M. P. KHARLAMOV
Volgograd Academy for Public Administration,
8, Gagarina St., Volgograd 400131, Russia
E-mail: mharlamov@vags.ru

BIFURCATION DIAGRAMS OF THE KOWALEVSKI TOP IN TWO CONSTANT FIELDS

Received April 09, 2005; accepted June 11, 2005 DOI: 10.1070/RD2005v010n04ABEH000321

The Kowalevski top in two constant fields is known as the unique profound example of an integrable Hamiltonian system with three degrees of freedom not reducible to a family of systems in fewer dimensions. As the first approach to topological analysis of this system we find the critical set of the integral map; this set consists of the trajectories with number of frequencies less than three. We obtain the equations of the bifurcation diagram in $\mathbf{R}^3$. A correspondence to the Appelrot classes in the classical Kowalevski problem is established. The admissible regions for the values of the first integrals are found in the form of some inequalities of general character and boundary conditions for the induced diagrams on energy levels.

Contents

1. Introduction

During the last 20 years the integrable case of S. Kowalevski [1] has received several generalizations. Among them a special place is given to the case [2] of rotation about a fixed point of a heavy electrically charged gyrostat in gravitational and electric fields of force. For a rigid body without gyrostatic effects the corresponding equations were first considered in [3] and interpreted as the equations of motion of a massive magnet subject to the gravity force and constant magnetic force fields. The mathematical model of superposition of such fields is referred to as *two constant fields* [4].

The case [2] does not have any explicit groups of symmetry and therefore provides an illustration of a physically realizable system with three degrees of freedom not admitting any obvious reduction to a family of systems with two degrees of freedom. The phase topology of irreducible systems has not been studied yet. The theory of n-dimensional integrable systems originated in [5] has not been further developed due to the absence, at that moment, of non-trivial natural examples.

The result [2] succeeded some previous publications dealing with rigid bodies and gyrostats satisfying the conditions of Kowalevski type: I. V. Komarov [6] has proved the complete integrability of

Mathematics Subject Classification: 70E17, 70G40
Key words and phrases: Kowalevski top, double field, critical set, bifurcation diagrams

the Kowalevski gyrostat in gravity force field by finding the first generalization of the Kowalevski integral K; the corresponding integral for the rigid body in two constant fields was pointed out by O. I. Bogoyavlensky [3]; this integral was upgraded to the case of gyrostat by H. Yehia [7]. Yet the analog of the Kowalevski case for two constant fields had not been considered integrable until A. G. Reyman and M. A. Semenov–Tian–Shansky [2] found the Lax representation with spectral parameter; this immediately led to the new integral generalizing the square of momentum integral for axially symmetric force fields.

Later, in a joint publication with A. I. Bobenko [4], the authors of [2] presented algebraic foundations for the integrability of multidimensional Kowalevski gyrostats and described a viable way of explicit integration using finite-band technique. For two constant fields this integration was never fulfilled.

This paper starts the investigation of three-dimensional phase topology of a rigid body of Kowalevski type in two constant fields.

2. Preliminaries

Consider a rigid body with fixed point O. Choose a trihedral at O rotating along with the body and refer to it all vector and tensor objects. Denote by $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ the canonical unit basis in $\mathbf{R}^3$; then the moving trihedral itself is represented as $O\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.

Constant field is a force field inducing the rotating moment about O of the form

$$\mathbf{r} \times \boldsymbol{\alpha} \tag{2.1}$$

with constant vector $\mathbf{r}$ and with $\boldsymbol{\alpha}$ corresponding to some physical vector fixed in inertial space; $\mathbf{r}$ points from O to the center of application of the field, $\boldsymbol{\alpha}$ is the field's intensity.

For two constant fields the rotating moment is $\mathbf{r}_1 \times \boldsymbol{\alpha} + \mathbf{r}_2 \times \boldsymbol{\beta}$. It can be represented as (2.1) if either $\mathbf{r}_1 \times \mathbf{r}_2 = 0$ or $\boldsymbol{\alpha} \times \boldsymbol{\beta} = 0$. In the sequel we suppose that

$$\mathbf{r}_1 \times \mathbf{r}_2 \neq 0, \quad \boldsymbol{\alpha} \times \boldsymbol{\beta} \neq 0. \tag{2.2}$$

Two constant fields satisfying (2.2) are said to be *independent.*

Introduce some notation.

Let $L(n,k)$ be the space of $n \times k$-matrices. Put $L(k) = L(k,k)$.

Identify $\mathbf{R}^6 = \mathbf{R}^3 \times \mathbf{R}^3$ with $L(3,2)$ by the isomorphism j that joins two columns

$$A = j(\mathbf{a}_1, \mathbf{a}_2) = \|\mathbf{a}_1 \ \mathbf{a}_2\| \in L(3,2), \qquad \mathbf{a}_1, \mathbf{a}_2 \in \mathbf{R}^3.$$

For the inverse map, we write

$$j^{-1}(A) = (\mathbf{c}_1(A), \mathbf{c}_2(A)) \in \mathbf{R}^3 \times \mathbf{R}^3, \qquad A \in L(3,2).$$

If $A, B \in L(3,2)$, $\mathbf{a} \in \mathbf{R}^3$, by definition, put

$$A \times B = \sum_{i=1}^{2} \mathbf{c}_i(A) \times \mathbf{c}_i(B) \in \mathbf{R}^3; \quad \mathbf{a} \times A = j(\mathbf{a} \times \mathbf{c}_1(A), \mathbf{a} \times \mathbf{c}_2(A)) \in L(3,2). \tag{2.3}$$

Lemma 1. *Let $\Lambda \in SO(3)$, $D \in GL(2,\mathbf{R})$, $\mathbf{a} \in \mathbf{R}^3$, $A, B \in L(3,2)$. Then*

$$\begin{aligned} &\Lambda(A \times B) = (\Lambda A) \times (\Lambda B); \quad (AD^{-1}) \times (BD^T) = A \times B; \\ &\Lambda(\mathbf{a} \times A) = (\Lambda \mathbf{a}) \times (\Lambda A); \quad \ \mathbf{a} \times (AD) = (\mathbf{a} \times A)D. \end{aligned}$$

The proof is by direct calculation.

Denote by $\mathbf{I}$ the inertia tensor of the body at O and by $\boldsymbol{\omega}$ the angular velocity. Using the notation (2.3) we write the Euler – Poisson equations of motion in the form

$$\mathbf{I}\boldsymbol{\omega}^{\cdot} = \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + A \times U, \qquad U^{\cdot} = -\boldsymbol{\omega} \times U. \tag{2.4}$$

Here $A = j(\mathbf{r}_1, \mathbf{r}_2)$ is a constant matrix, $U = j(\boldsymbol{\alpha}, \boldsymbol{\beta})$. The phase space of (2.4) is $\{(\boldsymbol{\omega}, U)\} = \mathbf{R}^3 \times L(3,2)$.

In fact, U in (2.4) is restricted by geometric integrals; that is, for some constant symmetric $C \in L(2)$

$$U^T U = C. \tag{2.5}$$

Let $\mathcal{O}$ represent the set (2.5) in $L(3,2)$. In order to emphasize the C-dependence, we write $\mathcal{O} = \mathcal{O}(C)$.

Let $S = (\mathbf{I}, A, C)$. Denote by X_S the vector field on $\mathbf{R}^3 \times \mathcal{O}(C)$ corresponding to the system (2.4).

Associate to $\Lambda \in SO(3)$, $D \in GL(2, \mathbf{R})$ the linear automorphisms $\Psi(\Lambda, D)$ and $\psi(\Lambda, D)$ of $\mathbf{R}^3 \times L(3,2)$ and $L(3) \times L(3,2) \times L(2)$

$$\begin{aligned} &\Psi(\Lambda, D)(\boldsymbol{\omega}, U) = (\Lambda\boldsymbol{\omega}, \Lambda U D^T), \\ &\psi(\Lambda, D)(\mathbf{I}, A, C) = (\Lambda \mathbf{I} \Lambda^T, \Lambda A D^{-1}, D C D^T). \end{aligned} \tag{2.6}$$

It is easy to see that (2.5) and (2.6) imply $\Psi(\Lambda, D)(\mathbf{R}^3 \times \mathcal{O}(C)) = \mathbf{R}^3 \times \mathcal{O}(DCD^T)$. Using Lemma 1 we obtain the following statement.

Lemma 2. *For each* $(\Lambda, D) \in SO(3) \times GL(2, \mathbf{R})$*, we have*

$$\Psi(\Lambda, D)_*(X_S(v)) = X_{\psi(\Lambda,D)(S)}(\Psi(\Lambda, D)(v)), \qquad v \in \mathbf{R}^3 \times \mathcal{O}(C).$$

Thus, any two problems of rigid body dynamics in two constant fields (for short, RBD-problems) determined by the sets of parameters S and $\psi(\Lambda, D)(S)$ are completely equivalent.

Let us call an RBD-problem *canonical* if the centers of application of forces lie on the first two axes of the moving trihedral at unit distance from O and the intensities of the forces are orthogonal to each other.

Proposition 1. *For each RBD-problem with independent forces there exists an equivalent canonical problem. Moreover, in both equivalent problems the centers of application of forces belong to the same plane in the body containing the fixed point.*

Proof. Let the RBD-problem determined by the set of parameters $S = (\mathbf{I}, A, C)$ satisfy (2.2). This means that the symmetric matrices $A_* = (A^T A)^{-1}$ and C are positively definite.

According to the well-known fact from linear algebra, A_* and C can be reduced, respectively, to the identity matrix and to a diagonal matrix via the same conjugation operator

$$DA_*D^T = E, \quad DCD^T = \mathrm{diag}\{a^2, b^2\}, \qquad D \in GL(2, \mathbf{R}),\ a, b \in \mathbf{R}_+.$$

Then $\mathbf{c}_1(AD^{-1})$ and $\mathbf{c}_2(AD^{-1})$ form an orthonormal pair in $\mathbf{R}^3$. There exists $\Lambda \in SO(3)$ such that $\Lambda \mathbf{c}_i(AD^{-1}) = \mathbf{e}_i\,(i = 1, 2)$. The first statement is obtained by applying Lemma 2 with the previously chosen Λ, D to the initial vector field X_S.

To finish the proof, notice that the transformation $A \mapsto AD^{-1}$ preserves the plane spanning $\mathbf{c}_1(A)$, $\mathbf{c}_2(A)$. The matrix Λ in (2.6) represents the change of the moving trihedral. Therefore, if $\mathbf{a} \in \mathbf{R}^3$ represents some physical vector in the initial problem, then $\Lambda\mathbf{a}$ is the same vector with respect to the body in the equivalent problem. ■

REMARK 1. The fact that any RBD-problem can be reduced to the problem with one of the pairs $\mathbf{r}_1, \mathbf{r}_2$ or $\boldsymbol{\alpha}, \boldsymbol{\beta}$ orthonormal is known from [8]. Simultaneous orthogonalization of both pairs crucially simplifies calculations below.

It follows from Proposition 1 that, without loss of generality, for independent forces we may suppose

$$\mathbf{r}_1 = \mathbf{e}_1, \quad \mathbf{r}_2 = \mathbf{e}_2, \tag{2.7}$$

$$\boldsymbol{\alpha} \cdot \boldsymbol{\alpha} = a^2, \ \boldsymbol{\beta} \cdot \boldsymbol{\beta} = b^2, \ \boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0. \tag{2.8}$$

Change, if necessary, the order of $\mathbf{e}_1, \mathbf{e}_2$ (with simultaneous change of the direction of $\mathbf{e}_3$) to obtain $a \geqslant b > 0$.

Consider a dynamically symmetric top in two constant fields with the centers of application of forces in the equatorial plane of its inertia ellipsoid. Choose a moving trihedral such that $O\mathbf{e}_3$ is the symmetry axis. Then the inertia tensor $\mathbf{I}$ becomes diagonal. Let $a = b$. For any $\Theta \in SO(2)$ denote by $\hat{\Theta} \in SO(3)$ the corresponding rotation of $\mathbf{R}^3$ about $O\mathbf{e}_3$. Take in (2.6) $\Lambda = \hat{\Theta}$, $D = \Theta$. Under the conditions (2.7), (2.8) $\psi = \mathrm{Id}$ and Ψ becomes the symmetry group. The system (2.4) has the cyclic integral $\mathbf{I}\boldsymbol{\omega} \cdot (a^2\mathbf{e}_3 - \boldsymbol{\alpha} \times \boldsymbol{\beta})$ pointed out in [7] for the analog of the Kowalevski case. Therefore it is possible to reduce such an RBD-problem to a family of systems with two degrees of freedom.

Let us call an RBD-problem *irreducible* if for its canonical representation (2.7), (2.8) the following inequality holds

$$a > b > 0. \tag{2.9}$$

The following statements are needed in the future; they also reveal some features of a wide class of RBD-problems.

Lemma 3. *In an irreducible RBD-problem, the body has exactly four equilibria.*

Proof. The set of singular points of (2.4) is defined by $\boldsymbol{\omega} = 0$, $A \times U = 0$. For the equivalent canonical problem with (2.7)

$$\mathbf{e}_1 \times \boldsymbol{\alpha} + \mathbf{e}_2 \times \boldsymbol{\beta} = 0. \tag{2.10}$$

Then the four vectors in (2.10) are parallel to the same plane and $|\mathbf{e}_1 \times \boldsymbol{\alpha}| = |\mathbf{e}_2 \times \boldsymbol{\beta}|$. With (2.8), (2.9) this equality yields

$$\boldsymbol{\alpha} = \pm a\mathbf{e}_1, \ \boldsymbol{\beta} = \pm b\mathbf{e}_2. \tag{2.11}$$

From mechanical point of view, the result is absolutely clear: none of the orthogonal forces with unequal intensities and 'orthonormal' centers of application can produce a non-zero moment at an equilibrium. ∎

Lemma 4. *Let an irreducible RBD-problem in its canonical form have the diagonal inertia tensor* $\mathbf{I}$*. Then the body has the following families of periodic motions of pendulum type*

$$\begin{gathered} \boldsymbol{\alpha} \equiv \pm a\mathbf{e}_1, \quad \boldsymbol{\omega} = \dot{\varphi}\mathbf{e}_1, \quad \boldsymbol{\beta} = b(\mathbf{e}_2 \cos\varphi - \mathbf{e}_3 \sin\varphi), \\ 2\ddot{\varphi} = -b\sin\varphi; \end{gathered} \tag{2.12}$$

$$\begin{gathered} \boldsymbol{\beta} \equiv \pm b\mathbf{e}_2, \quad \boldsymbol{\omega} = \dot{\varphi}\mathbf{e}_2, \quad \boldsymbol{\alpha} = a(\mathbf{e}_1 \cos\varphi + \mathbf{e}_3 \sin\varphi), \\ 2\ddot{\varphi} = -a\sin\varphi; \end{gathered} \tag{2.13}$$

$$\begin{gathered} \boldsymbol{\alpha} \times \boldsymbol{\beta} \equiv \pm ab\mathbf{e}_3, \quad \boldsymbol{\omega} = \dot{\varphi}\mathbf{e}_3, \\ \boldsymbol{\alpha} = a(\mathbf{e}_1 \cos\varphi - \mathbf{e}_2 \sin\varphi), \quad \boldsymbol{\beta} = \pm b(\mathbf{e}_1 \sin\varphi + \mathbf{e}_2 \cos\varphi), \\ \ddot{\varphi} = -(a \pm b)\sin\varphi. \end{gathered} \tag{2.14}$$

The proof is obvious. Note that in the case considered the pointed out families are the only motions with constant direction of the angular velocity. In particular, the body in two independent constant fields does not have any uniform rotations.

3. Critical set of the Kowalevski top in two constant fields

Suppose that the irreducible RBD-problem has a diagonal inertia tensor with principal moments of inertia satisfying the ratio 2:2:1; then we obtain the integrable case [2] of the Kowalevski top in two constant fields. By an appropriate choice of measurement units, we present equations (2.4) in scalar form

$$\begin{gathered} 2\dot{\omega_1} = \omega_2\omega_3 + \beta_3,\ 2\dot{\omega_2} = -\omega_1\omega_3 - \alpha_3,\ \dot{\omega_3} = \alpha_2 - \beta_1, \\ \dot{\alpha_1} = \alpha_2\omega_3 - \alpha_3\omega_2,\ \dot{\beta_1} = \beta_2\omega_3 - \beta_3\omega_2, \\ \dot{\alpha_2} = \alpha_3\omega_1 - \alpha_1\omega_3,\ \dot{\beta_2} = \beta_3\omega_1 - \beta_1\omega_3, \\ \dot{\alpha_3} = \alpha_1\omega_2 - \alpha_2\omega_1,\ \dot{\beta_3} = \beta_1\omega_2 - \beta_2\omega_1. \end{gathered} \tag{3.1}$$

The phase space is $P^6 = \mathbf{R}^3 \times \mathcal{O}$, where $\mathcal{O} \subset \mathbf{R}^3 \times \mathbf{R}^3$ is defined by (2.8); $\mathcal{O}$ is diffeomorphic to $SO(3)$.

The complete set of first integrals in involution on P^6 consists of the energy integral H, the generalized Kowalevski integral K [3], and the integral G found in [2]:

$$\begin{aligned} H &= \omega_1^2 + \omega_2^2 + \frac{1}{2}\omega_3^2 - (\alpha_1 + \beta_2), \\ K &= (\omega_1^2 - \omega_2^2 + \alpha_1 - \beta_2)^2 + (2\omega_1\omega_2 + \alpha_2 + \beta_1)^2, \\ G &= \frac{1}{4}(2\alpha_1\omega_1 + 2\alpha_2\omega_2 + \alpha_3\omega_3)^2 + \frac{1}{4}(2\beta_1\omega_1 + 2\beta_2\omega_2 + \beta_3\omega_3)^2 + \\ &\quad + \frac{1}{2}\omega_3(2\gamma_1\omega_1 + 2\gamma_2\omega_2 + \gamma_3\omega_3) - b^2\alpha_1 - a^2\beta_2. \end{aligned} \tag{3.2}$$

Here we denote by γ_i the components of $\boldsymbol{\gamma} = \boldsymbol{\alpha} \times \boldsymbol{\beta}$ relative to the moving basis.

Introduce the integral map

$$J = G \times K \times H : P^6 \to \mathbf{R}^3. \tag{3.3}$$

Let $\sigma \subset P^6$ be the set of critical points of J. By definition, the bifurcation diagram of J is the subset $\Sigma \subset \mathbf{R}^3$ over which J fails to be locally trivial; Σ determines the cases when the topological type of the integral manifolds

$$J_c = J^{-1}(c), \quad c = (g, k, h) \in \mathbf{R}^3 \tag{3.4}$$

changes. Finding the critical set σ and the bifurcation diagram is the necessary step in the topological analysis of the problem as a whole.

It follows from the Liouville – Arnold theorem that for $c \notin \Sigma$ the manifold (3.4), if not empty, is a union of three-dimensional tori. The considered Hamiltonian system is non-degenerate (at least for sufficiently small values of b); then the trajectories on such a torus are quasi-periodic with three almost everywhere independent frequencies. The critical set σ is invariant under the phase flow and consists of trajectories with number of frequencies less than three. These trajectories are called *critical motions*. For a generic value $c \in \Sigma$ the set $J_c \cap \sigma$ consists of two-dimensional tori. The dynamical system induced on the union of such tori for c in some open subset in Σ is a Hamiltonian system with two degrees of freedom. Vice versa, let M be a submanifold of P^6, $\dim M = 4$, and suppose that the induced system on M is Hamiltonian. Then, obviously, $M \subset \sigma$. This speculation gives a useful tool to find out whether a common level of functions consists of critical points of J.

Lemma 5. *Consider a system of equations*

$$f_1 = 0,\ f_2 = 0 \tag{3.5}$$

on a domain W open in P^6. Let X be the vector field on P^6 corresponding to (3.1) *and $M \subset W$ defined by* (3.5). *Suppose*

(i) f_1 and f_2 are smooth functions independent on M;

(ii) $Xf_1 = 0,\ Xf_2 = 0$ on M;

(iii) the Poisson bracket $\{f_1, f_2\}$ is non-zero almost everywhere on M.

Then M consists of critical points of the map J.

Proof. Conditions (i), (ii) imply that M is a smooth four-dimensional manifold invariant under the restriction of the phase flow to the open set W. Condition (iii) means that the closed 2-form induced on M by the symplectic structure on P^6 is almost everywhere non-degenerate. Thus the flow on M is almost everywhere Hamiltonian with two degrees of freedom. It inherits the property of complete integrability. Then almost all its integral manifolds consist of two-dimensional tori and necessarily lie in σ. Since M is closed in W and σ is closed in P^6, we conclude that $M \subset \sigma$. ■

Two systems of the type (3.5) are known. The first one was pointed out in [3]. It is the zero level of the integral K. The condition $K = 0$ leads to two independent equations defining the smooth four-dimensional manifold $\mathfrak{M} \subset \sigma$. It is shown in [9] that the 2-form induced on $\mathfrak{M}$ by the symplectic structure on P^6 is degenerate on the surface of codimension 1.

The second critical subset $\mathfrak{N} \subset \sigma$ was found in [10] in the form of a system of two equations satisfying the conditions of Lemma 5. The functions in these equations have essential singularities at the points

$$\alpha_1 = \beta_2, \quad \alpha_2 = -\beta_1. \tag{3.6}$$

The set $\mathfrak{N}$ was investigated in [11]. It was shown that $\mathfrak{N}$ is the set of critical points of some smooth function F on P^6. Then $\mathfrak{N}$ is stratified by the rank of Hesse's matrix of F and fails to be a smooth four-dimensional manifold at some points of the set (3.6). In particular, it cannot be defined by any global system of two independent equations. In this case the induced 2-form also has degenerate points even in the smooth part of $\mathfrak{N}$.

The following result completes the description of the critical set σ by adding a new invariant subset $\mathfrak{O} \subset P^6$; $\mathfrak{O}$ is almost everywhere a smooth four-dimensional manifold. Note that the sets $\mathfrak{M}$, $\mathfrak{N}$ and $\mathfrak{O}$ have pairwise nonempty intersections corresponding to bifurcations of critical integral manifolds of the induced 'almost Hamiltonian' systems with two degrees of freedom.

Let us introduce the following notation

$$\begin{gathered} p^2 = a^2 + b^2, \; r^2 = a^2 - b^2; \\ \xi_1 = \alpha_1 - \beta_2, \; \xi_2 = \alpha_2 + \beta_1, \; \eta_1 = \alpha_1 + \beta_2, \; \eta_2 = \alpha_2 - \beta_1. \end{gathered}$$

Theorem 1. *The set of critical points of the integral map* (3.3) *consists of the following subsets in P^6:*

1) the set $\mathfrak{M}$ defined by the system

$$Z_1 = 0, \; Z_2 = 0 \tag{3.7}$$

with

$$Z_1 = \omega_1^2 - \omega_2^2 + \xi_1, \; Z_2 = 2\omega_1\omega_2 + \xi_2; \tag{3.8}$$

2) the set $\mathfrak{N}$ defined by the system

$$F_1 = 0, \; F_2 = 0, \quad \xi_1^2 + \xi_2^2 \neq 0 \tag{3.9}$$

with

$$\begin{aligned} F_1 &= (\xi_1^2 + \xi_2^2)\omega_3 - 2[(\xi_1\omega_1 + \xi_2\omega_2)\alpha_3 + (\xi_2\omega_1 - \xi_1\omega_2)\beta_3], \\ F_2 &= (\xi_1^2 - \xi_2^2)(2\omega_1\omega_2 + \xi_2) - 2\xi_1\xi_2(\omega_1^2 - \omega_2^2 + \xi_1), \end{aligned} \tag{3.10}$$

and by the system

$$\begin{gathered} \xi_1 = \xi_2 = 0, \; \alpha_3 = \pm r, \; \beta_3 = 0, \; \eta_1^2 + \eta_2^2 = 2(p^2 - r^2), \\ (\omega_1^2 + \omega_2^2)(\alpha_3\omega_3 + \eta_1\omega_1 + \eta_2\omega_2) + r^2\omega_1 = 0; \end{gathered} \tag{3.11}$$

3) the set $\mathfrak{O}$ defined by the system

$$R_1 = 0, \; R_2 = 0 \tag{3.12}$$

with

$$\begin{aligned} R_1 &= (\alpha_3\omega_2 - \beta_3\omega_1)\omega_3 + 2\xi_1\omega_1\omega_2 - \xi_2(\omega_1^2 - \omega_2^2) + \eta_2(\omega_1^2 + \omega_2^2), \\ R_2 &= (\alpha_3\omega_1 + \beta_3\omega_2)\omega_3^2 + [\alpha_3^2 + \beta_3^2 + \xi_1(\omega_1^2 - \omega_2^2) + 2\xi_2\omega_1\omega_2 + \\ &\quad + \eta_1(\omega_1^2 + \omega_2^2)]\omega_3 + 2[\xi_1(\alpha_3\omega_1 - \beta_3\omega_2) + \xi_2(\alpha_3\omega_2 + \beta_3\omega_1)]. \end{aligned} \tag{3.13}$$

Proof. Introduce the change of variables [10] ($i^2 = -1$)

$$\begin{gathered} x_1 = \xi_1 + i\xi_2, \quad x_2 = \xi_1 - i\xi_2, \\ y_1 = \eta_1 + i\eta_2, \quad y_2 = \eta_1 - i\eta_2, \\ z_1 = \alpha_3 + i\beta_3, \quad z_2 = \alpha_3 - i\beta_3, \\ w_1 = \omega_1 + i\omega_2, \quad w_2 = \omega_1 - i\omega_2, \quad w_3 = \omega_3. \end{gathered} \tag{3.14}$$

The system (3.1) takes the form

$$\begin{gathered} x_1' = -x_1w_3 + z_1w_1, \quad x_2' = x_2w_3 - z_2w_2, \\ y_1' = -y_1w_3 + z_2w_1, \quad y_2' = y_2w_3 - z_1w_2, \\ 2z_1' = x_1w_2 - y_2w_1, \quad 2z_2' = -x_2w_1 + y_1w_2, \\ 2w_1' = -(w_1w_3 + z_1), \quad 2w_2' = w_2w_3 + z_2, \quad 2w_3' = y_2 - y_1. \end{gathered} \tag{3.15}$$

Here the prime stands for $d/d(it)$.

Denote by V^9 the subspace of $\mathbf{C}^9$ defined by (3.14). On V^9, equations (2.8) of the phase space P^6 become

$$\begin{gathered} z_1^2 + x_1y_2 = r^2, \quad z_2^2 + x_2y_1 = r^2, \\ x_1x_2 + y_1y_2 + 2z_1z_2 = 2p^2. \end{gathered} \tag{3.16}$$

By virtue of (3.14) and (3.16) the integrals (3.2) take the form

$$\begin{aligned} H &= \frac{1}{2}w_3^2 + w_1w_2 - \frac{1}{2}(y_1 + y_2), \\ K &= (w_1^2 + x_1)(w_2^2 + x_2), \\ G &= \frac{1}{4}(p^2 - x_1x_2)w_3^2 + \frac{1}{2}(x_2z_1w_1 + x_1z_2w_2)w_3 + \\ &\quad + \frac{1}{4}(x_2w_1 + y_1w_2)(y_2w_1 + x_1w_2) - \frac{1}{4}p^2(y_1 + y_2) + \frac{1}{4}r^2(x_1 + x_2). \end{aligned} \tag{3.17}$$

Let f be an arbitrary function on V^9. For brevity, the term 'critical point of f' will always mean a critical point of the restriction of f to P^6. Similarly, df means the restriction of the differential of f to the set of vectors tangent to P^6.

While calculating critical points of various functions (in the above sense), it is convenient to avoid introducing Lagrange multipliers for the restrictions (3.16). Notice that the following vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial w_1}, \; X_2 = \frac{\partial}{\partial w_2}, \; X_3 = \frac{\partial}{\partial w_3}, \\ Y_1 &= z_2\frac{\partial}{\partial x_2} + z_1\frac{\partial}{\partial y_2} - \frac{1}{2}x_1\frac{\partial}{\partial z_1} - \frac{1}{2}y_1\frac{\partial}{\partial z_2}, \\ Y_2 &= z_1\frac{\partial}{\partial x_1} + z_2\frac{\partial}{\partial y_1} - \frac{1}{2}y_2\frac{\partial}{\partial z_1} - \frac{1}{2}x_2\frac{\partial}{\partial z_2}, \\ Y_3 &= x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + y_1\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial y_2} \end{aligned}$$

are tangent to $P^6 \subset V^9$ and linearly independent at any point of P^6. Then the set of critical points of f is defined by the system of equations

$$X_1f = 0, \quad X_2f = 0, \quad X_3f = 0, \tag{3.18}$$

$$Y_1f = 0, \quad Y_2f = 0, \quad Y_3f = 0. \tag{3.19}$$

1. Apply (3.18) and (3.19) to $f = K$. Then a critical point of K satisfies either

$$w_1^2 + x_1 = 0,\ w_2^2 + x_2 = 0 \tag{3.20}$$

or

$$w_1 = w_2 = 0,\ z_1 = z_2 = 0. \tag{3.21}$$

The system (3.20) coincides with (3.7) and the only invariant set generated by (3.21) consists of all points of the trajectories (2.14). Such points satisfy (3.12).

2. Consider the regular points of K at which H and K are dependent. Applying (3.18) to $f = H + sK$ with Lagrange multiplier s we immediately obtain $w_3 = 0$. Then from (3.15) we come to solutions (2.12), (2.13). Along the corresponding trajectories both conditions (3.9), (3.12) are valid.

3. We now assume that H and K are independent. Introduce the function with Lagrange multipliers τ, s

$$L = 2G + (\tau - p^2)H + sK.$$

The multiplier of G is non-zero by assumption. The term with p^2 is added for convenience.

The set $\sigma_0 \subset \sigma$ of the points satisfying for some τ, s the condition

$$2dG + (\tau - p^2)dH + sdK = 0 \tag{3.22}$$

is preserved by the phase flow of (3.15). Applying the corresponding Lie derivative to (3.22) gives

$$\tau' dH + s' dK = 0.$$

Since dH and dK are supposed to be linearly independent, on σ_0 we obtain

$$\tau' = 0,\ s' = 0. \tag{3.23}$$

Hence τ, s are partial integrals of motion on the invariant surface σ_0.

Equations (3.18) with $f = L$ give

$$\begin{aligned} x_2 z_1 w_3 + x_2 y_2 w_1 + (\tau - z_1 z_2) w_2 + 2 s w_1 (w_2^2 + x_2) = 0,\\ x_1 z_2 w_3 + (\tau - z_1 z_2) w_1 + x_1 y_1 w_2 + 2 s w_2 (w_1^2 + x_1) = 0; \end{aligned} \tag{3.24}$$

$$(\tau - x_1 x_2) w_3 + x_2 z_1 w_1 + x_1 z_2 w_2 = 0. \tag{3.25}$$

First consider the case (3.6). From (3.16) we come to the following values

$$x_1 = x_2 = 0,\ z_1^2 = z_2^2 = r^2,\ y_1 y_2 = 2(p^2 - r^2). \tag{3.26}$$

Equations (3.24) and (3.25) hold if $w_1 w_2 = 0$ or $w_3 = 0$. If either of these equalities takes place on some interval of time (and hence identically), then we obtain one of the solutions (2.12)–(2.14).

Let $w_1 w_2 \neq 0, w_3 \neq 0$ at some point satisfying (3.26). Then (3.24) and (3.25) yield

$$\tau = 0,\ s = r^2/(2 w_1 w_2). \tag{3.27}$$

Since z_1 and z_2 are complex conjugates of each other, it follows from (3.26) that they are real and equal. Denote their value by $z = \pm r$.

With (3.26) and (3.27) the system (3.19) reduces to a single equation

$$w_1 w_2 [2 z w_3 + (w_2 y_1 + w_1 y_2)] + r^2 (w_1 + w_2) = 0, \tag{3.28}$$

which corresponds to (3.11).

Note that (3.28) is obtained from (3.9) as $\rho = \sqrt{\xi_1^2 + \xi_2^2}$ tends to zero only after dividing by the maximal available power of ρ. Thus, at the points (3.6) the system (3.9), without the assumption that $\rho \neq 0$, has extra solutions not belonging to σ.

Suppose $x_1 x_2 \neq 0$. The determinant of (3.24) with respect to τ, s equals $\delta = 2(x_1 w_2^2 - x_2 w_1^2)$. Let $\delta \equiv 0$ on some interval of time; calculating the derivatives of this identity in virtue of (3.15), we obtain one of the cases (3.20), (3.21). Therefore we may assume that $\delta \neq 0$. Then (3.24) implies

$$s = \frac{1}{2(x_1 w_2^2 - x_2 w_1^2)}[(x_2 z_1 w_1 - x_1 z_2 w_2) w_3 + x_2 y_2 w_1^2 - x_1 y_1 w_2^2], \tag{3.29}$$

$$\begin{aligned} \tau = z_1 z_2 + \frac{1}{x_1 w_2^2 - x_2 w_1^2}\{[x_1 x_2 (z_2 w_1 - z_1 w_2) - w_1 w_2 (x_2 z_1 w_1 - x_1 z_2 w_2)] w_3 - \\ - w_1 w_2 (x_2 y_2 w_1^2 - x_1 y_1 w_2^2) + x_1 x_2 w_1 w_2 (y_1 - y_2)\}. \end{aligned} \tag{3.30}$$

Eliminating τ from (3.25) and (3.30) we obtain $S_1 = 0$, where

$$\begin{aligned} S_1 = {} & [x_1 x_2 (z_2 w_1 - z_1 w_2) - w_1 w_2 (x_2 z_1 w_1 - x_1 z_2 w_2)] w_3^2 + \\ & + [(x_1 x_2 - z_1 z_2)(x_2 w_1^2 - x_1 w_2^2) - w_1 w_2 (x_2 y_2 w_1^2 - x_1 y_1 w_2^2) + \\ & + x_1 x_2 w_1 w_2 (y_1 - y_2)] w_3 - (x_2 w_1^2 - x_1 w_2^2)(x_2 z_1 w_1 + x_1 z_2 w_2). \end{aligned}$$

Next we solve (3.25) for τ and calculate the derivative τ' in virtue of (3.15). According to (3.23) we must have $S_2 = 0$, where

$$\begin{aligned} S_2 = {} & (x_2 z_1 w_1 - x_1 z_2 w_2) w_3^2 + (x_2 y_2 w_1^2 - x_1 y_1 w_2^2 + x_2 z_1^2 - x_1 z_2^2) w_3 - \\ & - (y_1 - y_2)(x_2 z_1 w_1 + x_1 z_2 w_2). \end{aligned}$$

Notice that

$$S_1 + w_1 w_2 S_2 = F_1 R.$$

Here

$$F_1 = x_1 x_2 w_3 - (x_2 z_1 w_1 + x_1 z_2 w_2) \tag{3.31}$$

is the first function from (3.10). The function

$$R = (z_2 w_1 - z_1 w_2) w_3 + x_2 w_1^2 - x_1 w_2^2 + w_1 w_2 (y_1 - y_2) \tag{3.32}$$

is a multiple of the first function from (3.13), precisely, $R = 2iR_1$. Thus on the trajectories consisting of critical points, we have either $F_1 \equiv 0$ or $R_1 \equiv 0$. Calculating the derivatives of these identities in virtue of (3.15) we obtain (3.9) and (3.12), respectively. Hence (3.9) and (3.12) provide necessary conditions for a point to belong to σ_0.

To prove sufficiency, it is enough to check (3.19). We avoid this technically complicated procedure and only notice that the systems (3.9) and (3.12) satisfy the assumptions of Lemma 5. ■

The phase topology of the induced system on $\mathfrak{M}$ was studied in [9]. The system of invariant relations (3.9) corresponds to that found in [10]. In the paper [11] the equations of motion on $\mathfrak{N}$ are separated and the initial phase variables are expressed via two auxiliary variables, the latter being elliptic functions of time. The motions on $\mathfrak{M}$ generalize those of the 1st Appelrot class (Delone class) of the Kowalevski problem [12]. As b tends to zero the motions on $\mathfrak{N}$, as shown in [10], convert to the so-called *especially marvellous* motions of the 2nd and 3rd classes of Appelrot [12]. The set defined by the system (3.12) was not pointed out earlier.

To find the classical analog of the set $\mathfrak{O}$, put $\boldsymbol{\beta} = 0$ in (3.13). Then $\xi_1 = \eta_1 = \alpha_1$, $\xi_2 = \eta_2 = \alpha_2$ and we obtain

$$R_1 = 2\ell\omega_2, \quad R_2 = 2\ell(\omega_1\omega_3 + \alpha_3),$$

where $2\ell = 2\alpha_1\omega_1 + 2\alpha_2\omega_2 + \alpha_3\omega_3$ is a constant of the momentum integral existing in the case of one force field. Therefore equations (3.12) yield either $\ell = 0$ or

$$\omega_2 = 0, \; \omega_1\omega_3 + \alpha_3 = 0. \tag{3.33}$$

The condition $\ell = 0$ follows from the fact that when $\boldsymbol{\beta} = 0$ the integral G takes the value ℓ^2. Equations (3.33) define *especially marvellous* motions of the 4th class of Appelrot [12].

4. Bifurcation diagram

Since all common levels of the first integrals (3.2) are compact, the bifurcation diagram Σ coincides with the set of critical values of the map (3.3), that is, $\Sigma = J(\sigma)$.

Let $\gamma = |\boldsymbol{\alpha} \times \boldsymbol{\beta}|$. According to (2.8), $\gamma = ab$.

Denote by Δ the region of existence of motions, that is, the set of $c = (g, k, h) \in \mathbf{R}^3$ for which the integral manifolds (3.4) are not empty.

Theorem 2. *The bifurcation diagram of the map $G \times K \times H$ is the intersection of Δ with the union of the surfaces*

$$\Gamma_1: \quad k = 0; \tag{4.1}$$

$$\Gamma_2: \quad p^2 h - 2g + r^2\sqrt{k} = 0; \tag{4.2}$$

$$\Gamma_3: \quad p^2 h - 2g - r^2\sqrt{k} = 0; \tag{4.3}$$

$$\Gamma_4: \quad \begin{cases} k = 3s^2 - 4hs + p^2 + h^2 - \dfrac{\gamma^2}{s^2} \\ g = -s^3 + hs^2 + \dfrac{\gamma^2}{s} \end{cases}, \quad s \in \mathbf{R}\backslash\{0\} \tag{4.4}$$

and the line segment

$$\Gamma_5: \quad g = \gamma h, \; k = p^2 - 2\gamma, \; h^2 \leqslant 4\gamma. \tag{4.5}$$

In the parametric representation of the surface Γ_4 the parameter s stands for a multiple root of the polynomial

$$\Phi(s) = s^4 - 2hs^3 + (h^2 + p^2 - k)s^2 - 2gs + \gamma^2. \tag{4.6}$$

Proof. 1. The equation of the surface (4.1) follows immediately from (3.7), (3.8), and the expression of K in (3.2).

2. Relations (4.2), (4.3) are equivalent to

$$(p^2 h - 2g)^2 - r^4 k = 0. \tag{4.7}$$

Introduce the function

$$F = (p^2 H - 2G)^2 - r^4 K.$$

For $x_1 x_2 \neq 0$ denote

$$U_1 = \sqrt{\frac{x_2}{x_1}(w_1^2 + x_1)}, \quad U_2 = \sqrt{\frac{x_1}{x_2}(w_2^2 + x_2)} \qquad (U_2 = \overline{U_1}). \tag{4.8}$$

From representations (3.17) and (3.31) we obtain

$$\begin{aligned} p^2 H - 2G + r^2\sqrt{K} &= \frac{1}{2x_1x_2}F_1^2 + 2r^2(\operatorname{Im} U_1)^2, \\ p^2 H - 2G - r^2\sqrt{K} &= \frac{1}{2x_1x_2}F_1^2 - 2r^2(\operatorname{Re} U_1)^2. \end{aligned} \tag{4.9}$$

Here $\sqrt{K}$ is the principal square root of K.

The equation of the zero level of F splits into two distinct equations

$$F_1^2 + 4r^2 x_1 x_2 (\operatorname{Im} U_1)^2 = 0, \tag{4.10}$$
$$F_1^2 - 4r^2 x_1 x_2 (\operatorname{Re} U_1)^2 = 0. \tag{4.11}$$

From (3.10) and (4.8) we have

$$F_2 = \frac{x_1 x_2}{2i}(U_1^2 - U_2^2) = 2x_1 x_2 \operatorname{Im} U_1 \operatorname{Re} U_1.$$

Thus the solutions of (3.9) satisfy either (4.10) or (4.11) and therefore lie on the zero level of the function F. The corresponding values of the first integrals satisfy (4.7).

From (3.17) it follows that (4.7) holds for all points of the phase space such that $x_1 x_2 = 0$ (regardless of their critical or regular nature). Hence (4.7) holds for the points (3.11).

3. Consider the system (3.12). In terms of the variables (3.14) it is equivalent to the following equations:

$$R = 0, \quad R_* = 0. \tag{4.12}$$

Here R is defined by (3.32) and

$$\begin{aligned} R_* = (z_2 w_1 + z_1 w_2) w_3^2 + [x_2 w_1^2 + x_1 w_2^2 + w_1 w_2 (y_1 + y_2) + 2 z_1 z_2] w_3 + \\ + 2(x_2 z_1 w_1 + x_1 z_2 w_2). \end{aligned} \tag{4.13}$$

Notice that, after several differentiations in virtue of the system (3.15), the possibility $z_2^2 w_1^2 - z_1^2 w_2^2 \equiv 0$ leads to the conditions (3.21), that is, to the critical motions (2.14). Assuming (3.21) we obtain from (3.16)

$$x_1 x_2 = p^2 - 2q, \quad y_1 y_2 = p^2 + 2q, \quad (x_1 + x_2) y_1 y_2 = r^2 (y_1 + y_2)$$
$$(q = \pm\gamma).$$

The corresponding values of the integrals (3.17) are

$$h = \frac{1}{2} w_3^2 - \frac{1}{2}(y_1 + y_2) \geqslant -\sqrt{p^2 + 2q}, \quad k = p^2 - 2q, \quad g = qh. \tag{4.14}$$

If $q = -\gamma$, then all of these values satisfy (4.4) with

$$s = \frac{1}{2}[h \pm \sqrt{h^2 + 4\gamma}]. \tag{4.15}$$

Let $q = \gamma$. Then the values (4.14) satisfy (4.4) with

$$s = \frac{1}{2}[h \pm \sqrt{h^2 - 4\gamma}], \tag{4.16}$$

that is, only for the energy range $h^2 \geqslant 4\gamma$. For $q = \gamma$ and $h^2 \leqslant 4\gamma$ the values (4.14) fill the segment (4.5).

Consider the trajectories for which the equalities (3.21) do not hold identically. Express w_3 from the first equation (4.12):

$$w_3 = -\frac{1}{z_2 w_1 - z_1 w_2}[x_2 w_1^2 - x_1 w_2^2 + w_1 w_2 (y_1 - y_2)]. \tag{4.17}$$

Replacing w_3 in $(z_2 w_1 - z_1 w_2)^2 R_*$ by (4.17), we obtain the expression $2 w_1 w_2 Q$ (the resultant of (3.32), (4.13) as polynomials in w_3), where Q is a non-homogeneous polynomial of third degree in w_1 and w_2 whose coefficients are polynomials in x_i, y_i and z_i of degree not greater than four. Since (3.21) is already excluded, the system (3.12) is replaced by (4.17) and the equation

$$Q = 0. \tag{4.18}$$

We claim that in virtue of (4.17) and (4.18), the values (3.29) and (3.30) satisfy the identities

$$\begin{aligned}
&\tau - p^2 - 2s(s - H) = 0, \\
&(\tau - p^2)^2 + 4(p^2 - K)s^2 - 8Gs + (p^4 - r^4) = 0, \\
&(\tau - p^2)(2s - H) + 2(p^2 - K)s - 2G = 0.
\end{aligned} \tag{4.19}$$

Here the calculation sequence is as follows.

We substitute (3.17), (3.29), (3.30), (4.17) in the left-hand side of each equation (4.19) and multiply the result by the denominator, which is already supposed to be non-zero. The expression thereby obtained appears to be the product of some polynomial in variables (3.14) and the polynomial Q, which equals zero due to (4.18).

Replace in (4.19) the functions G, K, H by their constant values g, k, h and exclude τ with the help of the first relation. The remaining two reduce to the form

$$\Phi(s) = 0, \quad d\Phi(s)/ds = 0, \tag{4.20}$$

where Φ is the polynomial (4.6). Equations (4.4) are equivalent to (4.20). ■

Remark 2. It is easy to see now that the relations (4.1)–(4.3) turn into corresponding relations of the 1st, 2nd, and 3rd classes of Appelrot as $\boldsymbol{\beta}$ tends to zero. Simultaneously, the polynomial (4.6) turns to $s\varphi(s)$, where $\varphi(s)$ is the Euler resolvent of the second polynomial of Kowalevski. This provides an alternative insight into the connection of the set Γ_4 with the 4th Appelrot class of motions. The part of the segment Γ_5 defined by the inequality $h^2 < 4\gamma$ for the classical case ($\gamma = 0$) disappears.

5. The region of existence of motions

The results of the previous section are not complete until we find some conditions that give a criterion to establish whether a point of $\tilde{\Sigma} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$ belongs to the region of existence of motions $\Delta = J(P^6) \subset \mathbf{R}^3$.

Three inequalities of general character can be obtained immediately from (3.2) and (4.9):

$$k \geqslant 0; \tag{5.1}$$

$$h \geqslant -(a + b); \tag{5.2}$$

$$p^2 h \geqslant 2g - r^2\sqrt{k}. \tag{5.3}$$

In case of the Kowalevski top in the gravity field ($p^2 = r^2$) the inequality obtained from (5.3) was established by Appelrot [12].

To get more precise estimations for $(g, k, h) \in \Delta$, restrict the problem to iso-energetic surfaces $E_h = \{v \in P^6 : H(v) = h\}$. Denote

$$J_h = G \times K\,\big|_{E_h} : E_h \to \mathbf{R}^2$$

and let $\tilde{\Sigma}_h, \Sigma_h, \Delta_h$ be the cross-sections of $\tilde{\Sigma}, \Sigma, \Delta$ by the plane parallel to and height h above the (g, k)-plane. For any h the set Σ_h is a bifurcation diagram of the map J_h.

Notice that all sets E_h are compact. As proved in [13], they are connected as well. Therefore the values of any continuous function on E_h fill a bounded and connected segment.

Let

$$\begin{array}{c} k_*(h) = \min K\,|E_h\ ,\ k^*(h) = \max K\,|E_h\ ; \\ g_*(h) = \min G\,|E_h\ ,\ g^*(h) = \max G\,|E_h\ . \end{array} \tag{5.4}$$

Then the rectangle

$$\Pi(h) = \{(k,g) : k_*(h) \leqslant k \leqslant k^*(h),\, g_*(h) \leqslant g \leqslant g^*(h)\}$$

cuts Σ_h out of $\tilde{\Sigma}_h$ and we hope that this operation is not ambiguous.

The following statements allow us to find explicitly the values (5.4) and give some more information about the sets $\Gamma_i \cap \Delta$.

We are going to investigate various maps constructed of combinations of the first integrals G, K, H and, possibly, restricted to invariant submanifolds in P^6. For each map $I : \mathfrak{C} \to \mathbf{R}^k$ of this type we call a point $c \in \mathbf{R}^k$ *an admissible value* if $I^{-1}(c) \neq \emptyset$.

In Sections 3, 4 we often referred to the motions (2.12)–(2.14). They will be also important in the sequel. Calculating the related values of G, K, H we obtain the sets λ_i in (h,g)-plane and μ_i in (h,k)-plane $(i = 1, ..., 6)$:

$$\begin{array}{lll} \lambda_1 :\ g = a^2h + a(a^2 - b^2), & \mu_1 :\ k = (h + 2a)^2, & h \geqslant -(a+b); \\ \lambda_2 :\ g = a^2h - a(a^2 - b^2), & \mu_2 :\ k = (h - 2a)^2, & h \geqslant a - b; \\ \lambda_3 :\ g = b^2h - b(a^2 - b^2), & \mu_3 :\ k = (h + 2b)^2, & h \geqslant -(a+b); \\ \lambda_4 :\ g = b^2h + b(a^2 - b^2), & \mu_4 :\ k = (h - 2b)^2, & h \geqslant -a + b; \\ \lambda_5 :\ g = abh, & \mu_5 :\ k = (a - b)^2, & h \geqslant -(a+b); \\ \lambda_6 :\ g = -abh, & \mu_6 :\ k = (a + b)^2, & h \geqslant -a + b. \end{array}$$

The existing pairwise intersections of the first four sets in either group correspond to the equilibria (2.11).

Recall that $\mathfrak{M} = \{K = 0\} \subset P^6$ and $J(\mathfrak{M}) = \Gamma_1 \cap \Delta$. Denote

$$M = p^2H - 2G : P^6 \to \mathbf{R}$$

and let $H^{(1)} = H|_{\mathfrak{M}}$, $M^{(1)} = M|_{\mathfrak{M}}$. The following result belongs to D. B. Zotev [9].

Proposition 2. *(i) The function $H^{(1)}$ has three critical values $h_1 = -2b$, $h_2 = 2b$ and $h_3 = 2a$. In particular,*

$$\min_{\mathfrak{M}} H = -2b.$$

(ii) The bifurcation diagram of $J^{(1)} = H^{(1)} \times M^{(1)} : \mathfrak{M} \to \mathbf{R}^2$ consists of the half-line

$$m = 0,\ h \geqslant -2b \tag{5.5}$$

and the set of solutions of the equation

$$\begin{array}{c} 27m^4 + 4h(h^2 - 18p^2)m^3 - 2[4p^2h^4 - (16p^4 + 15r^4)h^2 + 2p^2(8p^4 - 9r^4)]m^2 + \\ +4r^4h[h^4 - 4p^2h^2 + 2(2p^4 - 3r^4)]m - r^8[(h^2 - 2p^2)^2 - 4r^4] = 0 \end{array} \tag{5.6}$$

in the quadrant $\{m \geqslant 0, h \geqslant -2b\}$.

(iii) The set of admissible values of $J^{(1)}$ is

$$0 \leqslant m \leqslant m_0(h),\ h \geqslant -2b,$$

where $m_0(h)$ stands for the greatest positive root of (5.6)*, which is considered as an equation in m.*

The bifurcation diagram of $J^{(1)}$ is shown in Fig. 1. The admissible values fill the shaded region.

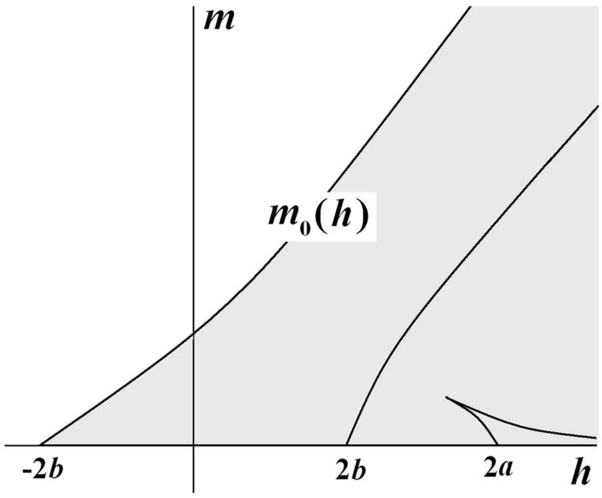

Fig. 1. The bifurcation diagram of $H^{(1)} \times M^{(1)}$

The proof given in [9] is based on an ingenious change of variables on $\mathfrak{M}$. Let us point out the relation between this result and Theorem 2.

Let

$$m = p^2 h - 2g. \tag{5.7}$$

It follows from (5.3) that $m \geqslant 0$ on $\mathfrak{M}$. This inequality explains (5.5). Moreover, the line $m = 0$ in the plane $k = 0$ is the intersection $\Gamma_1 \cap (\Gamma_2 \cup \Gamma_3)$ (in fact, along this line Γ_1 and $\Gamma_2 \cup \Gamma_3$ are tangent to each other).

The intersection $\Gamma_1 \cap \Gamma_4$ is defined by the system obtained from (4.1), (4.4)

$$\begin{array}{l} 3s^4 - 4hs^3 + (p^2 + h^2)s^2 - \gamma^2 = 0, \\ s^4 - hs^3 + gs - \gamma^2 = 0. \end{array} \tag{5.8}$$

By virtue of the notation (5.7) the left-hand side of (5.6) becomes the resultant of the polynomials in (5.8) with respect to s. Thus the set (5.6) corresponds to $\Gamma_1 \cap \Gamma_4$.

Recall that $\mathfrak{N} \subset P^6$ is defined by (3.9), (3.11) and $J(\mathfrak{N}) = (\Gamma_2 \cup \Gamma_3) \cap \Delta$. Let $H^{(2)} = H|_{\mathfrak{N}}$, $G^{(2)} = G|_{\mathfrak{N}}$. Introduce the map

$$J^{(2)} = H^{(2)} \times G^{(2)} : \mathfrak{N} \to \mathbf{R}^2.$$

The following statement is proved in [14].

Proposition 3. *(i) The bifurcation diagram of $J^{(2)}$ consists of the half-lines λ_1, λ_2, λ_3, λ_4, the half-line*

$$g = \frac{1}{2}p^2 h, \; h \geqslant -2b,$$

and the curve

$$2p^2(p^2h - 2g)^2 - 2r^4h(p^2h - 2g) + r^8 = 0, \; p^2h \geqslant 2g.$$

(ii) The admissible values of $J^{(2)}$ fill the region defined by the system of inequalities

$$\left\{ \begin{array}{l} b^2h - b(a^2 - b^2) \leqslant g \leqslant a^2h + a(a^2 - b^2), \; h \geqslant -(a+b), \\ 2p^2(p^2h - 2g)^2 - 2r^4h(p^2h - 2g) + r^8 \geqslant 0. \end{array} \right.$$

The bifurcation diagram of $J^{(2)}$ is shown in Fig. 2. The admissible values fill the shaded region.

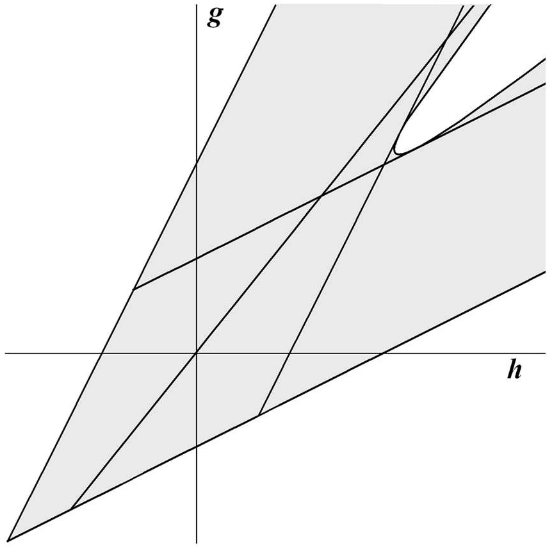

Fig. 2. The bifurcation diagram of $H^{(2)} \times G^{(2)}$

Propositions 2, 3 completely define those parts of Γ_1, Γ_2 and Γ_3 which correspond to real critical motions, that is, the sets $\Gamma_1 \cap \Delta$ and $(\Gamma_2 \cup \Gamma_3) \cap \Delta$.

Consider the map

$$J^{(3)} = H \times K : P^6 \to \mathbf{R}^2.$$

The critical set of $J^{(3)}$ is already found (see steps 1, 2 in the proof of Theorem 1). In addition to the manifold $\mathfrak{M}$ it contains all pendulum motions (2.12) – (2.14).

Proposition 4. *(i) The bifurcation diagram of the map $J^{(3)}$ consists of the parabolic curves μ_1, μ_3, μ_2, μ_4, the half-lines μ_5, μ_6, and the half-line*

$$k = 0, \ h \geqslant -2b. \tag{5.9}$$

(ii) Let

$$\begin{gathered} k_*(h) = \begin{cases} (h+2b)^2, & -(a+b) \leqslant h \leqslant -2b \\ 0, & h \geqslant -2b \end{cases}, \\ k^*(h) = (h+2a)^2. \end{gathered} \tag{5.10}$$

The admissible values of $J^{(3)}$ fill the region

$$k_*(h) \leqslant k \leqslant k^*(h), \ h \geqslant -(a+b). \tag{5.11}$$

The inequality in (5.9) follows from Proposition 2. The relationship $k_*(h)$ in (5.10) is built in accordance with (5.1). The range of h in (5.11) is defined by (5.2). The region of admissible values (shaded in Fig. 3) is found using the mentioned above fact that for each $h \geqslant -(a+b)$ the image of E_h under K is a bounded connected segment.

Finally, consider the map

$$J^{(4)} = H \times G : P^6 \to \mathbf{R}^2.$$

For $0 < |s| \leqslant b$ and $|s| \geqslant a$, let

$$\phi(s) = \sqrt{\frac{(s^2-a^2)(s^2-b^2)}{s^2}} \geqslant 0.$$

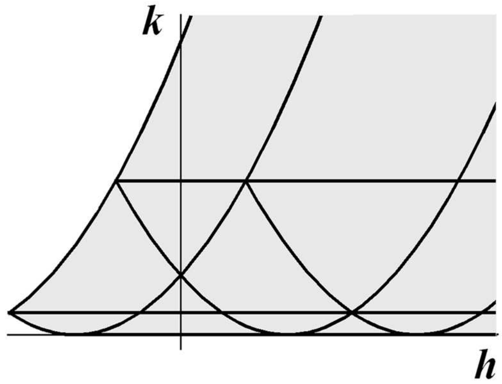

Fig. 3. The bifurcation diagram of $H \times K$

Proposition 5. *(i) The bifurcation diagram of the map* $J^{(4)}$ *consists of the half-lines* λ_1, λ_2, λ_3, λ_4, λ_5, λ_6 *and the curves*

$$C_1 : \left\{ \begin{array}{l} h = 2s + \phi(s) \\ g = \dfrac{\gamma^2}{s} + s^3 + s^2\phi(s) \end{array} \right. , \quad s \in [-b, 0), \tag{5.12}$$

$$C_2 : \left\{ \begin{array}{l} h = 2s + \phi(s) \\ g = \dfrac{\gamma^2}{s} + s^3 + s^2\phi(s) \end{array} \right. , \quad s \in (0, b], \tag{5.13}$$

$$C_3 : \left\{ \begin{array}{l} h = 2s - \phi(s) \\ g = \dfrac{\gamma^2}{s} + s^3 - s^2\phi(s) \end{array} \right. , \quad s \in [a, +\infty). \tag{5.14}$$

(ii) Denote by $g_0(h)$ *the one-valued function defined by* (5.12) *when* $h \geqslant -2b$. *Let*

$$\begin{array}{l} g_*(h) = \left\{ \begin{array}{ll} b^2 h - b(a^2 - b^2), & -(a+b) \leqslant h \leqslant -2b \\ g_0(h), & h \geqslant -2b \end{array} \right. , \\ g^*(h) = a^2 h + a(a^2 - b^2). \end{array} \tag{5.15}$$

Then the region of admissible values of $J^{(4)}$ *is defined by the inequalities*

$$g_*(h) \leqslant g \leqslant g^*(h), \; h \geqslant -(a+b).$$

The bifurcation diagram of $J^{(4)}$ with the shaded region of admissible values is shown in Fig. 4.

A straightforward proof of Proposition 5 can be obtained using the same technique as in the proof of Theorem 1. Here we just point out some general ideas that explain this result from the point of view of the geometry of Σ.

Let $v \in P^6$ be the critical point of $J^{(4)}$. If $dH(v) = 0$, then v is an equilibrium, that is, a singular point of the system (3.1). Such a point is a critical point for each first integral of (3.1). In particular, $dG(v) = 0$. The values (h, g) of $J^{(4)}$ at equilibria (2.11) are the points of pairwise intersection of the lines $\lambda_1 - \lambda_4$.

Let

$$\operatorname{rank}\{dG(v), dH(v)\} = 1, \tag{5.16}$$

and $c = (g, k, h) = J(v)$. Then $c \in \Sigma$. It follows from (4.1)–(4.5) that (5.16) necessarily implies

$$\operatorname{rank}\{dG(v), dH(v), dK(v)\} = 1. \tag{5.17}$$

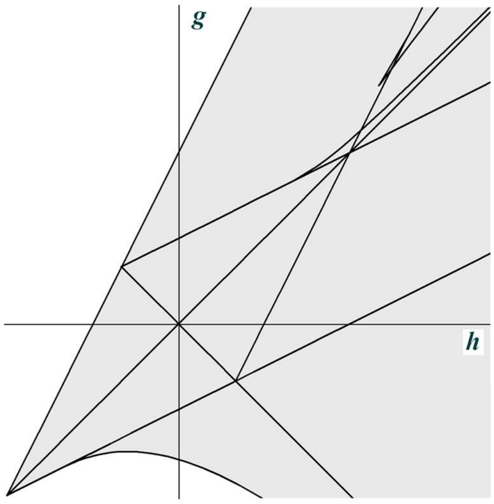

Fig. 4. The bifurcation diagram of $H \times G$

If the tangent plane to Σ at the point c is well defined, then the set of zero linear combinations of $dG(v), dH(v)$ and $dK(v)$ is one-dimensional. This fact contradicts to (5.17). Therefore, c belongs either to the segment Γ_5 or to the set of transversal intersections of two smooth leaves of Σ.

The intersection of Γ_1 and $\Gamma_2 \cup \Gamma_3$ is nowhere transversal.

Transversal intersections of Γ_1 and Γ_4 are given by the system (5.8). Solving it with respect to g and h and taking into account the admissible region established in Proposition 2, we arrive at the curves (5.12)–(5.14).

Consider intersections of $\Gamma_2 \cup \Gamma_3$ and Γ_4. Substitute (4.4) for k, g in (4.7):

$$(s^2 - a^2)(s^2 - b^2)[2s^2 - 2hs + p^2]^2 = 0. \tag{5.18}$$

Transversal intersections correspond to the values $s = \pm a, \; s = \pm b$ (the last multiplier in (5.18) is responsible for the tangency points of $\Gamma_2 \cup \Gamma_3$ and Γ_4). This implies the equations of $\lambda_1 - \lambda_4$. As shown in [11] the corresponding motions on $\mathfrak{N}$ are the pendulums (2.12), (2.13). From this fact the inequalities for h are obtained.

Suppose that Γ_4 has a point of self-intersection. Then for some h the curve defined by (4.4) in (g, k)-plane has a double point. Let s_1, s_2 be the corresponding values of s. It follows from (4.4) that

$$s_1 + s_2 = h, \quad s_1^2 s_2^2 = \gamma^2.$$

Hence s_1, s_2 form one of the pairs (4.15), (4.16). Substituting these pairs in (4.4) gives (4.14). The obtained set of points in (h, g)-plane united with the projection of the segment Γ_5 forms the half-lines λ_5 and λ_6.

The admissible region for (h, g) is established in the same way as in the previous case. Propositions 4 and 5 give the explicit formulae (5.10), (5.15) for the values (5.4). Then for each h we can compute the limits for the parameter s in (4.4) corresponding to $\Gamma_4 \cap \Delta$. Thus the set Σ_h is completely determined. Finally, Δ_h is obtained as the span of the curves $\Gamma_i \cap \Delta_h$.

6. Conclusion

At this point we can draw all bifurcation diagrams of the induced momentum maps on iso-energetic surfaces, which are typically five-dimensional. A lot of information on the stability of the critical integral manifolds may be immediately obtained for the tori in $\mathfrak{M}$ and $\mathfrak{N}$. The investigation of the new critical set $\mathfrak{O}$ waits to be fulfilled.

Since each E_h is a foliation into three-dimensional tori with some degenerations, we can construct the base B_h for such a foliation just by factorizing E_h, more exactly, by identifying points of the same connected component of $J_{g,k,h}$. Then B_h is a two-dimensional analog of Fomenko's graph [5] for an iso-energetic manifold of integrable system with two degrees of freedom. In its turn B_h is a bundle over Δ_h whose fibres are finite sets; the number of elements in any fibre is equal to the number of connected components of the corresponding integral manifold. The problem of finding this number for all possible situations seems solvable. Then we obtain a complete description of the 'coverings' $B_h \to \Delta_h$ and, consequently, establish the topology of B_h.

Naturally, the next step requires new mathematical ideas on how the tori in E_h glue together along the paths in the admissible regions.

If we consider B_h as a two-dimensional cell complex, then, for regular levels of energy, 0-cells correspond to closed orbits, a point of each 1-cell represents a two-dimensional torus, and a point of each 2-cell represents a three-dimensional torus. The union of the cells of dimensions 0 and 1 forms a graph, to which the method of marked molecules [15] can be applied without any modification. The question is what kind of a numeric mark should be attached to each two-dimensional cell to obtain from B_h the complete invariant of Liouville's foliation of the iso-energetic surface?

Another approach is to consider the set Σ_h^0 of singular points of Σ_h (self-intersections, tangency points, and cusps), which is easy to obtain from the above results, and associate to each $c \in \Sigma_h^0$ the marked loop molecule [16]. In this case, of course, the notion of a mark should be changed to suit increased dimensions of the tori.

We see that the Kowalevski top in two constant fields provides a highly non-trivial example of integrable Hamiltonian system and a complete description of its phase topology is really a challenging problem.

References

[1] *S. Kowalevski.* Sur le problème de la rotation d'un corps solide autour d'un point fixe. Acta Math. 1889. V. 12. P. 177–232.

[2] *A. G. Reyman, M. A. Semenov-Tian-Shansky.* Lax representation with a spectral parameter for the Kowalewski top and its generalizations. Lett. Math. Phys. 1987. V. 14. №1. P. 55–61.

[3] *O. I. Bogoyavlensky.* Euler equations on finite-dimension Lie algebras arising in physical problems. Commun. Math. Phys. 1984. V. 95. P. 307–315.

[4] *A. I. Bobenko, A. G. Reyman, M. A. Semenov-Tian-Shansky.* The Kowalewski top 99 years later: a Lax pair, generalizations and explicit solutions. Commun. Math. Phys. 1989. V. 122. №2. P. 321–354.

[5] *A. T. Fomenko.* Symplectic Geometry. Methods and Applications. Mosk. Univ. Publ., Moscow. 1988. (In Russian)

[6] *I. V. Komarov.* A generalization of the Kovalevskaya top. Phys. Letters. 1987. V. 123. №1. P. 14–15.

[7] *H. Yehia.* New integrable cases in the dynamics of rigid bodies. Mech. Res. Commun. 1986. V. 13. №3. P. 169–172.

[8] *A. V. Borisov, I. S. Mamaev.* Rigid Body Dynamics. Moscow-Izhevsk: SPC 'Regular & Chaotic Dynamics'. 2001. (In Russian)

[9] *D. B. Zotev.* Fomenko–Zieschang invariant in the Bogoyavlenskyi case. Reg. & Chaot. Dyn. 2000. V. 5. №4. P. 437–458.

[10] *M. P. Kharlamov.* One class of solutions with two invariant relations in the problem of motion of the Kowalevsky top in double constant field. Mekh. Tverd. Tela. 2002. №32. P. 32–38. (In Russian)

[11] *M. P. Kharlamov, A. Y. Savushkin.* Separation of variables and integral manifolds in one partial problem of motion of the generalized Kowalevski top. Ukr. Math. Bull. 2004. V. 1. №4. P. 548–565. (In Russian)

[12] *G. G. Appelrot.* Non-Completely Symmetric Heavy Gyroscopes. In: 'Motion of a Rigid Body about a Fixed Point', Collection of papers in memory of S. V. Kovalevskaya. Acad. Sci. USSR, Moscow-Leningrad. 1940. P. 61–156. (In Russian)

[13] *M. P. Kharlamov, D. B. Zotev.* Non-degenerate energy surfaces of rigid body in two constant fields. Reg. & Chaot. Dyn. 2005. V. 10. №1. P. 15–19.

[14] *M. P. Kharlamov, A. Y. Savushkin, E. G. Shvedov.* Bifurcation set in one problem of motion of the generalized Kowalevski top. Mekh. Tverd. Tela. 2003. №33. P. 10–19. (In Russian)

[15] *A. V. Bolsinov, A. T. Fomenko.* Integrable Hamiltonian Systems. Topology. Geometry. Classification. Moscow-Izhevsk: SPC 'Regular & Chaotic Dynamics'. 1999. V. 1, 2. (In Russian)

[16] *A. V. Bolsinov, P. H. Richter, A. T. Fomenko.* The method of loop molecules and the topology of the Kovalevskaya top. Matem. Sbornik. 2000. V. 191. №2. P. 3–42. (In Russian)