \mathcal{A}_3, \mathcal{A}_2})$ we have that $|\pi_1^{-1}(\{P\})| = 1$ and so J is multi-stratified.

It is easy to see that J is not weakly stratified. In fact, if we consider a projection $\pi_{3,1} : \mathcal{V}(J_{S, \mathcal{A}_{3,1}, \mathcal{A}_{3,2}}) \rightarrow \mathcal{V}(J_{S, \mathcal{A}_{3,1}})$, then $\pi_{3,2}^{-1}(\{(0, 0, 1, 1)\}) = \{(0, 0, 1, 1, 1), (0, 0, 1, 1, 2), (0, 0, 1, 1, 3)\}$, $\pi_{3,2}^{-1}(\{(1, 1, 2, 2)\}) = \{(1, 1, 2, 2, 1)\}$, $\pi_{3,2}^{-1}(\{(1, 1, 2, 0)\}) = \{(1, 1, 2, 0, 1)\}$, $\pi_{3,2}^{-1}(\{(2, 3, 0, 3)\}) = \{(2, 3, 0, 3, 3)\}$. So $\sum_3^{3,2} = \{(0, 0, 1, 1)\}$, but $\sum_2^{3,2} = \emptyset$.

Proposition 4.11. *Let J be a strongly multi-stratified ideal then J is a weakly stratified ideal.*

Proof. For any $1 \leq i \leq m$ and for any $j = 1, \dots, L-1$, let us consider the natural projection

$$\pi_{j,i} : \mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \bar{\mathcal{A}}_{j,1}, \dots, \bar{\mathcal{A}}_{j,i}}) \longrightarrow \mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \bar{\mathcal{A}}_{j,1}, \dots, \bar{\mathcal{A}}_{j,i-1}})$$

We will also use ρ_j and π_j as in Definition 4.5.

To avoid complications, we consider only the case $2 \leq i \leq m-1$, being the modifications in the $i = 1$ and $i = m$ obvious.

The first fact that we note is that $\eta(j, i) \leq j$, because if the pre-images at block level contain at most j elements, then at variable level they cannot contain more. Let $2 \leq l \leq \eta(j, i)$ such that $\Sigma_l^{j,i} \neq \emptyset$. It is enough to show that $\Sigma_{l-1}^{j,i} \neq \emptyset$.

Let \bar{R}, \bar{P} and Q such that $Q \in \Sigma_l^{j,i}$, $Q = (\bar{S}, \bar{\mathcal{A}}_L, \dots, \bar{\mathcal{A}}_{j+1}, \bar{\mathbf{a}}_{j,1}, \dots, \bar{\mathbf{a}}_{j,i-1})$, $\bar{P} = (\bar{S}, \bar{\mathcal{A}}_L, \dots, \bar{\mathcal{A}}_{j+1})$, $\bar{R} = (\bar{\mathbf{a}}_{j,1}, \dots, \bar{\mathbf{a}}_{j,i-1})$, so $Q = (\bar{P}, \bar{R})$.

Then $\pi_{j,i}^{-1}(\{Q\}) = \{(Q, \lambda_1), \dots, (Q, \lambda_l)\}$ and all λ_ℓ 's are distinct.

Let $\Gamma_1, \dots, \Gamma_l \in \mathcal{V}(J_{\mathbf{a}_{j,i+1}, \dots, \mathbf{a}_{j,m}})$ such that $(Q, \lambda_\ell, \Gamma_\ell) \in \mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_j})$. The Γ_ℓ 's do not have to be distinct. For any $1 \leq \ell \leq l$ at least one such Γ_ℓ must exist.

We choose one Γ_ℓ for any ℓ . So $\{(Q, \lambda_1, \Gamma_1), \dots, (Q, \lambda_l, \Gamma_l)\} \subset \pi_j^{-1}(\bar{P})$ and $\{(\bar{R}, \lambda_\ell, \Gamma_\ell)\}_{1 \leq \ell \leq l} \subset \rho_j(V(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathcal{A}_j}))$. Let

$$T = \{(\bar{R}, \lambda_1, \Gamma_1), \dots, (\bar{R}, \lambda_{l-1}, \Gamma_{l-1})\}.$$

Then $T \subset \rho_j(V(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathcal{A}_j}))$ and $|T| = l - 1 \leq \eta(j, i) - 1 \leq j - 1$.

Since J is strongly multi-stratified, there is $\tilde{P} \in \mathcal{V}(J_{\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}})$ such that

$$T = \rho_j(\pi_j^{-1}(\{\tilde{P}\})), \text{ so } \pi_j^{-1}(\{\tilde{P}\}) = \{(\tilde{P}, \bar{R}, \lambda_1, \Gamma_1), \dots, (\tilde{P}, \bar{R}, \lambda_{l-1}, \Gamma_{l-1})\}.$$

This implies that $\{(\tilde{P}, \bar{R}, \lambda_1), \dots, (\tilde{P}, \bar{R}, \lambda_{l-1})\} = \pi_{j,i}^{-1}(\{(\tilde{P}, \bar{R})\})$, and so $\Sigma_{l-1}^{j,i} \neq \emptyset$, as all λ_ℓ 's are distinct. \square

Let $<_{\text{lex}}$ be the lexicographic term order such that $\mathcal{A}_L <_{\text{lex}} \dots <_{\text{lex}} \mathcal{A}_1$ and $a_{j,1} <_{\text{lex}} \dots <_{\text{lex}} a_{j,m}$, for any $1 \leq j \leq L$. Let $<_{\mathcal{S}}$ be a term order on \mathcal{S} and $<_{\mathcal{T}}$ a term order on \mathcal{T} . Let $<$ be the block order $< = (<_{\mathcal{S}}, <_{\text{lex}}, <_{\mathcal{T}})$. We are now assuming that J is any zero-dimensional ideal in $\mathbb{K}[\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_1, \mathcal{T}]$. Let $G = \text{GB}(J)$. It is well-known that the elements of $G \cap (\mathbb{K}[\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_1] \setminus \mathbb{K}[\mathcal{S}])$ can be collected into non-empty blocks $\{G^j\}_{1 \leq j \leq L}$, where²

$$G^L = G \cap (\mathbb{K}[\mathcal{S}, \mathcal{A}_L] \setminus \mathbb{K}[\mathcal{S}])$$

and, for $1 \leq j \leq L-1$,

$$G^j = G \cap (\mathbb{K}[\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathcal{A}_j] \setminus \mathbb{K}[\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}]).$$

Then we denote by $G^{L,1}$, $G^{L,i}$, $G^{j,1}$ and $G^{j,i}$, $1 < j \leq L$, $1 < i \leq m$, respectively, the sets:

$$G^{L,1} = G \cap (\mathbb{K}[\mathcal{S}, \mathbf{a}_{L,1}] \setminus \mathbb{K}[\mathcal{S}])$$

$$G^{L,i} = G \cap (\mathbb{K}[\mathcal{S}, \mathbf{a}_{L,1}, \dots, \mathbf{a}_{L,i}] \setminus \mathbb{K}[\mathcal{S}, \mathbf{a}_{L,1}, \dots, \mathbf{a}_{L,i-1}])$$

$$G^{j,1} = G \cap (\mathbb{K}[\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}] \setminus \mathbb{K}[\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}])$$

$$G^{j,i} = G \cap (\mathbb{K}[\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i}] \setminus \mathbb{K}[\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}]).$$

In other words, let g be any polynomial in $G^{j,i}$. Then:

- g contains the variable $\mathbf{a}_{j,i}$,
- g does not contain any greater variable (i.e. no variables in blocks $\mathcal{A}_{j+1} \dots \mathcal{A}_1$ and none of the remaining variables in the j -th block $\mathbf{a}_{j,i+1}, \dots, \mathbf{a}_{j,m}$),
- g may contain lesser variables (the \mathcal{S} variables, the \mathbf{a} variables contained in blocks $L, \dots, j-1$ and the lesser \mathbf{a} variables in the same block: $\mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}$).

As the ideal under consideration is assumed zero-dimensional, the sets $G^{j,i}$ are non-empty. The polynomials in any $G^{j,i}$ can be grouped according to their degree δ with respect to $\mathbf{a}_{j,i}$.

For us it is essential to know the *maximum value* of δ in $G^{j,i}$, that we call

² In [OS05] we use the notation G_i .

$$\zeta(j, i) \tag{13}$$

So we can write:

$$G^{j,i} = \sqcup_{\delta=1}^{\zeta(j,i)} G_{\delta}^{j,i}, \quad j = 1, \dots, L, \quad i = 1, \dots, m, \quad \text{with } G_{\zeta(j,i)}^{j,i} \neq \emptyset,$$

but some $G_{\delta}^{j,i}$ could be empty. In this way, if $g \in G_{\delta}^{j,i}$ we have:

- $g \in \mathbb{K}[\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}][\mathbf{a}_{j,i}] \setminus \mathbb{K}[\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}]$
- $\deg_{\mathbf{a}_{j,i}}(g) = \delta$.

Note that we can view ζ as a function $\zeta : \{1 \dots L\} \times \{1 \dots m\} \rightarrow \mathbb{N}$, that is, as a function with exactly the same range of η .

If $g \in G_{\delta}^{j,i}$, then we can write uniquely g as

$$g = a_{\delta} \mathbf{a}_{j,i}^{\delta} + a_{\delta-1} \mathbf{a}_{j,i}^{\delta-1} + \dots + a_0,$$

with $a_j \in \mathbb{K}[\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}]$. and a_{δ} is the leading polynomial of g .

We name the elements of $G_{\delta}^{j,i}$ according to the term order of their leading terms, i.e. $G_{\delta}^{j,i} = \{g_{j,\delta,1}^{(i)}, \dots, g_{j,\delta,|G_{\delta}^{j,i}|}^{(i)}\}$, with $\mathbf{T}(g_{j,\delta,h}^{(i)}) < \mathbf{T}(g_{j,\delta,h+1}^{(i)})$ for any h .

We note the following lemma.

Lemma 4.12. *For any $j = 1, \dots, L$ and $i = 1, \dots, m$, $G_{\zeta(j,i)}^{j,i} = \{g_{j,\zeta(j,i),1}^{(i)}\}$, i.e. there exists only one polynomial in $G_{\zeta(j,i)}^{j,i}$ such that $\deg_{\mathbf{a}_{j,i}} = \zeta(j, i)$.*

Proof. From elementary properties of Gröbner bases of zero-dimensional ideals, for any variable $\mathbf{a}_{j,i}$, G must contain a polynomial g with leading term $\mathbf{a}_{j,i}^k$, for some $k \geq 1$. Note that $g \in G^{j,i}$, because variable $\mathbf{a}_{j,i}^k$ is present in g and any greater variable cannot be present. If there is a $\bar{g} \in G^{j,i}$ with $\deg_{\mathbf{a}_{j,i}} \bar{g} \geq k$, then $\mathbf{a}_{j,i} | \mathbf{T}(\bar{g})$ and so \bar{g} can be removed (recall that G is reduced). As a consequence, g has the highest possible degree in $\mathbf{a}_{j,i}$, i.e. $k = \zeta(j, i)$, and so $g = g_{j,\zeta(j,i),1}^{(i)}$. \square

We are ready for the main result of this section (whose proof is given in Section 5). Compare with Theorem 32 in [GS09].

Proposition 4.13. *Let G be a reduced Gröbner basis of a radical weakly stratified ideal J with respect to $<$ as previously described. Let $\mathcal{V}(J) \subset \mathbb{A}$. Then for any $j = 1, \dots, L$ and $i = 1, \dots, m$,*

$$G^{j,i} = \sqcup_{\delta=1}^{\zeta(j,i)} G_{\delta}^{j,i}, \text{ with}$$

1. $\zeta(j, i) = \eta(j, i)$, i.e. ζ is the level function of J ;
2. $G_{\delta}^{j,i} \neq \emptyset$ for any $1 \leq \delta \leq \zeta(j, i)$;

3. $G_{\zeta(j,i)}^{j,i} = \{g_{j,\zeta(j,i),1}^{(i)}\}$, i.e. there exists only one polynomial in $G_{\zeta(j,i)}^{j,i}$ such that $\deg_{a_{j,i}} = \zeta(j,i)$;

4. we have that

$$\mathbf{T}(g_{j,\zeta(j,i),1}^{(i)}) = a_{j,i}^{\zeta(j,i)}.$$

Note that it is the radicality that ensures 1., but in later situations we will have 1. also without radicality.

5 Proof of Proposition 4.13

5.1 Preliminaries of proof

To prove Proposition 4.13 we use this classical theorem:

Theorem 5.1 (Buchberger-Möller, [MB82,Mor09a]). *Let $H' \subset H$ be ideals in $\mathbb{K}[V_1, \dots, V_N]$ such that:*

- (i) *there is a \mathbb{K} -linear map $\theta : H \rightarrow \mathbb{K}$ s.t. $\ker(\theta) = H'$,*
- (ii) *there are \mathcal{N} field elements $\{\beta_k\}_{1 \leq k \leq N} \subset \mathbb{K}$ s.t. $(V_k - \beta_k)H \subset H'$ for $1 \leq k \leq \mathcal{N}$, that is, $\theta((V_k - \beta_k)f) = 0$ for all $f \in H$.*

Let W be a strictly ordered Gröbner basis of H relative to a term order $<$, then a Gröbner basis W' of H' w.r.t $<$ can be constructed as follows:

- (1) *compute $\alpha_g = \theta(g)$ for all $g \in W$.*
- (2) *if $\alpha_g = 0$ for all g , then $W = W'$, which happens if and only if $H = H'$ and $\theta = 0$ in $\text{Hom}_{\mathbb{K}}(H, \mathbb{K})$.*
- (3) *otherwise, let g^* be the least g such that $\alpha_g \neq 0$.*

We have $W' = W_1 \cup W_2 \cup W_3$, with

- $W_1 = \{g \mid g < g^*\},$
- $W_2 = \{(V_k - \beta_k)g^* \mid 1 \leq k \leq \mathcal{N}\},$
- $W_3 = \{g - \frac{\alpha_g}{\alpha_{g^*}}g^* \mid g > g^*\}.$

Remark 5.2. In the proof of Theorem 5.1, the hypothesis (ii) is used only to prove that $W_2 \subset H'$. Therefore, Theorem 5.1 still holds if we replace (ii) with a much weaker hypothesis, that is,

- (iii) *there are \mathcal{N} field elements $\{\beta_k\}_{1 \leq k \leq N} \subset \mathbb{K}$ s.t. $(V_k - \beta_k)g^* \in H'$, $1 \leq k \leq \mathcal{N}$, where g^* is as in (3).*

Remark 5.3. Let G be a Gröbner basis of an ideal I with respect to a term ordering $>$ and let $g_1, g_2 \in G$ be such that $\mathbf{T}(g_1) \mid \mathbf{T}(g_2)$. Then $G \setminus \{g_2\}$ is again a Gröbner basis of I . Therefore, any time there is a redundant basis element, we can remove it.

From the remainder of this section, we fix $1 \leq i \leq m$ and $1 \leq j \leq L$, and we extend the projection

$$\pi : \mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}, \mathbf{a}_{j,i}) \rightarrow \mathcal{V}(J_{S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}}) \quad (14)$$

to

$$\pi : \overline{\mathbb{K}}^{N+(L-j)m+i} \rightarrow \overline{\mathbb{K}}^{N+(L-j)m+i-1}$$

Coherently, we consider only the variable $\mathbf{a}_{j,i}$ in the block \mathcal{A}_j .

Remark 5.4. To simplify the notation in the proof, we use τ as a symbol with a special meaning, as follows. We introduce τ to single out the contribution of variable $\mathbf{a}_{j,i}$. Any non-zero element of $\mathbb{K}[S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}]$ may be written as τ and we use \cong to express this unconventional identification. For example, $\mathbf{a}_{L,1} \cong \tau$ and $1 \cong \tau$ but also $\tau \mathbf{a}_{L,1} \cong \tau$ and $\mathbf{a}_{L,1} \mathbf{a}_{j,i} \cong \tau \mathbf{a}_{j,i} \not\cong \tau$ and $s_1 \mathbf{a}_{j,i}^2 \cong \tau \mathbf{a}_{j,i}^2 \cong \mathbf{a}_{L,2} \mathbf{a}_{j,i}^2$.

Let H be a zero-dimensional ideal in $\mathbb{K}[S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i}]$.

Let W be its Gröbner basis. Denote with

$\overline{W} = W \cap (\mathbb{K}[S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i}] \setminus \mathbb{K}[S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}])$ and $\widehat{W} = W \cap (\mathbb{K}[S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}])$, so that $W = \overline{W} \sqcup \widehat{W}$. With the τ notation, we have

$$\widehat{W} \cong \{\tau, \dots, \tau\} \text{ and } \overline{W} \subset \{\tau \mathbf{a}_{j,i} + \tau, \dots, \tau \mathbf{a}_{j,i} + \tau, \tau \mathbf{a}_{j,i}^2 + \tau \mathbf{a}_{j,i} + \tau, \dots\}.$$

In the same way we can denote

$$\overline{H} = H \cap (\mathbb{K}[S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i}] \setminus \mathbb{K}[S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}])$$

$$\text{and } \widehat{H} = H \cap (\mathbb{K}[S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}]).$$

Remark 5.5. Suppose we want to compute the ideal H' from H by adding a point $Q = (P, \bar{\mathbf{a}}_{j,i})$, with $P = (\bar{s}_1, \dots, \bar{s}_N, \bar{\mathbf{a}}_{L,1}, \dots, \bar{\mathbf{a}}_{j,i-1})$. We apply Theorem 5.1 to compute W' from W using the point evaluation $\theta(g) = g(Q)$. In this case it is easy to see that we can take as β_i the i -th component of Q . There are two distinct cases:

- (1) *either* for all $g \in \widehat{W}$, $g(Q) = g(P) = 0$,
- (2) *or* there exists $g \in \widehat{W}$ such that $g(Q) = g(P) \neq 0$.

The first case implies $g^* \in \overline{W}$, the second case implies $g^* \in \widehat{W}$. Since these are logically distinct, we can conclude that there are only two (distinct) cases:

- (1) *either* for all $g \in \widehat{W}$, $g(Q) = g(P) = 0$, and this happens if and only if $g^* \in \overline{W}$,
- (2) *or* there exists $g \in \widehat{W}$ such that $g(Q) = g(P) \neq 0$, this happens if and only if $g^* \in \widehat{W}$.

5.2 Sketch of proof

Let us consider $g = g_{j,\zeta(j,i),1}^{(i)}$ and $\Delta = \eta(j, i)$.
 Let $I = J \cap (\mathbb{K}[S, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i}])$. Since $\mathcal{V}(I) \subset \mathbb{A}_{j,i}$ and I is radical and zero-dimensional, $I = \mathcal{I}(\mathcal{V}(I)) = \mathcal{I}(\Sigma_1^{j,i} \sqcup \dots \sqcup \Sigma_\Delta^{j,i})$. Since J is weakly-stratified, we will have $\Sigma_h^{j,i} \neq \emptyset$ for all $1 \leq h \leq \Delta$.

Our proof needs several steps:

- **Step I.**

We consider $P_1 \in \Sigma_1^{j,i}$, $P_2 \in \Sigma_2^{j,i}$ and π as in (14). We are interested in the leading terms of the Gröbner basis of $\mathcal{I}(\pi^{-1}(P_1))$ and of $\mathcal{I}(\pi^{-1}(P_1) \cup \pi^{-1}(P_2))$. However, the exact knowledge of these leading terms is unnecessary and it is sufficient for us to determine their expression in the τ notation. We perform this step in Subsection 5.3.

- **Step II.**

Generalising the previous argument, in Subsection 5.4 (Lemma 5.6) we take any $2 \leq t \leq \Delta$ and consider any point P_h in $\Sigma_h^{j,i}$ for all $1 \leq h \leq t$. We describe the leading terms of the Gröbner basis of $\mathcal{I}(\pi^{-1}(P_1) \cup \dots \cup \pi^{-1}(P_t))$. Since we need an induction on the number of points to prove Lemma 5.6, we give an intermediate lemma: Lemma 5.7.

- **Step III.**

As the leading terms of the Gröbner basis of $\mathcal{I}(\pi^{-1}(P_1) \cup \dots \cup \pi^{-1}(P_\Delta))$ are already in the desired shape, in Lemma 5.9 we show that adding more points does not change the shape of the leading terms of the Gröbner basis, as long as the points come from some $\Sigma_h^{j,i}$ with $h \leq \Delta$.

5.3 First part of the proof

We use the approach of Remark 5.5.

- Let $P_1 = (\bar{s}_1, \dots, \bar{s}_N, \bar{\mathbf{a}}_{L,1}, \dots, \bar{\mathbf{a}}_{j,i-1}) \in \Sigma_1^{j,i}$ and $H = \mathcal{I}(\pi^{-1}(P_1))$ be the vanishing ideal of $\pi^{-1}(P_1)$.
 Then $\pi^{-1}(P_1) = \{(\bar{s}_1, \dots, \bar{s}_N, \bar{\mathbf{a}}_{L,1}, \dots, \bar{\mathbf{a}}_{j,i-1}, \bar{\mathbf{a}}_{j,i})\}$. The basis $W = \text{GB}(H)$ is $W = \{s_1 - \bar{s}_1, \dots, s_N - \bar{s}_N, \mathbf{a}_{L,1} - \bar{\mathbf{a}}_{L,1}, \dots, \mathbf{a}_{j,i} - \bar{\mathbf{a}}_{j,i}\}$. Using our notation we have

$$\mathbf{T}(W) = \{\tau, \dots, \tau, \mathbf{a}_{j,i}\}. \quad (15)$$

- We consider a point $P_2 \in \Sigma_2^{j,i}$ that, with abuse of notation³, we write

³ Where we do not imply that the components of P_2 are the same as those of P_1 , although we use the same symbols.

$P_2 = (\bar{s}_1, \dots, \bar{s}_N, \bar{a}_{L,1}, \dots, \bar{a}_{j,i-1})$. We can write

$$\pi^{-1}(P_2) = \begin{cases} Q_1 = (\bar{s}_1, \dots, \bar{s}_N, \bar{a}_{L,1}, \dots, \bar{a}_{j,i-1}, \bar{a}_{j,i}^{(1)}) \\ Q_2 = (\bar{s}_1, \dots, \bar{s}_N, \bar{a}_{L,1}, \dots, \bar{a}_{j,i-1}, \bar{a}_{j,i}^{(2)}) \end{cases}$$

* We add the point Q_1 .

Using Theorem 5.1 we can build W' from W in (15). If $\forall g \in \widehat{W}$, $g(Q_1) = 0$, then $\pi(Q_1) \in \mathcal{V}(\widehat{H})$. But $\pi(Q_1) = P_2$ and $\mathcal{V}(\widehat{H}) = \{P_1\}$, so $P_1 = P_2$ and $|\pi^{-1}(P_2)| = 3$, which is impossible because $P_2 \in \Sigma_2^{j,i}$. Therefore, for Remark 5.5, $g^* \in \widehat{W}$.

So the Gröbner basis $W' = W_1 \sqcup W_2 \sqcup W_3$, where

- $W_1 = \{g \in \widehat{W} \mid g < g^*\}$ because $g^* \in \widehat{W}$, so we have $W_1 \cong \{\tau, \dots, \tau\}$ and $\mathbf{T}(W_1) \cong \{\tau, \dots, \tau\}$.
- W_2 is composed by the following polynomials

$$\begin{aligned} &g^*(s_1 - \bar{s}_1), \dots, g^*(s_N - \bar{s}_N) \\ &g^*(a_{L,1} - \bar{a}_{L,1}), \dots, g^*(a_{j,i-1} - \bar{a}_{j,i-1}) \\ &g^*(a_{j,i} - \bar{a}_{j,i}^{(1)}) \end{aligned}$$
- $W_3 = \{g - \frac{g(Q_1)}{g^*(Q_1)}g^* \mid g > g^*\}$.

We have $\mathbf{T}(W_2) \cong \{\tau, \dots, \tau, \tau a_{j,i}\}$ and $\mathbf{T}(W_3) \subseteq \{\tau, \dots, \tau, a_{j,i}\}$ and $a_{j,i} \in \mathbf{T}(W_3)$. With $\mathbf{T}(W_3) \subseteq \{\tau, \dots, \tau, a_{j,i}\}$ we actually mean that $\mathbf{T}(W_3)$ is a subset of a set S such that $S \approx \{\tau, \dots, \tau, a_{j,i}\}$. We will write similarly from now on without any further comment. Observe that $\mathbf{T}(W') \cong \{\tau, \dots, \tau, \tau a_{j,i}, a_{j,i}\}$. By Remark 5.3, we have $\mathbf{T}(W') \cong \{\tau, \dots, \tau, a_{j,i}\}$.

* We add the point Q_2 .

Let⁴ $W := W'$ and let us use again Theorem 5.1.

We have to find a polynomial $g^* \in W$ such that $g^*(Q_2) \neq 0$. Of course $g^* \notin \widehat{W}$, because $\pi(Q_1) = \pi(Q_2) = P_2$. Thus $g^* \in \overline{W}$ and $g^* = a_{j,i} + \tau$. W' is formed by $W' = W_1 \sqcup W_2 \sqcup W_3$, where

- $W_1 \cong \{\tau, \dots, \tau\}$,
- $W_2 = W_{2,1} \cup W_{2,2}$ where

$$\begin{aligned} &W_{2,1} = \{g^*(s_1 - \bar{s}_1), \dots, g^*(s_N - \bar{s}_N), g^*(a_{L,1} - \bar{a}_{L,1}), \dots, g^*(a_{j,i-1} - \bar{a}_{j,i-1}), \text{ so} \\ &\mathbf{T}(W_{2,1}) \cong \{\tau a_{j,i}, \dots, \tau a_{j,i}\}. \\ &W_{2,2} = \{g^*(a_{j,i} - \bar{a}_{j,i}^{(2)})\} \implies \mathbf{T}(W_{2,2}) = \{a_{j,i}^2\}. \end{aligned}$$
- $W_3 = \emptyset$.

So

$$\mathbf{T}(W') = \{\tau, \dots, \tau, \tau a_{j,i}, \dots, \tau a_{j,i}, a_{j,i}^2\} \quad (16)$$

⁴ With "let $W := W'$ " we mean that in this proof step we remove all elements in set W and instead we insert into W all elements from W' . After that, we remove all elements from W' . We also forget the values of g^* and W_1, W_2, W_3 .

5.4 Second part of proof

If $\Delta \leq 2$, we have finished our proof. Otherwise, i.e. $\Delta \geq 3$, we want to prove, using induction on t with $1 \leq t \leq \Delta$, the following

Lemma 5.6. *The Gröbner basis W of $H = \mathcal{I}(\pi^{-1}(P_1) \cup \dots \cup \pi^{-1}(P_t))$, where $1 \leq t \leq \Delta$ and P_h is any point in $\Sigma_h^{j,i}$ for $1 \leq h \leq t$, is such that*

$$\mathbf{T}(W) \cong \{\tau, \dots, \tau, \tau \mathbf{a}_{j,i}, \dots, \tau \mathbf{a}_{j,i}, \dots, \tau \mathbf{a}_{j,i}^{t-1}, \dots, \tau \mathbf{a}_{j,i}^{t-1}, \mathbf{a}_{j,i}^t\} \quad (17)$$

Proof. The Gröbner basis with $t = 1$ and $t = 2$ were just shown in (15) and (16) respectively.

By induction we suppose to have $t - 1$ points $\{P_1, \dots, P_{t-1}\}$ and to have a Gröbner basis W such that:

$$\mathbf{T}(W) \cong \{\tau, \dots, \tau, \tau \mathbf{a}_{j,i}, \dots, \tau \mathbf{a}_{j,i}, \dots, \tau \mathbf{a}_{j,i}^{t-2}, \dots, \tau \mathbf{a}_{j,i}^{t-2}, \mathbf{a}_{j,i}^{t-1}\} \quad (18)$$

Now we can prove the t -th step. In order to do it, we prove the following lemma, with its long proof between horizontal lines.

Lemma 5.7. *Let $3 \leq t \leq \Delta$ and $P_t \in \Sigma_t^{j,i}$ with $\pi^{-1}(P_t) = \{Q_1, \dots, Q_t\}$. For any $1 \leq u \leq t - 1$, let H^u be the vanishing ideal*

$$H^u = \mathcal{I}(\pi^{-1}(P_1) \cup \dots \cup \pi^{-1}(P_{t-1}) \cup \{Q_1, \dots, Q_u\}) \text{ and}$$

$$H^0 = \mathcal{I}(\pi^{-1}(P_1) \cup \dots \cup \pi^{-1}(P_{t-1})).$$

Let W^u be its reduced Gröbner basis, then

$$\mathbf{T}(W^u) \text{ has the same structure as } \mathbf{T}(W) \text{ in (18).}$$

Proof. Let $P_t = (\bar{s}_1, \dots, \bar{s}_N, \bar{\mathbf{a}}_{L,1}, \dots, \bar{\mathbf{a}}_{j,i-1})$. Since $P_t \in \Sigma_t^{j,i}$, then

$$\pi^{-1}(P_t) = \begin{cases} Q_1 = (\bar{s}_1, \dots, \bar{s}_N, \bar{\mathbf{a}}_{L,1}, \dots, \bar{\mathbf{a}}_{j,i-1}, \bar{\mathbf{a}}_{j,i}^{(1)}) \\ Q_2 = (\bar{s}_1, \dots, \bar{s}_N, \bar{\mathbf{a}}_{L,1}, \dots, \bar{\mathbf{a}}_{j,i-1}, \bar{\mathbf{a}}_{j,i}^{(2)}) \\ \vdots \\ Q_t = (\bar{s}_1, \dots, \bar{s}_N, \bar{\mathbf{a}}_{L,1}, \dots, \bar{\mathbf{a}}_{j,i-1}, \bar{\mathbf{a}}_{j,i}^{(t)}) \end{cases}$$

We prove the lemma by induction on u .

(a) We know that W^0 is as in (18). We add point Q_1 to H^0 .

Using Theorem 5.1 we can build W^1 from W^0 as usual. We adopt the " W, W' " notation. If $\forall g \in \widehat{W}$, $g(Q_1) = 0$, then $\pi(Q_1) \in \mathcal{V}(\widehat{H})$. But $\pi(Q_1) = P_t$ and $\mathcal{V}(\widehat{H}) = \{P_1, \dots, P_{t-1}\}$, so $P_t = P_k$ for some $1 \leq k \leq t-1$, and $|\pi^{-1}(P_k)| = k + 1$ which is impossible because $P_k \in \Sigma_k^{j,i}$. Therefore, for Remark 5.5, $g^* \in \widehat{W}$.

So the Gröbner basis W' is formed by the union of these sets:

- $W_1 = \{g \in \widehat{W} \mid g < g^*\}$. Since $g^* \in \widehat{W}$ then $\mathbf{T}(W_1) = \{\tau, \dots, \tau\}$,
- $W_2 = W_{2,1} \cup W_{2,2}$ where

$$W_{2,1} = \{g^*(s_1 - \bar{s}_1), \dots, g^*(s_N - \bar{s}_N), g^*(a_{L,1} - \bar{a}_{L,1}), \dots, g^*(a_{j,i-1} - \bar{a}_{j,i-1}),$$

$$\text{so } \mathbf{T}(W_{2,1}) = \{\tau, \dots, \tau\}.$$

$$W_{2,2} = \{g^*(a_{j,i} - \bar{a}_{j,i}^{(1)})\} \implies \mathbf{T}(W_{2,2}) = \{\tau a_{j,i}\}.$$
- $W_3 = \{g - \frac{g(Q_1)}{g^*(Q_1)} g^* \mid g > g^*\}$ and so the leading terms of W_3 are those in $\mathbf{T}(W)$, except possibly for τ .

Therefore $W^1 = W'$ has the same structure of $W^0 = W$ in (18) (because $\tau a_{j,i}$ is already present in (18)).

- (b) We add the point Q_2 to H^1 and we compute W^2 .

Let $W := W'$ and we use again Theorem 5.1.

We find $g^* \in W$ such that $g^*(Q_2) \neq 0$. We are sure that $g^* \notin \widehat{G}$, because $\pi(Q_1) = \pi(Q_2) = P_t$, and so $g^* \in \overline{G}$. We can claim:

Claim $\mathbf{T}(g^*) \cong \tau a_{j,i}$.

Proof. The Gianni-Kalkbrener theorem ([Gia89], [Kal89]) says that there exists a polynomial $g \in \overline{W}$ such that

$$g(P_t, a_{j,i}) = g(\bar{s}_1, \dots, \bar{s}_N, \bar{a}_{L,1}, \dots, \bar{a}_{j,i-1}, a_{j,i}) \neq 0 \text{ in } \mathbb{K}[a_{j,i}]$$

and the solutions of $g(P_t, a_{j,i})$ are exactly the extensions of P_t . In $\mathcal{V}(H)$ we have only one extension of P_t (which is Q_1), so the degree of g w.r.t. $a_{j,i}$ must be 1 and so $g \cong \tau a_{j,i} + \tau$.

Let g be the smallest polynomial of this kind. We have that $g^* = g$, because $g(Q_1) = 0$, $g(Q_2) \neq 0$ and all smaller polynomials vanish at Q_2 . \square

So W' is the union of

- $W_1 = \{g \in \widehat{W} \mid g < g^*\}$.
Since $g^* = \tau a_{j,i} + \tau$ then $\mathbf{T}(W_1) = \{\tau, \dots, \tau\}$ or $\mathbf{T}(W_1) = \{\tau, \dots, \tau, \tau a_{j,i}, \dots, \tau a_{j,i}\}$,
- $W_2 = W_{2,1} \cup W_{2,2}$ where

$$W_{2,1} = \{g^*(s_1 - \bar{s}_1), \dots, g^*(s_N - \bar{s}_N), g^*(a_{L,1} - \bar{a}_{L,1}), \dots, g^*(a_{j,i-1} - \bar{a}_{j,i-1}),$$

$$\text{so } \mathbf{T}(W_{2,1}) = \{\tau a_{j,i}, \dots, \tau a_{j,i}\}. \quad W_{2,2} = \{g^*(a_{j,i} - \bar{a}_{j,i}^{(2)})\} \implies \mathbf{T}(W_{2,2}) = \{\tau a_{j,i}^2\}.$$
- $W_3 = \{g - \frac{g(Q_2)}{g^*(Q_2)} g^* \mid g > g^*\}$ so the leading terms of W_3 are those in $\mathbf{T}(W)$, except possibly for τ and $\tau a_{j,i}$.

If $t = 3$, we have that $a_{j,i}^2 \in \mathbf{T}(W)$, and so any leading term $\tau a_{j,i}^2$ can be removed (by Remark 5.3) and we obtain again that the structure of $W' = W^2$ is as in (18). Otherwise ($t \geq 4$), the leading term $\tau a_{j,i}^2$ remains and we still have the structure of (18).

- (c) We proceed inductively on u until we are left to add the point Q_{t-1} .

- (d) We add Q_{t-1} .

In this case $H = \mathcal{I}(\pi^{-1}(P_1) \cup \dots \cup \pi^{-1}(P_{t-1}) \cup \{Q_1, \dots, Q_{t-2}\})$ and W^{t-2} has (by induction on u) the structure of (18). Let $W = W^{t-2}$ and $W' = W^{t-1}$. We apply Theorem 5.1.

We have to find $g^* \in W$ such that $g^*(Q_{t-1}) \neq 0$. Exactly as before, $g^* \notin \widehat{W}$. We know that $\mathbf{T}(g^*) \cong \tau \mathbf{a}_{j,i}^{t-2}$. To prove it we might use the Gianni-Kalkbrener theorem ([Gia89], [Kal89]) repeating the reasoning of our Claim on page 37.

So W' is the union of the following sets:

- $W_1 = \{g \in W \mid g < g^*\}$. Since $\mathbf{T}(g^*) \cong \tau \mathbf{a}_{j,i}^{t-2}$,
 $\mathbf{T}(W_1) = \{\tau, \dots, \tau, \tau \mathbf{a}_{j,i}, \dots, \tau \mathbf{a}_{j,i}, \dots, \tau \mathbf{a}_{j,i}^{t-3}, \dots, \tau \mathbf{a}_{j,i}^{t-3}\}$ or possibly also $\tau \mathbf{a}_{j,i}^{t-2} \in \mathbf{T}(W_1)$,
- $W_2 = W_{2,1} \cup W_{2,2}$ where
 $W_{2,1} = \{g^*(s_1 - \bar{s}_1), \dots, g^*(s_N - \bar{s}_N), g^*(\mathbf{a}_{L,1} - \bar{\mathbf{a}}_{L,1}), \dots, g^*(\mathbf{a}_{j,i-1} - \bar{\mathbf{a}}_{j,i-1})\}$
 $W_{2,2} = \{g^*(\mathbf{a}_{j,i} - \bar{\mathbf{a}}_{j,i}^{(r+1)})\}$
- $W_3 = \{g - \frac{g(Q_{r+1})}{g^*(Q_{r+1})} g^* \mid g > g^*\}$.

Since $g^* = \tau \mathbf{a}_{j,i}^{t-2} + \dots$, we have $\mathbf{T}(W_{2,1}) = \{\tau \mathbf{a}_{j,i}^{t-2}, \dots, \tau \mathbf{a}_{j,i}^{t-2}\}$ and $\mathbf{T}(W_{2,2}) = \{\tau \mathbf{a}_{j,i}^{t-1}\}$. But in the Gröbner basis W in (18) there exists a polynomial \bar{g} such that $\mathbf{T}(\bar{g}) = \mathbf{a}_{j,i}^{t-1}$. So $\mathbf{T}(\bar{g}) \mid \tau \mathbf{a}_{j,i}^{t-1}$ and we can remove the new term. Hence $\mathbf{T}(W')$ does not change and it remains as in (18).

Lemma 5.7 is proved. \square

Now we know $\mathbf{T}(W^{t-1})$, which are the leading term for the basis of $H = \mathcal{I}(\pi^{-1}(P_1) \cup \dots \cup \pi^{-1}(P_{t-1}) \cup \{Q_1, \dots, Q_{t-1}\})$. We can add the point Q_t and we use our " W, W' " notation. Using Gianni-Kalkbrener's theorem we may prove as usual that $\mathbf{T}(g^*) = \mathbf{a}_{j,i}^{t-1}$. So the leading terms of

$$g^*(s_1 - \bar{s}_1), \dots, g^*(s_N - \bar{s}_N), g^*(\mathbf{a}_{L,1} - \bar{\mathbf{a}}_{L,1}), \dots, g^*(\mathbf{a}_{j,i-1} - \bar{\mathbf{a}}_{j,i-1})$$

are all of the type $\tau \mathbf{a}_{j,i}^{t-1}$, while $g^*(\mathbf{a}_{j,i} - \bar{\mathbf{a}}_{j,i}^{(t)}) = \mathbf{a}_{j,i}^t + \dots$, so its leading term is $\mathbf{a}_{j,i}^t$. The new leading terms are $\{\tau \mathbf{a}_{j,i}^{t-1}, \dots, \tau \mathbf{a}_{j,i}^{t-1}, \mathbf{a}_{j,i}^t\}$. Therefore (Remark 5.3), the structure of W' becomes the same as in (17), because there are no other new terms, since $\{g > g^*\} = \emptyset$.

This concludes the proof of Lemma 5.6. \square

Corollary 5.8. *With the above notation, if $H = \mathcal{I}(\pi^{-1}(P_1) \cup \dots \cup \pi^{-1}(P_\Delta))$, then*

$$\mathbf{T}(W) \cong \{\tau, \dots, \tau, \tau \mathbf{a}_{j,i}, \dots, \tau \mathbf{a}_{j,i}, \dots, \tau \mathbf{a}_{j,i}^{\Delta-1}, \dots, \tau \mathbf{a}_{j,i}^{\Delta-1}, \mathbf{a}_{j,i}^\Delta\} \quad (19)$$

Proof. Apply Lemma 5.6 with $t = \Delta$. \square

5.5 Third part of the proof

Lemma 5.9. *Let $\mathcal{I}(\pi^{-1}(P_1) \cup \dots \cup \pi^{-1}(P_\Delta)) \supset H \supset J$ be a radical zero-dimensional ideal. Suppose that the leading terms of its reduced Gröbner basis satisfy (19). Let $\dot{P}_h \in \Sigma_h^{j,i}$, $1 \leq h \leq \Delta$ and let $H' = \mathcal{I}(\mathcal{V}(H) \cup \pi^{-1}(\dot{P}_h))$. Then $\mathcal{I}(\pi^{-1}(P_1) \cup \dots \cup \pi^{-1}(P_\Delta)) \supset H \supset H' \supset J$ and the leading terms of its reduced Gröbner basis satisfy (19).*

Proof. We use our " W, W' " notation, so that $W = \text{GB}(H)$ and $W' = \text{GB}(H')$. Let us take a point ⁵ $\dot{P}_k = (\bar{s}_1, \dots, \bar{s}_N, \bar{a}_{L,1}, \dots, \bar{a}_{j,i-1}) \in \Sigma_k^{j,i}$ with $1 \leq k \leq \Delta$.

$$\pi^{-1}(\dot{P}_k) = \begin{cases} Q_1 = (\bar{s}_1, \dots, \bar{s}_N, \bar{a}_{L,1}, \dots, \bar{a}_{j,i-1}, \bar{a}_{j,i}^{(1)}) \\ \vdots \\ Q_k = (\bar{s}_1, \dots, \bar{s}_N, \bar{a}_{L,1}, \dots, \bar{a}_{j,i-1}, \bar{a}_{j,i}^{(k)}) \end{cases}$$

* We add the point Q_1 .

We build W' using Theorem 5.1. We know that $g^* \in \widehat{W}$ (as in (a) of Lemma 5.7). So $W' = W_1 \sqcup W_2 \sqcup W_3$ where

- $W_1 \cong \{\tau, \dots, \tau\}$, because $g^* \in \widehat{W}$. So $\mathbf{T}(W_1) \cong \{\tau, \dots, \tau\}$,
- $W_2 = W_{2,1} \cup W_{2,2}$ where

$$W_{2,1} = \{g^*(s_1 - \bar{s}_1), \dots, g^*(s_N - \bar{s}_N), g^*(a_{L,1} - \bar{a}_{L,1}), \dots, g^*(a_{j,i-1} - \bar{a}_{j,i-1}),$$
 so $\mathbf{T}(W_{2,1}) = \{\tau, \dots, \tau\}$.

$$W_{2,2} = \{g^*(a_{j,i} - \bar{a}_{j,i}^{(1)})\}$$
 so $\mathbf{T}(W_{2,2}) = \{\tau a_{j,i}\}$.
- $W_3 = \{g - \frac{g(Q_1)}{g^*(Q_1)} g^* \mid g > g^*\}$ and so the leading terms of W_3 are those in $\mathbf{T}(W)$, except possibly for new τ 's.

Therefore the structure of W' is the same as that of W .

* We add Q_{r+1} with $2 \leq r+1 \leq k$. We assume, using induction on r , that W verifies (19).

Let $W := W'$ and we use again Theorem 5.1.

To construct W' we have to find $g^* \in W$ such that $g^*(Q_{r+1}) \neq 0$. Exactly as in case (d) of Lemma 5.7, $\mathbf{T}(g^*) = \tau a_{j,i}^r$. So $W' = W_1 \sqcup W_2 \sqcup W_3$, where

- $W_1 = \{g \in \widehat{W} \mid g < g^*\}$ where $\mathbf{T}(g^*) = \tau a_{j,i}^r$.
So $\mathbf{T}(W_1) = \{\tau, \dots, \tau, \tau a_{j,i}, \dots, \tau a_{j,i}, \dots, \tau a_{j,i}^{r-1}, \dots, \tau a_{j,i}^{r-1}\}$ or possibly also $\tau a_{j,i}^r \in \mathbf{T}(W_1)$,
- $W_2 = W_{2,1} \cup W_{2,2}$ where

$$W_{2,1} = \{g^*(s_1 - \bar{s}_1), \dots, g^*(s_N - \bar{s}_N), g^*(a_{L,1} - \bar{a}_{L,1}), \dots, g^*(a_{j,i-1} - \bar{a}_{j,i-1})\},$$

$$W_{2,2} = \{g^*(a_{j,i} - \bar{a}_{j,i}^{(r+1)})\}$$
- $W_3 = \{g - \frac{g(Q_{r+1})}{g^*(Q_{r+1})} g^* \mid g > g^*\}$.

Now

- If $r+1 \leq k \leq \Delta - 1$, then the structure of $\mathbf{T}(W')$ does not change. In fact $\mathbf{T}(W') = \mathbf{T}(W_1) \cup \mathbf{T}(W_2) \cup \mathbf{T}(W_3)$, where
 - $\mathbf{T}(W_1) = \{\tau, \dots, \tau, \tau a_{j,i}, \dots, \tau a_{j,i}, \dots, \tau a_{j,i}^{r-1}, \dots, \tau a_{j,i}^{r-1}\}$ or possibly also $\tau a_{j,i}^r \in \mathbf{T}(W_1)$,
 - $\mathbf{T}(W_2) = \mathbf{T}(W_{2,1}) \cup \mathbf{T}(W_{2,2})$ where $\mathbf{T}(W_{2,1}) = \{\tau a_{j,i}^r, \dots, \tau a_{j,i}^r\}$ and $\mathbf{T}(W_{2,2}) = \{\tau a_{j,i}^{r+1}\}$.
 - The leading terms of W_3 are those in $\mathbf{T}(W)$ with degree (in $a_{j,i}$) at least $r+1$, plus possibly some terms in $\mathbf{T}(W)$ of degree r , that is, those greater than $\mathbf{T}(g^*)$.

⁵ With our usual abuse of notation.

- If $r + 1 = \Delta$ then $\mathbf{T}(g^*) \cong \tau \mathbf{a}_{j,i}^{\Delta-1}$, so the leading terms of

$$g^*(s_1 - \bar{s}_1), \dots, g^*(s_N - \bar{s}_N), g^*(\mathbf{a}_{L,1} - \bar{\mathbf{a}}_{L,1}), \dots, g^*(\mathbf{a}_{j,i-1} - \bar{\mathbf{a}}_{j,i-1})$$

remain $\tau \mathbf{a}_{j,i}^{\Delta-1}$, but

$$g^*(\mathbf{a}_{j,i} - \bar{\mathbf{a}}_{j,i}^{(\Delta)}) \cong \tau \mathbf{a}_{j,i}^{\Delta} + \dots$$

Since in W there is a \bar{g} such that $\mathbf{T}(\bar{g}) = \mathbf{a}_{j,i}^{\Delta}$ and $\mathbf{T}(\bar{g}) | \tau \mathbf{a}_{j,i}^{\Delta}$, then (Remark 5.3) the structure of W' does not change and verifies (19).

□

We reiterate Lemma 5.9 starting from $H = \mathcal{I}(\pi^{-1}(P_1) \cup \dots \cup \pi^{-1}(P_{\Delta}))$ and adding all the sets $\pi^{-1}(\dot{P}_h)$ until all points in $\mathcal{V}(J)$ have been added. When we obtain J , we will have that its leading terms satisfy (19), so point (1) and (2) of Proposition 4.13 are proved. In particular, (19) proves also (3) and (4). The proof of Proposition 4.13 is complete.

6 Multi-dimensional general error locator polynomials

The following theorem ensures that our weak multi-dimensional general error locator polynomials (see Definition 3.9) exist for any code.

Theorem 6.1. *Let $C = C^{\perp}(I, L)$ be an affine-variety code with $d \geq 3$. Then*

- i) $J_*^{C,t}$ is a radical strongly multi-stratified ideal w.r.t. the X variables.
- ii) A Gröbner basis of $J_*^{C,t}$ contains a set of weak multi-dimensional general error locator polynomials for C .

Proof. i) We recall that $J_*^{C,t}$ is the ideal in $\mathbb{F}_q[s_1, \dots, s_r, X_t, \dots, X_1, e_1, \dots, e_t]$ as defined in (8). We set $H = J_*^{C,t}$. We want to show that H is a radical strongly multi-stratified ideal with respect to the X variables. The radicality of H is obvious since it contains the field equations for all variables. Let us consider π_j and ρ_i , $1 \leq j \leq t$ as in Definition 4.5

$$\begin{aligned} \pi_t : \mathcal{V}(H_{S,X_t}) &\rightarrow \mathcal{V}(H_S), & \pi_j : \mathcal{V}(H_{S,X_t,\dots,X_j}) &\rightarrow \mathcal{V}(H_{S,X_t,\dots,X_{j+1}}) \\ \rho_j : \mathcal{V}(H_{S,X_t,\dots,X_{j+1},X_j}) &\longrightarrow \mathcal{V}(H_{X_j}), & j = 1, \dots, L. \end{aligned}$$

By Definition 4.5, H is a strongly multi-stratified ideal with respect to the X variables if:

- a0) Let $Z_j = \rho_j(\mathcal{V}(H_{S,X_t,\dots,X_{j+1},X_j}))$, then $Z_j = Z_{\bar{j}}$ for any $1 \leq j \neq \bar{j} \leq t$. In this case we use $Z = Z_j$. Since the locations are only $\mathcal{V}(I) \cup \{P_0\}$, then $Z = \mathcal{V}(I) \cup \{P_0\}$.

- a1) Let $1 \leq j \leq t-1$. For any $T \subset Z$ with $1 \leq |T| \leq j$, there is $\tilde{v} \in \mathcal{V}(H_{S, X_t, \dots, X_{j+1}})$ such that $|\rho_j(\pi_j^{-1}\{\tilde{v}\})| = T$.
- a2) Moreover, for any $T \subset Z$, $1 \leq |T| \leq t$ there is $\bar{s} \in \mathcal{V}(H_S)$ such that $\rho_t(\pi_t^{-1}\{\bar{s}\}) = T$.
- b1) For any $1 \leq j \leq t-1$ and for any $u \in \mathcal{V}(H_{S, X_t, \dots, X_{j+1}, X_j})$ we have that $|\pi_j^{-1}(\{u\})| \leq j$.
- b2) Moreover, for any $\bar{s} \in \mathcal{V}(H_S)$ we have that $|\pi_t^{-1}(\{\bar{s}\})| \leq t$.

Let $\mathbf{s} = (\bar{s}_1, \dots, \bar{s}_r)$ be a correctable syndrome corresponding to an error e of weight $\mu \leq t$. Let Q be a point in $\mathcal{V}(H)$ corresponding to \mathbf{s} . We have

$$Q = (\bar{s}_1, \dots, \bar{s}_r, \bar{A}_t, \dots, \bar{A}_1, \bar{e}_1, \dots, \bar{e}_t).$$

We note that for any permutation $\sigma \in S_t$, there is $\tilde{Q} \in \mathcal{V}(H)$,

$$\tilde{Q} = (\tilde{s}_1, \dots, \tilde{s}_r, \bar{A}_{\sigma(t)}, \dots, \bar{A}_{\sigma(1)}, \bar{e}_{\sigma(1)}, \dots, \bar{e}_{\sigma(t)}). \quad (20)$$

So (20) gives immediately a0).

We want to prove a1) and a2). Let $1 \leq j \leq t-1$ and let $T \subset Z$, $1 \leq |T| \leq j$. Let $k = |T|$. There are two cases to consider: either $P_0 \in T$ or $P_0 \notin T$.

- $P_0 \in T$. Let $Q \in \mathcal{V}(H)$ corresponding to an error with weight $\mu = t - j + k - 1$. Thanks to (20) we can assume that

$$Q = (\bar{s}_1, \dots, \bar{s}_r, \bar{A}_t, \dots, \bar{A}_{j+1}, \bar{A}_j, \dots, \bar{A}_1, \bar{e}_1, \dots, \bar{e}_t)$$

where $\{\bar{A}_t, \dots, \bar{A}_{j+1}\}$ are $t-j$ elements in Z that are different from P_0 , $\{\bar{A}_j, \dots, \bar{A}_1\}$ are $(j-k+1)$ P_0 's and $(k-1)$ is the number of the elements of T different from P_0 . Let $u = (\bar{s}_1, \dots, \bar{s}_r, \bar{A}_t, \dots, \bar{A}_{j+1})$. At this point, we will obviously have $\rho_j(\pi_j^{-1}(u)) = T$.

- $P_0 \notin T$. Let $Q \in \mathcal{V}(H)$ corresponding to an error with weight $\mu = t - j + k - 1$. Similarly to the previous case, thanks to (20), we can assume that $Q = (\bar{s}_1, \dots, \bar{s}_r, \bar{A}_t, \dots, \bar{A}_{j+1}, \bar{A}_j, \dots, \bar{A}_1, \bar{e}_1, \dots, \bar{e}_t)$, where $\{\bar{A}_t, \dots, \bar{A}_{j+1}\}$ are $(t-j)$ elements of $\mathcal{V}(I) = Z \setminus \{P_0\}$, $\{\bar{A}_j, \dots, \bar{A}_1\}$ contains $(j-k)$ points equal to P_0 and k points forming T . Let $u = (\bar{s}_1, \dots, \bar{s}_r, \bar{A}_t, \dots, \bar{A}_{j+1})$, then we have $\rho_j(\pi_j^{-1}(u)) = T$.

The proof of a2) is similar and is omitted.

To prove b1) and b2) it is enough to observe that if $t-j$ locations (including possibly the ghost point) are fixed, then at most j distinct locations can exist for that error.

- ii) Since H is strongly multi-stratified, H is weakly stratified (for Proposition 4.11), and so we can apply Proposition 4.13. As weak locators, we take $\mathcal{P}_i = g_{t, \zeta(t, i), 1}^{(i)}$, where $\zeta(t, i) = \eta(t, i) \leq t$ and $\mathbf{T}(\mathcal{P}_i) = x_i^{t_i}$. In fact, the number of possible extensions is bounded by both t_i and

$|\{\hat{\pi}_i(P) \mid P \in \mathcal{V}(I) \cup P_0\}|$. The first condition of Def. 3.9 is satisfied.

In order to prove the second condition we note that $\mathcal{P}_i(\mathbf{s}, \bar{x}_1, \dots, \bar{x}_{i-1}, x_i)$ has among its solutions the \bar{x}_i 's such that $(\bar{x}_1, \dots, \bar{x}_i)$ are the first i components of an error location corresponding to \mathbf{s} (or $P_{0,i}$ value).

□

We can summarize our findings so far.

Using weak locators does not work because $\mathcal{P}_i(S, x_1, \dots, x_i)$ depends also on $i - 1$ x -variables. Thus, the point $(S, x_1, \dots, x_{i-1}) \in \mathcal{V}(I)$ has the right multiplicity if and only if $t_i = 1$. If this fail, it is very likely to have parasite solutions.

On the other hand, if we use the general error evaluator polynomial \mathcal{E} , we can proceed in two ways (see Example 3.12), but both require an additional choice to discover parasite solutions. With non-trivial codes, this choice is very computationally expensive.

The strategy we propose here is to force point $(S, x_1, \dots, x_{i-1}) \in \mathcal{V}(I)$ to have the right multiplicity. See Definition 3.9 for the t_i 's.

Definition 6.2. Let $C = C^\perp(I, L)$ be an affine-variety code.

Let $P_0 = (\bar{x}_{0,1}, \dots, \bar{x}_{0,m}) \in (\mathbb{F}_q)^m \setminus \mathcal{V}(I)$ be a ghost point. For any $1 \leq i \leq m$, let \mathcal{L}_i be a polynomial in $\mathbb{F}_q[S, x_1, \dots, x_i]$, where $S = \{s_1, \dots, s_r\}$. Then $\{\mathcal{L}_i\}_{1 \leq i \leq m}$ is a set of **multi-dimensional general error locator polynomials** for C if for any i

- $\mathcal{L}_i(S, x_1, \dots, x_i) = x_i^{t_i} + a_{t_i-1}x_i^{t_i-1} + \dots + a_0$, $a_j \in \mathbb{F}_q[S, x_1, \dots, x_{i-1}]$ for $0 \leq j \leq t_i - 1$. In other words, \mathcal{L}_i is a monic polynomial with degree t_i with respect to the variable x_i and its coefficients are in $\mathbb{F}_q[S, x_1, \dots, x_{i-1}]$.
- Given a syndrome $\bar{\mathbf{s}} = (\bar{s}_1, \dots, \bar{s}_r) \in (\mathbb{F}_q)^r$, corresponding to an error vector of weight $\mu \leq t$ and μ error locations $(\bar{x}_{1,1}, \dots, \bar{x}_{1,m}), \dots, (\bar{x}_{\mu,1}, \dots, \bar{x}_{\mu,m})$. If we evaluate the S variables at $\bar{\mathbf{s}}$ and the variables (x_1, \dots, x_{i-1}) at the truncated locations $\bar{\mathbf{x}}^j = (\bar{x}_{j,1}, \dots, \bar{x}_{j,i-1})$ for any $0 \leq j \leq \mu$, then the roots of $\mathcal{L}_i(\bar{\mathbf{s}}, \bar{\mathbf{x}}^j, x_i)$ are $\{\bar{x}_{h,i} \mid \bar{\mathbf{x}}^h = \bar{\mathbf{x}}^j, 1 \leq h \leq \mu\}$ when $\mu = t$, and $\{\bar{x}_{h,i} \mid \bar{\mathbf{x}}^h = \bar{\mathbf{x}}^j, 0 \leq h \leq \mu\}$ when $\mu \leq t - 1$. That is, the polynomial $\mathcal{L}_i(\bar{\mathbf{s}}, \bar{\mathbf{x}}^j, x_i)$ does not have parasite solutions.

Note that the number of distinct first components of error locations could be lower than μ and t_i .

To show how multi-dimensional general error locator polynomials can be applied, we redo the example on p. 155 of [FL98]. We postpone for the moment the problem of the existence of these polynomials and of the method to compute them.

Example 6.3. Let us consider the Hermitian code $C = C^\perp(I, L)$ from the curve $y^2 + y = x^3$ over \mathbb{F}_4 and with defining monomials $\{1, x, y, x^2, xy\}$, as in the Example 3.7. Let us consider the lex term-ordering with $s_1 < s_2 < s_3 < s_4 <$

$s_5 < x_2 < y_2 < x_1 < y_1 < e_2 < e_1$ in $\mathbb{F}_4[s_1, s_2, s_3, s_4, s_5, x_2, y_2, x_1, y_1, e_1, e_2]$.

We consider the ideal $J_*^{C,t}$. In this ideal we are lucky enough to find the two multi-dimensional general error locator polynomials that are $\mathcal{L}_{2,1}(s_1, \dots, s_5, x_2)$ and $\mathcal{L}_{2,2}(s_1, \dots, s_5, x_2, y_2)$, which are respectively the polynomials \mathcal{L}_x and \mathcal{L}_{xy} of degree two in x_2 and y_2 . In this case $t_1 = t_2 = t = 2$ (a_x, b_x, a_y, b_y, c_y are in the Appendix).

$$\mathcal{L}_x = \mathbf{x}^2 + \mathbf{x} a_x + b_x \text{ and } \mathcal{L}_{xy} = \mathbf{y}^2 + \mathbf{y} a_y + \mathbf{x} b_y + c_y$$

Also in this example, we consider the three cases of Example 3.7.

- We suppose that two errors occurred in the points $P_6 = (\alpha, \alpha + 1)$ and $P_7 = (\alpha + 1, \alpha)$, so the syndrome vector corresponding to $(0, 0, 0, 0, 0, 1, 1, 0)$ is $\mathbf{s} = (0, 1, 1, 1, 0)$. In order to find the error positions we evaluate \mathcal{L}_x in $\bar{\mathbf{s}}$ and we obtain the correct values of x , in fact $\mathcal{L}_x(\bar{\mathbf{s}}, x) = x^2 + x + 1 = (x - \alpha)(x - (\alpha + 1))$. Now we have to evaluate \mathcal{L}_y in $(\bar{\mathbf{s}}, \alpha)$ and in $(\bar{\mathbf{s}}, \alpha + 1)$. Also in this case we obtain the correct solutions (with the highest possible multiplicity)

$$\begin{aligned} \mathcal{L}_{xy}(\bar{\mathbf{s}}, \alpha, y) &= y^2 + \alpha = (y - (\alpha + 1))^2 \\ \mathcal{L}_{xy}(\bar{\mathbf{s}}, \alpha + 1, y) &= y^2 + \alpha + 1 = (y - \alpha)^2. \end{aligned}$$

- We consider the syndrome $(\alpha + 1, 0, \alpha, 0, 0)$, corresponding to the error vector $(1, \alpha, 0, 0, 0, 0, 0, 0)$, we obtain

$$\mathcal{L}_x(\bar{\mathbf{s}}, x) = x^2 \text{ and } \mathcal{L}_{xy}(\bar{\mathbf{s}}, 0, y) = y^2 + y = y(y - 1).$$

The solutions of the above system are $(0, 0), (0, 1)$. Also in this case the solutions of the equation $\mathcal{L}_x(\bar{\mathbf{s}}) = 0$ are correct.

- Again, when there is only one error of value $\alpha + 1$ in the third point, we have the correct answers, in fact

$$\begin{aligned} \mathcal{L}_x(\alpha + 1, \alpha + 1, 1, \alpha + 1, 1, x) &= x^2 + 1 = (x + 1)^2 \\ \mathcal{L}_{xy}(\alpha + 1, \alpha + 1, 1, \alpha + 1, 1, 1, y) &= y^2 + (\alpha + 1)y + \alpha = (y - 1)(y - \alpha). \end{aligned}$$

so the solutions are $(1, 1)$, which is the ghost point, and $(1, \alpha)$ i.e. the coordinates of the right location.

The main difference between $\mathcal{L}_x, \mathcal{L}_{xy}$ of Example 3.7 and $\mathcal{L}_x, \mathcal{L}_{xy}$ of this example is that now we do not have spurious solutions, that is, now the roots of our locators are exactly the error locations and no more ambiguity exists.

As evident from the previous example, multidimensional general error locator polynomials are very convenient for decoding. However, to prove their existence we cannot use the theoretical methods developed so far, because these methods do not deal with multiplicities. In the next section we will develop more advanced theoretical methods, that will permit to construct ideals where these polynomials lie and can be easily spotted.

7 Stuffed ideals

Let G be a reduced Gröbner basis of a radical weakly stratified ideal J as in Proposition 4.13. From now on we consider the ordering as in Proposition 4.13. In this section we fix $1 \leq i \leq m$ and $1 \leq j \leq L$ and we consider the projection

$$\pi : \mathcal{V}(J_{\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}, \mathbf{a}_{j,i}}) \rightarrow \mathcal{V}(J_{\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}})$$

We consider the variable $\mathbf{a}_{j,i}$ in block A_j .

Let $\mathcal{R} = \mathbb{K}[\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}]$. Let g be a polynomial in $G^{j,i}$ such that the degree in $\mathbf{a}_{j,i}$ of g is $\Delta = \zeta(j, i) = \eta(j, i)$. By Proposition 4.13, we know that this polynomial exists and it can be assumed to be monic in $\mathcal{R}[\mathbf{a}_{j,i}]$. Let $P_h \in \Sigma_h^{j,i}$ where $1 \leq h \leq \Delta - 1$, then

$$g(P_h, \mathbf{a}_{j,i}) = \mathbf{a}_{j,i}^\Delta + \alpha_{\Delta-1} \mathbf{a}_{j,i}^{\Delta-1} + \dots + \alpha_0 \in \mathbb{K}[\mathbf{a}_{j,i}] \text{ where } \alpha_i \in \mathbb{K}.$$

We are interested in solutions of the equation

$$g(P_h, \mathbf{a}_{j,i}) = 0. \tag{21}$$

Since $P_h \in \Sigma_h^{j,i}$, there exist distinct Q_1, \dots, Q_h such that $\pi^{-1}(P_h) = \{Q_1, \dots, Q_h\}$, with $Q_l = (P_h, \lambda_l)$ for any $1 \leq l \leq h$. So $\lambda_1, \dots, \lambda_h$ are some solutions of (21). But there exist other $\Delta - h$ solutions (counting multiplicities) of (21), say $\lambda_{h+1}, \dots, \lambda_\Delta$. There are two cases:

- (a) It may be that $\lambda_{h+l} \notin \{\lambda_1, \dots, \lambda_h\}$ for some l . In this case, point (P_h, λ_{h+l}) is not an extension of P_h , because $(P_h, \lambda_{h+l}) \notin \mathcal{V}(J_{\mathcal{S}, \mathcal{A}_L, \dots, \mathcal{A}_{j+1}, \mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i-1}, \mathbf{a}_{j,i}})$, and so λ_{h+l} is a parasite solution.
- (b) But it may also be that $\{\lambda_{h+1}, \dots, \lambda_\Delta\} \subset \{\lambda_1, \dots, \lambda_h\}$, depending on the multiplicities of the $\{\lambda_1, \dots, \lambda_h\}$. In this case, if we solve (21), we have exactly the extensions and we are not confused by parasite solutions.

We want to change slightly our variety in order to force case (b). To do that, we need that the sum of multiplicities of $\{\lambda_l\}_{1 \leq l \leq h}$ is equal to Δ . To increase the multiplicity of any λ_l , we can use the Hasse derivative (see Sec. 2.3 and in particular Theorem 2.8).

Definition 7.1. Let $K \subset \mathcal{R}[\mathbf{a}_{j,i}]$ be a zero-dimensional ideal such that $\mathcal{V}(K) \subset \mathbb{A}_{j,i}$. Let $\Delta = \eta(j, i)$. Let $G = \text{GB}(K)$ and $g = g_\Delta^{(i)}$. We say that K is **stuffed**

if for any $1 \leq h \leq \Delta - 1$ and for any $P_h \in \Sigma_h^{j,i}$, the equation (21) has h distinct solutions in \mathbb{K} .

Definition 7.2. Let $H \subset \mathbb{K}[V_1, \dots, V_N]$ be a zero-dimensional ideal. Let $n \geq 1$. Let $f \in H$ and $Q \in \mathcal{V}(H)$ where $Q = (P, \bar{V}_N)$. Let $\vartheta_1 : H \rightarrow \mathbb{K}$ such that

$$\vartheta_1(f) = \varphi^{(1)}(f(P, V_N)) \Big|_{V_N = \bar{V}_N}$$

and let $H^{[Q,1]} = \ker \vartheta_1$. We define inductively $\vartheta_n : H^{[Q,n-1]} \rightarrow \mathbb{K}$ such that

$$\vartheta_n(f) = \varphi^{(n)}(f(P, V_N)) \Big|_{V_N = \bar{V}_N}$$

and we write $H^{[Q,n]} = \ker \vartheta_n$.

We note that $H^{[Q,1]}$ is an ideal. In fact, if $f \in H^{[Q,1]}$, $g \in \mathbb{K}[V_N]$ and $\bar{V}_N \in \mathcal{V}(f)$ then we claim that

$$\varphi^{(1)}(fg)(\bar{V}_N) = \varphi^{(1)}(f)(\bar{V}_N)g(\bar{V}_N) + \varphi^{(1)}(g)(\bar{V}_N)f(\bar{V}_N) = 0.$$

In fact, $\varphi^{(1)}(f)(\bar{V}_N) = 0$, since $f \in \ker \vartheta_1$ and $f(\bar{V}_N) = 0$, since $\bar{V}_N \in \mathcal{V}(f)$. Inductively, we can similarly prove that $H^{[Q,n]}$ is an ideal.

Let us consider a zero-dimensional ideal $K \subset \mathcal{R}[\mathbf{a}_{j,i}]$. It is convenient to call our variables also as $\{V_1, \dots, V_N\} = \mathcal{S} \cup \mathcal{A}_L \cup \dots \cup \mathcal{A}_{j+1} \cup \{\mathbf{a}_{j,1}, \dots, \mathbf{a}_{j,i}\}$, in such a way that $V_1 < \dots < V_N$ and $V_N = \mathbf{a}_{j,i}$.

We suppose that $G = \text{GB}(K)$ satisfies (19), that is

$$\mathbf{T}(G) \cong \{\tau, \dots, \tau, \tau V_N, \dots, \tau V_N, \dots, \tau V_N^{\Delta-1}, \dots, \tau V_N^{\Delta-1}, V_N^{\Delta}\}$$

and τ is any elements in $V_1 < \dots < V_{N-1}$. In particular there is a polynomial $g \in G^{j,i}$ s.t. $\mathbf{T}(g) = \mathbf{a}_{j,i}^{\Delta} = V_N^{\Delta}$.

For each $1 \leq h \leq \Delta - 1$ we perform the following operations:

- (a) If for any $P_h \in \Sigma_h^{j,i}$ equation (21) has h distinct solutions in \mathbb{K} , we do nothing. Otherwise, we take any $P_h \in \Sigma_h^{j,i}$ such that (21) has more than h solutions.
- (b) We consider $Q = (P_h, \bar{V}_N)$ which is any extension of P_h . We want to compute $H^{[Q,\Delta-h]}$. In order to do that, we iteratively compute $\ker \vartheta_n$ (see Definition 7.2) from $n = 1$ to $n = \Delta - h$.
- (c) For any such n , we apply Theorem 5.1 to $H = H^{[Q,n-1]}$ and $H' = H^{[Q,n]}$, so that $H' = \ker \vartheta_n$. The hypotheses of Theorem 5.1 are trivially satisfied, because ϑ_n is \mathbb{K} -linear and $\ker \vartheta_n$ is an ideal. In the subsequent step (d), we get ready to apply Theorem 5.1.
- (d) We consider the point $Q = (\bar{V}_1, \dots, \bar{V}_N) = (P, \bar{\mathbf{a}}_{j,i})$ with $P = (\bar{V}_1, \dots, \bar{V}_{N-1}) = (\bar{s}_1, \dots, \bar{s}_N, \bar{\mathbf{a}}_{L,1}, \dots, \bar{\mathbf{a}}_{j,i-1})$, Q is a solution of ideal H . To apply Theo-

rem 5.1, we consider the smallest polynomial $g^* \in W$, with $W = \text{GB}(H)$, such that

$$\vartheta_n(g^*) \neq 0, \text{ that is, } \varphi^{(n)}(g^*((P, V_{\mathcal{N}})))|_{V_{\mathcal{N}}=\bar{V}_{\mathcal{N}}} \neq 0.$$

We compute W' from W . To apply Theorem 5.1, we need to identify β_k 's such that

$$\vartheta_n((V_k - \beta_k)g^*) = 0 \text{ where } 1 \leq k \leq \mathcal{N},$$

where we consider the weaker form (iii) in Remark 5.2.

We solve previous equation as follows

$$\vartheta_n((V_k - \beta_k)g^*) = (\bar{V}_k - \beta_k)\vartheta_n(g^*) = 0 \quad (1 \leq k \leq \mathcal{N} - 1)$$

$$\implies \beta_k \text{ is the } k\text{-th component of } Q.$$

$$\vartheta_n((V_{\mathcal{N}} - \beta_{\mathcal{N}})g^*) = g^*(Q) + \bar{V}_{\mathcal{N}}\vartheta_n(g^*) - \beta_{\mathcal{N}}\vartheta_n(g^*) = 0$$

$$\implies \beta_{\mathcal{N}} = \frac{g^*(Q) + \bar{V}_{\mathcal{N}}\vartheta_n(g^*)}{\vartheta_n(g^*)} = \frac{g^*(Q)}{\vartheta_n(g^*)} + \bar{V}_{\mathcal{N}}.$$

Lemma 7.3. *We claim that*

$$g^* \in G_r^{j,i}, \text{ i.e. } \mathbf{T}(g^*) \cong \tau \mathbf{a}_{j,i}^r \text{ where } n-1 \leq r \leq \Delta-1.$$

Proof. Recall that we use \cong to express a unconventional identification (see Remark 5.4). If $\mathbf{T}(g^*) \cong \tau \mathbf{a}_{j,i}^r$ with $r < n-1$, then $\vartheta_n(g^*) = 0$. So $\mathbf{T}(g^*) \cong \tau \mathbf{a}_{j,i}^r$ with $r \geq n-1$. However, $r \neq \Delta$, otherwise we have already finished. So $n-1 \leq r \leq \Delta-1$. \square

(e) We build W' using Theorem 5.1. By Lemma 7.3 we have that

$$\mathbf{T}(g^*) \cong \tau \mathbf{a}_{j,i}^r \quad n-1 \leq r \leq \Delta-1.$$

So $W' = W_1 \cup W_2 \cup W_3$ where

$$- W_1 = \{g \mid g < g^*\}.$$

So $\mathbf{T}(W_1) \cong \{\tau, \dots, \tau \mathbf{a}_{j,i}, \dots, \tau \mathbf{a}_{j,i}^{r-1}\}$, or possibly also $\tau \mathbf{a}_{j,i}^r \in \mathbf{T}(W_1)$.

$$- W_2 = W_{2,1} \cup W_{2,2} \text{ where } W_{2,1} = \{g^*(s_1 - \bar{s}_1), \dots, g^*(\mathbf{a}_{j,i-1} - \bar{\mathbf{a}}_{j,i-1}),$$

$$\text{so } \mathbf{T}(W_{2,1}) = \{\tau \mathbf{a}_{j,i}^r, \dots, \tau \mathbf{a}_{j,i}^r\}.$$

$$W_{2,2} = \{g^*(\mathbf{a}_{j,i} - \frac{g^*}{\vartheta_r(g^*)} - \bar{\mathbf{a}}_{j,i})\}. \text{ Then } \mathbf{T}(W_{2,2}) = \{\tau \mathbf{a}_{j,i}^{r+1}\}.$$

$$- W_3 = \{g - \frac{\vartheta_r(g)}{\vartheta_r(g^*)} g^* \mid g > g^*\} \text{ and hence the leading terms of } W_3 \text{ are those in } \mathbf{T}(W), \text{ except for } \tau, \dots, \tau \mathbf{a}_{j,i}^{r-1} \text{ and possibly } \tau \mathbf{a}_{j,i}^r.$$

Therefore, the structure of W' is the same as that of W , except possibly if $r+1 = \Delta$. In that case

$$- \mathbf{T}(W_1) \cong \{\tau, \dots, \tau \mathbf{a}_{j,i}, \dots, \tau \mathbf{a}_{j,i}, \tau \mathbf{a}_{j,i}^{\Delta-2}, \dots, \tau \mathbf{a}_{j,i}^{\Delta-2}\}, \text{ or possibly also } \tau \mathbf{a}_{j,i}^{\Delta-1} \in \mathbf{T}(W_1),$$

$$- \mathbf{T}(W_2) = \mathbf{T}(W_{2,1}) \cup \mathbf{T}(W_{2,2}) \text{ where } \mathbf{T}(W_{2,1}) = \{\tau \mathbf{a}^{\Delta-1}, \dots, \tau \mathbf{a}^{\Delta-1}\}$$

- and $\mathbf{T}(W_{2,2}) = \{\tau \mathbf{a}_{j,i}^\Delta\}$.
 - $\mathbf{T}(W_3) = \emptyset$.

In the Gröbner basis W in (19) there exists a polynomial \bar{g} such that $\mathbf{T}(\bar{g}) = \mathbf{a}_{j,i}^\Delta$. So $\mathbf{T}(\bar{g})|\tau \mathbf{a}_{j,i}^\Delta$ and we can remove the new term. Thus $\mathbf{T}(W')$ does not change and it remains as in (19).

- (f) Once all the above operations have been concluded, for any $1 \leq h \leq \Delta - 1$ and for any $P_h \in \Sigma_h^{j,i}$, (21) will have exactly h distinct solutions and the resulting ideal will be stuffed.

We have thus proved the following theorem:

Theorem 7.4. *Let $K \subset \mathcal{R}[\mathbf{a}_{j,i}]$ be a zero-dimensional ideal such that $G = \text{GB}(K)$ verifies (19). Let g be the polynomial in G such that $\mathbf{T}(g) = \mathbf{a}_{j,i}^\Delta$, with $\Delta = \eta(j, i)$. We can obtain an ideal $\tilde{K} \subset \mathcal{R}[\mathbf{a}_{j,i}]$ such that*

1. \tilde{K} is stuffed.
2. $\text{GB}(\tilde{K})$ verifies (19).
3. $\mathcal{V}(\tilde{K}) = \mathcal{V}(K)$.

Although, in Theorem 7.4 we obtain \tilde{K} as in the procedure above, there are other ways to obtain \tilde{K} , for example by simultaneously increasing the multiplicity of more λ_h 's.

Note that, generally speaking, \tilde{K} will lose the radicality, but its Gröbner basis will retain (19), which is what we need.

Theorem 7.5. *If K and \tilde{K} are as in Theorem 7.4, then if K is, respectively, strongly multi-stratified, multi-stratified and weakly stratified, then \tilde{K} is so.*

Proof. The stuffing procedure does not change the number of pre-images at any level. \square

Now, we are finally able to prove the existence of our multi-dimensional general error locator polynomials for any code. Note that this is another constructive proof, since it tells us how to compute our polynomials, that is, simply by computing a suitable Gröbner bases of the corresponding stuffed ideal.

Theorem 7.6. *Let $C = C^\perp(I, L)$ be an affine-variety code with $d \geq 3$. Let $\tilde{J}_*^{C,t}$ be a stuffed ideal of $J_*^{C,t}$. Then*

- i) $\tilde{J}_*^{C,t}$ is a strongly multi-stratified ideal with respect to the X variables.
- ii) A Gröbner basis of $\tilde{J}_*^{C,t}$ contains a set of multi-dimensional general error locator polynomials for C .

Proof. i) We can use Th.7.5 and so $\tilde{J}_*^{C,t}$ is a strongly multi-stratified ideal.

ii) As locators we can take for any i

$$\mathcal{L}_i = g_{t, \zeta(t, i), 1}^{(i)},$$

where $\zeta(t, i) = \eta(t, i)$ and $\mathbf{T}(\mathcal{L}_i) = x_i^{t_i}$, $t_i = \zeta(t, i) = \eta(t, i)$, thanks to Theorem 7.4. So the first condition of Definition 6.2 is satisfied.

Let $H = J_*^{C, t}$ and $\tilde{H} = \tilde{J}_*^{C, t}$. In order to prove the second condition we note that, since \mathcal{L}_i is a polynomial of $\tilde{H}_{S, x_{t, 1}, \dots, x_{t, i}}$ it will vanish at $(\mathbf{s}, \bar{x}_1, \dots, \bar{x}_i)$, where $\mathbf{s} = (\bar{s}_1, \dots, \bar{s}_r)$ and $(\mathbf{s}, \bar{x}_1, \dots, \bar{x}_i)$ can be extended to a point in $\mathcal{V}(H) = \mathcal{V}(\tilde{H})$. Since \tilde{H} is stuffed, $\mathcal{L}_i(\mathbf{s}, \bar{x}_1, \dots, \bar{x}_{i-1}, x_i)$ has as solutions only the \bar{x}_i 's such that $(\bar{x}_1, \dots, \bar{x}_i)$ are the first i components of an error location corresponding to \mathbf{s} (or $P_{0, i}$).

□

8 Families of affine-variety codes

In this section we consider some families of affine-variety codes.

8.1 SDG curves

We discuss codes from some curves introduced in [SDG06].

Definition 8.1 ([SDG06]). *Let \mathbb{F}_s be a subfield of \mathbb{F}_q . A polynomial f in $\mathbb{F}_q[x]$ is called an $(\mathbb{F}_q, \mathbb{F}_s)$ -polynomial if for each $\gamma \in \mathbb{F}_q$ we have $f(\gamma) \in \mathbb{F}_s$.*

Proposition 8.2 ([SDG06]).

1. *The polynomial $f(x) = b_3x^3 + b_2x^2 + b_1x + b_0 \in \mathbb{F}_4[x]$ is an $(\mathbb{F}_4, \mathbb{F}_2)$ -polynomial if and only if $b_0, b_3 \in \mathbb{F}_2$ and $b_2 = b_1^2$.*
2. *The polynomial $g(x) = b_7x^7 + \dots + b_1x + b_0 \in \mathbb{F}_8[x]$ is an $(\mathbb{F}_8, \mathbb{F}_2)$ -polynomial if and only if $b_0, b_7 \in \mathbb{F}_2$, $b_2 = b_1^2$, $b_4 = b_2^2$, $b_6 = b_3^2$ and $b_3 = b_5^2$.*

Let $\mathcal{F} = \{f(x) + g(y) \mid f, g \text{ are } (\mathbb{F}_8, \mathbb{F}_2)\text{-polynomials, } \deg(f) = 4, \deg(g) = 6\}$. In [SDG06] it is shown that the family \mathcal{F} has 784 members and that each member of this family has 32 roots in $(\mathbb{F}_8)^2$. Let us consider the polynomial $\mathcal{G} = f(x) + g(y)$, with $f(x) = x^4 + x^2 + x$ and $g(y) = y^6 + y^5 + y^3 + 1$, so that $\mathcal{G} \in \mathcal{F}$. Let $I = \langle \mathcal{G} \rangle$ and $J_*^{C, t}$ be the ideal associated to the $C = C^\perp(I, L)$ code over \mathbb{F}_8 that can correct up to $t = 1$ errors and with defining monomials $L = \{1, y, x, y^2\}$.

Ideal $J_*^{C, t}$ is generated by:

$$\{x_1^8 - x_1, y_1^8 - y_1, e_1^7 - 1, x_1^4 + x_1^2 + x_1 + y_1^6 + y_1^5 + y_1^3 + 1, e_1 - s_1, e_1 y_1 - s_2, e_1 x_1 - s_3, e_1 y_1^2 - s_4\}$$

and the reduced Gröbner basis G with respect to the lex ordering with

$s_1 < s_2 < s_3 < s_4 < x_1 < y_1 < e_1$ is

$$\{s_1^7 + 1, s_2^8 + s_2, s_3^4 + s_3^2 s_1^2 + s_3 s_1^3 + s_2^6 s_1^5 + s_2^5 s_1^6 + s_2^3 s_1 + s_1^4, s_4 + s_2^2 s_1^6, \\ \mathbf{x}_1 + s_3 s_1^6, \mathbf{y}_1 + s_2 s_1^6, e_1 + s_1\}$$

and then

$$\mathcal{L}_2 = \mathbf{y}_1 + s_3 s_1^6, \quad \mathcal{L}_1 = \mathbf{x}_1 + s_2 s_1^6.$$

8.2 SDG surfaces I

We discuss codes from some surfaces introduced in [SDG06].

Let $\mathcal{F} = \{f(x) + g(y) + h(z) \mid f, g, h \text{ are } (\mathbb{F}_4, \mathbb{F}_2)\text{-polynomials, } \deg(f) = \deg(h) = 3, \deg(g) = 2\}$. In [SDG06] it is shown that the family \mathcal{F} has 96 members and that each member of this family has 32 roots in $(\mathbb{F}_4)^3$. Let us consider the polynomial $\mathcal{G} = f(x) + g(y) + h(z)$, with $f(x) = x^3$, $g(y) = y^2 + y + 1$ and $h(z) = z^3 + 1$, so that $\mathcal{G} \in \mathcal{F}$. Let $I = \langle \mathcal{G} \rangle$ and $J_*^{C,t}$ be the ideal associated to the code $C = C^\perp(I, L)$ over \mathbb{F}_4 that can correct up to $t = 1$ error and with defining monomials $L = \{1, x, z, y\}$.

The ideal $J_*^{C,t} \subset \mathbb{F}_4[s_1, s_2, s_3, s_4, x_1, y_1, z_1, e_1]$ is generated by $\{x_1^4 - x_1, y_1^4 - y_1, z_1^4 - z_1, e_1^3 - 1, g + f + h, e_1 - s_1, e_1 z_1 - s_3, e_1 x_1 - s_2, e_1 y_1 - s_4\}$ and the reduced Gröbner basis G with respect to the lex ordering with $s_1 < s_2 < s_3 < s_4 < x_1 < y_1 < z_1 < e_1$ is

$$\{s_1^3 + 1, s_2^4 + s_2, s_3^4 + s_3, s_4^2 + s_4 s_1 + s_3^3 s_1^2 + s_2^3 s_1^2, \mathbf{y}_1 + s_4 s_1^2, \mathbf{x}_1 + s_2 s_1^2, \mathbf{z}_1 + s_3 s_1^2, e_1 + s_1\},$$

then

$$\mathcal{L}_1 = \mathbf{x}_1 + s_2 s_1^2, \quad \mathcal{L}_2 = \mathbf{y}_1 + s_4 s_1^2, \quad \mathcal{L}_3 = \mathbf{z}_1 + s_3 s_1^2.$$

8.3 SDG surfaces II

We discuss codes from another family of surfaces introduced in [SDG06].

Let $\mathcal{F} = \{\beta x^2 z + \beta^2 x z^2 + f(x) + g(y) + h(z) \mid \beta \neq 0, f, g, h \text{ are } (\mathbb{F}_4, \mathbb{F}_2)\text{-polynomials, } \deg(f) \leq 2, \deg(h) \leq 3, \deg(g) = 2\}$. In [SDG06] it is shown that the family \mathcal{F} has 576 members and that each member of this family has 32 roots in $(\mathbb{F}_4)^3$. Let us consider the polynomial $\mathcal{G} = x^2 z + x z^2 + f(x) + g(y) + h(z)$, with $\beta = 1$, $f(x) = 1$, $g(y) = y^2 + y + 1$ and $h(z) = z^3 + 1$, so that $\mathcal{G} \in \mathcal{F}$. Let $I = \langle \mathcal{G} \rangle$ and $J_*^{C,t}$ be the ideal associated to the code $C = C^\perp(I, L)$ over \mathbb{F}_4 that can correct one error and with defining monomials $L = \{1, z, z^2, z^3, x, y\}$. The ideal $J_*^{C,t} \subset \mathbb{F}_4[s_1, s_2, s_3, s_4, s_5, s_6, x_1, y_1, z_1, e_1]$ is generated by $\{x_1^4 - x_1, y_1^4 - y_1, z_1^4 - z_1, e_1^3 - 1, x_1^2 z_1 + x_1 z_1^2 + f + g + h, e_1 - s_1, e_1 z_1 - s_2, e_1 z_1^2 - s_3, e_1 z_1^3 - s_4, e_1 x_1 - s_5, e_1 y_1 - s_6\}$ and the reduced Gröbner basis G with respect to the lex ordering with $s_1 < s_2 < s_3 < s_4 < s_5 < s_6 < x_1 < y_1 < z_1 < e_1$ is

$$\{s_1^3 + 1, s_2^4 + s_2, s_3 + s_2^2 s_1^2, s_4 + s_2^3 s_1, s_5^4 + s_5, s_6^2 + s_6 s_1 + s_5^2 s_2 s_1^2 + s_5 s_2^2 s_1^2 + s_2^3 s_1^2 + s_1^2, \\ \mathbf{x}_1 + s_5 s_1^2, \mathbf{y}_1 + s_6 s_1^2, \mathbf{z}_1 + s_2 s_1^2, e_1 + s_1\}$$

and then

$$\mathcal{L}_1 = \mathbf{x}_1 + s_5 s_1^2, \quad \mathcal{L}_2 = \mathbf{y}_1 + s_6 s_1^2, \quad \mathcal{L}_3 = \mathbf{z}_1 + s_2 s_1^2.$$

8.4 Norm-trace curves

We now give an example for codes coming from a family of curves ([Gei03]), which are a natural generalization of Hermitian curves.

Let $C = C^\perp(I, L)$ be the code from the norm-trace curve $x^7 = y^4 + y^2 + y$ over \mathbb{F}_8 and with defining monomials $\{1, x, x^2, y\}$. This code ([Gei03]) can correct $t = 1$ error. Let $J_*^{C,t}$ be the ideal generated by:

$$\{x_1^8 - x_1, y_1^8 - y_1, e_1^7 - 1, e_1 - s_1, e_1 x_1 - s_2, e_1 x_1^2 - s_3, e_1 y_1 - s_4, x_1^7 - y_1^4 - y_1^2 - y_1\}$$

and the reduced Gröbner basis G with respect to the lex ordering with $s_1 < s_2 < s_3 < s_4 < x_1 < y_1 < e_1$ is

$$\{s_1^7 + 1, s_2^8 + s_2, s_3 + s_2^2 s_1^6, s_4^4 + s_4^2 s_1^2 + s_4 s_1^3 + s_2^7 s_1^4, \mathbf{x}_1 + s_2 s_1^6, \mathbf{y}_1 + s_4 s_1^6, e_1 + s_1\}.$$

Then

$$\mathcal{L}_1 = \mathbf{x}_1 + s_2 s_1^6, \quad \mathcal{L}_2 = \mathbf{y}_1 + s_4 s_1^6.$$

Observe that in all our examples so far no stuffing was required, because we were considering the case $t = 1$, which clearly cannot contain multiplicities.

8.5 Hermitian curves

Let q be a power of a prime, then the Hermitian curve \mathcal{H} over \mathbb{F}_{q^2} is defined by the affine equation $\mathcal{G} : x^{q+1} = y^q + y$. Each member of this family has $n = q^3$ points in \mathbb{F}_{q^2} and it is well-known that the function space is generated by monomials.

In Example 3.7 we considered the case $q = 2$ and $t = 2$, we now consider the code C corresponding to the case $q = 3$ and $t = 2$. The defining monomials are $L = \{1, x, y, x^2, xy, y^2, x^3\}$. As before, we choose as ghost point $(1, 1)$.

Our ideal $J_*^{C,2}$ is generated by

$$\begin{aligned} &\{x_1^9 - x_1, y_1^9 - y_1, e_1^9 - e_1, e_2^9 - e_2, x_2^9 - x_2, y_2^9 - y_2, y_1^3 x_1 - y_1^3 + y_1 x_1 - y_1 - x_1^5 + x_1^4, \\ &y_2^3 x_2 - y_2^3 + y_2 x_2 - y_2 - x_2^5 + x_2^4, y_1^4 - y_1^3 + y_1^2 - y_1 - y_1 x_1^4 + x_1^4, y_2^4 - y_2^3 + y_2^2 - y_2 - y_2 x_2^4 + x_2^4, \\ &e_1 + e_2 - s_1, e_1 x_1 + e_2 x_2 - s_2, e_1 y_1 + e_2 y_2 - s_3, e_1 x_1^2 + e_2 x_2^2 - s_4, e_1 x_1 y_1 + e_2 x_2 y_2 - s_5, \\ &e_1 y_1^2 + e_2 y_2^2 - s_6, e_1 x_1^3 + e_2 x_2^3 - s_7, e_1((x_1 - 1)^8 - 1)((y_1 - 1)^8 - 1), e_2((x_2 - 1)^8 - 1)((y_2 - 1)^8 - 1), \\ &(e_1^8 - 1)(x_1 - 1), (e_1^8 - 1)(y_1 - 1), (e_2^8 - 1)(x_2 - 1), (e_2^8 - 1)(y_2 - 1), e_1 e_2((x_1 - x_2)^8 - 1)((y_1 - y_2)^8 - 1)\}. \end{aligned}$$

We calculate the Gröbner basis G with respect to the usual lex ordering with $s_1 < \dots < s_7 < x_2 < y_2 < x_1 < y_1 < e_2 < e_1$. The general error evaluator polynomial of \mathcal{C} contains 134 monomials and it is reported in the Appendix.

The first weak locator \mathcal{P}_2 contains 172 monomials, while the second weak locators \mathcal{P}_1 contains 494 monomials (see Appendix for all polynomials). However, these polynomials are by far not random. Indeed, we can prove the

following general structure result for any $q \geq 2$ and $t = 2$.

Theorem 8.3. *Let p be any prime number and $m \in \mathbb{N}$ such that $q = p^m \geq 2$. Let $C = C^\perp(I, L)$ be any Hermitian code with $t = 2$ over \mathbb{F}_q . Then all sets of multi-dimensional general error locator polynomials for C are of the form*

$$\begin{aligned} \{\mathcal{L}_2 = \mathcal{L}_x = x^2 + ax + b, \mathcal{L}_1 = \mathcal{L}_{xy} = y^2 + cy + d\} \\ \{\mathcal{L}_2 = \mathcal{L}_y = y^2 + Ay + B, \mathcal{L}_1 = \mathcal{L}_{yx} = x^2 + Cx + D\} \end{aligned} \quad (22)$$

with $a, b, A, B \in \mathbb{F}_p[S]$, $c, d \in \mathbb{F}_p[S, x]$ and $C, D \in \mathbb{F}_p[S, y]$. Moreover,

$$q \geq 2 \implies as_2 + bs_1 = -s_4, \quad (23)$$

$$q \geq 3 \implies As_3 + Bs_1 = -s_6. \quad (24)$$

Let $q \geq 2$ and $s_1 = s_2 = 0$. We have $e_1 = -e_2$, $x_1 = x_2$, $b = x_1^2$, $a = 2x_1$.
Let $q \geq 3$ and $s_1 = s_3 = 0$. We have $e_1 = -e_2$, $y_1 = y_2$, $B = y_1^2$, $A = 2y_1$.

All the results above hold also for any set of weak multi-dimensional general error locator polynomials

$$\begin{aligned} \{\mathcal{P}_2 = \mathcal{P}_x = x^2 + ax + b, \mathcal{P}_1 = \mathcal{P}_{xy} = y^2 + cy + d\} \\ \{\mathcal{P}_2 = \mathcal{P}_y = y^2 + Ay + B, \mathcal{P}_1 = \mathcal{P}_{yx} = x^2 + Cx + D\} \end{aligned} \quad (25)$$

Proof. Let $H = J_*^{C,2}$ be the non-stuffed ideal for C and \tilde{H} its stuffed ideal as in Theorem 7.6. There are two Gröbner bases of H and \tilde{H} that are relevant for us. If the order has $S < x_2 < y_2$ then we get G_x for H and \tilde{G}_x for \tilde{H} . If the order has $S < y_2 < x_2$ then we get G_y for H and \tilde{G}_y for \tilde{H} . As in Theorem 7.6, \tilde{G}_x contains polynomials $\mathbf{p}_x \in \mathbb{F}_q[S, x_2]$ and $\mathbf{p}_{x,y} \in \mathbb{F}_q[S, x_2, y_2]$ such that, once we replace x_2 with x and y_2 with y , we get a set of locators $\{\mathcal{L}_2 = \mathcal{L}_x, \mathcal{L}_1 = \mathcal{L}_{xy}\}$.

The degree of \mathbf{p}_x in x_2 is, *a priori*, 1 or 2. However, since there are at least two points $\{P_1, P_2\}$ on the curve with two different x , then $\deg_{x_2} \mathbf{p}_x = 2$, since \mathbf{p}_x must have two distinct roots once evaluated on a syndrome corresponding to a weight-2 error with $\{P_1, P_2\}$ as locations.

The degree of $\mathbf{p}_{x,y}$ in y_2 is, *a priori*, 1 or 2. However, for any $\bar{x} \in \mathbb{F}_q$ there are at least two points $\{P_1 = (\bar{x}, \bar{y}_1), P_2 = (\bar{x}, \bar{y}_2)\}$ on the curve with $\bar{y}_1 \neq \bar{y}_2$. Then $\deg_{y_2} \mathbf{p}_y = 2$, since it must have the two distinct roots $\{\bar{y}_1, \bar{y}_2\}$ once evaluated on a syndrome corresponding to a weight-2 error with $\{P_1, P_2\}$ as locations.

The previous argument can be trivially adapted to show that $\deg_{y_2}(\mathbf{p}_y) = 2$ and $\deg_{x_2}(\mathbf{p}_{y,x}) = 2$, where $\mathbf{p}_y \in \mathbb{F}_q[S, y_2]$ and $\mathbf{p}_{y,x} \in \mathbb{F}_q[S, y_2, x_2]$ come from \tilde{G}_y , and so (22) is proved, except for our claim that all the coefficients of these polynomials actually lie in the base field \mathbb{F}_p , which follows from Remark 3.8.

To prove (23), we first claim that

$$f \in H \implies f^2 \in \tilde{H}. \quad (26)$$

To see (26) we note that in the creation of \tilde{H} from H we only impose the vanishing of the first-order derivative at points of $\mathcal{V}(H)$, but if we take any point $Q \in \mathcal{V}(H)$ we have (see Definition 7.2 for θ_1)

$$\theta_1(f^2) = 2f(Q)\theta_1(f) = 0\theta_1(f) = 0.$$

Since $s_1 - e_1 - e_2, s_2 - e_1x_1 - e_2x_2, s_4 - e_1x_1^2 - e_2x_2^2 \in H$, we have that $(s_1 - e_1 - e_2)^2, (s_2 - e_1x_1 - e_2x_2)^2, (s_4 - e_1x_1^2 - e_2x_2^2)^2 \in \tilde{H}$ for (26). Passing from variables to values we observe that

$$\bar{s}_1 = \bar{e}_1 + \bar{e}_2, \quad \bar{s}_2 = \bar{e}_1\bar{x}_1 + \bar{e}_2\bar{x}_2, \quad \bar{s}_4 = \bar{e}_1\bar{x}_1^2 + \bar{e}_2\bar{x}_2^2 \quad (27)$$

and that

$$\bar{a} = a(\bar{S}) = -(\bar{x}_1 + \bar{x}_2), \quad \bar{b} = \bar{x}_1\bar{x}_2.$$

So

$$-(\bar{x}_1 + \bar{x}_2)(\bar{e}_1\bar{x}_1 + \bar{e}_2\bar{x}_2) + \bar{x}_1\bar{x}_2(\bar{e}_1 + \bar{e}_2) = -(\bar{e}_1\bar{x}_1^2 + \bar{e}_2\bar{x}_2^2), \text{ which proves (23).}$$

In the same way, we can calculate the set of locators $\{\mathcal{L}_2 = \mathcal{L}_y, \mathcal{L}_1 = \mathcal{L}_{yx}\}$. If $q \geq 3$, we have also $s_1 - e_1 - e_2, s_3 - e_1y_1 - e_2y_2, s_6 - e_1y_1^2 - e_2y_2^2 \in H$, so we have that $(s_1 - e_1 - e_2)^2, (s_3 - e_1y_1 - e_2y_2)^2, (s_6 - e_1y_1^2 - e_2y_2^2)^2 \in \tilde{H}$ for (26). Again, we pass from variables to values, and we obtain

$$\bar{s}_1 = \bar{e}_1 + \bar{e}_2, \quad \bar{s}_3 = \bar{e}_1\bar{y}_1 + \bar{e}_2\bar{y}_2, \quad \bar{s}_6 = \bar{e}_1\bar{y}_1^2 + \bar{e}_2\bar{y}_2^2 \quad (28)$$

and that

$$\bar{A} = A(\bar{S}) = -(\bar{x}_1 + \bar{x}_2), \quad \bar{B} = B(\bar{S}) = \bar{x}_1\bar{x}_2.$$

So

$$-(\bar{x}_1 + \bar{x}_2)(\bar{e}_1\bar{y}_1 + \bar{e}_2\bar{y}_2) + \bar{y}_1\bar{y}_2(\bar{e}_1 + \bar{e}_2) = -(\bar{e}_1\bar{y}_1^2 + \bar{e}_2\bar{y}_2^2) \implies \bar{A}s_3 + \bar{B}s_1 = -s_6.$$

The last part of theorem comes from direct computations, as follows.

From (27), in the case $\bar{s}_1 = \bar{s}_2 = 0$, we note $\bar{e}_1 = -\bar{e}_2$, $\bar{x}_1 = \bar{x}_2$. And so

1. If $p = 2$ then $\bar{a} = -(\bar{x}_1 + \bar{x}_2) = 2\bar{x}_1 = 0$ and $\bar{b} = \bar{x}_1\bar{x}_2 = \bar{x}_1^2$.
2. If $p \neq 2$ then $\bar{a} = -(\bar{x}_1 + \bar{x}_2) = 2\bar{x}_1 \implies \bar{x}_1 = \frac{\bar{a}}{2}$.

From (28), if $s_1 = s_3 = 0$ then $e_1 = -e_2$ and $y_1 = y_2$. And thus

1. If $p = 2$ then $\bar{A} = 0$ e $\bar{B} = y_1^2$.
2. If $p \neq 2$ then $\bar{A} = 2y_1 \implies y_1 = \frac{\bar{A}}{2}$ and $\bar{B} = y_1^2$.

Since in the proof so far we have used the relations on the syndromes coming from the non-stuffed ideal H , everything that we proved up to now holds also

for the weak locators. \square

The locator \mathcal{P}_2 computed for the Hermitian code with $q = 3$ and $t = 2$ is indeed of the form $\mathcal{P}_2 = \mathcal{P}_x = x^2 + ax + b$, with $|a| = 82$ and $|b| = 91$, so, for example when $s_1 \neq 0$, it is enough to evaluate $a(\bar{S})$ and then we obtain $b(\bar{S})$ as

$$b(\bar{S}) = -\frac{s_4 + a(\bar{S})s_2}{s_1}.$$

Also \mathcal{P}_1 is as above, that is, of the form $\mathcal{P}_1 = \mathcal{P}_{xy} = y^2 + cy + d$.

Regrettably, we have not been able to compute explicitly \mathcal{L}_2 and \mathcal{L}_1 for $q = 3$, due to the high computation cost of the stuffing procedure.

9 Conclusions and open problems

Assuming we are able to compute the relevant Gröbner basis, we have identified a very easy decoding procedure for any affine-variety code: we evaluate our polynomials $\{\mathcal{L}_i\}_{1 \leq i \leq m}$ in the received syndromes and we use some simple root-finding to get the error locations. As it is traditional in coding theory, once we have the error locations we can directly get the error values and hence the decoding problem is completely solved. This apparently idyllic situation is marred by two serious issues:

- the computation of the associated Gröbner basis can be quite beyond present means already for medium-size codes;
- even if we compute our locators, they could be so dense that their use would be impractical.

These two apparently different problems may have one common solution: to identify our polynomials *without* computing any Gröbner basis, but using the “structure of the code”. This is indeed a desperate goal, if tried for general codes, but we believe that some code families have locators which are easy to describe explicitly and very sparse. Our belief stems from our results in [OS07] (and [MOS06]), where we explicitly give locators for families of cyclic codes, which apparently have no special structure, simultaneously proving their sparsity (see also [CL10], [LCCC10], [LCJC10] for recent results on the structure of locators).

We then suggest the following problems.

Problem 9.1. For any q and t write formally $\{\mathcal{L}_i\}$ for the Hermitian code.

Problem 9.2. For any q , r and t write formally $\{\mathcal{L}_i\}$ for the code from norm-trace curves.

Problem 9.3. For any admissible parameter, write formally $\{\mathcal{L}_i\}$ for the codes from [SDG06] curves.

Problem 9.4. For any admissible parameter, write formally $\{\mathcal{L}_i\}$ for the codes from [SDG06] surfaces I.

Problem 9.5. For any admissible parameter, write formally $\{\mathcal{L}_i\}$ for the codes from [SDG06] surfaces II.

An interesting problem comes from the definition of the t_i 's in Definition 6.2. Clearly, we have $t_i < t$ only if any t points on the variety have necessarily less than t distinct values for their i -th component. For example, you might think of two parallel lines in the plane $(\mathbb{F}_q)^2$, $x = a$ and $x = b$, any defining a Reed Solomon code. In this case, whatever $t \geq 2$ can be, we will always have $t_1 = 2$. This example is very special, since the variety is reducible. We then ask the following problem.

Problem 9.6. To identify (easy to check) conditions on a curve and on the function space such $t_i = t$ for any i .

For special cases this is quite obvious. For example when $t = 1$ this is always true. It would be very nice to get a generalization of the former problem.

Problem 9.7. To identify conditions on a curve and on the function space such either $t_i = t$ for any i or to find a (projective?affine?) transformation of $(\mathbb{F}_q)^m$ such that the same holds.

In [OS05] we studied also locators able to correct simultaneously errors and erasures (Definition 2.10). It is obvious how to extend Definition 3.9, 3.10, 6.2 to cover also simultaneous error-and-erasure decoding. A suitable ideal modified from $J_*^{C,t}$ will again be strongly multi-stratified and so Theorem 8.3 can be extended accordingly, to cover the new case. We do not give explicitly the related definitions and results, due to triviality of the extensions.

Our decoding works well with the case of one ghost point, which plays the same role as that played by zero in the zero-free n -th root codes (being cyclic codes a special case). However, there is no reason why we should restrict to the use of *one* ghost point, since a clever choice of multiple ghost points could give easier decoding. Indeed the geometric structure of the union of $\mathcal{V}(I)$ and the ghost points influences the shape of the Gröbner bases of our ideals and hence the shape of our polynomials.

Problem 9.8. To identify a clever choice of (possibly) multiple ghost points, in order to minimize the corresponding locators.

We have recently known of a promising new approach to the decoding of one-point geometric Goppa codes ([Lax11]), where *generic* versions of locators are proposed. The nice idea behind it is that trying to correct all correctable syndromes forces the locator polynomials to be dense, while it could be possible in some cases to identify a very large subset of syndromes (*generic syndromes*) such that the locator for those is small.

Problem 9.9. To define rigorously sets of *generic* multi-dimensional locators.

Acknowledgements

The first two authors would like to thank their supervisor: the third author. For useful suggestions and discussions, the authors would like to thank: P. Fitzpatrick, O. Geil, T. Mora, M. Piva and C. Traverso.

Suggestions and criticism by an anonymous referee helped us greatly to improve the presentation.

This work has been partially presented at the Claude Shannon Institute Workshop in Cork (Ireland), May 2007 and the Mini workshop on error correcting codes and network coding in Aalborg (Denmark), September 2007.

Some of these results are present in the second author's PHD thesis ([Ors08])

We have run our computer simulations using the software package Singular (<http://www.singular.uni-kl.de>).

References

- [ABF09] D. Augot, M. Bardet, and J.-C. Faugère, *On the decoding of cyclic codes with Newton identities*, J. Symbolic Comput. **44** (2009), 1608–1625.
- [BP09] S. Bulygin and R. Pellikaan, *Bounded decoding of linear error-correcting codes with Gröbner bases*, J. Symbolic Comput. **44** (2009), 1626–1643.
- [CL10] Y. Chang and C. Lee, *Algebraic decoding of a class of binary cyclic codes via Lagrange interpolation formula*, IEEE Trans. on Inf. Th. **56** (2010), 130–139.
- [CM02] M. Caboara and T. Mora, *The Chen-Reed-Helleseth-Truong decoding algorithm and the Gianni-Kalkbrenner Gröbner shape theorem*, Appl. Algebra Engrg. Comm. Comput. **13** (2002), no. 3, 209–232.
- [Coo90] A. B. III Cooper, *Direct solution of BCH decoding equations*, Comm., Cont. and Sign. Proc. (1990), 281–286.
- [Coo91] ———, *Finding BCH error locator polynomials in one step*, Electronic Letters **27** (1991), no. 22, 2090–2091.
- [Coo93] ———, *Toward a new method of decoding algebraic codes using Gröbner bases*, Transactions of the Tenth Army Conference on Applied Mathematics and Computing (1992), vol. 93, U.S. Army, 1993, pp. 1–11.
- [CRHT94a] X. Chen, I. S. Reed, T. Helleseth, and T. K. Truong, *Algebraic decoding of cyclic codes: a polynomial ideal point of view*, Finite fields, Contemp. Math., vol. 168, Amer. Math. Soc., 1994, pp. 15–22.
- [CRHT94b] X. Chen, I. S. Reed, T. Helleseth, and T. K. Truong, *Use of Gröbner bases to decode binary cyclic codes up to the true minimum distance*, IEEE Trans. on Inf. Th. **40** (1994), no. 5, 1654–1661.
- [FL98] J. Fitzgerald and R. F. Lax, *Decoding affine variety codes using Gröbner bases*, Des. Codes Cryptogr. **13** (1998), no. 2, 147–158.

-
- [Gei03] O. Geil, *On codes from norm-trace curves*, Finite Fields Appl. **9** (2003), 351–371.
- [Gia89] P. Gianni, *Properties of Gröbner bases under specializations*, Proc. of EUROCAL1987, LNCS, vol. 378, Springer, 1989, pp. 293–297.
- [Gio06] Marta Giorgetti, *On some algebraic interpretation of classical codes*, Ph.D. thesis, University of Milan, 2006.
- [GR09] E. Guerrini and A. Rimoldi, *FGLM-like decoding: from Fitzpatrick’s approach to recent developments*, Gröbner Bases, Coding, and Cryptography (M. Sala, T. Mora, L. Perret, S. Sakata, and C. Traverso, eds.), RISC Book Series, Springer, Heidelberg, 2009, pp. 197–218.
- [GS06] M. Giorgetti and M. Sala, *A commutative algebra approach to linear codes*, BCRI preprint, www.bcri.ucc.ie, 58, UCC, Cork, Ireland, 2006.
- [GS09] ———, *A commutative algebra approach to linear codes*, Journal of Algebra **321** (2009), no. 8, 2259–2286.
- [Kal89] M. Kalkbrener, *Solving systems of algebraic equations by using Gröbner bases*, Proc. of EUROCAL 1987, LNCS, vol. 378, 1989, pp. 282–292.
- [Lax11] R. F. Lax, *Generic interpolation polynomial for list decoding*, to appear (2011), 15.
- [LCCC10] C.D. Lee, Y. Chang, H.H. Chang, and J.H. Chen, *Unusual general error locator polynomial for the (23, 12, 7) golay code*, Communications Letters, IEEE **14** (2010), no. 4, 339–341.
- [LCJC10] C.D. Lee, Y. Chang, M.H. Jing, and J.H. Chen, *Decoding binary cyclic codes with irreducible generator polynomials up to actual minimum distance*, Communications Letters, IEEE **14** (2010), no. 11, 1050–1052.
- [LN97] R. Lidl and H. Niederreiter, *Finite fields*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1997.
- [LY97] P. Loustau and E. V. York, *On the decoding of cyclic codes using Gröbner bases*, AAECC **8** (1997), no. 6, 469–483.
- [MB82] H. M. Möller and B. Buchberger, *The construction of multivariate polynomials with preassigned zeros*, LNCS **144** (1982), 24–31.
- [MO09] T. Mora and E. Orsini, *Decoding cyclic codes: the Cooper philosophy*, Gröbner Bases, Coding, and Cryptography (M. Sala, T. Mora, L. Perret, S. Sakata, and C. Traverso, eds.), RISC Book Series, Springer, Heidelberg, 2009, pp. 69–91.
- [Mor05] T. Mora, *Solving polynomial equation systems. II, Macaulay’s paradigm and Gröbner technology*, Encyclopedia of Mathematics and its Applications, vol. 99, Cambridge University Press, 2005.
-

- [Mor09a] ———, *The FGLM problem and Moeller's algorithm on zero-dimensional ideals*, Gröbner Bases, Coding, and Cryptography (M. Sala, T. Mora, L. Perret, S. Sakata, and C. Traverso, eds.), RISC Book Series, Springer, Heidelberg, 2009, pp. 27–45.
- [Mor09b] ———, *Gröbner technology*, Gröbner Bases, Coding, and Cryptography (M. Sala, T. Mora, L. Perret, S. Sakata, and C. Traverso, eds.), RISC Book Series, Springer, Heidelberg, 2009, pp. 11–26.
- [Morar] ———, *Solving polynomial equation systems. III, algebraic solving and beyond*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, to appear.
- [MOS06] T. Mora, E. Orsini, and M. Sala, *General error locator polynomials for binary cyclic codes with $t \leq 2$ and $n < 63$* , BCRI preprint, www.bcric.ucc.ie 43, UCC, Cork, Ireland, 2006.
- [Ors08] Emmanuela Orsini, *On the decoding and distance problems for algebraic codes*, Ph.D. thesis, University of Milan, 2008.
- [OS05] E. Orsini and M. Sala, *Correcting errors and erasures via the syndrome variety*, J. Pure Appl. Algebra **200** (2005), 191–226.
- [OS07] ———, *General error locator polynomials for binary cyclic codes with $t \leq 2$ and $n < 63$* , IEEE Trans. on Inf. Th. **53** (2007), 1095–1107.
- [SDG06] G. Salazar, D. Dunn, and S. B. Graham, *An improvement of the Feng-Rao bound on minimum distance*, Finite Fields Appl. **12** (2006), no. 3, 313–335.

Appendix

In Example 3.5 we have the following polynomial

$$\begin{aligned} \mathcal{L}_{xy} = & \mathbf{y}^2 + \mathbf{y}(2s_4^6 + 2s_4^5s_3 + 2s_4^5s_2s_1^6 - 2s_4^5s_2 + 2ys_4^4s_3^2 - 2s_4^4s_2^2 + 2s_4^4s_2s_1 + 2s_4^3s_3^3 - 2s_4^3s_2^2 + 2s_4^3s_2s_1^2 + \\ & 2s_4^2s_3^4 - 2s_4^2s_2^4 + 2s_4^2s_2s_1^3 + 2s_4s_3^5 - s_4s_3s_2^4 - s_4s_3s_2s_1^3 + 2s_4s_3s_1^4 - s_4s_2^5 + s_4s_2s_1^4 - 2s_3^6 + 2s_3^5s_2s_1^6 - \\ & 2s_3^5s_2 + 2s_3^4s_2^2 - 2s_3^4s_2s_1 + 3s_3^3s_2^3 - 3s_3^3s_2s_1^2 + 3s_3^3s_2^6s_1^4 + 2s_3^2s_2^5s_1^5 - 3s_3^2s_2^4s_1^6 - s_3^2s_2^4 + 2s_3^2s_2^3s_1 - \\ & s_3^2s_2^2s_1^2 - 2s_3^2s_1^4 - s_3s_2^6s_1^5 + s_3s_2^5s_1^6 + s_3s_2^4s_1 - 2s_3s_2^3s_1^2 + s_3s_2^2s_1^3 - 3s_2^6 - s_2^5s_1 - s_2^4s_1^2 - s_2^3s_1^3 - s_2^2s_1^4 - \\ & s_2s_1^5 - s_1^6) - s_4^6s_3^6 + 2s_4^6s_2^6 + 3s_4^6s_2s_1^5 - 2s_4^6s_1^6 - s_4^5s_3 - s_4^5s_2s_1^6 + s_4^5s_2 - s_4^4s_3^2 + s_4^4s_2^2 - s_4^4s_2s_1 - \\ & s_4^3s_3^3 + s_4^3s_2^2 - s_4^3s_2s_1^2 - s_4^2s_3^4 + s_4^2s_2^4 - s_4^2s_2s_1^3 - s_4s_3^5 + 2s_4s_3s_2^4 - s_4s_3s_1^4 - s_4s_2^5 + s_4s_2s_1^4 - 3s_3^6s_2^6 + \\ & 2s_3^6s_2s_1^5 - s_3^6s_1^6 - s_3^6 - 2s_3^5s_2s_1^6 + 2s_3^5s_2 + 3s_3^3s_2^3 - 3s_3^3s_2s_1^2 - s_3^2s_2^6s_1^4 - s_3^2s_2^5s_1^5 + s_3^2s_2^4s_1^6 + 3s_3^2s_2^3s_1 - \\ & s_3^2s_2^2s_1^2 - 3s_3^2s_2s_1^3 + s_3^2s_1^4 - 3s_3s_2^6s_1^5 + 3s_3s_2^5s_1^6 - 3s_3s_2^5 + 2s_3s_2^4s_1 + 2s_3s_2^3s_1^2 + s_3s_2^2s_1^3 - 2s_3s_2s_1^4 + \\ & 2s_2^6s_1^6 + 2s_2^6 + 2s_2^5s_1 + 2s_2^4s_1^2 + 2s_2^3s_1^3 + 2s_2^2s_1^4 + 2s_2s_1^5. \end{aligned}$$

In Subsection 8.5 we have the following polynomials

$$\begin{aligned} \mathcal{E} = & \mathbf{e}_2^2 - \mathbf{e}_2 s_1 - s_7 s_5^3 s_2^6 - s_7 s_5 - s_7 s_4 s_3^3 s_2^7 s_1^6 - s_7 s_4 s_3 s_2^7 - s_7 s_4 s_2^3 s_1^5 + s_7 s_4^3 s_2^5 s_1^8 - s_7 s_4^3 s_2^5 - s_7 s_3^3 s_2 s_1^5 - \\ & s_7 s_3 s_2 s_1^7 + s_7 s_2^5 s_1^4 - s_6 s_1 - s_5^4 s_4 s_3^3 s_2^6 s_1^4 - s_5^4 s_4 s_3 s_2^6 s_1^6 + s_5^4 s_4 s_2^3 s_1^3 + s_5^4 s_3^3 s_2^8 s_1^3 + s_5^4 s_3 s_2^8 s_1^5 + \\ & s_4^4 s_3 s_1^5 + s_4^4 s_2^4 s_1^2 + s_3^3 s_2^3 s_3 s_2 s_1 + s_3^3 s_2^4 s_3 s_2 s_1^3 - s_3^3 s_2^4 s_2^5 + s_3^3 s_4 s_3^4 s_2^7 s_1^3 + s_3^3 s_4 s_3^3 s_2^7 s_1^5 - s_3^3 s_4 s_3 s_2^3 s_1^2 + \\ & s_3^3 s_4 s_2^7 s_1^7 - s_3^3 s_3^6 s_2 s_1^8 + s_3^3 s_3^6 s_2 + s_3^3 s_3^6 s_2 s_1^2 - s_3^3 s_3^3 s_2^5 s_1^7 + s_3^3 s_2^3 s_2^4 + s_3^3 s_3 s_2^5 s_1 + s_3^3 s_2 s_1^6 - s_2^5 s_4^3 s_3 s_1^8 - \\ & s_2^5 s_4^3 s_3 s_1^2 + s_2^5 s_4^3 s_2^4 s_1^7 - s_2^5 s_4^3 s_3 s_2^2 s_1^7 - s_2^5 s_4^3 s_3 s_2^2 s_1 - s_2^5 s_4^3 s_2^6 s_1^6 + s_2^5 s_4^3 s_3 s_1^7 - s_2^5 s_4^3 s_2^4 s_1 + s_2^5 s_4^3 s_3^3 s_1^3 + \\ & s_2^5 s_4^3 s_2^8 s_1^5 - s_2^5 s_4^3 s_3^4 s_1^4 - s_2^5 s_4 s_3 s_1^6 - s_2^5 s_3^6 s_2^4 s_1^4 + s_2^5 s_3^4 s_2^6 s_1^6 - s_2^5 s_3^3 s_2^8 s_1^8 + s_2^5 s_2^6 s_1^2 - s_5 s_4^5 s_3^4 s_2 s_1^7 - \\ & s_5 s_4^5 s_3^3 s_2 s_1 + s_5 s_4^5 s_3 s_2^5 s_1^6 + s_5 s_4^5 s_2 s_1^3 - s_5 s_4^4 s_3^4 s_2^3 s_1^6 + s_5 s_4^4 s_3^3 s_2^7 s_1^3 - s_5 s_4^4 s_3^3 s_2^3 s_1^8 + s_5 s_4^4 s_3^3 s_2 s_1^6 - \\ & s_5 s_4^3 s_3^5 s_2 s_1^8 + s_5 s_4^3 s_3^3 s_2 s_1^2 + s_5 s_4^3 s_3 s_2 s_1^4 + s_5 s_4^3 s_2^5 s_1 + s_5 s_4^2 s_2^8 s_1^8 - s_5 s_4^2 s_2^7 + s_5 s_4 s_3^6 s_2 s_1 + s_5 s_4 s_3^3 s_2^5 s_1^8 - \\ & s_5 s_4 s_3^3 s_2 s_1^5 - s_5 s_5 s_4 s_3 s_2^5 s_1^2 - s_7 s_2^7 s_1^3 - s_5 s_3^6 s_2^3 s_1^8 + s_5 s_3^6 s_2^3 + s_5 s_3^5 s_2^7 s_1^5 - s_5 s_3^4 s_3^2 s_1^2 - s_5 s_3^3 s_2^3 s_1^4 - \\ & s_8^3 s_6 s_1^4 + s_8^3 s_3 s_1^6 - s_4 s_3^3 s_2^4 s_1^3 - s_4 s_2^8 s_1^8 - s_4 s_3 s_2^4 s_1^5 - s_4 s_8 s_2^8 + s_4 s_1^2 - s_4 s_3 s_2^2 s_1^3 + s_4 s_3 s_2^2 s_1^5 - s_7 s_3^3 s_2 s_1^2 - \\ & s_7 s_3^2 s_2^7 s_1 - s_4 s_3 s_2^6 s_1^4 - s_4 s_3^6 s_2 s_1^2 + s_4 s_3^4 s_2^4 s_1^4 - s_4 s_3^3 s_2 s_1 + s_4 s_3 s_1 - s_4 s_3^2 s_2^4 s_1^6 - s_4 s_3 s_2^8 s_1^3 + s_4 s_3 s_1^3 - \\ & s_4 s_2^4 s_1^8 + s_4 s_2^4 - s_4 s_3^5 s_2^6 s_1 - s_4 s_3^3 s_2^8 s_1 - s_4 s_3^3 s_2^2 + s_4 s_3^2 s_2^5 s_1 + s_4 s_3 s_2^2 s_1 - s_4 s_3 - s_4 s_3^5 s_2^4 s_1^5 - s_4 s_3^4 s_2^8 s_1^2 + \\ & s_4 s_3 s_1^2 + s_4 s_3^2 s_2^8 s_1^4 - s_4 s_2^3 s_1^4 + s_4 s_3 s_2^4 s_1 + s_4 s_2^6 s_1 + s_4 s_1^6 + s_4 s_3^3 s_2^5 s_1 - s_4 s_3^2 s_2^3 s_1 + s_4 s_3 s_2^6 s_1 + s_4 s_3 s_2^2 + \\ & s_4 s_2^5 s_1^5 + s_4 s_3^4 s_2 s_1^8 + s_4 s_3^3 s_2 s_1^2 - s_4 s_3 s_2^8 s_1^7 - s_4 s_2^4 s_1^4 + s_4 s_3^2 s_2^8 s_1 + s_4 s_3 s_2^2 - s_4 s_3^6 s_2^5 s_1 + s_4 s_3^5 s_2 s_1^2 - \\ & s_4 s_3^3 s_2^4 s_1^4 + s_4 s_3 s_2^5 s_1^6 + s_4 s_2^3 s_1^3 - s_3^8 s_2^2 s_1^2 - s_3^6 s_2^4 s_1^4 + s_3^5 s_2 s_1 - s_3^3 s_2^8 s_1^6 - s_3^3 s_2^4 s_1^3 - s_3^3 s_2 s_1^8 + s_3^3 - s_3 s_4^5 s_1^5. \end{aligned}$$

$$\begin{aligned} P_x = & \mathbf{x}_2^2 + \mathbf{x}_2 (s_7 s_4 s_3^3 s_1^3 + s_7 s_4 s_3 s_1^5 - s_7 s_4 s_2^4 s_1^2 - s_7 s_3^3 s_2^2 s_1^2 - s_7 s_3 s_2^2 s_1^4 + s_7 s_2^6 s_1 - s_3^3 s_4^5 - s_3^3 s_4 s_2^2 s_1^7 + \\ & s_3^3 s_1^3 s_3 s_1^7 + s_3^3 s_3 s_3 s_1 + s_3^3 s_2^4 s_1^6 + s_3^3 s_2^6 s_1^5 - s_3^3 s_4 s_3^3 s_1^6 + s_3^3 s_4 s_3 - s_3^3 s_4 s_3^3 s_2^4 s_1^5 - s_3^3 s_4 s_2^3 s_1^2 - \\ & s_3^3 s_4 s_3 s_2^4 s_1^7 + s_3^3 s_3^6 s_2^5 s_1 - s_3^3 s_3^4 s_2^2 s_1^7 + s_3^3 s_3^3 s_2^2 s_1 - s_5 s_4^6 s_1 + s_5 s_4^5 s_2^2 s_1^8 - s_5 s_4^5 s_2^2 + s_5 s_4^4 s_3 s_1^8 + s_5 s_4^4 s_3 s_1^2 - \\ & s_5 s_4^4 s_2^7 s_1^7 - s_5 s_4^3 s_3^3 s_2^2 s_1^7 - s_5 s_4^3 s_3 s_2^2 s_1 - s_5 s_4^2 s_3^6 s_1^7 + s_5 s_4^2 s_3^4 s_1 - s_5 s_4^2 s_3^3 s_2^4 s_1^6 - s_5 s_4^2 s_3^3 s_2 s_1^8 - \\ & s_5 s_4^2 s_3^3 s_1^5 - s_5 s_4 s_3^6 s_2^2 s_1^6 + s_5 s_4 s_3^4 s_2^2 + s_5 s_4 s_3^3 s_2^5 s_1^5 - s_5 s_4 s_3^3 s_2^2 s_1^2 + s_5 s_4 s_3 s_2^9 s_1^7 - s_5 s_3^6 s_2^4 s_1^5 + s_5 s_3^4 s_2^5 s_1^7 - \\ & s_5 s_3^3 s_2^4 s_1 + s_4 s_1^8 + s_4 s_2^3 s_1^7 + s_4 s_2^2 s_1^7 - s_4 s_3^3 s_2^6 s_1 + s_4 s_2^4 s_1^6 + s_4 s_3^3 s_2^7 s_1^8 - s_4 s_3^3 s_2^7 - s_4 s_3 s_2^3 s_1^7 + s_4 s_2^2 s_1^4 + \\ & s_4 s_2^6 s_1^5 + s_4 s_6 s_2 s_1^5 + s_4 s_3 s_2^7 s_1 + s_4 s_3^3 s_2^5 s_1^4 + s_4 s_3 s_2^6 s_1 + s_4 s_8 s_1^4 - s_4 s_3^6 s_2 s_1^4 + s_4 s_3^3 s_2^8 s_1 + s_4 s_3 s_2^7 s_1^5 + \\ & s_4 s_2^3 s_1^2 + s_4 s_2^2 s_1^3 + s_4 s_2 s_2 s_1^6 - s_4 s_3^5 s_2 s_1^8 + s_4 s_3^4 s_2^5 s_1^5 + s_4 s_3^3 s_2 s_1^2 + s_4 s_2^3 s_2^5 s_1^7 - s_4 s_3 s_2 s_1^4 - s_4 s_2^5 s_1 + s_4 s_2^2 s_1^2 + \\ & s_4 s_3^8 s_2 s_1^8 - s_4 s_3^8 s_2 + s_4 s_3^4 s_2^7 s_1^4 + s_4 s_3^3 s_2^7 s_1^6 - s_4 s_2^7 s_1^8 + s_4 s_2^7 + s_4 s_2^6 s_1 - s_3^7 s_2^5 s_1^4 + s_3^5 s_2^5 s_1^6 - s_3^3 s_2^5 s_1^8 + s_2^8 - 1) - \\ & s_7 s_4 s_3^3 s_2 s_1^2 - s_7 s_4 s_3 s_2 s_1^4 + s_7 s_4 s_2^5 s_1 + s_7 s_3^3 s_2 s_1^3 + s_7 s_3 s_2^3 s_1^3 - s_7 s_2^7 + s_3^3 s_4^5 s_2 s_1^7 + s_3^3 s_4^4 s_2 s_1^6 - s_3^3 s_4^3 s_3 s_2 s_1^6 - \\ & s_3^3 s_4 s_3 s_2 - s_3^3 s_4 s_2^5 s_1^5 - s_3^3 s_2^7 s_1^4 + s_3^3 s_4 s_3^6 s_2 s_1^5 - s_3^3 s_4 s_3^4 s_2 s_1^7 + s_3^3 s_4 s_3^3 s_2^5 s_1^4 + s_3^3 s_4 s_3^3 s_2 s_1 + s_3^3 s_4 s_3 s_2^5 s_1^6 - \\ & s_3^3 s_3^6 s_2^4 s_1 + s_3^3 s_4 s_2 s_1^6 - s_3^3 s_2^3 s_2 - s_2^5 s_3^6 s_2 s_1^8 + s_2^5 s_3^6 s_2 + s_2^5 s_3^6 s_1 - s_2^5 s_3^6 + s_5 s_4^6 s_2 s_1^8 - s_5 s_4^4 s_3^3 s_2 s_1^7 - s_5 s_4^4 s_3 s_2 s_1 + \\ & s_5 s_4 s_2^5 s_1^6 + s_5 s_3^3 s_3^3 s_2 s_1^6 - s_5 s_4 s_3 s_3^3 s_1^8 - s_5 s_4^3 s_3 s_2^3 + s_5 s_2^6 s_3 s_2 s_1^6 - s_5 s_4^2 s_3 s_2 s_1^8 + s_5 s_4^2 s_3 s_2^5 s_1^5 + s_5 s_4^2 s_3 s_2 s_1^2 + \\ & s_5 s_4 s_3 s_2^5 s_1^7 + s_5 s_4 s_2 s_1^4 + s_5 s_4 s_3^6 s_2 s_1^5 - s_5 s_4 s_3^4 s_2 s_1^7 - s_5 s_4 s_3^3 s_2^7 s_1^4 + s_5 s_4 s_3^3 s_2 s_1 - s_5 s_4 s_3 s_2^7 s_1^6 + s_5 s_3^7 s_8 s_1^8 - \\ & s_5 s_3^7 s_8 - s_5 s_3^7 s_1^8 + s_5 s_3^7 + s_5 s_3^6 s_2^5 s_1^4 - s_5 s_4^4 s_2^5 s_1^6 - s_5 s_3^2 s_2^5 s_1^8 - s_5 s_3^2 s_2^5 - s_4 s_2 s_1^7 + s_4 s_3 s_1^8 - s_4 s_3 - s_4 s_2^4 s_1^5 - \\ & s_4 s_3^3 s_1^6 + s_4 s_3^3 s_2 s_1^5 - s_4 s_2^5 s_1^5 + s_4 s_3^2 s_1^6 - s_4 s_2^8 s_1^3 - s_4 s_2^7 s_1^4 - s_4 s_3^6 s_2 s_1^4 - s_4 s_3^4 s_2^2 s_1^6 - s_4 s_3^3 s_2 s_1^3 + s_4 s_3^2 s_2^8 s_1^8 - \\ & s_4 s_3^2 s_2 - s_4 s_3 s_2^5 s_1^5 - s_4 s_2 s_1^3 + s_4 s_3^6 s_2 s_1^3 - s_4 s_3^5 s_1^8 + s_4 s_3^5 - s_4 s_3^3 s_2^4 s_1^7 - s_4 s_3 s_2^8 s_1^4 - s_4 s_2 s_1 - s_4 s_2^3 s_1^2 - \\ & s_4 s_3^2 s_2^5 s_1 + s_4 s_3^2 s_2 s_1^7 - s_4 s_4^4 s_2^6 s_1^4 - s_4 s_3^3 s_2 s_1 - s_4 s_2^3 s_2^6 s_1 + s_4 s_2^3 s_2^3 s_1 + s_4 s_2^2 - s_4 s_2^5 s_1 - s_4 s_3^4 s_2 s_1^3 - \\ & s_4 s_3^3 s_2 s_1^5 - s_4 s_2^7 s_1^8 - s_4 s_1^7 + s_3^3 s_2 s_1^3 - s_3^6 s_2 s_1^8 + s_3^6 s_2^2 - s_3^5 s_2 s_1^5 + s_3^3 s_2 s_1^7. \end{aligned}$$

$$\begin{aligned} P_{xy} = & \mathbf{y}_2^2 - \mathbf{y}_2 (s_6^2 s_3^3 s_1^3 - s_6^2 s_3 s_1^5 + s_6^2 s_2^4 s_1^2 - s_6 s_3^7 + s_6 s_3^6 s_1 - s_6 s_3^4 s_2^4 s_1^7 - s_6 s_3^4 s_1^3 + s_6 s_3^3 s_2^4 + s_6 s_2^2 s_1^5 - s_6 s_3 s_2^8 s_1^6 + \\ & s_6 s_3 s_2^4 s_1^2 + s_6 s_2^8 s_1^7 + s_6 s_2^4 s_1^3 - s_6 s_1^7 - s_8^3 + s_3^6 s_3^2 s_1^6 - s_3^6 s_1^2 - s_3^5 s_2^4 s_1^7 - s_3^5 s_1^3 + s_3^4 s_2^4 + s_3^4 s_1^4 + s_3^3 s_2^5 s_1^5 - s_3^3 s_1^5 - \\ & s_3^3 s_2^8 s_1^6 + s_3^3 s_2^2 s_1^7 - s_3^3 s_1^6 - s_3 s_2^8 s_1^7 + s_3 s_2^4 s_1^3 + s_2^8 + s_1^8 + 1) + \mathbf{x}_2 (s_7 s_4 s_3^3 s_2^6 s_1^5 + s_7 s_4 s_3 s_2^6 s_1^7 - s_7 s_4 s_2^4 s_1^4 - \\ & s_7 s_3^3 s_2^8 s_1^4 - s_7 s_3 s_2^8 s_1^6 + s_7 s_2^4 s_1^3 + s_4^4 s_2 s_1 + s_3^3 s_4 s_1 - s_3^3 s_4 s_2^2 - s_3^3 s_2^4 s_3 s_1^4 + s_3^3 s_2^4 s_3 s_1^5 + s_3^3 s_2^4 s_3 s_1^6 - s_3^3 s_2^4 s_3 s_1^7 - \\ & s_3^3 s_2^4 s_3 s_1^8 + s_3^3 s_2^4 s_3 s_1^9 - s_3^3 s_2^4 s_3 s_1^{10} - s_3^3 s_2^4 s_3 s_1^{11} - s_3^3 s_2^4 s_3 s_1^{12} - s_3^3 s_2^4 s_3 s_1^{13} - s_3^3 s_2^4 s_3 s_1^{14} - s_3^3 s_2^4 s_3 s_1^{15} - s_3^3 s_2^4 s_3 s_1^{16} - \\ & s_3^3 s_2^4 s_3 s_1^{17} - s_3^3 s_2^4 s_3 s_1^{18} - s_3^3 s_2^4 s_3 s_1^{19} - s_3^3 s_2^4 s_3 s_1^{20} - s_3^3 s_2^4 s_3 s_1^{21} - s_3^3 s_2^4 s_3 s_1^{22} - s_3^3 s_2^4 s_3 s_1^{23} - s_3^3 s_2^4 s_3 s_1^{24} - s_3^3 s_2^4 s_3 s_1^{25} - \\ & s_3^3 s_2^4 s_3 s_1^{26} - s_3^3 s_2^4 s_3 s_1^{27} - s_3^3 s_2^4 s_3 s_1^{28} - s_3^3 s_2^4 s_3 s_1^{29} - s_3^3 s_2^4 s_3 s_1^{30} - s_3^3 s_2^4 s_3 s_1^{31} - s_3^3 s_2^4 s_3 s_1^{32} - s_3^3 s_2^4 s_3 s_1^{33} - s_3^3 s_2^4 s_3 s_1^{34} - \\ & s_3^3 s_2^4 s_3 s_1^{35} - s_3^3 s_2^4 s_3 s_1^{36} - s_3^3 s_2^4 s_3 s_1^{37} - s_3^3 s_2^4 s_3 s_1^{38} - s_3^3 s_2^4 s_3 s_1^{39} - s_3^3 s_2^4 s_3 s_1^{40} - s_3^3 s_2^4 s_3 s_1^{41} - s_3^3 s_2^4 s_3 s_1^{42} - s_3^3 s_2^4 s_3 s_1^{43} - \\ & s_3^3 s_2^4 s_3 s_1^{44} - s_3^3 s_2^4 s_3 s_1^{45} - s_3^3 s_2^4 s_3 s_1^{46} - s_3^3 s_2^4 s_3 s_1^{47} - s_3^3 s_2^4 s_3 s_1^{48} - s_3^3 s_2^4 s_3 s_1^{49} - s_3^3 s_2^4 s_3 s_1^{50} - s_3^3 s_2^4 s_3 s_1^{51} - s_3^3 s_2^4 s_3 s_1^{52} - \\ & s_3^3 s_2^4 s_3 s_1^{53} - s_3^3 s_2^4 s_3 s_1^{54} - s_3^3 s_2^4 s_3 s_1^{55} - s_3^3 s_2^4 s_3 s_1^{56} - s_3^3 s_2^4 s_3 s_1^{57} - s_3^3 s_2^4 s_3 s_1^{58} - s_3^3 s_2^4 s_3 s_1^{59} - s_3^3 s_2^4 s_3 s_1^{60} - s_3^3 s_2^4 s_3 s_1^{61} - \\ & s_3^3 s_2^4 s_3 s_1^{62} - s_3^3 s_2^4 s_3 s_1^{63} - s_3^3 s_2^4 s_3 s_1^{64} - s_3^3 s_2^4 s_3 s_1^{65} - s_3^3 s_2^4 s_3 s_1^{66} - s_3^3 s_2^4 s_3 s_1^{67} - s_3^3 s_2^4 s_3 s_1^{68} - s_3^3 s_2^4 s_3 s_1^{69} - s_3^3 s_2^4 s_3 s_1^{70} - \\ & s_3^3 s_2^4 s_3 s_1^{71} - s_3^3 s_2^4 s_3 s_1^{72} - s_3^3 s_2^4 s_3 s_1^{73} - s_3^3 s_2^4 s_3 s_1^{74} - s_3^3 s_2^4 s_3 s_1^{75} - s_3^3 s_2^4 s_3 s_1^{76} - s_3^3 s_2^4 s_3 s_1^{77} - s_3^3 s_2^4 s_3 s_1^{78} - s_3^3 s_2^4 s_3 s_1^{79} - \\ & s_3^3 s_2^4 s_3 s_1^{80} - s_3^3 s_2^4 s_3 s_1^{81} - s_3^3 s_2^4 s_3 s_1^{82} - s_3^3 s_2^4 s_3 s_1^{83} - s_3^3 s_2^4 s_3 s_1^{84} - s_3^3 s_2^4 s_3 s_1^{85} - s_3^3 s_2^4 s_3 s_1^{86} - s_3^3 s_2^4 s_3 s_1^{87} - s_3^3 s_2^4 s_3 s_1^{88} - \\ & s_3^3 s_2^4 s_3 s_1^{89} - s_3^3 s_2^4 s_3 s_1^{90} - s_3^3 s_2^4 s_3 s_1^{91} - s_3^3 s_2^4 s_3 s_1^{92} - s_3^3 s_2^4 s_3 s_1^{93} - s_3^3 s_2^4 s_3 s_1^{94} - s_3^3 s_2^4 s_3 s_1^{95} - s_3^3 s_2^4 s_3 s_1^{96} - s_3^3 s_2^4 s_3 s_1^{97} - \\ & s_3^3 s_2^4 s_3 s_1^{98} - s_3^3 s_2^4 s_3 s_1^{99} - s_3^3 s_2^4 s_3 s_1^{100} - s_3^3 s_2^4 s_3 s_1^{101} - s_3^3 s_2^4 s_3 s_1^{102} - s_3^3 s_2^4 s_3 s_1^{103} - s_3^3 s_2^4 s_3 s_1^{104} - s_3^3 s_2^4 s_3 s_1^{105} - \\ & s_3^3 s_2^4 s_3 s_1^{106} - s_3^3 s_2^4 s_3 s_1^{107} - s_3^3 s_2^4 s_3 s_1^{108} - s_3^3 s_2^4 s_3 s_1^{109} - s_3^3 s_2^4 s_3 s_1^{110} - s_3^3 s_2^4 s_3 s_1^{111} - s_3^3 s_2^4 s_3 s_1^{112} - s_3^3 s_2^4 s_3 s_1^{113} - \\ & s_3^3 s_2^4 s_3 s_1^{114} - s_3^3 s_2^4 s_3 s_1^{115} - s_3^3 s_2^4 s_3 s_1^{116} - s_3^3 s_2^4 s_3 s_1^{117} - s_3^3 s_2^4 s_3 s_1^{118} - s_3^3 s_2^4 s_3 s_1^{119} - s_3^3 s_2^4 s_3 s_1^{120} - s_3^3 s_2^4 s_3 s_1^{121} - \\ & s_3^3 s_2^4 s_3 s_1^{122} - s_3^3 s_2^4 s_3 s_1^{123} - s_3^3 s_2^4 s_3 s_1^{124} - s_3^3 s_2^4 s_3 s_1^{125} - s_3^3 s_2^4 s_3 s_1^{126} - s_3^3 s_2^4 s_3 s_1^{127} - s_3^3 s_2^4 s_3 s_1^{128} - s_3^3 s_2^4 s_3 s_1^{129} - \\ & s_3^3 s_2^4 s_3 s_1^{130} - s_3^3 s_2^4 s_3 s_1^{131} - s_3^3 s_2^4 s_3 s_1^{132} - s_3^3 s_2^4 s_3 s_1^{133} - s_3^3 s_2^4 s_3 s_1^{134} - s_3^3 s_2^4 s_3 s_1^{135} - s_3^3 s_2^4 s_3 s_1^{136} - s_3^3 s_2^4 s_3 s_1^{137} - \\ & s_3^3 s_2^4 s_3 s_1^{138} - s_3^3 s_2^4 s_3 s_1^{139} - s_3^3 s_2^4 s_3 s_1^{140} - s_3^3 s_2^4 s_3 s_1^{141} - s_3^3 s_2^4 s_3 s_1^{142} - s_3^3 s_2^4 s_3 s_1^{143} - s_3^3 s_2^4 s_3 s_1^{144} - s_3^3 s_2^4 s_3 s_1^{145} - \\ & s_3^3 s_2^4 s_3 s_1^{146} - s_3^3 s_2^4 s_3 s_1^{147} - s_3^3 s_2^4 s_3 s_1^{148} - s_3^3 s_2^4 s_3 s_1^{149} - s_3^3 s_2^4 s_3 s_1^{150} - s_3^3 s_2^4 s_3 s_1^{151} - s_3^3 s_2^4 s_3 s_1^{152} - s_3^3 s_2^4 s_3 s_1^{153} - \\ & s_3^3 s_2^4 s_3 s_1^{154} - s_3^3 s_2^4 s_3 s_1^{155} - s_3^3 s_2^4 s_3 s_1^{156} - s_3^3 s_2^4 s_3 s_1^{157} - s_3^3 s_2^4 s_3 s_1^{158} - s_3^3 s_2^4 s_3 s_1^{159} - s_3^3 s_2^4 s_3 s_1^{160} - s_3^3 s_2^4 s_3 s_1^{161} - \\ & s_3^3 s_2^4 s_3 s_1^{162} - s_3^3 s_2^4 s_3 s_1^{163} - s_3^3 s_2^4 s_3 s_1^{164} - s_3^3 s_2^4 s_3 s_1^{165} - s_3^3 s_2^4 s_3 s_1^{166} - s_3^3 s_2^4 s_3 s_1^{167} - s_3^3 s_2^4 s_3 s_1^{168} - s_3^3 s_2^4 s_3 s_1^{169} - \\ & s_3^3 s_2^4 s_3 s_1^{170} - s_3^3 s_2^4 s_3 s_1^{171} - s_3^3 s_2^4 s_3 s_1^{172} - s_3^3 s_2^4 s_3 s_1^{173} - s_3^3 s_2^4 s_3 s_1^{174} - s_3^3 s_2^4 s_3 s_1^{175} - s_3^3 s_2^4 s_3 s_1^{176} - s_3^3 s_2^4 s_3 s_1^{177} - \\ & s_3^3 s_2^4 s_3 s_1^{178} - s_3^3 s_2^4 s_3 s_1^{179} - s_3^3 s_2^4 s_3 s_1^{180} - s_3^3 s_2^4 s_3 s_1^{181} - s_3^3 s_2^4 s_3 s_1^{182} - s_3^3 s_2^4 s_3 s_1^{183} - s_3^3 s_2^4 s_3 s_1^{184} - s_3^3 s_2^4 s_3 s_1^{185} - \\ & s_3^3 s_2^4 s_3 s_1^{186} - s_3^3 s_2^4 s_3 s_1^{187} - s_3^3 s_2^4 s_3 s_1^{188} - s_3^3 s_2^4 s_3 s_1^{189} - s_3^3 s_2^4 s_3 s_1^{190} - s_3^3 s_2^4 s_3 s_1^{191} - s_3^3 s_2^4 s_3 s_1^{192} - s_3^3 s_2^4 s_3 s_1^{193} - \\ & s_3^3 s_2^4 s_3 s_1^{194} - s_3^3 s_2^4 s_3 s_1^{195} - s_3^3 s_2^4 s_3 s_1^{196} - s_3^3 s_2^4 s_3 s_1^{197} - s_3^3 s_2^4 s_3 s_1^{198} - s_3^3 s_2^4 s_3 s_1^{199} - s_3^3 s_2^4 s_3 s_1^{200} - s_3^3 s_2^4 s_3 s_1^{201} - \\ & s_3^3 s_2^4 s_3 s_1^{202} - s_3^3 s_2^4 s_3 s_1^{203} - s_3^3 s_2^4 s_3 s_1^{204} - s_3^3 s_2^4 s_3 s_1^{205} - s_3^3 s_2^4 s_3 s_1^{206} - s_3^3 s_2^4 s_3 s_1^{207} - s_3^3 s_2^4 s_3 s_1^{208} - s_3^3 s_2^4 s_3 s_1^{209} - \\ & s_3^3 s_2^4 s_3 s_1^{210} - s_3^3 s_2^4 s_3 s_1^{211} - s_3^3 s_2^4 s_3 s_1^{212} - s_3^3 s_2^4 s_3 s_1^{213} - s_3^3 s_2^4 s_3 s_1^{214} - s_3^3 s_2^4 s_3 s_1^{215} - s_3^3 s_2^4 s_3 s_1^{216} - s_3^3 s_2^4 s_3 s_1^{217} - \\ & s_3^3 s_2^4 s_3 s_1^{218} - s_3^3 s_2^4 s_3 s_1^{219} - s_3^3 s_2^4 s_3 s_1^{220} - s_3^3 s_2^4 s_3 s_1^{221} - s_3^3 s_2^4 s_3 s_1^{222} - s_3^3 s_2^4 s_3 s_1^{223} - s_3^3 s_2^4 s_3 s_1^{224} - s_3^3 s_2^4 s_3 s_1^{225} - \\ & s_3^3 s_2^4 s_3 s_1^{226} - s_3^3 s_2^4 s_3 s_1^{227} - s_3^3 s_2^4 s_3 s_1^{228} - s_3^3 s_2^4 s_3 s_1^{229} - s_3^3 s_2^4 s_3 s_1^{230} - s_3^3 s_2^4 s_3 s_1^{231} - s_3^3 s_2^4 s_3 s_1^{232} - s_3^3 s_2^4 s_3 s_1^{233} - \\ & s_3^3 s_2^4 s_3 s_1^{234} - s_3^3 s_2^4 s_3 s_1^{235} - s_3^3 s_2^4 s_3 s_1^{236} - s_3^3 s_2^4 s_3 s_1^{237} - s_3^3 s_2^4 s_3 s_1^{238} - s_3^3 s_2^4 s_3 s_1^{239} - s_3^3 s_2^4 s_3 s_1^{240} - s_3^3 s_2^4 s_3 s_1^{241} - \\ & s_3^3 s_2^4 s_3 s_1^{242} - s_3^3 s_2^4 s_3 s_1^{243} - s_3^3 s_2^4 s_3 s_1^{244} - s_3^3 s_2^4 s_3 s_1^{245} - s_3^3 s_2^4 s_3 s_1^{246} - s_3^3 s_2^4 s_3 s_1^{247} - s_3^3 s_2^4 s_3 s_1^{248} - s_3^3 s_2^4 s_3 s_1^{249} - \\ & s_3^3 s_2^4 s_3 s_1^{250} - s_3^3 s_2^4 s_3 s_1^{251} - s_3^3 s_2^4 s_3 s_1^{252} - s_3^3 s_2^4 s_3 s_1^{253} - s_3^3 s_2^4 s_3 s_1^{254} - s_3^3 s_2^4 s_3 s_1^{255} - s_3^3 s_2^4 s_3 s_1^{256} - s_3^3 s_2^4 s_3 s_1^{257} - \\ & s_3^3 s_2^4 s_3 s_1^{258} - s_3^3 s_2^4 s_3 s_1^{259} - s_3^3 s_2^4 s_3 s_1^{260} - s_3^3 s_2^4 s_3 s_1^{261} - s_3^3 s_2^4 s_3 s_1^{262} - s_3^3 s_2^4 s_3 s_1^{263} - s_3^3 s_2^4 s_3 s_1^{264} - s_3^3 s_2^4 s_3 s_1^{265} - \\ & s_3^3 s_2^4 s_3 s_1^{266} - s_3^3 s_2^4 s_3 s_1^{267} - s_3^3 s_2^4 s_3 s_1^{268} - s_3^3 s_2^4 s_3 s_1^{269} - s_3^3 s_2^4 s_3 s_1^{270} - s_3^3 s_2^4 s_3 s_1^{271} - s_3^3 s_2^4 s_3 s_1^{272} - s_3^3 s_2^4 s_3 s_1^{273} - \\ & s_3^3 s_2^4 s_3 s_1^{274} - s_3^3 s_2^4 s_3 s_1^{275} - s_3^3 s_2^4 s_3 s_1^{276} - s_3^3 s_2^4 s_3 s_1^{277} - s_3^3 s_2^4 s_3 s_1^{278} - s_3^3 s_2^4 s_3 s_1^{279} - s_3^3 s_2^4 s_3 s_1^{280} - s_3^3 s_2^4 s_3 s_1^{281} - \\ & s_3^3 s_2^4 s_3 s_1^{282} - s_3^3 s_2^4 s_3 s_1^{283} - s_3^3 s_2^4 s_3 s_1^{284} - s_3^3 s_2^4 s_3 s_1^{285} - s_3^3 s_2^4 s_3 s_1^{286} - s_3^3 s_2^4 s_3 s$$

