Holographic Principle,Minimal Length and Measurability

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Abstract

In this paper the Holographic Principle is used with the involvement of a minimal length. It is shown that two alternative approaches to the minimal length involving theory are possible. By the first approach, the minimal length is actualized only at high energies, whereas at low energies it is zero, the theory being continuous. In this approach there is generalization of the entropic approach to gravity for the ultraviolet region. By the second approach, the minimal length is nonzero at all energies scales. Then the entropic approach to gravity in its present form is impossible in this case because the theory must be free from such infinitesimal quantities as infinitely small variations in the surface of the holographic screen, its volume, and entropy.

1 Introduction

This paper presents a study of the Holographic principle in a theory with the minimal length l_{min}.
It is known that a minimal length on the order of the Planck length $l_{min} \sim l_p$ becomes important only at very high (Planck's) energies $E \propto E_P$. At low energies $E \ll E_P$ it is practically not apparent. What results from this fact?
1.1 Approach I: provided the minimal length l_{min} is involved, then on going

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from high to low energies the corresponding low-energy theory, to a high accuracy, may be considered to be continuous in agreement with the limiting transition

$$l_{min} \rightarrow 0 \quad (1)$$

1.2 Approach II: according to this approach that is an alternative to the first one, the minimal length $l_{min} \neq 0$ is involved both at high and low energies. In this work the proposed approaches are considered in view of the Holographic Principle.
The first approach is demonstrated using as an example the generalization of the well-known results by E.Verlinde [1] to high (Planck's) energies. The starting point for such generalization is the holographic screen $\mathcal{S}$ and its information content. This material is given in Section 2.
For the second approach based on $l_{min} \neq 0$, (i.e. on the absence of the formula (1)) at all the energy scales, both high and low, we first introduce the definition of a **measurable quantity**. Then it is shown that, if a theory operates only with the **measurable quantities**, the method in [1] in its present formalism is invalid within the scope of the minimal length theory,because the theory must be free from such infinitesimal quantities as infinitely small variations in the surface of the holographic screen, its volume, and entropy. This material is given in Section 3.
This paper is based on the previous works [2], [3] and partially – on [4].

2 Holographic Principle, Entropic Aprroach to Gravity and Minimal Length

2.1 High-Energy Deformation of Main Quantities

This Section is based on the generalization to high energies [2] of the results by E.Verlinde [1] associated with his entropic approach to gravity.
In what follows in this Section we use the results from [2]. The starting point is the spherically-symmetric holographic screen $\mathcal{S}$ from [1] and its information content.
As known, a formula for the ≪**bit density**≫ dN on $\mathcal{S}$ is given as ((5.32) in

[1]):

$$N = \frac{A}{G\hbar}, dN = \frac{dA}{G\hbar}, \tag{2}$$

where N – «bit» number on $\mathcal{S}$.
However, when the holographic principle [5]–[9] is valid, N is actually the entropy S up to the factor $S \sim N$ and hence from (2) it follows directly that $dS \propto dA/G\hbar$.
What are the changes in S on going to high (Planck) energies? The answer to this question is already known owing to the fact that at these energies the Heisenberg Uncertainty Principle (HUP) is replaced by GUP [10]–[18]:

$$\triangle x \geq \frac{\hbar}{\triangle p} + \ell^2 \frac{\triangle p}{\hbar}, \tag{3}$$

where $\ell^2 = \alpha' l_p^2$ and α' – dimensionless numerical factor. (3) leads to the minimal length $l_{min} = \xi l_P = 2\sqrt{\alpha'} l_P$.
The well-known Bekenstein-Hawking formula for the black hole entropy in the semiclassical approximation [19],[20]

$$S^{BH} = \frac{A}{4l_p^2} \tag{4}$$

is modified by the corresponding quantum corrections on going from HUP to GUP [21]–[24].
In particular, [22]:

$$S_{GUP}^{BH} = \frac{A}{4l_p^2} - \frac{\pi\alpha'^2}{4} \ln\left(\frac{A}{4l_p^2}\right) + \sum_{n=1}^{\infty} c_n \left(\frac{A}{4l_p^2}\right)^{-n} + \text{const} , \tag{5}$$

where the expansion coefficients $c_n \propto \alpha'^{2(n+1)}$ can always be computed to any desired order of accuracy.
The general form of quantum corrections for the black hole entropy derived in (5) remains valid for any horizon spaces and, in particular, for the holographic screen $\mathcal{S}$ from [1].
Higher-order corrections may be derived using the Taylor-series expansion in terms of the small parameter l_p^2/A

$$S_{GUP} = \frac{A}{4l_p^2} + \frac{\widetilde{\alpha}}{4} \ln\left(\frac{A}{l_p^2}\right) + \sum_{n=1}^{\infty} \widetilde{c}_n \left(\frac{A}{l_p^2}\right)^{-n} + \text{const} \tag{6}$$

in a similar way to the Taylor-series expansion of the right-hand side in (6) in terms of the small parameter $4l_p^2/A$. This is valid as GUP gives the ultraviolet cutoff at the level of $l_{min} \sim l_p$.
Assuming that at high energy $S \to S_G$ and hence $N \to N_G$ and in consideration of that $S = N/4$, we obtain

$$N_{GUP} = \frac{A}{l_p^2} + \widetilde{\alpha} \ln \left(\frac{A}{l_p^2}\right) + 4 \sum_{n=1}^{\infty} \widetilde{c}_n \left(\frac{A}{l_p^2}\right)^{-n} + \text{const},. \tag{7}$$

In terms of N_{GUP} we can define the holographic screen area, as measured at high energies, by $A_{GUP} \equiv G\hbar N_{GUP}$, where G and $\hbar$ – gravitational and Planck constants, respectively, and N_{GUP} is given (7). Considering that we, similar to [1], assume that the speed of light $c = 1$, then, according to $l_p^2 = G\hbar$, from (7) we have

$$A_{GUP} = A + G\hbar\widetilde{\alpha} \ln \left(\frac{A}{G\hbar}\right) + 4G\hbar \sum_{n=1}^{\infty} \widetilde{c}_n \left(\frac{A}{G\hbar}\right)^{-n} + \text{const} . \tag{8}$$

So, (2) has a fairly definite analog at high energies

$$dN_{GUP} = \frac{dA_{GUP}}{G\hbar} \tag{9}$$

that on going to the known low energies gives (2). There is a single considerable difference, in [1] the quantity N was defined in terms of A and dN was defined in terms of dA but in the case under study the situation is opposite: A_{GUP} is defined in terms of N_{GUP} and dA_{GUP} in terms of dN_{GUP}. The logic series is here as follows: $A \Rightarrow N \Rightarrow N_{GUP} \Rightarrow A_{GUP}$

The high-energy (for GUP) redefinition problem of the temperature $T \to T_{GUP}$ for the holographic screen $\mathcal{S}$ has been studied in [25]. In [25] T_{GUP} was derived as a series

$$T_{GUP} = T + \Theta_T T^3 + ... = T + \widetilde{T}_{GUP}, \tag{10}$$

where the factors in the right-hand side(10) may be computed in the explicit form and at low energies $\widetilde{T}_{GUP} \to 0$.

Then, under (7)–(10), we can have a GUP - analog of Komar's mass in ((5.33) from [1])

$$M_{GUP} \equiv \frac{1}{2}\int_{\mathcal{S}} T_{GUP} dN_{GUP} = \frac{1}{2}\int_{\mathcal{S}} (T+\widetilde{T}_{GUP}) dN_{GUP} = \frac{1}{2G\hbar}\int_{\mathcal{S}} T_{GUP} dA_{GUP}, \tag{11}$$

that in the low-energy limit gives the well-known Komar formula [26],([27], p.289).

It is clear that the «GUP-deformed Komar's mass» M_{GUP} in the first term (11) as a summand has the known Komar's mass [26],((11.2.9 - 11.2.10),[27]) ((5.34), [1])

$$M = \frac{1}{4\pi G}\int_{\mathcal{S}} T dA. \tag{12}$$

If feasible, it is desirable to express all the above-derived fundamental quantities in terms of a unified parameter. As shown by the author in [28], [29], this is possible for black holes within the scope of GUP and a role of the unified small parameter is played by the parameter introduced previously in [30]–[41] as follows:

$$\alpha_x = l_{min}^2/x^2, \tag{13}$$

where x is the measuring scale, $l_{min} \sim l_p$ by virtue of GUP (3), and $0 < \alpha \leq 1/4$.

Obviously, the principal results obtained in [28], [29] remain in force for an arbitrary screen $\mathcal{S}$ and may be applied to the quantities N_{GUP},A_{GUP}, M_{GUP}.

In particular, we have

$$N_{GUP} = N + \widetilde{\alpha}\ln(\sigma\alpha_R^{-1}) + 4\sum_{n=1}^{\infty}\widetilde{c}_n\sigma^{-n}\alpha_R^n + \text{const},$$

$$A_{GUP} = A + \widetilde{\alpha}G\hbar\ln(\sigma\alpha_R^{-1}) + 4G\hbar\sum_{n=1}^{\infty}\widetilde{c}_n\sigma^{-n}\alpha_R^n + \text{const}, \tag{14}$$

where R – characteristic linear size (radius) of the screen $\mathcal{S}$; α_R – value of α parameter at the point R, σ is a dimensionless computational factor.

It is convenient to refer to the form N_{GUP} and A_{GUP} derived in (14) as to the α-representation. Also, it is clear that M_{GUP} (11) may be derived in terms of α_R.
It should be noted that α_x is considered as **a deformation parameter** for the Heisenberg algebra on going from HUP to GUP.

Definition 1.
Deformation is understood as an extension of a particular theory by inclusion of one or several additional parameters in such a way that the initial theory appears in the limiting transition [42].

Generally speaking, initially the construction of such a Deformation was realized with other parameters (e.g. [16],[17],([43])). But it is easily shown that QFT parameter of the deformations associated with GUP may be expressed in terms of the parameter α that has been introduced in the approach to the density matrix deformation [28], [29]. Here the notation of [43] is used. Then from [43] we have

$$[\vec{x},\vec{p}] = i\hbar(1+\beta^2\vec{p}^2+...) \tag{15}$$

and

$$\Delta x_{\min} \approx \hbar\sqrt{\beta} \sim l_p. \tag{16}$$

As shown in [28], [29], the right-hand side of (15) may be completed with an expansion in terms of the small parameter α_x:

$$[\vec{x},\vec{p}] = i\hbar(1+\beta^2\vec{p}^2+...) = i\hbar(1+a_1\alpha_x+a_2\alpha_x^2+...). \tag{17}$$

In the case under study convenience of using α_x stems from its smallness, its dimensionless character, and ability to test changes in the radius R of the holographic screen $\mathcal{S}$.
Based on the aforesaid, we can proceed to generalization of the results from Section 5.2 of [1] and to derivation of equations for a gravitational field within the scope of GUP. We must consider two absolutely different cases.

2.2 Transition to Higher Energies

We must consider two absolutely different cases.

2.2.1 Quantum Corrections to the Principal Result

It is assumed that the screen radius R is given by $R \gg l_{min} \propto l_p$ and then

$$\alpha_R \ll 1/4, \tag{18}$$

where α_R is the deformation parameter introduced in the previous subsection and corresponding R.
Then the principal result from the final part of Section 5.2 in [1] remains valid owing to the replacement of M (formula (5.34) from [1]) by $M_{GUP} = M_{GUP}[\alpha_R]$ (11). The ≪α_R – complement≫ (i.e. the difference $\widetilde{M}[\alpha_R] = M_{GUP}[\alpha_R] - M$) to M will be simply a (small) quantum correction for the principal result.
Because of (18), it is supposed that α_R is continuously varying from R and all the quantities in Section 5.2 of [1] are also continuously dependent on α_R (18). Then we can write down the (≪α_R – analog≫ of formula (5.37) in [1]) as

$$2\int_\Sigma \left(T_{ab}[\alpha] - \frac{1}{2}T[\alpha]g_{ab}[\alpha]\right) n^a \xi^b dV = \frac{1}{4\pi G}\int_\Sigma R_{ab}[\alpha] n^a \xi^b dV, \tag{19}$$

where the dependence of $T_{ab}[\alpha]$ and $R_{ab}[\alpha]$ on $\alpha = \alpha_R$ is completely determined, in accordance with [27],[1], by the integral $M_{GUP}[\alpha]$ (11).
Besides, it is assumed that n^a and ξ^b are dependent on α, though the dependence is dropped.
Next, similar to [1], from (19) we can derive the α**-deformed** Einstein Equations using the method from [45]. Note that both this method and its minor modification given in ([1], end of Section 5.2) in this case are valid because α_R is small and continuous, the whole system being continuously dependent on it.
Solutions of the α**-deformed** Einstein Equations represent a series in α_R, and for $\alpha_R \to 0$ or for $\alpha' = 0$ become the corresponding solutions of (Section

5.2 in [1]).
Using the result obtained in [44], we can easily extend the above result to the case with a nonzero cosmological term $\Lambda \neq 0$. In [44] Komar's formula was generalized to the case of a nonzero Λ. All the arguments from (Section 5.2 of [1]) in this case remain valid and formula (5.37) takes the following form:

$$2\int_{\Sigma}\left(T_{ab}-\frac{1}{2}Tg_{ab}\right)n^a\xi^b dV=\frac{1}{4\pi G}\int_{\Sigma}(R_{ab}+\Lambda g_{ab})n^a\xi^b dV. \tag{20}$$

We can easily obtain the α - analog of the last formula with the dynamic cosmological term $\Lambda(\alpha)$ as a corresponding complement to the right-hand side (19).

2.2.2 Transition to Ultraviolet Limit

This case has been considered in detail in [29] and [2].
Then the screen $\mathcal{S}$ has a radius on the order of several Planck's lengths $R \approx \xi l_{min} = 2\alpha'\xi l_p$, where ξ – number on the order of 1 or

$$\alpha_R \approx 1/4. \tag{21}$$

The problem is which object puts the limit for such a screen $\mathcal{S}$. It may be assumed that if $T_{ab} \neq 0$ then the object may be represented only by Planck's black hole or by a micro-black hole with a radius on the order of several Planck's lengths.
Clearly, the methods of [1] and [45] are not in force for such screen $\mathcal{S}$ because it is impossible to use the result of [45] as ≪a very small region the space-time≫ is no longer ≪an approximate Minkowski space-time≫ [1].
Also, such micro-black hole is a horizon space, jet at high energies (Planck scales). As is known, for horizon spaces, black holes in particular, at low energies (semiclassical approximation) the results of [46] are valid.
At the horizon (and we are interested in this case only) Einstein's field Equations in differential form may be written as ([46] formula (119)):

$$(\frac{\hbar c f'(a)}{4\pi})\frac{c^3}{G\hbar}d\left(\frac{1}{4}4\pi a^2\right)-\frac{1}{2}\frac{c^4 da}{G}=Pd\left(\frac{4\pi}{3}a^3\right) \tag{22}$$

where $R = a$ – radius of a black hole (i.e. of the screen $\mathcal{S}$), $P = T_R^R$ is the trace of the momentum-energy tensor and radial pressure, and the horizon location will be given by simple zero of the function $f(R)$, at $R = a$.
As shown in [29] the equations (22) may be written in terms of the deformation parameter α with the coefficients containing only the numerical factors and fundamental constants.
Also, the work [29] presents two possible variants of high-energy (Planck) α -deformation $\alpha \rightarrow 1/4$ (22).
Hereinafter, we assume that the energy – momentum tensor of matter fields is not traceless

$$T_a^a \neq 0, \tag{23}$$

similar, in particular, to the case under study (22) $P = T_R^R \neq 0$

2.2.2.A Case of equilibrium thermodynamics ([29], section (6.1))

In this case it is assumed that in the high-energy (ultraviolet (UV))limit the thermodynamic identity (22) is retained but now all the quantities involved in this identity become α-deformed ($\alpha \rightarrow 1/4$). All the quantities Ξ in (22) are replaced by the corresponding quantities Ξ_{GUP} with the subscript GUP. Then the high-energy α-deformation of equation (22) takes the form

$$k_B T_{GUP}(\alpha) dS_{GUP}(\alpha) - dE_{GUP}(\alpha) = P(\alpha) dV_{GUP}(\alpha). \tag{24}$$

Substituting into (24) the corresponding quantities
$T_{GUP}(\alpha), S_{GUP}(\alpha), E_{GUP}(\alpha), V_{GUP}(\alpha), P(\alpha)$ and expanding them into a Laurent series in terms of α, close to high values of α, specifically close to $\alpha = 1/4$, we can derive a solution for the high energy α-deformation of the general relativity (24) as a function of $P(\alpha)$. Provided at high energies the generalization of (22) to (24) is possible, we can have the high-energy α-deformation of the metric.
It is noteworthy that in (24) T_{GUP} this time is calculated from ([21], formula

(10))

$$
T_{GUP}^{BH} = \frac{1}{4\pi}\frac{\hbar R}{2\alpha'^2 l_p^2}[1-\sqrt{1-\frac{\alpha'^2 l_p^2}{R^2}}] =
$$

$$
\frac{\hbar \alpha_R^{-1}}{4\pi\alpha' l_p}[1-(1-\alpha_R)^{1/2}] \tag{25}
$$

with subsequent replacement of l_p by $\sqrt{G\hbar}$ for $c = 1$.
It is especially interesting to consider the following case.

2.2.2.B Case of nonequilibrium thermodynamics ([29], section (6.2))

In this case the α - dependent dynamic cosmological term $\Lambda(\alpha) \neq 0$ appears in the right-hand side of (24). Then, with the addition of $\Lambda(\alpha) \neq 0$, the α – representation (24) (for $\hbar = 1$) is given as follows ((([29], formula (53)):

$$
-\alpha^2 f'(\alpha) - \frac{1}{2}\alpha = 16\pi\alpha'^2 P(\alpha)G^2 - G\Lambda(\alpha),
$$

$$
f'(\alpha) = 4\pi k_B T_{GUP}(\alpha) \tag{26}
$$

where $\alpha = \alpha_R \approx 1/4$ and the derivative $f'(\alpha)$ of the $f(\alpha)$ is taken with respect to α.
$\Lambda(\alpha)$ in the right-hand side of (26) may be subjected to a series expansion in terms of α, in compliance with the holographic principle [5]–[9] as applied to the Universe [47]. In [48],[41], [28],[29] in the leading order this expansion results in the first power, i.e. we have

$$
\Lambda(\alpha_R) \sim \alpha_R \Lambda_p, \tag{27}
$$

where Λ_p – initial value of $\Lambda \approx \Lambda_{1/4}$ derived using the well-known procedure of ≪summation over all zero modes≫ and the Planck momentum cutoff [49],[50]. Actually, (27) is in a good agreement with the observable $\Lambda = \Lambda_{observ}$. Because a radius of the visible part of the Universe is given as $R = R_{Univ} \approx 10^{28} cm$, it is clear that $\alpha_R \approx 10^{-122}$ and (27) is completely consistent with the experiment [50].

Note that, proceeding directly from a quantum field theory but without the use of the holographic principle, we can have only a rough estimate of Λ that, on the whole, is at variance with Λ_{observ}. Such an estimate may be obtained in different ways: by simulation [51]; using the cutoff [49] but now in the infrared limit; with the use of the Generalized Uncertainty Principle for the pair (Λ, V), where V – four-dimensional volume [41], [28]. In the α–representation in this case the expansion in terms of α results in the second leading order

$$\Lambda(\alpha_R) \sim \alpha_R^2 \Lambda_p, \tag{28}$$

that, obviously, is at variance with the accepted facts.
In the following Section the static spherically-symmetric horizon space considered in [46] is treated from a new point of view.

Afterword to Section 2

A.1. In this Section it has been implied that at low energies $E \ll E_P$ the theory is continuous and the minimal length $l_{min} \neq 0$ is involved only at the energies $E \approx E_P$. And in the following Section it is demonstrated that an alternative viewpoint is also possible: the minimal length $l_{min} \neq 0$ is involved at all the ≪energy levels≫.

A.2. According to A.1, at low energies, for $x \gg l_{min}$, as follows from (13), α_x is varying practically continuously and hence, for α_x close to zero, the formulas (24),(26) are absolutely reasonable.
At the same time, it has been supposed that the above-mentioned formulas (24) and (26) give an adequate description for the situation in the high-energy case $\alpha_x \to 1/4$ too.
Then, due to the fact that at high energies $E \approx E_P$, $l_{min} \neq 0$, we should have not ≪continuous equations≫ (24) and (26) but the equations discrete for α_x.
In greater detail this problem is considered in the second part of the following Section.

3 Minimal Length and Measurability

3.1 Minimal Length and Measurable and Nonmeasurable Quantities

In the previous Section it has been supposed that the minimal length $l_{min} \propto l_P$ is involved only at high energies E on the order of Planck's energies $E \propto E_P$, whereas at low energies $E \ll E_P$ the theory, to a high accuracy, may be considered continuous, the following limit being the case: $l_{min} \rightarrow 0$. An alternative view is that, provided a minimal length exists, it is existent at all the energy scales and not at high (Planck's) scales only.
What is inferred on this basis for real physics? At least, it is suggested that the use of infinitesimal quantities dx_μ in a mathematical apparatus of both quantum theory and gravity is incorrect, despite the fact that both these theories give the results correlating well with the experiment (for example, [52]).
Indeed, in all cases the infinitesimal quantities dx_μ bring about an infinitely small length ds [27]

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu \tag{29}$$

that is inexistent because of l_{min}.
The same is true for any function Υ dependent only on different parameters L_i whose dimensions of length of the exponents are equal to or greater than 1 $\nu_i \geq 1$

$$\Upsilon \equiv \Upsilon(L_i^{\nu_i}). \tag{30}$$

Obviously, the infinitely small variation $d\Upsilon$ of Υ is senseless as, according to (30), we have

$$d\Upsilon \equiv d\Upsilon(\nu_i L_i^{\nu_i - 1} dL_i). \tag{31}$$

But, because of l_{min}, the infinitesimal quantities dL_i make no sense and hence $d\Upsilon$ makes no sense too.
Instead of these infinitesimal quantities, reasonable are «minimal variations possible» Δ_{min} of the quantity L having the dimension of length, i.e. the quantity

$$\Delta_{min} L = l_{min}. \tag{32}$$

And then

$$\Delta_{min}\Upsilon \equiv \Delta_{min}\Upsilon(\nu_i L_i^{\nu_i-1}\Delta_{min}L_i) = \Delta_{min}\Upsilon(\nu_i L_i^{\nu_i-1}l_{min}). \quad (33)$$

However, the «minimal variations possible» of any quantity having the dimensions of length (32) which are equal to $l_{min} \propto l_P$ require, according to the Heisenberg Uncertainty Principle (HUP) [53]:

$$\Delta x \geq \frac{\hbar}{\Delta p} \quad (34)$$

maximal momentum $p_{max} \propto P_{Pl}$ and energy $E_{max} \propto E_P$. Here l_P, P_{Pl}, E_P – Planck's length, momentum, and energy, respectively.(Note that the normalization $\Delta x \Delta p \geq \hbar$ is used rather than $\Delta x \Delta p \geq \hbar/2$.)
But at low energies (far from the Planck energy) there are no such quantities and hence in essence $\Delta_{min}L = l_{min} \propto l_P$ (32) corresponds to the high-energy (Planck's) case only.
For the energies lower than Planck's energy, the «minimal variations possible» $\Delta_{min}L$ of the quantity L having the dimensions of length must be greater than l_{min} and dependent on the present E

$$\Delta_{min} \equiv \Delta_{min,E}, \Delta_{min,E}L > l_{min}. \quad (35)$$

Besides, as we have a minimal length unit l_{min}, it is clear that any quantity having the dimensions of length is «quantized», i.e. its value measured in the units l_{min} equals an integer number and we have

$$L = N_L l_{min}, \quad (36)$$

where N_L– positive integer number.
The problem is, how the «minimal variations possible» $\Delta_{min,E}$ (35) are dependent on the energy or, what is the same, on the scales of the measured lengths?
Then assuming that HUP is to a high accuracy derived from GUP on going to low energies or that HUP is a special case of GUP at low values of the momentum, we have

$$(GUP, \Delta p \rightarrow 0) = (HUP). \quad (37)$$

By the language of N_L from (36), (37) is nothing else but a change-over to the following:

$$(N_{\Delta x} \approx 1) \rightarrow (N_{\Delta x} \gg 1). \tag{38}$$

The assumed equalities in (34) and (3) may be conveniently rewritten in terms of l_{min} with the use of the deformation parameter $\alpha_{\Delta x}$ (13).
Then with the equality $(\Delta p \Delta x = \hbar)$ (3) is of the form

$$\Delta x = \frac{\hbar}{\Delta p} + \frac{\alpha_{\Delta x}}{4} \Delta x. \tag{39}$$

In this case due to formulae (36) and (38) the equation (39) takes the following form:

$$N_{\Delta x} l_{min} = \frac{\hbar}{\Delta p} + \frac{1}{4 N_{\Delta} x} l_{min} \tag{40}$$

or

$$(N_{\Delta x} - \frac{1}{4 N_{\Delta} x}) l_{min} = \frac{\hbar}{\Delta p}. \tag{41}$$

That is

$$\Delta p = \frac{\hbar}{(N_{\Delta x} - \frac{1}{4 N_{\Delta} x}) l_{min}}. \tag{42}$$

From (40)–(42) it is clear that HUP (34) in the case of the equality appears to a high accuracy in the limit $N_{\Delta} x \gg 1$ in conformity with (38).
It is easily seen that the parameter α_a from (13) is discrete as it is nothing else but

$$\alpha_a = l_{min}^2 / a^2 = \frac{l_{min}^2}{N_a^2 l_{min}^2} = \frac{1}{N_a^2}. \tag{43}$$

At the same time, from (43) it is evident that α_a is irregularly discrete.
It is clear that from formula (42) at low energies $(N_{\Delta} x \gg 1)$, up to a constant

$$\frac{\hbar^2}{l_{min}^2} = \frac{\hbar c^3}{4 \alpha' G} \tag{44}$$

we have

$$\alpha_{\Delta x} = (\Delta p)^2. \tag{45}$$

But all the above computations are associated with the nonrelativistic case. As known, in the relativistic case, when the total energy of a particle with the mass m and with the momentum p equals [54]:

$$E = \sqrt{p^2c^2 + m^2c^4}, \tag{46}$$

a minimal value for Δx takes the form [55]:

$$\Delta x \approx \frac{c\hbar}{E}. \tag{47}$$

And in the **ultrarelativistic case**

$$E \approx pc \tag{48}$$

this means simply that

$$\Delta x \approx \frac{\hbar}{p}. \tag{49}$$

Provided the minimal length l_{min} is involved and considering the ≪**Integrality Condition**≫ **(IC)** (36), in the general case for (47) at the energies considerably lower than the Planck energies $E \ll E_P$ we obtain the following:

$$\Delta x = N_{\Delta x} l_{min} \approx \frac{c\hbar}{E},$$
$$or$$
$$E \approx \frac{c\hbar}{N_{\Delta x}}. \tag{50}$$

Similarly, at the same energy scale in the ultrarelativistic case we have

$$p \approx \hbar / N_{\Delta x}. \tag{51}$$

Note that all the foregoing results associated with GUP and with its limiting transition to HUP for the pair $(\Delta x, \Delta p)$, as shown in [32], may be in **ultrarelativistic case** easily carried to the ≪energy - time≫ pair $(\Delta t, \Delta E)$. Indeed (3) gives [32]:

$$\frac{\Delta x}{c} \geq \frac{\hbar}{\Delta pc} + \alpha' l_P^2 \frac{\Delta p}{c\hbar}, \tag{52}$$

then

$$\Delta t \geq \frac{\hbar}{\Delta E} + \alpha' \frac{l_p^2}{c^2} \frac{\Delta pc}{\hbar} = \frac{\hbar}{\Delta E} + \alpha' t_p^2 \frac{\Delta E}{\hbar}. \tag{53}$$

where according to (48) the difference between ΔE and $\Delta(pc)$ can be neglected and t_P is the Planck time $t_P = L_P/c = \sqrt{G\hbar/c^5} \simeq 0,54\ 10^{-43} sec.$ From whence it follows that we have a maximum energy of the order of Planck's:

$$E_{max} \sim E_P$$

Then the foregoing formulas (34)–(41) are rewritten by substitution as follows:

$$\Delta x \rightarrow \Delta t; \Delta p \rightarrow \Delta E; l_{min} \rightarrow t_{min}; N_L \rightarrow N_{t=L/c} \tag{54}$$

Specifically, (41) takes the form

$$(N_{\Delta t} - \frac{1}{4N_{\Delta}t})t_{min} = \frac{\hbar}{\Delta E}. \tag{55}$$

As shown, for the ultrarelativistic case there is t_{min}.
Next we assume that for **all cases** there is a minimal measuring unit of time

$$t_{min} = l_{min}/v_{max} = l_{min}/c. \tag{56}$$

Then, similar to (36), we get the ≪**Integrality Condition**≫ **(IC)** for any time t:

$$t \equiv t(N_t) = N_t t_{min}, \tag{57}$$

for certain $|N_t| \geq 0$ – integer.
According to (55), let us define the corresponding energy E

$$E \equiv E(N_t) = \frac{\hbar}{|N_t - \frac{1}{4N_t}|t_{min}}. \tag{58}$$

Note that at low energies $E \ll E_P$, that is for $|N_t| \gg 1$, the formula (58) naturally takes the following form:

$$E \equiv E(N_t) = \frac{\hbar}{|N_t|t_{min}} = \frac{\hbar}{|t(N_t)|}. \tag{59}$$

Definition 2.
Let us define the quantity having the dimensions of length L or time t **measurable**, *when it satisfies the relation (36) (and respectively (57)).*

Thus, **measurable infinitesimal changes** in length (and hence in time) are **impossible** and any such changes are dependent on the existing energies.
In particular, a minimal possible **measurable** change of length is l_{min}. It corresponds to some maximal value of the energy E_{max} or momentum P_{max}, If $l_{min} \propto l_P$, then $E_{max} \propto E_P$,$P_{max} \propto P_{Pl}$, where $P_{max} \propto P_{Pl}$, where P_{Pl} is where the Planck momentum. Then denoting in **nonrelativistic** case with $\triangle_p(w)$ a **minimal measurable** change every spatial coordinate w corresponding to the energy E we obtain

$$\triangle_{P_{max}}(w) = \triangle_{E_{max}}(w) = l_{min}. \tag{60}$$

Evidently, for lower energies (momentums) the corresponding values of $\triangle_p(w)$ are higher and, as the quantities having the dimensions of length are quantized (36), for $p \equiv p(N_p) < p_{max}$, $\triangle_p(w)$ is transformed to

$$|\triangle_{p(N_p)}(w)| = |N_p| l_{min}. \tag{61}$$

where $|N_p| > 1$-integer so that we have

$$|N_p - \frac{1}{4N_p}| l_{min} = \frac{\hbar}{|p(N_p)|}. \tag{62}$$

In the relativistic case the formula (60) holds, whereas (61) and (62) for $E \equiv E(N_E) < E_{max}$ are replaced by

$$|\triangle_{E(N_E)}(w)| = |N_E| l_{min}, \tag{63}$$

where $|N_E| > 1$-integer.
Next we assume that at high energies $E \propto E_P$ there is a possibility only for the **nonrelativistic** case or **ultrarelativistic** case.

Then for the **ultrarelativistic** case, with regard to (48)–(55), formula (62) takes the form

$$|N_E - \frac{1}{4N_E}|l_{min} = \frac{\hbar c}{E(N_E)} = \frac{\hbar}{|p(N_p)|}, \tag{64}$$

where $N_E = N_p$.
In the relativistic case at low energies we have

$$E \ll E_{max} \propto E_P. \tag{65}$$

In accordance with (46),(47) formula (61) is of the form

$$|\triangle_{E(N_E)}(w)| = |N_E|l_{min} = \frac{\hbar c}{E(N_E)}, |N_E| \gg 1 - integer. \tag{66}$$

In the nonrelativistic case at low energies (65) due to (62) we get

$$|\triangle_{p(N_p)}(w)| = |N_p|l_{min} = \frac{\hbar}{|p(N_p)|}, |N_p| \gg 1 - integer. \tag{67}$$

In a similar way for the time coordinate t, by virtue of formulas (57)–(59), at the same conditions we have similar formulas (60),(61),(62)

$$\triangle_{E_{max}}(t) = t_{min}. \tag{68}$$

For $E \equiv E(N_t) < E_{max}$

$$|\triangle_{E(N_t)}(t)| = |N_t|t_{min}, \tag{69}$$

where $|N_E| >$ 1-integer, so that we obtain

$$|N_t - \frac{1}{4N_t}|t_{min} = \frac{\hbar c}{E(N_t)}. \tag{70}$$

In the relativistic case at low energies

$$E \ll E_{max} \propto E_P, \tag{71}$$

in accordance with (46),(47), formula (61) takes the form

$$|\triangle_{E(N_t)}(w)| = |N_t| l_{min} = \frac{\hbar c}{E(N_t)}, |N_t| \gg 1 - integer. \tag{72}$$

Comment 1
Obviously, when l_{min} is involved, the foregoing formulas for the momentums $p(N_p)$ and for the energies $E(N_E), E(N_t)$ may **certainly** give the highly accurate result that is close to the experimental one only at the verified low energies: $|N_p| \gg 1, |N_E| \gg 1, |N_t| \gg 1$.
In the case of high energies $E \propto E_{max} \propto E_P$ or, what is the same $|N_p| \to 1, |N_E| \to 1, |N_t| \to 1$, we have a certain, experimentally unverified, model with a correct low-energy limit

In what follows, within the scope of the above definitions, we consider, unless stated otherwise, **only measurable** increments (variations) of the space-time quantities and the corresponding momentums and energies.
Proceeding from all the above, this simply means that all minimal increments (variations) of the space-time quantities are dependent on the present energies and coincident with the corresponding **minimal uncertainties** from the **Uncertainty Principle at the All Scales Energies.**

3.2 Gravity for the Static Spherically-Symmetric Space With Horizon in Terms of Measurable Quantities

Now let us return to the example of a horizon space given in Subsection 2.2.2 (formula (22)), considering it in greater detail and in view of **Definition 2**. Gravity and thermodynamics of horizon spaces and their interrelations are currently most actively studied [46], [56]–[67]. Let us consider a relatively simple illustration – the case of a static spherically-symmetric horizon in space-time, the horizon being described by the metric

$$ds^2 = -f(r)c^2dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2. \tag{73}$$

The horizon location will be given by a simple zero of the function $f(r)$, at the radius $r = a$.

This case is studied in detail by T.Padmanabhan in his works [56, 46]. We use the notation system of [46]. Let, for simplicity, the space be denoted as $\mathcal{H}$.
It is known that for horizon spaces one can introduce the temperature that can be identified with an analytic continuation to imaginary time. In the case under consideration ([46], eq.(116))

$$k_B T = \frac{\hbar c f'(a)}{4\pi}. \tag{74}$$

Therewith, the condition $f(a) = 0$ and $f'(a) \neq 0$ must be fulfilled.
Then at the horizon $r = a$ Einstein's field equations

$$\frac{c^4}{G}\left[\frac{1}{2}f'(a)a - \frac{1}{2}\right] = 4\pi P a^2 \tag{75}$$

where $P = T^r_r$ is the trace of the momentum-energy tensor and radial pressure.
Now we proceed to the variables «α» from the formula (13) to consider (75) in a new notation, expressing a in terms of the corresponding deformation parameter α. In what follows we omit the subscript in formula (13) of α_x, where the context implies which index is the case. In particular, here we use α instead of α_a. Then we have [29]

$$a = l_{min}\alpha^{-1/2}. \tag{76}$$

Therefore,

$$f'(a) = -2l_{min}^{-1}\alpha^{3/2}f'(\alpha). \tag{77}$$

Substituting this into (75) we obtain in the considered case of Einstein's equations in the «α–representation» the following [29]:

$$\frac{c^4}{G}(-\alpha f'(\alpha) - \frac{1}{2}) = 4\pi P\alpha^{-1}l_{min}^2. \tag{78}$$

Multiplying the left- and right-hand sides of the last equation by α, we get

$$\frac{c^4}{G}(-f'(\alpha)\alpha^2 - \frac{1}{2}\alpha) = 4\pi P l_{min}^2. \tag{79}$$

L.h.s. of (79) is dependent on α. Because of this, r.h.s. of (79) must be dependent on α as well, i. e. $P = P(\alpha)$, i.e

$$\frac{c^4}{G}(-f'(\alpha)\alpha^2 - \frac{1}{2}\alpha) = 4\pi P(\alpha) l_{min}^2. \tag{80}$$

Note that in this specific case the parameter α within constant factors is coincident with the Gaussian curvature K_a [68] corresponding to a:

$$\frac{l_{min}^2}{a^2} = l_{min}^2 K_a. \tag{81}$$

Substituting r.h.s of (81) into (80), we obtain the Einstein equation on horizon, in this case in terms of the Gaussian curvature

$$\frac{c^4}{G}(-f'(K_a)K_a^2 - \frac{1}{2}K_a) = 4\pi P(K_a). \tag{82}$$

This means that up to the constants

$$-f'(K_a)K_a^2 - \frac{1}{2}K_a = P(K_a), \tag{83}$$

i.e. the Gaussian curvature K_a is a solution of Einstein equations in this case.
Then we examine different cases of the solution (83) within the scope of **Definition 2** – in the case when a is a measurable quantity (36).

3.2.1) First, let us assume that $a \gg l_{min}$. As, according (36), the radius a is quantized, we have $a = N_a l_{min}$ with the natural number $N_a \gg 1$. Then it is clear that the Gaussian curvature $K_a = 1/a^2 \approx 0$ takes a (nonuniform) discrete series of values close to zero, and, within the factor $1/l_{min}^2$, this series represents inverse squares of natural numbers

$$(K_a) = (\frac{1}{N_a^2}, \frac{1}{(N_a \pm 1)^2}, \frac{1}{(N_a \pm 2)^2}, ...). \tag{84}$$

Note that $N_a \gg 1$ is associated with the low-energy case $E \ll E_P$ and hence all variations (increments) of the radius $R = a$ are given by the formulas

(66) or (67).
For definiteness, without the loss in generality, we suggest that the case under consideration (66) is relativistic, and the minimal increment a is given by

$$\begin{aligned} & a \rightarrow a \pm N_E l_{min} \\ & 1 \ll |N_E| \ll N_a. \end{aligned} \tag{85}$$

Then it is clear that

$$a_{\pm \triangle_{E(N_E)}} \equiv a \pm \triangle_{E(N_E)}(w) = N_a(1 \pm \frac{|N_E|}{N_a}) l_{min}. \tag{86}$$

But, as $N_a \gg 1$, for sufficiently large N_a and fixed E, the bracketed expression in r.h.s. (86) is close to 1 under (85):

$$1 \pm \frac{|N_E|}{N_a} \approx 1. \tag{87}$$

Obviously, we get for fixed E

$$\lim_{N_a \to \infty} (1 \pm \frac{|N_E|}{N_a}) \rightarrow 1. \tag{88}$$

As a result, the Gaussian curvature $K_{a_{\pm \triangle_{E(N_E)}}}$ corresponding to $r = a_{\pm \triangle_{E(N_E)}}$

$$K_{a_{a_{\pm \triangle_{E(N_E)}}}} = 1/a^2_{\pm \triangle_{E(N_E)}} \propto \frac{1}{N_a^2 (1 \pm \frac{|N_E|}{N_a})^2} \tag{89}$$

in the case under study is only slightly different from K_a.
Thus, the Gaussian curvature K_a, for fixed E, due to its smallness ($K_a \ll 1$, is practically continuously dependent on the increments $\triangle_{E(N_E)}$ (formulas (86)–(89)). Then in terms of K_a, up to the constant factor, we can obtain the following:

$$d[L(K_a)] = d[P(K_a)], \tag{90}$$

Where have

$$L(K_a) = -f'(K_a) K_a^2 - \frac{1}{2} K_a, \tag{91}$$

i. e. l.h.s of (82) (or (83)).
But in fact, as in this case the energies are low, it is more correct to consider

$$L(K_{a\pm\triangle_{E(N_E)}}) - L(K_a) = P(K_{a\pm\triangle_{E(N_E)}}) - P(K_a) \equiv F_E[P(K_a)], \qquad (92)$$

with fixed E and $\triangle_{E(N_E)}$ from (65),rather than (90).
In view of the foregoing arguments 3.2.1), the difference between (92) and (90) is insignificant and it is perfectly correct to use (90) instead of (92).
3.2.2) Now we consider the opposite case or the transition to the **ultraviolet limit**

$$a = \kappa l_{min}. \qquad (93)$$

Here κ is on the order of 1.
Then it is clear that formulas (87) and (88) are no longer valid as the energies $E \propto E_P$, $N_a = \kappa$ are close to 1 and $|\triangle_{E(N_E)}|/a = |N_E|/N_a \approx 1$.
In this case the Gaussian curvature K_a is not a «small value» continuously dependent on a and takes, according to (89), a discrete series of values $K_a, K_{a\pm\eta l_{min}}, K_{a_{\pm\eta' l_{min}}}, ..,$ where $\eta, \eta^{'}, ...$– integers on the order of 1.
Yet (75), similar to (82) (83), is valid in the semiclassical approximation only, i.e. at **low energies**.
In accordance with the above arguments, the limiting transition to **high energies** (93) gives a discrete chain of equations or a single equation with a discrete set of solutions as follows:

$$-f'(K_a)K_a^2 - \frac{1}{2}K_a = \Theta(K_a);$$

$$-f'(K_{a\pm\eta l_{min}})K^2_{a\pm\eta l_{min}} - \frac{1}{2}K_{a\pm\eta l_{min}} = \Theta(K_{a\pm\eta l_{min}});$$

and so on. Here $\Theta(K_a)$ – some function that in the limiting transition to low energies must reproduce the low-energy result to a high degree of accuracy, i.e. $P(K_a)$ appears for $a \gg l_{min}$ from formula (83)

$$\lim_{K_a\rightarrow 0} \Theta(K_a)) = P(K_a). \qquad (94)$$

In general, $\Theta(K_a)$ may lack coincidence with the high-energy limit of the momentum-energy tensor trace (if any):

$$\lim_{a \to l_{min}} P(K_a). \tag{95}$$

At the same time, when we naturally assume that the Static Spherically-Symmetric Horizon Space-Time with the radius of several Planck's units (93) is nothing else but a micro black hole, then the high-energy limit (95) is existing and the replacement of $\Theta(K_a)$ by $P(K_a)$ in r.h.s. of the foregoing equations is possible to give a hypothetical gravitational equation for the event horizon micro black hole.
In all the cases under study, 3.2.1) and 3.2.2), the deformation parameter α_a (13) is, within the constant factor, coincident with the Gaussian curvature K_a that is in essence continuous in the low-energy case and discrete in the high-energy case.
In this way the above-mentioned example shows that, despite the absence of infinitesimal spatial-temporal increments owing to the existence of l_{min} and the essential ≪discreteness≫ of a theory, this discreteness at low energies is not ≪felt≫, the theory being actually continuous. The indicated discreteness is significant only in the case of high (Planck's) energies.

It is seen that **the infinitesimal increment of entropy** dS of the spherically symmetric holographic screen $\mathcal{S}$ with the radius R and with the surface area A is a **nonmeasurable quantity**.
Really, it is obvious that infinitesimal variations of the screen surface area dA are possible only in a continuous theory involving no l_{min}.
When $l_{min} \propto l_P$ is involved, the minimal variation $\triangle A$ is evidently associated with a minimal variation in the radius R

$$R \to R \pm l_{min} = R \pm \triangle_{E_{max}}(R) \tag{96}$$

it is dependent on R and growing with $\sim R$ for $R \gg l_{min}$ (possible only at the maximum energy $E_{max} \propto E_P$)

$$\triangle_{\pm} A(R) = (A(R \pm l_{min}) - A(R)) \propto (\pm 2Rl_{min} + l_{min}^2) \propto (\pm 2N_R + 1), \tag{97}$$

where $N_R = R/l_{min}$, as indicated above in (36).
But if $E \ll E_{max} \propto E_P$, then a minimal variation in the radius R is obviously greater than l_{min}

$$labelbitdensity1.1 R \rightarrow R \pm \triangle_{E(N_E)}(R) = R \pm |N_E| l_{min}, \qquad (98)$$

and in this case in the right-hand side of (97), within the constant l^2_{min}, we have the number quickly growing at low energies as well:

$$\begin{aligned} \triangle_{\pm} A(R) = (A(R \pm l_{min}) - A(R)) \propto (\pm 2 R N_E l_{min} + N_E^2 l_{min}^2) \\ \propto N_E(\pm 2 N_R + N_E). \end{aligned} \qquad (99)$$

In any case from this it follows that dA has no chance to be a **measurable quantity**, as its measurability suggests measurability of the quantity dR, and this is impossible.
Since dS, within a multiplicative constant, equals dA [19],[20]: $dS \propto dA/4$, dS is also a **nonmeasurable quantity**.
Because of this, the «main instrument» in the well-known paper [1] that is the infinitesimal variation dN in the bit numbers N on the holographic screen $\mathcal{S}$ is also a **nonmeasurable quantity** [3] as $dN \propto dS$ to within an integer factor.

Let us return to equation (22) of Section 2, but now in the new context.
In [46] it is more generally shown that the Einstein Equation for horizon spaces in the differential form may be written as a thermodynamic identity (the first principle of thermodynamics) ([46], formula (119)):

$$\underbrace{\frac{\hbar c f'(a)}{4\pi}}_{k_B T} \underbrace{\frac{c^3}{G\hbar} d\left(\frac{1}{4} 4\pi a^2\right)}_{dS} \underbrace{- \frac{1}{2}\frac{c^4 da}{G}}_{-dE} = \underbrace{P d\left(\frac{4\pi}{3} a^3\right)}_{P\,dV}. \qquad (100)$$

where, as noted above, T – temperature of the horizon surface, S –corresponding entropy, E– internal energy, V – space volume.
Similar to the previous example from [1], with the use of formulas (96)–(99) we can demonstrate that, in terms of the **measurable quantities** of **Definition 2**, there are no infinitesimal increments da, dV, dS.

In fact, we have **no-go theorems**.
It should be noted that the example given in this Subsection has been studied in [3], with a single but significant difference – instead of the energy dependent increments of the radius $R = a$, in [3] the space fluctuations R [69]–[92] have been considered.

4 Conclusion

The last statements concerning dS, dN may be explicitly interpreted using the language of a quantum information theory as follows:
due to the existence of the minimal length l_{min}, the minimal area l^2_{min} and volume l^3_{min} are also involved, and that means ≪quantization≫ of the areas and volumes. As, up to the known constants, the ≪bit number≫ N from (2) and the entropy S from (4) are nothing else but

$$S = \frac{A}{4l^2_{min}}, N = \frac{A}{l^2_{min}}, \tag{101}$$

it is obvious that there is a ≪minimal measure≫ for the ≪amount of data≫ that may be referred to as ≪one bit≫ (or ≪one qubit≫).
The statement that there is no such quantity as dN (and respectively dS) is equivalent to claiming the absence of $0,25$ bit, $0,001$ bit, and so on.
This inference completely conforms to the Hooft-Susskind Holographic Principle (HP) [5]–[8] that includes two main statements:

(a)All information contained in a particular spatial domain is concentrated at the boundary of this domain.
(b)A theory for the boundary of the spatial domain under study should contain maximally one degree of freedom per Planck's area l^2_P.

In fact (but not explicitly) HP implicates the existence of $l_{min} = l_P$. The existence of $l_{min} \propto l_P$ totally conforms to HP, providing its generalization. Specifically, without the loss of generality, l^2_P in point (b) may be replaced by l^2_{min}.

So, as demonstrated in the previous Section for the particular cases, provided a theory involves the minimal length $l_{min} \propto l_P$, gravity is almost independent of the parameters associated with this length, specifically α_l i.e. the dependence is weak, and so the theory is practically continuous. This stems from the fact that these parameters are very small due to remoteness of the energies characterizing them from the Planck energies and almost **insensitive** to the corresponding change in measuring scales.

Despite a **discrete** nature of the theory owing to the existence of l_{min}, to a high degree of accuracy we can use infinitesimal variations of dx_μ, coincident in the case under study with l_{min} and t_{min}. In this way in the case considered in Section 3.2 the **Conformity Principle** stating that (*on going to low energies the known theory (in particular GR) must be reproduced to a high degree of accuracy, at least its experimentally verified part*) holds **to a high accuracy**.

Still it is clear that, as formally GR has no additional parameters associated with l_{min} and the low-energy for now hypothetical variant of the minimal length theory denoted as $Grav^{l_{min}}$ has such parameters, there is also the **high accuracy** limit indicated above. This limit in every case determines the «gap» between GR and $Grav^{l_{min}}$. Evaluation of this gap is a real challenge for those trying to construct a unified theory at all energy levels.

As noted in 3.2, for high energies, i.e. for $l \rightarrow l_{min}$, a discrete chain of equations (or a single equation with a discrete set of solutions) is derived that is numbered by inverse squares of the integers $1; 1/4; 1/9; .....$

We have used GR to demonstrate that the above models in 3.2 at low energies are actually insensitive to variations of the discrete parameters α_l associated with the minimal length. Of course, it is more correct to use $Grav^{l_{min}}$ and to compare the obtained results with GR. But, as yet there is no $Grav^{l_{min}}$, it is connived that at low energies GR and $Grav^{l_{min}}$ differ insignificantly and the indicated parameters, provided l_{min} is involved, are introduced into GR similarly to $Grav^{l_{min}}$.

As the Planck length $l_P = (\hbar G/c^3)^{1/2}$ is expressed in terms of the fundamental constants, the proportionality coefficient ξ from formula (3), relating l_{min} and l_P, in a minimal-length theory l_{min} should also be a fundamental constant because it (along with G, $\hbar$, and c) must be involved in all the basic formulae of this theory. Then the question arises: what is its value? In [13] for the coefficient α' in GUP (3) the substantiated value was equal to 1. Provided this is true, $\xi = 2$ and hence the Bekenstein-Hawking formula for the black hole entropy S^{BH} may be written most naturally and elegantly as follows:

$$S^{BH} = \frac{A}{l_{min}^2}. \qquad (102)$$

So, the principal inference of Section 3 this work is as follows:
provided the minimal length l_{min} is involved, its existence must be taken into consideration not only at high but also at low energies, both in a quantum theory and in gravity. This becomes apparent by rejection of the infinitesimal quantities associated with the spatial-temporal variations $dx_\mu, \ldots$. In other words, with the involvement of l_{min}, the General Relativity (GR) must be replaced by a (still unframed) minimal-length gravitation theory that may be denoted as $Grav^{l_{min}}$. In their results GR and $Grav^{l_{min}}$ should be very close but, as regards their mathematical apparatus (instruments), these theories are absolutely different.
Besides, $Grav^{l_{min}}$ should offer a rather natural transition from high to low energies

$$[N_L \approx 1] \rightarrow [N_L \gg 1] \qquad (103)$$

and vice versa

$$[N_L \gg 1] \rightarrow [N_L \approx 1], \qquad (104)$$

where N_L – integer from formula (36) determining the characteristics scale of the lengths L (energies $E \sim 1/L \propto 1/N_L$).

References

[1] Verlinde, E. On the Origin of Gravity and the Laws of Newton, *JHEP* **2011**, *1104*, 029, [ArXiv:1001.0785]

[2] Shalyt-Margolin,A. E. Probable Entropic Nature of Gravity in Ultraviolet and Infrared Limits—Part I: An Ultraviolet Case. *Advances in High Energy Physics.* **2013**, *2013*, Article ID 384084, 7 pages

[3] Shalyt-Margolin, A. E. Minimal Length and the Existence of Some Infinitesimal Quantities in Quantum Theory and Gravity. *Advances in High Energy Physics.* **2014**, *2014*, Article ID 195157, 8 pages

[4] Shalyt-Margolin, A. E. Space-Time Fluctuations, Quantum Field Theory with UV-cutoff and Einstein Equations.*Nonlinear Phenomena in Complex Systems.* **2014**, *17*,138–146.

[5] 't. Hooft,G. Dimensional reduction in quantum gravity.Essay dedicated to Abdus Salam. **1993**,arXiv: gr-qc/9310026.

[6] 't. Hooft,G. The Holographic Principle.**2000**,arXiv: hep-th/0003004.

[7] Susskind,L. The World as a hologram. *J. Math. Phys.***1995**,*36*,6377–6396 .

[8] Bousso,R. The Holographic principle. *Rev. Mod. Phys.***2002**, *74*,825–874 .

[9] Bousso,R. Covariant entropy conjecture. *JHEP.***1999**, *07*, 004.

[10] Veneziano, G. A. Stringy Nature Needs Just Two Constants. *Europhys.Lett.***1986**, *2*, 199–211 .

[11] Amati, D., Ciafaloni, M., Veneziano,G. Can Space-Time Be Probed Below the String Size? *Phys.Lett.B.***1989**, *216*, 41–47.

[12] Witten, E. Reflections on the Fate of Spacetime. *Phys. Today* **1996**, *49*, 24–28.

[13] Adler,R. J.; Santiago,D. I. On gravity and the uncertainty principle. *Mod. Phys. Lett. A* **1999**, *14*, 1371–1378.

[14] Ahluwalia,D.V. Wave particle duality at the Planck scale: Freezing of neutrino oscillations. *Phys.Lett* **2000**, *A275*, 31–35.

[15] Ahluwalia,D.V. *Mod.Phys.Lett* **2002**, Interface of gravitational and quantum realms. *A17*, 1135–1146.

[16] Maggiore,M. The Algebraic structure of the generalized uncertainty principle. *Phys.Lett.B* **1993**, *319*, 83–86.

[17] Kempf,A.; Mangano,G.; Mann,R.B. Hilbert space representation of the minimal length uncertainty relation. *Phys.Rev.D* **1995**, *52*, 1108–1118.

[18] Nozari, Kourosh.; Etemadi, Amir. Minimal length, maximal momentum and Hilbert space representation of quantum mechanics, *Physical Review D.* **2012**, *85*, 104029.

[19] Bekenstein,J.D. Black Holes and Entropy. *Phys.Rev.D* **1973**, *7*, 2333–2345.

[20] Hawking,S. Black Holes and Thermodynamics. *Phys.Rev.D* **1976**, *13*, 191–204.

[21] Park, M.-I. The Generalized Uncertainty Principle in (A)dS Space and the Modification of Hawking Temperature from the Minimal Length. *Phys.Lett.B.* **2008**, *659*, 698–702.

[22] Medved, A.J.M.; Vagenas, E.C. When conceptual worlds collide: The GUP and the BH entropy. *Phys.Rev.D* **2004**, *70*, 124021.

[23] Kim,Wontae.; Son,Edwin J.; Yoon, Myungseok. Thermodynamics of a black hole based on a generalized uncertainty principle. *JHEP* **2008**, *08*, 035.

[24] Nouicer,K.: Quantum-corrected black hole thermodynamics to all orders in the Planck length. *Phys.Lett.B.* **2007**, *646*, 63–71.

[25] Setare,M. R., Momeni,D., Myrzakulov, R., Entropic corrections to Newton's law. *Phys. Scr.* **2012**, *85*, 065007.

[26] Komar,A. Covariant Conservation Laws in General Relativity. *Phys. Rev.* **1959**, *113*, 934–936..

[27] Wald, Robert. M. *General Relativity*; The University Chicago Press Chicago and London, 1984.

[28] Shalyt-Margolin,A.E. Entropy In The Present And Early Universe: New Small Parameters And Dark Energy Problem *Entropy* **2010**,*12*, 932-952.

[29] Shalyt-Margolin,A.E. Quantum Theory at Planck Scale, Limiting Values, Deformed Gravity and Dark Energy Problem, *Intern. J. Mod. Phys. D* **2012**,*21*, 1250013.

[30] Shalyt-Margolin, A.E.; Suarez, J.G. Quantum mechanics of the early universe and its limiting transition. *gr-qc/0302119*, 16pp.

[31] Shalyt-Margolin, A.E.; Suarez, J.G. Quantum mechanics at Planck's scale and density matrix. *Intern. Journ. Mod. Phys D* **2003**, *12*, 1265–1278.

[32] Shalyt-Margolin, A.E.; Tregubovich, A.Ya. Deformed density matrix and generalized uncertainty relation in thermodynamics. *Mod. Phys.Lett. A* **2004**, *19*, 71–82.

[33] Shalyt-Margolin, A.E. Nonunitary and unitary transitions in generalized quantum mechanics, new small parameter and information problem solving. *Mod. Phys. Lett. A* **2004**, *19*, 391–404.

[34] Shalyt-Margolin, A.E. Pure states, mixed states and Hawking problem in generalized quantum mechanics. *Mod. Phys. Lett. A* **2004**, *19*, 2037–2045.

[35] Shalyt-Margolin, A.E. The Universe as a nonuniform lattice in finite volume hypercube. I. Fundamental definitions and particular features. *Intern. Journ. Mod.Phys D* **2004**, *13*, 853–864.

[36] Shalyt-Margolin, A.E. The Universe as a nonuniform lattice in the finite-dimensional hypercube. II. Simple cases of symmetry breakdown and restoration. *Intern.Journ.Mod.Phys.A* **2005**, *20*, 4951–4964.

[37] Shalyt-Margolin, A.E.; Strazhev,V.I. The Density Matrix Deformation in Quantum and Statistical Mechanics in Early Universe. In *Proc. Sixth International Symposium "Frontiers of Fundamental and Computational Physics"*,edited by B.G. Sidharth at al. Springer,2006, pp.131–134.

[38] Shalyt-Margolin, A.E. The Density matrix deformation in physics of the early universe and some of its implications. In *Quantum Cosmology Research Trends*,edited by A. Reimer, Horizons in World Physics. **246**, Nova Science Publishers, Inc., Hauppauge, NY,2005, pp. 49–91.

[39] Shalyt-Margolin, A.E. Deformed density matrix and quantum entropy of the black hole. *Entropy* **2006**, *8*, 31–43.

[40] Shalyt-Margolin,A.E. Entropy in the Present and Early Universe. *Symmetry.* **2007**, *18*, 299–320.

[41] Shalyt-Margolin,A.E. Entropy in the Present and Early Universe and Vacuum Energy.*AIP Conference Proceedings.* **2010**, *1205*, 160–167.

[42] Faddeev, L., Mathematical View on Evolution of Physics. *Priroda* **1989**, *5*, 11–18.

[43] Kim,Y. W., Lee,H. W., Myung,Y. S. Entropy bound of local quantum field theory with generalized uncertainty principle. *Phys.Lett.B.* **2009**, *673*, 293–296.

[44] Magnon, A. On Komar Integrals in Asymptotically Anti de Sitter Spacetimes. *J. Math. Phys.* **1985**, *26*, 3112–3117.

[45] Jacobson,T. Thermodynamics of space-time: The Einstein equation of state. *Phys. Rev. Lett* **1995**, *75*, 1260–1263.

[46] Padmanabhan,T. Thermodynamical Aspects of Gravity: New insights. *Rep. Prog. Phys.* **2010**, *74*, 046901. arXiv:0911.5004.

[47] Fischler,W., Susskind, L. Holography and cosmology. *hep-th/9806039*, 6pp.

[48] Balazs,C.; Szapudi,I. Naturalness of the vacuum energy in holographic theories. *hep-th/0603133*, 4pp.

[49] Zel'dovich, Y.B. The Cosmological Constant and the Theory of Elementary Particles. *Sov. Phys. Uspehi.* **1968**, *11*, 381–393.

[50] Weinberg, S. The Cosmological Constant Problem. *Rev. Mod. Phys* **1989**, *61*, 1–23.

[51] Markov,M. A.: Ultimate Matter Density as the Universal Low of Nature.*JETP letters.*, **1982** *36*, 214–216.

[52] Penrose,R. *Quantum Theory and Space-Time*, Fourth Lecture in *Stephen Hawking and Roger Penrose, The Nature of Space and Time*; Prinseton University Press: Prinseton, 1996.

[53] Heisenberg,W. Uber den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Zeitschrift fur Physik.* **1927** *43*, 172–198; English translation: Wheeler, J. A.; Zurek,H. it Quantum Theory and Measurement; Princeton Univ. Press: Princeton, 1983; pp.62–84.

[54] Landau,L.D.; Lifshits, E.M.*Theoretical Physics,vol.2. Field Theory*; Nauka: Moskow,1988.

[55] Berestetskii,V.B.;Lifshits,E.M.; Pitaevskii,L.P. *; Theoretical Physics,vol.4,Relativistic Quantum Theory* Nauka: Moskow, 1971.

[56] Padmanabhan,T. Classical and Quantum Thermodynamics of horizons in spherically symmetric spacetimes *Class.Quant.Grav.* **2002**, *19*, 5387–5408.

[57] Padmanabhan,T. A New perspective on gravity and the dynamics of spacetime. *Int.Jorn.Mod.Phys* **2005**, *D14*, 2263–2270.

[58] Padmanabhan,T. The Holography of gravity encoded in a relation between entropy, horizon area and action for gravity. *Gen.Rel.Grav.* **2002**, *34*, 2029–2035.

[59] Padmanabhan,T. Holographic Gravity and the Surface term in the Einstein-Hilbert Action. *Braz.J.Phys.* **2005**, *35*, 362–372.

[60] Padmanabhan,T. Gravity: A New holographic perspective. *Int.J.Mod.Phys.D* **2006**, *15*, 1659–1676.

[61] Mukhopadhyay,A.; Padmanabhan,T. Holography of gravitational action functionals. *Phys.Rev.D.* **2006**, 124023.

[62] Padmanabhan,T. Dark energy and gravity. *Gen.Rel.Grav.* **2008**,*40*, 529–564.

[63] Padmanabhan,T.; Paranjape Aseem. Entropy of Null Surfaces and Dynamics of Spacetime. *Phys.Rev.D.* **2007**, *75*, 064004.

[64] Padmanabhan,T. Gravity as an emergent phenomenon: A conceptual description. *AIP Conference Proceedings.* **2007**,*939* 114–123.

[65] Padmanabhan,T. Gravity and the thermodynamics of horizons. *Phys.Rept* **2005**, *406*, 49–125.

[66] Paranjape,A.; Sarkar,S.; Padmanabhan,T.: Thermodynamic route to field equations in Lancos-Lovelock gravity. *Phys.Rev.D.* **2006**, *74*, 104015.

[67] Padmanabhan,T. Equipartition of energy in the horizon degrees of freedom and the emergence of gravity. *Mod.Phys.Lett.A.* **2010**, *25*, 1129–1136.

[68] S. Kobayashi, K. Nomozu, *Foundations of Differential Geometry*; V.II, Interscience Publishers: New York-London-Sydney 1969.

[69] Wheeler, J. A. *Geometrodynamics*; Academic Press: New York and London, 1962.

[70] Misner,C. W.; Thorne,K. S.; Wheeler,J. A. *Gravitation*; Freeman: San Francisco,1973.

[71] Garattini,Remo. A Spacetime Foam approach to the cosmological constant and entropy. *Int. J. Mod. Phys. D.,* **2002**, *4*, 635–652.

[72] Garattini,Remo. A Spacetime Foam Approach to the Schwarzschild-de Sitter Entropy. *Entropy.* **2000**, *2*, 26–38.

[73] Garattini,Remo. Entropy and the cosmological constant: a spacetime-foam approach. *Nucl.Phys.Proc.Suppl.* **2000**, *88*, 297–300.

[74] Garattini,Remo. Entropy from the foam. *Phys.Lett.B.* **1999**, *459*, 461–467.

[75] Scardigli,Fabio. Black Hole Entropy: a spacetime foam approach. *Class.Quant.Grav.* **1997**, *14*, 1781–1793.

[76] Scardigli,Fabio. Generalized Uncertainty Principle in Quantum Gravity from Micro-Black Hole Gedanken Experiment. *Phys.Lett.B.* **1999**, *452*, 39–44.

[77] Scardigli,Fabio. Gravity coupling from micro-black holes. *Nucl.Phys.Proc.Suppl.* **2000**, *88*, 291–294.

[78] Garay,Luis J. Thermal properties of spacetime foam. *Phys.Rev. D.* **1998**, *58*, 124015.

[79] Garay,Luis J. Spacetime foam as a quantum thermal bath. *Phys.Rev.Lett.* **1998**, *80*, 2508–2511.

[80] Ng,Y. J.; van Dam,H. Measuring the foaminess of space-time with gravity-wave interferometers. *Found. Phys.* **2000**, *30*, 795–805.

[81] Ng,Y. J. Spacetime foam, *Int. J. Mod. Phys. D.* **2002**, *11*, 1585–1590.

[82] Ng,Y. J. Selected topics in Planck-scale physics *Mod.Phys.Lett.A.* **2003**, *18*, 1073–1098.

[83] Ng,Y. J. Quantum Foam. **2004**, arXiv:gr-qc/0401015.

[84] Ng,Y. J.; van Dam,H. Spacetime Foam, Holographic Principle, and Black Hole Quantum Computers. *Int.J.Mod.Phys.A.* **2005**, *20*, 1328–1335.

[85] W.A. Christiansen, Ng,Y. J.; van Dam,H. Probing spacetime foam with extragalactic sources. *Phys.Rev.Lett.* **2006**, *96*, 051301.

[86] Ng, Y. Jack. Holographic Foam, Dark Energy and Infinite Statistics. *Phys.Lett.B.* **2007**, *657*, 10–14.

[87] Ng, Y. Jack. Spacetime Foam and Dark Energy. *AIP Conf.Proc.* **2009**, *1115*, 74–79.

[88] Christiansen,A. Wayne; Floyd, Y.David J. E.;Ng, Y. Jack; Perlman, Eric S. Limits on Spacetime Foam. *Phys.Rev.D.* **2011**, *83*, 084003.

[89] Amelino-Camelia,G. An interferometric gravitational wave detector as a quantum-gravity apparatus. *Nature* **1999**, *398*, 216–218.

[90] Diosi,L. and Lukacs,B. On the minimum uncertainties of space-time geodesics. *Phys. Lett. A.* **1989**, *142*, 331–334.

[91] Wigner,E.P. Relativistic Invariance and Quantum Phenomena. *Rev. Mod. Phys.* **1987**, *29*, vol.29, 255–281.

[92] Salecker,H. and Wigner,E.P. Quantum Limitations of the Measurement of Space-Time Distances. *Phys.Rev.* **1958**, *109*, 571–584.