Uncertainty Principle at All Scales Energies, Measurability and Mathematical Formalism of Quantum Theory and General Relativity

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Abstract

This paper is a continuation of the earlier studies conducted by the author and of his latest publication devoted to the inferences concerning the introduction of a minimal length in a quantum theory and in gravity. It is shown that, when Heisenberg's Uncertainty Principle is considered as a low-energy limit of the Generalized Uncertainty Principle, a minimal length is inevitably brought about even at low energies. In this case new parameters associated with this length are defined in the explicit form. Based on the pair of well-known gravitational models, it is demonstrated that the indicated parameters determine low and high-energy dynamics of these models. Various inferences are considered.

1 Introduction. Main Motivation.

This work is a direct continuation of the recently published paper [1] and is interlaced with the publication at some points. As shown in [1], provided the theory involves the minimal length l_{min} as a minimal measurement unit for the quantities having the dimensions of length, this theory must not have infinitesimal spatial-temporal quantities dx_μ because the latter lead to the infinitely small length ds [2]

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu \qquad (1)$$

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that is inexistent because of l_{min}.
Of course, in this case only **measurable** quantities are meant. As a mathematical notion, the quantity ds is naturally existent but, due to the involvement of l_{min}, it is **immeasurable**.
However it is well known that at high energies (on the order of the quantum gravity energies) the minimal length l_{min} to which the indicated energies are ≪sensitive≫, as distinct from the low ones, should inevitably become apparent in the theory. But if l_{min} is really present, it must be present at all the ≪Energy Levels≫ of the theory, low energies including. And this, in addition to the above arguments, points to the fact that the mathematical formalism of the theory should not involve any infinitesimal spatial-temporal quantities. Besides, some new parameters become involved, which are dependent on l_{min} [3]–[11].
What are the parameters of interest in the case under study? It is obvious that, as the quantum-gravitational effects will be revealed at very small (possibly Planck's) scales, these parameters should be dependent on some limiting values, e.g., $l_P \propto l_{min}$ and hence Planck's energy E_P.
This means that in a high-energy gravitation theory the energy- or, what is the same, measuring scales-dependent parameters should be necessarily introduced.
But, on the other hand, these parameters could hardly disappear totally at low energies, i.e. for General Relativity (GR) too. However, since the well-known canonical (and in essence the classical) statement of GR has no such parameters [2], the inference is as follows: their influence at low energies is so small that it may be disregarded at the modern stage in evolution of the theory and of the experiment.
Still this does not imply that they should be ignored in future evolution of the theory, especially on going to its high-energy limit.
But at the present time, the mathematical apparatus of both special and general relativity theories (and of a quantum theory as well) is based on the concept of continuity and on analysis of infinitesimal spatial-temporal quantities. This is a corner stone for the Minkowski space geometry (MS) and also for the Riemannian geometry (RG) [2].
However, this approach involves a problem when we proceed to a quantum description of nature. Even at a level of the heuristic understanding, it is

clear that, as measuring procedures in a quantum theory are fundamental, the description with the use of infinitesimal quantities is problematic because in its character the measuring procedure is discrete.
At a level of the mathematical formalism and physical principles, incompatibility of both the Minkowski space geometry and Riemannian geometry with the uncertainty principle is expected in any ≪format≫, in relativistic and nonrelativistic cases. This problem is considered in greater detail in the following section of this work.
Thus, if the matter concerns the **measurable** quantities only, the Quantum Theory (QT) and Gravity formalism should be changed: at leas, a new formalism should not involve the infinitesimal spatial-temporal quantities dx_μ. Naturally, because of the involved l_{min} (initially assuming that $l_{min} \propto l_P$) new theories should involve new parameters associated with l_{min}. Presently, such parameters are inexplicitly involved (for example, E/E_P in a quantum gravity phenomenology [3]).
But there is no need to discard the modern formalism of QT and Gravity, since it is clear that at low energies it offers an excellent approximation, experimentally supported to a high accuracy (see [12]). However, proceeding from the above, a change-over to high energies is impossible as, by author's opinion, this formalism is used in an effort to **combine uncombinable things**.
This work makes the arguments of [1] more forcible with the added reasons from the viewpoint of the Uncertainty Principle at all scales energies, on the one hand, and presents a study of the additional parameters associated with the involvement of l_{min} , in terms of which one can develop a new formalism for a quantum theory and for gravity at all the scales energies too, on the other hand.

2 Uncertainty Principle at All Scales Energies and Some of its Consequences

We begin not with Heisenberg's Uncertainty Principle (HUP) [13]

$$\Delta x \geq \frac{\hbar}{\Delta p} \tag{2}$$

but with its widely known high-energy generalization the Generalized Uncertainty Principle (GUP) [14]– [26]:

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_P^2 \frac{\Delta p}{\hbar}. \quad (3)$$

Here α' is the model-dependent dimensionless numerical factor and l_P is the Planckian length. (Note that the normalization $\Delta x \Delta p \geq \hbar$ is used rather than $\triangle x \Delta p \geq \hbar/2$.)

Note also that initially GUP (3) was derived within a string theory [14]– [17] and, subsequently, in a series of works far from this theory [18] – [24] it has been demonstrated that on going to high (Planck's) energies in the right-hand side of HUP (2) an additional ≪high-energy≫ term $\propto l_P^2 \frac{\triangle p}{\hbar}$ appears. Of particular interest is the work [18], where by means of a simple gedanken experiment it has been demonstrated that with regard to the gravitational interaction (3) is the case.

As (3) – quadratic inequality, then it naturally leads to the minimal length $l_{min} = \xi l_P = 2\sqrt{\alpha'} l_P$.

This means that the theory for the quantities with a particular dimension has a **minimal measurement unit**. At least, all the quantities with such a dimension should be ≪quantized≫, i. e. be measured by an integer number of these **minimal units** of measurement.

Specifically, if l_{min} – **minimal unit** of length, then for any length L we have the ≪**Integrality Condition**≫ **(IC)**

$$L = N_L l_{min}, \quad (4)$$

where $N_L \geq 0$ – integer.

What are the consequences for GUP (3) and HUP (2)?

Assuming that HUP is to a high accuracy derived from GUP on going to low energies or that HUP is a special case of GUP at low values of the momentum, we have

$$(GUP, \Delta p \rightarrow 0) = (HUP). \quad (5)$$

By the language of N_L from(4), (5) is nothing else but a change-over to the following:

$$(N_{\Delta x} \approx 1) \rightarrow (N_{\Delta x} \gg 1). \quad (6)$$

The assumed equalities in (2) and (3) may be conveniently rewritten in terms of l_{min} with the use of the deformation parameter α_a. This parameter has been introduced earlier in the papers [27]–[36] as a deformation parameter on going from the canonical quantum mechanics to the quantum mechanics at Planck's scales (early Universe) that is considered to be the quantum mechanics with the minimal length (QMML):

$$\alpha_a = l_{min}^2/a^2, \tag{7}$$

where a is the measuring scale.
Here deformation is understood as an extension of a particular theory by inclusion of one or several additional parameters in such a way that the initial theory appears in the limiting transition [37]. Then with the equality ($\Delta p \Delta x = \hbar$) (3) is of the form

$$\Delta x = \frac{\hbar}{\Delta p} + \frac{\alpha_{\Delta x}}{4}\Delta x. \tag{8}$$

In this case due to formulae (4) and (6) the equation (8) takes the following form:

$$N_{\Delta x} l_{min} = \frac{\hbar}{\Delta p} + \frac{1}{4N_{\Delta} x} l_{min} \tag{9}$$

or

$$(N_{\Delta x} - \frac{1}{4N_{\Delta} x}) l_{min} = \frac{\hbar}{\Delta p}. \tag{10}$$

That is

$$\Delta p = \frac{\hbar}{(N_{\Delta x} - \frac{1}{4N_{\Delta} x}) l_{min}}. \tag{11}$$

From (9)–(11) it is clear that HUP (2) in the case of the equality appears to a high accuracy in the limit $N_{\Delta} x \gg 1$ in conformity with (6).
It is easily seen that the parameter α_a from (7) is discrete as it is nothing else but

$$\alpha_a = l_{min}^2/a^2 = \frac{l_{min}^2}{N_a^2 l_{min}^2} = \frac{1}{N_a^2}. \tag{12}$$

At the same time, from (12) it is evident that α_a is irregularly discrete.
It is evident that from formula (11) at low energies ($N_{\Delta} x \gg 1$), up to a

constant

$$\frac{\hbar^2}{l_{min}^2} = \frac{\hbar c^3}{4\alpha' G} \tag{13}$$

we have

$$\alpha_{\Delta x} = (\Delta p)^2. \tag{14}$$

Note that all the foregoing results associated with GUP and with its limiting transition to HUP for the pair $(\Delta x, \Delta p)$, as shown in [29], may be easily carried to the "energy - time" pair $(\Delta t, \Delta E)$. Indeed (3) gives [29]:

$$\frac{\Delta x}{c} \geq \frac{\hbar}{\Delta pc} + \alpha' l_P^2 \frac{\Delta p}{c\hbar}, \tag{15}$$

then

$$\Delta t \geq \frac{\hbar}{\Delta E} + \alpha' \frac{l_p^2}{c^2} \frac{\Delta pc}{\hbar} = \frac{\hbar}{\Delta E} + \alpha' t_p^2 \frac{\Delta E}{\hbar}. \tag{16}$$

where the smallness of l_P is taken into account so that the difference between ΔE and $\Delta(pc)$ can be neglected and t_P is the Planck time $t_P = L_P/c = \sqrt{G\hbar/c^5} \simeq 0,54\ 10^{-43} sec$. From whence it follows that we have a maximum energy of the order of Planck's:

$$E_{max} \sim E_P$$

Then the foregoing formulae (2)–(10) are rewritten by substitution as follows:

$$\Delta x \to \Delta t; \Delta p \to \Delta E; l_{min} \to t_{min}; N_L \to N_{t=L/c} \tag{17}$$

Specifically, (10) takes the form

$$(N_{\Delta t} - \frac{1}{4 N_\Delta t}) t_{min} = \frac{\hbar}{\Delta E}. \tag{18}$$

In this way in the above-presented formalism a minimal length is meaningful at all the energy levels and not only at high energies, from where it actually originated. In other words, the length is «quantized» at all the energy levels without exceptions. But then in all cases the infinitesimal quantities dx_μ should be **removed** from the theory as in all cases **the infinitesimal**

quantities dx_μ bring about an infinitely small length ds (1) inexistent because of l_{min}.
Earlier HUP has been considered as a low-energy limit of GUP (5) with the minimal length attribute $l_{min} \propto l_P$. However, it is easily seen that even if we have no notion about the existence of GUP (3) (i. e. of the high-energy term $\propto l_P^2 \Delta p/\hbar$ in the right-hand side of (3)), still the use of **the infinitesimal quantities** dx_μ from the viewpoint of their **measurability** is problematic as at low energies, where HUP (2)) is valid, we have ≪great≫ Δx_μ, certainly higher than infinitesimal dx_μ. Because of this, to ≪measure≫ dx_μ we should go to high energies or to ≪small≫ Δx_μ.
At the same time, even at the ultimate (Planck's) energies a minimal ≪detected≫ (i. e. measurable) space-time volume is, within the known constants, restricted to

$$V_{min} \propto l_P^4. \tag{19}$$

Consequently, ≪detectability≫ of the infinitesimal space-time volume

$$V_{dx_\mu} = (dx_\mu)^4 \tag{20}$$

is impossible as this necessitates going to infinitely high energies

$$E \to \infty. \tag{21}$$

In the relativistic case for any probe particle with the mass m, if it is considered as a ≪**point object**≫, there is its Compton wavelength [38]

$$\bar{\lambda}_C = \frac{\lambda_C}{2\pi} = \frac{\hbar}{mc} \tag{22}$$

setting the ultimate accuracy for the determination of its coordinates. But, due to the infinitesimal special-temporal variations in MS, this minimum is easily gone beyond.

3 Minimum Spatial and Temporal Changes and Spacetime Quantum Fluctuations

As follows from the previous section, **measurable** infinitesimal changes in length (and hence in time) are impossible and such changes are dependent

on the existing energies.
In particular, a minimal possible **measurable** change of l_{min} corresponds to some maximal value of the energy E_{max}. If, similar to the previous section, $l_{min} \propto l_P$,$E_{max} \propto l_P$, then denoting with $\triangle_(L)$ a **minimal measurable** change in length corresponding to the energy E we obtain

$$\triangle_{E_{max}}(L) = l_{min}. \tag{23}$$

Evidently, for lower energies the corresponding values of $\triangle_E(L)$ are higher and, as the quantities having the dimensions of length are quantized (4), for $E < E_{max}$, $\triangle_E(L)$ is transformed to

$$\Delta_E(L) = N_E l_{min}, N_E > 1 - integer. \tag{24}$$

At low energies $N_E \gg 1$ but in any case in the suggested formulation N_E is independent of L.
The length dependence appears in the definition of **space-time quantum fluctuations** or, in a different way, of **space-time foam**.
The notion ≪space-time foam≫, introduced by J. A. Wheeler about 60 years ago for the description and investigation of physics at Planck's scales (Early Universe) [39],[40], is fairly settled. Despite the fact that in the last decade numerous works have been devoted to physics at Planck's scales within the scope of this notion, for example [41]–[60], by this time still their no clear understanding of the ≪space-time foam≫ as it is.
On the other hand, it is undoubtful that a quantum theory of the Early Universe should be a deformation of the well-known quantum theory.
In my works with the colleagues [27]–[36] I has put forward one of the possible approaches to resolution of a quantum theory at Planck's scales on the basis of the density matrix deformation.
In accordance with the modern concepts, the space-time foam [40] notion forms the basis for space-time at Planck's scales (Big Bang). This object is associated with the quantum fluctuations generated by uncertainties in measurements of the fundamental quantities, inducing uncertainties in any distance measurement. A precise description of the space-time foam is still lacking along with an adequate quantum gravity theory. But for the description of quantum fluctuations we have a number of interesting methods

(for example,[50]–[60], [61],[62]).
In what follows, we use the terms and symbols from [52]. Then for the fluctuations $\widetilde{\delta l}$ of the distance l we have the following estimate:

$$(\widetilde{\delta l})_{\gamma} \gtrsim l_{P}^{\gamma} l^{1-\gamma} = l_{P}(\frac{l}{l_{P}})^{1-\gamma} = l(\frac{l_{P}}{l})^{\gamma} = l\lambda_{l}^{\gamma}, \tag{25}$$

or that same

$$|(\widetilde{\delta l})_{\gamma}|_{min} = \beta l_{P}^{\gamma} l^{1-\gamma} = \beta l_{P}(\frac{l}{l_{P}})^{1-\gamma} = \beta l\lambda_{l}^{\gamma}, \tag{26}$$

where $0 < \gamma \leq 1$, coefficient β is of order 1 and $\lambda_{l} \equiv l_{P}/l$.
From (25),(26), we can derive the quantum fluctuations for all the primary characteristics, specifically for the time $(\widetilde{\delta t})_{\gamma}$, energy $(\widetilde{\delta E})_{\gamma}$, and metrics $(\widetilde{\delta g}_{\mu\nu})_{\gamma}$. In particular, for $(\widetilde{\delta g}_{\mu\nu})_{\gamma}$ we can use formula (10) in [52]

$$(\widetilde{\delta g}_{\mu\nu})_{\gamma} \gtrsim \lambda^{\gamma}. \tag{27}$$

Further in the text is assumed that the theory involves a minimal length on the order of Planck's length

$$l_{min} \propto l_{P}$$

or that is the same

$$l_{min} = \xi l_{P}, \tag{28}$$

where the coefficient ξ is on the order of unity too.
In this case it is unimportant which is the origin of this minimal length. In particular, it can assume that it comes from the Generalized GUP (3).
As stated in the previous section GUP (3) leads to the minimal length $l_{min} = \xi l_{P} = 2\sqrt{\alpha'} l_{P}$.
Therefore, in this case replacement of Planck's length by the minimal length in all the above formulae is absolutely correct and is used without detriment to the generality

$$l_{P} \rightarrow l_{min}. \tag{29}$$

Thus, $\lambda_{l} \equiv l_{min}/l$ and then (25)– (27) upon the replacement (29) are read unchanged.

So, (26) may be written as

$$|(\widetilde{\delta l})_{\gamma}|_{min} = \beta l \lambda_l^{\gamma} = \beta N_l (N_l^{-\gamma}) = \beta N_l^{1-\gamma} l_{min}. \qquad (30)$$

Here one should take into account the following consideration: due to the (Integrality Condition) (4) in the right-hand side of (30) for the factor $\beta N_l^{1-\gamma}$ before l_{min} its integer part is always meant

$$\beta N_l^{1-\gamma} \mapsto [\beta N_l^{1-\gamma}] \qquad (31)$$

and this goes without special mentioning for the whole text.
The following points of importance should be noted [63]:
3.1)It is clear that **at Planck's scales, i.e. at the minimal length scales**

$$l \rightarrow l_{min} \qquad (32)$$

models for different values of the parameter γ are coincident.

3.2)**Provided some quantity has a minimal measuring unit, values of this quantity are multiples of this unit**.
Naturally, any quantity having a minimal measuring unit is uniformly discrete.
The latter property is not met, in particular, by the energy E.
As $E \sim 1/l$, where l – measurable scale, **the energy E is a discrete quantity but the irregularly discrete one**. It is clear that the difference between the adjacent values of E is the less the lower E. In other words, for

$$E \ll E_P \qquad (33)$$

E becomes a practically continuous quantity.

3.3) In fact, the parameter λ_l nothing like

$$\lambda_l = \sqrt{\alpha_l}, \qquad (34)$$

where α_l is the deformation parameter introduced earlier in formula (7) and in [27]–[36].
The parameter α_l has the following clear physical meaning:

$$\alpha_l^{-1} \sim S^{BH}, \qquad (35)$$

where

$$S^{BH} = \frac{A}{4l_p^2} \tag{36}$$

is the well-known Bekenstein-Hawking formula for the black hole entropy in the semiclassical approximation [64],[65] for the black-hole event horizon surface A, with the characteristics linear dimension (≪radius≫) $R = l$. This is especially obvious in the spherically-symmetric case.
In what follows we use both parameters: λ_x and α_x.

4 Certain Significant Examples

4.1 Heuristic Markov's Model

This heuristic model was introduced in the work [66] at the early eighties of the last century. In [66], it is assumed that ≪by the universal decree of nature a quantity of the material density ϱ is always bounded by its upper value given by the expression that is composed of fundamental constants≫ ([66], p.214):

$$\varrho \leq \varrho_p = \frac{c^5}{G^2\hbar}, \tag{37}$$

with ϱ_p as ≪Planck's density≫.
Then the quantity

$$\wp_\varrho = \varrho/\varrho_p \leq 1 \tag{38}$$

is the **deformation parameter** as it is used in [66] to construct the following **of Einstein's equations deformation or $\wp_\varrho$-deformation** ([66],formula (2)):

$$R_\mu^\nu - \frac{1}{2}R\delta_\mu^\nu = \frac{8\pi G}{c^4}T_\mu^\nu(1-\wp_\varrho^2)^n - \Lambda\wp_\varrho^{2n}\delta_\mu^\nu, \tag{39}$$

where $n \geq 1/2$, T_μ^ν–energy-momentum tensor, Λ– cosmological constant.
The case of the parameter $\wp_\varrho \ll 1$ or $\varrho \ll \varrho_p$ correlates with the classical Einstein equation, and the case when $\wp_\varrho = 1$ – with the de Sitter Universe. In this way (39) may be considered as $\wp_\varrho$-deformation of the General Relativity.
As shown in [67], $\wp_\varrho$-of Einstein's equations deformation (39) is nothing else

but α-deformation of GR for the parameter α_l at $x = l$ from (7).
If $\varrho = \varrho_l$ is the average material density for the Universe of the characteristic linear dimension l, i.e. of the volume $V \propto l^3$, we have

$$\wp_{l,\varrho} = \frac{\varrho_l}{\varrho_p} \propto \alpha_l^2 = \omega \alpha_l^2, \tag{40}$$

where ω is some computable factor.
Then it is clear that α_l-representation (39) is of the form

$$R_\mu^\nu - \frac{1}{2} R \delta_\mu^\nu = \frac{8\pi G}{c^4} T_\mu^\nu (1 - \omega^2 \alpha_l^4)^n - \Lambda \omega^{2n} \alpha_l^{4n} \delta_\mu^\nu, \tag{41}$$

or in the general form we have

$$R_\mu^\nu - \frac{1}{2} R \delta_\mu^\nu = \frac{8\pi G}{c^4} T_\mu^\nu (\alpha_l) - \Lambda(\alpha_l) \delta_\mu^\nu. \tag{42}$$

But, as r.h.s. of (42) is dependent on α_l of any value and particularly in the case $\alpha_l \ll 1$, i.e. at $l \gg l_{min}$, l.h.s of (42) is also dependent on α_l of any value and (42) may be written as

$$R_\mu^\nu(\alpha_l) - \frac{1}{2} R(\alpha_l) \delta_\mu^\nu = \frac{8\pi G}{c^4} T_\mu^\nu (\alpha_l) - \Lambda(\alpha_l) \delta_\mu^\nu. \tag{43}$$

Thus, in this specific case we obtain the explicit dependence of GR on the available energies $E \sim 1/l$, that is insignificant at low energies or for $l \gg l_{min}$ and, on the contrary, significant at high energies, $l \to l_{min}$.

4.1.1) At low energies with the use of formulae (7), (12) for $a = l$ (and hence for $N_l \gg 1$) we get a ≪**nearly continuous theory**≫ practically similar to the General Relativity with the slowly (smoothly) varying parameter $\alpha_{l(t)}$, where t – time.

4.1.2) Clearly, at high energies the parameter $\alpha_{l(t)}$ is discrete and for the limiting value $\alpha_{l(t)} = 1$ we get a discrete series of equations of the form (42)(or a single equation of this form met by a discrete series of solutions) corresponding to $\alpha_{l(t)} = 1; 1/4; 1/9; ...$
As this takes place, $T_\mu^\nu(\alpha_l) \approx 0$ and in both cases, 4.1.1) and 4.1.2), $\Lambda(\alpha_l)$ is not longer a cosmological constant, being a dynamical cosmological term.

4.2 Static Spherically-Symmetric Space-Time with Horizon

This example thoroughly studied in the above-mentioned publication [1] is given here to complete the picture.

Gravity and thermodynamics of horizon spaces and their interrelations are currently most actively studied [68]–[80]. Let us consider a relatively simple illustration – the case of a static spherically-symmetric horizon in space-time, the horizon being described by the metric

$$ds^2 = -f(r)c^2dt^2 + f^{-1}(r)dr^2 + r^2d\Omega^2. \quad (44)$$

The horizon location will be given by a simple zero of the function $f(r)$, at the radius $r = a$.
This case is studied in detail by T. Padmanabhan in his works [68, 79] and by the author of this paper in [67]. We use the notation system of [79]. Let, for simplicity, the space be denoted as $\mathcal{H}$.
It is known that for horizon spaces one can introduce the temperature that can be identified with an analytic continuation to imaginary time. In the case under consideration ([79], eq.(116))

$$k_BT = \frac{\hbar c f'(a)}{4\pi}. \quad (45)$$

Therewith, the condition $f(a) = 0$ and $f'(a) \neq 0$ must be fulfilled.
Then at the horizon $r = a$ Einstein's field equations

$$\frac{c^4}{G}\left[\frac{1}{2}f'(a)a - \frac{1}{2}\right] = 4\pi Pa^2 \quad (46)$$

where $P = T^r_r$ is the trace of the momentum-energy tensor and radial pressure.
Now we proceed to the variables «α» from the Section 2 (formula (7)) to consider (46) in a new notation, expressing a in terms of the corresponding deformation parameter α. In what follows we omit the subscript in formula (7) of α_a, where the context implies which index is the case. In particular, here we use α instead of α_a. Then we have

$$a = l_{min}\alpha^{-1/2}. \quad (47)$$

Therefore,

$$f'(a) = -2l_{min}^{-1}\alpha^{3/2}f'(\alpha). \tag{48}$$

Substituting this into (46) we obtain in the considered case of Einstein's equations in the «α–representation» the following [67]:

$$\frac{c^4}{G}(-\alpha f'(\alpha) - \frac{1}{2}) = 4\pi P\alpha^{-1}l_{min}^2. \tag{49}$$

Multiplying the left- and right-hand sides of the last equation by α, we get

$$\frac{c^4}{G}(-f'(\alpha)\alpha^2 - \frac{1}{2}\alpha) = 4\pi Pl_{min}^2. \tag{50}$$

L.h.s. of (50) is dependent on α. Because of this, r.h.s. of (50) must be dependent on α as well, i. e. $P = P(\alpha)$, i.e

$$\frac{c^4}{G}(-f'(\alpha)\alpha^2 - \frac{1}{2}\alpha) = 4\pi P(\alpha)l_{min}^2. \tag{51}$$

Note that in this specific case the parameter α within constant factors is coincident with the Gaussian curvature K_a [**?**] corresponding to a:

$$\frac{l_{min}^2}{a^2} = l_{min}^2 K_a. \tag{52}$$

Substituting r.h.s of (52) into (51), we obtain the Einstein equation on horizon, in this case in terms of the Gaussian curvature

$$\frac{c^4}{G}(-f'(K_a)K_a^2 - \frac{1}{2}K_a) = 4\pi.P(K_a). \tag{53}$$

This means that up to the constants

$$-f'(K_a)K_a^2 - \frac{1}{2}K_a = P(K_a), \tag{54}$$

i.e. the Gaussian curvature K_a is a solution of Einstein equations in this case.

Then we examine different cases of the solution (54) with due regard for considerations of Sections 2,3.

4.2.1) First, let us assume that $a \gg l_{min}$. As, according to Section 2, the radius a is quantized, we have $a = N_a l_{min}$ with the natural number $N_a \gg 1$. Then it is clear that the Gaussian curvature $K_a = 1/a^2 \approx 0$ takes a(nonuniform)discrete series of values close to zero, and, within the factor $1/l_{min}^2$, this series represents inverse squares of natural numbers

$$(K_a) = (\frac{1}{N_a^2}, \frac{1}{(N_a \pm 1)^2}, \frac{1}{(N_a \pm 2)^2}, ...). \quad (55)$$

Let us return to formula (26) in Section 3 for $l = a$

$$|((\widetilde{\delta a})_\gamma)_{min}| = \beta N_a l_{min} N_a^{-\gamma} = \beta N_a^{1-\gamma} l_{min}, \quad (56)$$

where β in this case contains the proportionality factor that relates l_{min} and l_P.
Then, according to Section 3, $a_{\pm 1}$ is a measurable value of the radius r following after a, and we have

$$(a_{\pm 1})_\gamma \equiv a \pm ((\widetilde{\delta a})_\gamma)_{min} = a \pm \beta N_a^{1-\gamma} l_{min} = N_a(1 \pm \beta N_a^{-\gamma}) l_{min}. \quad (57)$$

But, as $N_a \gg 1$, for sufficiently large N_a and fixed γ, the bracketed expression in r.h.s. (57) is close to 1:

$$1 \pm \beta N_a^{-\gamma} \approx 1. \quad (58)$$

Obviously, we get

$$\lim_{N_a \to \infty} (1 \pm \beta N_a^{-\gamma}) \to 1. \quad (59)$$

As a result, the Gaussian curvature $K_{a_{\pm 1}}$ corresponding to $r = a_{\pm 1}$

$$K_{a_{\pm 1}} = 1/a_{\pm 1}^2 \propto \frac{1}{N_a^2(1 \pm \beta N_a^{-\gamma})^2} \quad (60)$$

in the case under study is only slightly different from K_a.
And this is the case for sufficiently large values of N_a, for any value of the parameter γ , for $\gamma = 1$ as well, corresponding to the absolute minimum of fluctuations $\approx l_{min}$,or more precisely – to βl_{min}. However, as all the

quantities of the length dimension are quantized and the factor β is on the order of 1, actually we have $\beta = 1$.
Because of this, provided the minimal length is involved, l_{min} (26) is read as

$$|(\widetilde{\delta l})_1|_{min} = l_{min}. \tag{61}$$

But, according to (28), $l_{min} = \xi l_P$ is on the order of Planck's length, and it is clear that the fluctuation $|(\widetilde{\delta l})_1|_{min}$ corresponds to Planck's energies and Planck's scales. The Gaussian curvature K_a, due to its smallness ($K_a \ll 1$ up to the constant factor l_{min}^{-2}) and smooth variations independent of γ (formulas (57)–(60)), is **insensitive** to the differences between various values of γ.
Consequently, for sufficiently small Gaussian curvature K_a we can take any parameter from the interval $0 < \gamma \leq 1$ as γ.
It is obvious that the case $\gamma = 1$, i. e. $|(\widetilde{\delta l})_1|_{min} = l_{min}$, is associated with infinitely small variations da of the radius r in the Riemannian geometry.
Since then K_a is varying practically continuously, in terms of K_a up to the constant factor we can obtain the following:

$$d[L(K_a)] = d[P(K_a)], \tag{62}$$

Where have

$$L(K_a) = -f'(K_a)K_a^2 - \frac{1}{2}K_a, \tag{63}$$

i. e. l.h.s of (53) (or (54)).
But in fact, as in this case the energies are low, it is more correct to consider

$$L((K_{a_{\pm 1}})_\gamma) - L(K_a) = [P(K_{a_{\pm 1}})_\gamma] - [P(K_a)] \equiv F_\gamma[P(K_a)], \tag{64}$$

where $\gamma < 1$,rather than (62).
In view of the foregoing arguments (4.2.1), the difference between (64) and (62)is insignificant and it is perfectly correct to use (62) instead of (64).
In [79] it is shown that the Einstein Equation for horizon spaces in the differential form may be written as a thermodynamic identity (the first principle of thermodynamics) ([79], formula (119)):

$$\underbrace{\frac{\hbar c f'(a)}{4\pi}}_{k_B T} \underbrace{\frac{c^3}{G\hbar} d\left(\frac{1}{4} 4\pi a^2\right)}_{dS} \underbrace{- \frac{1}{2}\frac{c^4 da}{G}}_{-dE} = \underbrace{P d\left(\frac{4\pi}{3} a^3\right)}_{P\,dV}. \tag{65}$$

However, this is questionable on account of the existing minimal length l_{min}. As the quantity l_{min} is fixed, it is obvious that «dS» and «dV» in (65)will be growing as a and a^2, respectively. And at low energies, i.e. for large values of $a \gg l_{min}$, this naturally leads to infinitely large rather than infinitesimal values.
4.2.2) Now we consider the opposite case or the transition to the **ultraviolet limit**

$$a \rightarrow \kappa l_{min}, \tag{66}$$

i.e.

$$a \approx \kappa l_{min}. \tag{67}$$

Here κ is on the order of 1.
Taking into consideration point 3.1) of Section 3 stating that in this case models for different values of the parameter γ are coincident, by formula (61) for any γ we have

$$|(\widetilde{\delta l})_{\gamma}|_{min}| = (\widetilde{\delta l})_{1}|_{min} = l_{min}. \tag{68}$$

But in this case the Gaussian curvature K_a is not a «small value» continuously dependent on a , taking, according to (60), a discrete series of values $K_a, K_{a\pm 1}, K_{a\pm 2}, ...$
Yet (46), similar to (53) ((54)), is valid in the semiclassical approximation only, i.e. at **low energies**.
Then in accordance with the above arguments, the limiting transition to **high energies**(66) gives a discrete chain of equations or a single equation with a discrete set of solutions as follows:

$$-f'(K_a)K_a^2 - \frac{1}{2}K_a = \Theta(K_a);$$

$$-f'(K_{a\pm 1})K_{a\pm 1}^2 - \frac{1}{2}K_{a\pm 1} = \Theta(K_{a\pm 1});$$

and so on. Here $\Theta(K_a)$ – some function that in the limiting transition to low energies must reproduce the low-energy result to a high degree of accuracy, i.e. $P(K_a)$ appears for $a \gg l_{min}$ from formula (54)

$$\lim_{K_a \rightarrow 0} \Theta(K_a)) = P(K_a). \tag{69}$$

In general, $\Theta(K_a)$ may lack coincidence with the high-energy limit of the momentum-energy tensor trace(if any):

$$\lim_{a \rightarrow l_{min}} P(K_a). \tag{70}$$

At the same time, when we naturally assume that the Static Spherically-Symmetric Horizon Space-Time with the radius of several Planck's units (67) is nothing else but a micro black hole, then the high-energy limit (70) is existing and the replacement of $\Theta(K_a)$ by $P(K_a)$ in r.h.s. of the foregoing equations is possible to give a hypothetical gravitational equation for the event horizon micro black hole. But a question arises, for which values of the parameter a (67) (or K_a) this is valid and what is a minimal value of the parameter $\gamma = \gamma(a)$ in this case?
In all the cases under study, 3.1.1) and 3.1.2), the deformation parameter α_a (7)(λ_a(34))is, within the constant factor, coincident with the Gaussian curvature K_a (respectively $\sqrt{K_a}$) that is in essence continuous in the low-energy case and discrete in the high-energy case.

What features are ≪**common**≫ for these two examples?

I. Provided the minimal length l_{min} is involved, in both examples the gravitational equations begin to be dependent on the dimensionless discrete parameter α that at low energies is close to 0 and is varying very slowly (smoothly) so that in fact the theory can be considered continuous but for high energies only, and at $\alpha \rightarrow 1$ the theory becomes really discrete.

II. According to the basic formulae of Section 2 and, in particular, to (14), the α-dependence of the gravitational equations reflects the relationship between the gravitational equations and the existent energies.

5 Some Comments and General Considerations

5.1. So, as demonstrated in the previous Section for the particular cases, provided a theory involves the minimal length $l_{min} \propto l_P$, gravity is almost independent of the parameters associated with this length, specifically α_l and γ (and hence λ_l and γ), i.e. the dependence is weak, and so the theory is practically continuous. This stems from the fact that these parameters are very small due to remoteness of the energies characterizing them from the Planck energies and almost **insensitive** to the corresponding change in measuring scales.
Despite a **discrete** nature of the theory owing to the existence of l_{min}, to a high degree of accuracy we can use infinitesimal variations of dx_μ, coincident in the case under study with l_{min} and t_{min}. In this way in the cases considered in Section 4 the **Conformity Principle** stating that (*on going to low energies the known theory (in particular GR) must be reproduced to a high degree of accuracy, at least its experimentally verified part*) holds **to a high accuracy**.
Still it is clear that, as formally GR has no additional parameters associated with l_{min} and the low-energy (for now hypothetical variant of the minimal length theory denoted as $Grav^{l_{min}}$ has such parameters, there is also the **high accuracy** limit indicated above. This limit in every case determines the «gap» between GR and $Grav^{l_{min}}$. Evaluation of this gap is a real challenge for those trying to construct a unified theory at all energy levels.
As noted in 4.1.2 and 4.2.2, for high energies, i.e. for $l \to l_{min}$, (or what is the same $\lambda_l \to 1, \gamma \to 1$) a discrete chain of equations (or a single equation with a discrete set of solutions) is derived that is numbered by inverse squares of the integers $1; 1/4; 1/9; ....$ to represent the parameter λ_l^2 at high (Planck's) energies.

5.2.We have used GR to demonstrate that the above models 4.1,4.2 at low energies are actually insensitive to variations of the discrete parameters $(\alpha_l(or\lambda_l), \gamma)$ associated with the minimal length. Of course, it is more correct to use $Grav^{l_{min}}$ and to compare the obtained results with GR. But, as yet there is no $Grav^{l_{min}}$, it is connived that at low energies GR and

$Grav^{l_{min}}$ differ insignificantly and the indicated parameters, provided l_{min} is involved, are introduced into GR similarly to $Grav^{l_{min}}$.

5.3. It is easily seen that the ≪Entropic Approach to Gravity≫ [81] in the present formalism is invalid within the scope of the minimal length theory. This was noted in [1]. In fact, the ≪main instrument≫ in [81] is a formula for the infinitesimal variation dN in the bit numbers N on the holographic screen $\mathcal{S}$ with the radius R and with the surface area A ([81],formula (4.18)):

$$dN = \frac{c^3}{G\hbar}\,dA = \frac{dA}{l_P^2}. \tag{71}$$

But it is obvious that infinitesimal variations of the screen surface area dA are possible only in a continuous theory involving no l_{min}.
When $l_{min} \propto l_P$ is involved, the minimal variation $\triangle A$ is evidently associated with a minimal variation in the radius R

$$R \to R \pm l_{min} \tag{72}$$

is dependent on R and growing as R for $R \gg l_{min}$ (formula (54) in [1]):

$$\triangle_{\pm} A(R) = A(R \pm l_{min})) - A(R) \propto (\frac{\pm 2R}{l_{min}} + 1) = \pm 2N_R + 1, \tag{73}$$

where $N_R = R/l_{min}$, as indicated above.
So, if l_{min} is involved, formula (4.18) from [81] has no sense similar to other formulae derived on its basis (4.19),(4.20),(4.22),(5.32)–(5.34), ... in [81] and similar to the derivation method for Einstein's equations proposed in this work.
Proceeding from the principal parameters of this work $\alpha_l(or\lambda_l)$, the fact is obvious and is supported by the formula (35)given in this paper, meaning that

$$\alpha_R^{-1} \sim A, \tag{74}$$

i.e. small variations of α_R (low energies) result in large variations of α_R^{-1}, as indicated by formula (73). In fact, we have a **no-go theorem**.

5.4. As the Planck length $l_P = (\hbar G/c^3)^{1/2}$ is expressed in terms of the

fundamental constants, the proportionality coefficient ξ from formula (28), relating l_{min} and l_P, in a minimal-length theory l_{min} should also be a fundamental constant because it (along with G, $\hbar$, and c) must be involved in all the basic formulae of this theory. Then the question arises: what is its value?
In [18] for the coefficient α' in GUP (3) the substantiated value was equal to 1. Provided this is true, $\xi = 2$ and hence the Bekenstein-Hawking formula for the black hole entropy S^{BH} may be written most naturally and elegantly as follows:

$$S^{BH} = \frac{A}{l_{min}^2}. \tag{75}$$

6 Conclusion

6.1. Thus, it has been shown that some models for GR (cosmology) involve the discrete parameters associated with the minimal length, while at low energies, due to their smallness, a theory is **insensitive** to their variations and may be considered almost continuous, independent of these parameters.

6.2. As at low energies $\alpha_l(\lambda_l)$–small parameter, the gap between GR and a hypothetical minimal length theory $Grav^{l_{min}}$ (mentioned in subsection 5.1) is determined by a series expansion in terms of this parameter close to 0 and by confinement of the leading terms in this series.
As in this case the cosmological term Λ is no longer a constant $\Lambda \neq const$, (and the Bianchi identity $\nabla^\mu G_{\mu\nu} \approx 0$ [2] will appear to a high degree of accuracy only in the low-energy limit), this term is dependent on $\alpha_l(\lambda_l)$ and we have [82],[67] with the known quantum field theory

$$\Lambda(\alpha) \propto (\alpha^2 + \eta_1\alpha^2 + ...)\Lambda_p, \tag{76}$$

and, provided the holographic principle is valid, we get [83]–[86]

$$\Lambda^{Hol}(\alpha) \propto (\alpha + \xi_1\alpha^2 + ...)\Lambda_p, \tag{77}$$

where Λ_p –cosmological term at Planck's scales.

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