Fusion procedure for Coxeter groups of type B and complex reflection groups G(m,1,n)

O. V. Ogievetsky[◦⋄1] **and L. Poulain d'Andecy**[◦]

◦ Center of Theoretical Physics[2], Luminy
13288 Marseille, France

⋄ J.-V. Poncelet French-Russian Laboratory, UMI 2615 du CNRS, Independent University of Moscow, 11 B. Vlasievski per., 119002 Moscow, Russia

Abstract

A complete system of primitive pairwise orthogonal idempotents for the Coxeter groups of type B and, more generally, for the complex reflection groups $G(m,1,n)$ is constructed by a sequence of evaluations of a rational function in several variables with values in the group ring. The evaluations correspond to the eigenvalues of the two arrays of Jucys–Murphy elements.

1. Introduction

A. Jucys [12] gave a construction of a complete system of pairwise orthogonal primitive idempotents of the group ring of the symmetric group; the construction, called now *fusion procedure*, involves a rational function in several variables and the idempotents are obtained by taking certain limiting values of this function. We refer to [1, 2, 4, 9, 15, 16, 18] for different aspects and applications of the fusion procedure for the symmetric groups. There are analogues of the fusion procedure for the Hecke algebra of type A [3, 17] and the spinor extension of the symmetric group [14, 10] (see [11] for its q-analogue).

A version of the fusion procedure for the symmetric group was given by A. Molev in [13]. Here the idempotents are obtained by consecutive evaluations of the rational function. An analogue of this fusion procedure was developed for the Hecke algebra [6], the Brauer algebra [5, 7] and the Birman–Murakami–Wenzl algebra [8].

The aim of this paper is to give a fusion procedure, in the spirit of [13], for the complex reflection groups $G(m,1,n)$. As in [13], and later [5, 6, 7, 8], we use the Jucys–Murphy elements. They were introduced for $G(m,1,n)$ independently in [21] and [23]. The Jucys–Murphy elements of $G(m,1,n)$ form two arrays, j_i and $\tilde{j}_i$, $i=1,\dots,n$, and their union is the maximal commutative set in $\mathbb{C}G(m,1,n)$,

[1]On leave of absence from P. N. Lebedev Physical Institute, Leninsky Pr. 53, 117924 Moscow, Russia

[2]Unité Mixte de Recherche (UMR 6207) du CNRS et des Universités Aix–Marseille I, Aix–Marseille II et du Sud Toulon – Var; laboratoire affilié à la FRUMAM (FR 2291)

see [20, 21]. An irreducible representation of the group $G(m,1,n)$ is coded by an m-tuple of partitions and the elements of the semi-normal basis correspond to standard m-tuples of tableaux; the eigenvalues of j_i carry information about the "position" - the place of a tableau in the m-tuple - while the eigenvalues of $\tilde{j}_i$ are related to the classical contents of nodes. In the work [20] both sets appeared as classical limits of simple expressions involving the single set of the Jucys–Murphy elements of the cyclotomic Hecke algebra, the flat deformation of $\mathbb{C}G(m,1,n)$. By the maximality, all diagonal matrix units of $\mathbb{C}G(m,1,n)$ can be expressed in terms of the Jucys–Murphy elements j_i and $\tilde{j}_i$, $i=1,\dots,n$. Then we translate this expression as a fusion procedure: any diagonal matrix unit can be obtained by a sequence of evaluations of a certain rational function with values in $\mathbb{C}G(m,1,n)$. The arrays j_i and $\tilde{j}_i$ play different roles: the positions can be evaluated simultaneously while the contents should then be evaluated consequently from 1 to n.

The group $G(1,1,n)$ is isomorphic to the symmetric group S_n and our fusion procedure for $m=1$ reproduces the fusion procedure of [13].

The group $G(2,1,n)$ is isomorphic to the hyperoctahedral group B_n, the Coxeter group of type B. Thus, in particular, we obtain a fusion procedure for the Coxeter group of type B and a description of a complete set of pairwise orthogonal primitive idempotents of B_n in terms of a single rational function with values in the group algebra $\mathbb{C}\mathsf{B}_n$.

For the clarity of the exposition, we first describe the fusion procedure for the Coxeter group B_n. This is done not only for aesthetic reasons: the rational function with values in $\mathbb{C}\mathsf{B}_n$ leading to the complete set of idempotents can be viewed as a word for the longest element of B_n in which certain entries are "Baxterized", similarly to the rational function for the type A. Although $G(m,1,n)$ is not a Coxeter group for $m>2$ and the notion of length of an element is not defined, there is an analogue of the longest element: it is longest with respect to the normal form of [20] (the classical limit of the normal form for the cyclotomic Hecke algebra $H(m,1,n)$ [19]); the rational function with values in $\mathbb{C}G(m,1,n)$ leading to the complete set of idempotents can again be viewed as a word for the longest element with certain entries Baxterized. We stress that the groups $G(m,1,n)$ admit a fusion procedure for any positive integer m, and our construction is uniform for all m.

The paper is organized as follows. Section 2 contains necessary definitions and notations about the groups B_n and their representations. The Jucys–Murphy elements for B_n were defined in [21, 22, 23]; we describe the diagonal matrix units in terms of them. In Section 3 we prove the Theorem 2 which gives the fusion procedure for the groups B_n. In Section 4, in the Theorem 6, we extend the results to the general case of the groups $G(m,1,n)$. As the proofs work mainly along the same lines as for B_n, we only indicate the necessary modifications.

2. Idempotents and Jucys–Murphy elements of the group B_n

2.1 Definitions

The Coxeter group A_n of type A (the symmetric group on $n+1$ letters) is generated by the elements $s_1,\dots,s_n$ with the defining relations:

$$\begin{array}{ll} s_i^2 = 1 & \text{for } i=1,\dots,n, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } i=1,\dots,n-1, \\ s_i s_j = s_j s_i & \text{for } i,j=1,\dots,n \text{ such that } |i-j|>1. \end{array} \qquad (1)$$

The Coxeter group B_{n+1} of type B (also called the hyperoctahedral group) is generated by the elements $s_1, \dots, s_n$ and t with the defining relations (1),

$$\begin{aligned} &ts_1ts_1 = s_1ts_1t, \\ &s_it = ts_i \qquad\qquad \text{for } i = 2, \dots, n \end{aligned} \tag{2}$$

and

$$t^2 = 1. \tag{3}$$

For any $i = 1, \dots, n$, set

$$\mathbf{s}_i(p, p', a, a') := s_i + \frac{\delta_{p,p'}}{a - a'}, \tag{4}$$

where $\delta_{p,p'}$ is the Kronecker delta. For $p = p'$ the elements (4) are called Baxterized elements; the parameters a and a' are referred to as spectral parameters. We also define

$$\mathbf{t}(p) := \frac{1}{2}(1 + pt). \tag{5}$$

The following relation is satisfied and will be used later:

$$\mathbf{s}_i(p, p', a, a')\mathbf{s}_i(p', p, a', a) = \frac{(a - a')^2 - \delta_{p,p'}}{(a - a')^2} \quad \text{for } i = 1, \dots, n. \tag{6}$$

Define the elements j_i, $i = 1, \dots, n+1$, and $\tilde{j}_i$, $i = 1, \dots, n+1$, of the group algebra $\mathbb{C}\mathsf{B}_{n+1}$ by the following initial conditions and recursions:

$$j_1 = t \ , \quad j_{i+1} = s_i j_i s_i \quad \text{and} \quad \tilde{j}_1 = 0 \ , \quad \tilde{j}_{i+1} = s_i \tilde{j}_i s_i + \frac{1}{2}(s_i + j_i s_i j_i). \tag{7}$$

The elements j_i and $\tilde{j}_i$ are analogues, for the group B_{n+1}, of the Jucys–Murphy elements. The elements j_i, $i = 1, \dots, n+1$, and $\tilde{j}_i$, $i = 1, \dots, n+1$, form a maximal commutative set in $\mathbb{C}\mathsf{B}_{n+1}$, see [20, 21, 22]; in addition, j_i and $\tilde{j}_i$ commute with all generators s_k, except s_i and s_{i-1}:

$$j_i s_k = s_k j_i \text{ and } \tilde{j}_i s_k = s_k \tilde{j}_i \quad \text{if } k \neq i-1, i. \tag{8}$$

Let $\lambda \vdash n+1$ be a partition of length $n+1$, that is, $\lambda = (\lambda_1, \dots, \lambda_k)$, where λ_j, $j = 1, \dots, k$, are positive integers, $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_k$ and $n+1 = \lambda_1 + \dots + \lambda_k$. We identify partitions with their Young diagrams: the Young diagram of λ is a left-justified array of rows of nodes containing λ_j nodes in the j-th row, $j = 1, \dots, k$; the rows are numbered from top to bottom.

A 2-partition, or a Young 2-diagram, of length $n+1$ is a pair of partitions such that the sum of their lengths equals $n+1$. A 2-node $\alpha^{(2)}$ is a pair (α, k) consisting of a usual node α and an integer $k = 1, 2$. The integer k will be called *position* of the 2-node. A set of 2-nodes can be equivalently described by an ordered pair of sets of nodes (the integer k of a 2-node (α, k) indicates to which set the node α belongs). A 2-partition $\lambda^{(2)}$ is a set of 2-nodes such that the subset consisting of the 2-nodes having position p is a usual partition, $p = 1, 2$.

For a 2-node $\alpha^{(2)} = (\alpha, k)$ lying in the line x and the column y of the k-th diagram, we denote by $c(\alpha^{(2)})$ the classical content of the node α, $c(\alpha^{(2)}) := c(\alpha) = y - x$. Let $\{\xi_1, \xi_2\}$ be the set of distinct square roots of unity, ordered arbitrarily; we define also $p(\alpha^{(2)}) := \xi_k$.

For a 2-partition $\lambda^{(2)}$, a 2-node $\alpha^{(2)}$ of $\lambda^{(2)}$ is called *removable* if the set of 2-nodes obtained from $\lambda^{(2)}$ by removing $\alpha^{(2)}$ is still a 2-partition. A 2-node $\beta^{(2)}$ not in $\lambda^{(2)}$ is called *addable* if the set of 2-nodes obtained from $\lambda^{(2)}$ by adding $\beta^{(2)}$ is still a 2-partition. For a 2-partition $\lambda^{(2)}$, we denote by $\mathcal{E}_-(\lambda^{(2)})$ the set of removable 2-nodes of $\lambda^{(2)}$ and by $\mathcal{E}_+(\lambda^{(2)})$ the set of addable 2-nodes of $\lambda^{(2)}$.

Let $\lambda^{(2)}$ be a 2-partition of length $n+1$. A standard 2-tableau of shape $\lambda^{(2)}$ is obtained by placing the numbers $1, \dots, n+1$ in the 2-nodes of the diagrams of $\lambda^{(2)}$ in such a way that the numbers in the nodes ascend along rows and columns in every diagram. For a standard 2-tableau $\mathcal{T}$ of shape $\lambda^{(2)}$ let $\alpha_i^{(2)}$ be the 2-node of $\mathcal{T}$ with number i, $i = 1, \dots, n+1$; we set $c(\mathcal{T}|i) := c(\alpha_i^{(2)})$ and $p(\mathcal{T}|i) := p(\alpha_i^{(2)})$.

The hook of a node α of a partition ν is the set of nodes of ν consisting of the node α and the nodes which lie either under α in the same column or to the right of α in the same row; the hook length $h_\nu(\alpha)$ of α is the cardinality of the hook of α. We extend this definition to 2-nodes. For a 2-node $\alpha^{(2)} = (\alpha, k)$ of a 2-partition $\nu^{(2)}$, the hook length of $\alpha^{(2)}$ in $\nu^{(2)}$, which we denote by $h_{\nu^{(2)}}(\alpha^{(2)})$, is the hook length of the node α in the k-th partition of $\nu^{(2)}$. Let

$$f_{\nu^{(2)}} := \Big(\prod_{\alpha^{(2)} \in \nu^{(2)}} h_{\nu^{(2)}}(\alpha^{(2)}) \Big)^{-1}. \tag{9}$$

2.2 Idempotents of the group B_{n+1}

The representation theory of the Coxeter groups of type B was developed by A. Young [24], see also [20, 21, 22]. The irreducible representations of B_{n+1} are in bijection with the 2-partitions of length $n+1$. The elements of the semi-normal basis of the irreducible representation of B_{n+1} corresponding to the 2-partition $\lambda^{(2)}$ are parameterized by the standard 2-tableaux of shape $\lambda^{(2)}$. For a standard 2-tableau $\mathcal{T}$, we denote by $E_\mathcal{T}$ the primitive idempotent of B_{n+1} corresponding to $\mathcal{T}$. The Jucys–Murphy elements are diagonal in the semi-normal basis; moreover, we have, for any $i = 1, \dots, n+1$,

$$j_i E_\mathcal{T} = E_\mathcal{T} j_i = p_i E_\mathcal{T} \quad \text{and} \quad \tilde{j}_i E_\mathcal{T} = E_\mathcal{T} \tilde{j}_i = c_i E_\mathcal{T}. \tag{10}$$

Here we set $p_i := p(\mathcal{T}|i)$ and $c_i := c(\mathcal{T}|i)$ for all $i = 1, \dots, n+1$ for brevity. Due to the maximality of the commutative set $\{j_i, \tilde{j}_i\}_{i=1,\dots,n+1}$ of Jucys–Murphy elements, the idempotent $E_\mathcal{T}$ can be expressed in terms of $j_i, \tilde{j}_i$, $i = 1, \dots, n+1$. Let $\alpha^{(2)}$ be the 2-node of $\mathcal{T}$ with the number $n+1$. As the tableau $\mathcal{T}$ is standard, the 2-node $\alpha^{(2)}$ of $\lambda^{(2)}$ is removable. Let $\mathcal{U}$ be the standard 2-tableau obtained from $\mathcal{T}$ by removing the 2-node $\alpha^{(2)}$ and let $\mu^{(2)}$ be the shape of $\mathcal{U}$. The inductive formula for $E_\mathcal{T}$ in terms of the Jucys–Murphy elements reads:

$$E_\mathcal{T} = E_\mathcal{U} \prod_{\substack{\beta^{(2)}:\ \beta^{(2)} \in \mathcal{E}_+(\mu^{(2)}) \\ c(\beta^{(2)}) \neq c(\alpha^{(2)})}} \frac{\tilde{j}_{n+1} - c(\beta^{(2)})}{c(\alpha^{(2)}) - c(\beta^{(2)})} \prod_{\substack{\beta^{(2)}:\ \beta^{(2)} \in \mathcal{E}_+(\mu^{(2)}) \\ p(\beta^{(2)}) \neq p(\alpha^{(2)})}} \frac{j_{n+1} - p(\beta^{(2)})}{p(\alpha^{(2)}) - p(\beta^{(2)})}\ . \tag{11}$$

Note that the second product in the right hand side of (11) contains only one term (it will not be so for the cyclotomic groups $G(m, 1, n)$ with $m > 2$). We have $\mathsf{B}_0 \cong \mathbb{C}$ and $E_{\mathcal{U}_0} = 1$ for the (unique) 2-tableau $\mathcal{U}_0$ of length 0.

Let $\{\mathcal{T}_1, \dots, \mathcal{T}_k\}$ be the set of pairwise different standard 2-tableaux that can be obtained from $\mathcal{U}$ by adding a 2-node with the number $n+1$. The following formula:

$$E_{\mathcal{U}} = \sum_{i=1}^{k} E_{\mathcal{T}_i},$$

together with (10) implies that the rational function

$$E_{\mathcal{U}} \; \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \; \frac{v - p_{n+1}}{v - j_{n+1}}$$

is non-singular at $u = c_{n+1}$ and $v = p_{n+1}$, and, moreover,

$$E_{\mathcal{U}} \; \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \; \frac{v - p_{n+1}}{v - j_{n+1}} \Big|_{\substack{u = c_{n+1} \\ v = p_{n+1}}} = E_{\mathcal{T}} \; . \tag{12}$$

Since j_{n+1} takes values ± 1, the rational function $\frac{v-p_{n+1}}{v-j_{n+1}}$ is non-singular at $v = p_{n+1}$ and

$$\frac{v - p_{n+1}}{v - j_{n+1}} \Big|_{v=p_{n+1}} = \frac{1}{2}(1 + p_{n+1} j_{n+1}) \; . \tag{13}$$

For the clarity of the calculations in the sequel, we define, generalizing (5),

$$\mathbf{j}_i(p) := \frac{1}{2}(1 + p j_i) \quad \text{for } i = 1, \dots, n+1. \tag{14}$$

Combining (12) and (13), we obtain the following formula for the idempotent $E_{\mathcal{T}}$:

$$E_{\mathcal{T}} = E_{\mathcal{U}} \; \mathbf{j}_{n+1}(p_{n+1}) \; \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \Big|_{u=c_{n+1}} . \tag{15}$$

3. Fusion formula for idempotents of B_{n+1}

We start with the following Lemma which will be useful in the sequel.

Lemma 1. *For any integer l, $1 \leqslant l \leqslant n$, we have*

(i) $\tilde{j}_{n+1} = s_n s_{n-1} \dots s_l \tilde{j}_l s_l \dots s_{n-1} s_n + \dfrac{1}{2} \displaystyle\sum_{i=l}^{n} s_n \dots s_{i+1} s_i s_{i+1} \dots s_n (1 + j_{n+1} j_i),$

(ii) $\mathbf{j}_l(p) s_l \dots s_{n-1} s_n \tilde{j}_{n+1} = \mathbf{j}_l(p) \tilde{j}_l s_l \dots s_{n-1} s_n + \dfrac{1}{2} \displaystyle\sum_{i=l}^{n} s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \mathbf{j}_i(p)(1 + j_{n+1} j_i);$

the product $s_l s_{l+1} \dots s_{i-1}$ in the right-hand-side of (ii) is understood to be equal to 1 if $i = l$.

Proof. We prove (i) by induction on $n-l$. The basis of the induction is, for $l=n$, the formula $\tilde{j}_{n+1}=s_n\tilde{j}_n s_n+\frac{1}{2}(s_n+s_n j_{n+1} j_n)$ which follows from the definition (7) of the Jucys–Murphy elements, namely from $\tilde{j}_{n+1}=s_n\tilde{j}_n s_n+\frac{1}{2}(s_n+j_n s_n j_n)$ and $j_n s_n=s_n j_{n+1}$. Now assume that

$$\tilde{j}_{n+1}=s_n s_{n-1}\dots s_{l+1}\tilde{j}_{l+1}s_{l+1}\dots s_{n-1}s_n+\frac{1}{2}\sum_{i=l+1}^{n} s_n\dots s_{i+1}s_i s_{i+1}\dots s_n(1+j_{n+1}j_i),$$

and replace $\tilde{j}_{l+1}$ by $s_l\tilde{j}_l s_l+\frac{1}{2}(s_l+j_l s_l j_l)$. One obtains the assertion (i) using that $j_l s_{l+1}\dots s_{n-1}s_n=s_{l+1}\dots s_{n-1}s_n j_l$ and that $j_l s_l s_{l+1}\dots s_{n-1}s_n=s_l s_{l+1}\dots s_{n-1}s_n j_{n+1}$.

Using (i), we find

$$\mathbf{j}_l(p)s_l\dots s_{n-1}s_n\tilde{j}_{n+1}=\mathbf{j}_l(p)\Big(\tilde{j}_l s_l\dots s_{n-1}s_n+\frac{1}{2}\sum_{i=l}^{n} s_l s_{l+1}\dots s_{i-1}\cdot s_{i+1}\dots s_{n-1}s_n(1+j_{n+1}j_i)\Big),$$

and (ii) follows since $j_l s_l s_{l+1}\dots s_{i-1}=s_l s_{l+1}\dots s_{i-1}j_i$ and j_i commutes with $s_{i+1}\dots s_{n-1}s_n$. □

Let $\phi_1(v,u):=\mathbf{t}(v)$; for $k=1,\dots,n$ define

$$\begin{aligned}\phi_{k+1}(v_1,\dots,v_k,v,u_1,\dots,u_k,u)&:=\mathbf{s}_k(v,v_k,u,u_k)\phi_k(v_1,\dots,v_{k-1},v,u_1,\dots,u_{k-1},u)s_k\\ &=\mathbf{s}_k(v,v_k,u,u_k)\mathbf{s}_{k-1}(v,v_{k-1},u,u_{k-1})\dots\mathbf{s}_1(v,v_1,u,u_1)\mathbf{t}(v)s_1\dots s_{k-1}s_k\ .\end{aligned}\tag{16}$$

Define the following rational function with values in the group ring of B_{n+1}:

$$\Phi(v_1,\dots,v_{n+1},u_1,\dots,u_{n+1}):=\overleftarrow{\prod_{k=0,\dots,n}}\phi_{k+1}(v_1,\dots,v_k,v_{k+1},u_1,\dots,u_k,u_{k+1})\ ;\tag{17}$$

the arrow over $\prod$ indicates that the (non-commuting) factors are taken in the descending order.

Let $\lambda^{(2)}$ be a 2-partition of length $n+1$ and $\mathcal{T}$ a standard 2-tableau of shape $\lambda^{(2)}$. For $i=1,\dots,n+1$, set $p_i:=p(\mathcal{T}|i)$ and $c_i:=c(\mathcal{T}|i)$.

Theorem 2. *The idempotent $E_{\mathcal{T}}$ corresponding to the standard 2-tableau $\mathcal{T}$ of shape $\lambda^{(2)}$ can be obtained by the following consecutive evaluations*

$$E_{\mathcal{T}}=f_{\lambda^{(2)}}\Phi(v_1,\dots,v_{n+1},u_1,\dots,u_{n+1})\Big|_{v_i=p_i,\ i=1,\dots,n+1}\Big|_{u_1=c_1}\dots\Big|_{u_n=c_n}\Big|_{u_{n+1}=c_{n+1}}\ .\tag{18}$$

Proof. Define

$$F_{\mathcal{T}}(u):=\frac{u-c_{n+1}}{u}\prod_{i=1}^{n}\frac{(u-c_i)^2}{(u-c_i)^2-\delta_{p_i,p_{n+1}}}\ .\tag{19}$$

Let $\mathcal{U}$ be the standard 2-tableau obtained from $\mathcal{T}$ by removing the 2-node with the number $n+1$ and let $\mu^{(2)}$ be the shape of $\mathcal{U}$.

Proposition 3. *We have*

$$F_{\mathcal{T}}(u)\phi_{n+1}(p_1,\dots,p_n,p_{n+1},c_1,\dots,c_n,u)E_{\mathcal{U}}=\frac{u-c_{n+1}}{u-\tilde{j}_{n+1}}\,\mathbf{j}_{n+1}(p_{n+1})E_{\mathcal{U}}.\tag{20}$$

Proof. We prove (20) by induction on n. As $c_1 = 0$ and $\tilde{j}_1 = 0$, the basis of induction for $n = 0$ is the formula $\mathbf{t}(p_1) = \mathbf{j}_1(p_1)$ which is satisfied by definition, see (5) and (14).

If $p_{n+1} \neq p_i$, $i = 2, \dots, n$, then fix $l = 1$. Otherwise fix l such that $p_{n+1} = p_l$ and $p_{n+1} \neq p_i$, $i = l+1, \dots, n$.

Define $\mathcal{V}$ to be the standard 2-tableau obtained from $\mathcal{U}$ by removing the 2-nodes containing the numbers $l+1, \dots, n$ and $\mathcal{W}$ to be the standard 2-tableau obtained from $\mathcal{V}$ by removing the 2-node with the number l. We will use that $E_{\mathcal{W}} E_{\mathcal{U}} = E_{\mathcal{U}}$ and that $E_{\mathcal{W}}$ commutes with s_i, for $i = l, l+1, \dots, n$. We rewrite the left-hand side of (20) as

$$F_{\mathcal{T}}(u) s_n \dots s_{l+1} \mathbf{s}_l(p_{n+1}, p_l, u, c_l) \cdot \phi_l(p_1, \dots, p_{l-1}, p_{n+1}, c_1, \dots, c_{l-1}, u) E_{\mathcal{W}} \cdot s_l s_{l+1} \dots s_n E_{\mathcal{U}} \ .$$

If $l > 1$ then we use the induction hypothesis, namely

$$\phi_l(p_1, \dots, p_{l-1}, p_l, c_1, \dots, c_{l-1}, u) E_{\mathcal{W}} = (F_{\mathcal{V}}(u))^{-1} \frac{u - c_l}{u - \tilde{j}_l} \mathbf{j}_l(p_l) E_{\mathcal{W}} \ ,$$

and we notice that $p_{n+1} = p_l$.
If $l = 1$ then $E_{\mathcal{W}} = 1$, $F_{\mathcal{V}}(u) = 1$ and, by definition, $\phi_1(p_{n+1}, u)$ is equal to $\mathbf{j}_1(p_{n+1})$. Thus, in both situations (for $l = 1$ and for $l > 1$), we obtain for the left-hand-side of (20):

$$F_{\mathcal{T}}(u)(F_{\mathcal{V}}(u))^{-1} s_n \dots s_{l+1} \mathbf{s}_l(p_{n+1}, p_l, u, c_l) \frac{u - c_l}{u - \tilde{j}_l} \mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n E_{\mathcal{U}} \ .$$

Therefore, the equality (20) is equivalent to

$$\begin{aligned} &F_{\mathcal{T}}(u)(F_{\mathcal{V}}(u))^{-1}(u - c_l) \mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n (u - \tilde{j}_{n+1}) E_{\mathcal{U}} \\ &\quad = \frac{(u - c_{n+1})(u - c_l)^2}{(u - c_l)^2 - \delta_{p_l, p_{n+1}}} (u - \tilde{j}_l) \mathbf{s}_l(p_l, p_{n+1}, c_l, u) s_{l+1} \dots s_n \mathbf{j}_{n+1}(p_{n+1}) E_{\mathcal{U}}, \end{aligned} \tag{21}$$

where, in moving $s_n \dots s_{l+1} \mathbf{s}_l(p_{n+1}, p_l, u, c_l) \dfrac{1}{u - \tilde{j}_l}$ to the right-hand-side and $\dfrac{1}{u - \tilde{j}_{n+1}}$ to the left-hand-side, we have used that $\tilde{j}_{n+1}$ commutes with j_{n+1} and $E_{\mathcal{U}}$, and also the formula (6) to take the inverse of $\mathbf{s}_l(p_{n+1}, p_l, u, c_l)$.

To prove the equality (21), first notice that we have

$$F_{\mathcal{T}}(u)(F_{\mathcal{V}}(u))^{-1}(u - c_l) = (u - c_{n+1}) \prod_{i=1}^{n} \frac{(u - c_i)^2}{(u - c_i)^2 - \delta_{p_i, p_{n+1}}} \prod_{i=1}^{l-1} \left(\frac{(u - c_i)^2}{(u - c_i)^2 - \delta_{p_i, p_l}} \right)^{-1} ,$$

which gives, since $p_i \neq p_{n+1}$ if $i > l$ and $p_l = p_{n+1}$ if $l > 1$,

$$F_{\mathcal{T}}(u)(F_{\mathcal{V}}(u))^{-1}(u - c_l) = \frac{(u - c_{n+1})(u - c_l)^2}{(u - c_l)^2 - \delta_{p_l, p_{n+1}}}. \tag{22}$$

So it remains to prove that

$$\mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n (u - \tilde{j}_{n+1}) E_{\mathcal{U}} = (u - \tilde{j}_l) \mathbf{s}_l(p_l, p_{n+1}, c_l, u) s_{l+1} \dots s_n \mathbf{j}_{n+1}(p_{n+1}) E_{\mathcal{U}}. \tag{23}$$

Expand $\mathbf{s}_l(p_l,p_{n+1},c_l,u)$ in the right hand side of (23):

$$((u-\tilde{j}_l)s_l-\delta_{p_l,p_{n+1}}\frac{\tilde{j}_l-u}{c_l-u})s_{l+1}\dots s_n\mathbf{j}_{n+1}(p_{n+1})E_{\mathcal{U}}\ . \tag{24}$$

As $\tilde{j}_l$ commutes with $s_{l+1}\dots s_n$ and j_{n+1} and $\tilde{j}_lE_{\mathcal{U}}=c_lE_{\mathcal{U}}$, we find that the expression (24) equals

$$\Big((u-\tilde{j}_l)s_ls_{l+1}\dots s_n\mathbf{j}_{n+1}(p_{n+1})-\delta_{p_l,p_{n+1}}s_{l+1}\dots s_n\mathbf{j}_{n+1}(p_{n+1})\Big)E_{\mathcal{U}}\ .$$

Then using that $s_ls_{l+1}\dots s_nj_{n+1}=j_ls_ls_{l+1}\dots s_n$, we obtain for the right-hand-side of (23):

$$\big((u-\tilde{j}_l)\mathbf{j}_l(p_{n+1})s_ls_{l+1}\dots s_n-\delta_{p_l,p_{n+1}}s_{l+1}\dots s_n\mathbf{j}_{n+1}(p_{n+1})\big)E_{\mathcal{U}}. \tag{25}$$

Using the Lemma 1, (ii), we write the left hand side of (23) in the form

$$\Big(\mathbf{j}_l(p_{n+1})(u-\tilde{j}_l)s_ls_{l+1}\dots s_n-\frac{1}{2}\sum_{i=l}^{n}s_ls_{l+1}\dots s_{i-1}\cdot s_{i+1}\dots s_{n-1}s_n\mathbf{j}_i(p_{n+1})(1+j_{n+1}j_i)\Big)E_{\mathcal{U}}\ . \tag{26}$$

As $j_iE_{\mathcal{U}}=p_iE_{\mathcal{U}}$, $i=1,\dots,n$, the expression (26) is equal to

$$\Big(\mathbf{j}_l(p_{n+1})(u-\tilde{j}_l)s_ls_{l+1}\dots s_n-\frac{1}{2}\sum_{i=l}^{n}s_ls_{l+1}\dots s_{i-1}\cdot s_{i+1}\dots s_{n-1}s_n\frac{1}{2}(1+p_ip_{n+1})(1+p_ij_{n+1})\Big)E_{\mathcal{U}}\ . \tag{27}$$

Since $p_kp_{n+1}=-1$, $k=l+1,\dots,n$, we finally obtain for the left-hand-side of (23):

$$\big(\mathbf{j}_l(p_{n+1})(u-\tilde{j}_l)s_ls_{l+1}\dots s_n-\delta_{p_l,p_{n+1}}s_{l+1}\dots s_{n-1}s_n\mathbf{j}_{n+1}(p_l)\big)E_{\mathcal{U}}\ . \tag{28}$$

The comparison of (28) and (25) proves the equality (23). □

Proposition 4. *The rational function $F_{\mathcal{T}}(u)$, defined by (19), is regular at $u=c_{n+1}$ and moreover*

$$F_{\mathcal{T}}(c_{n+1})=f_{\lambda^{(2)}}(f_{\mu^{(2)}})^{-1}. \tag{29}$$

Proof. The Proposition will directly follow from the result, used in [13], concerning usual tableaux. Let λ be a partition of length $m+1$, with $m\le n$. let S be a subset of $\{1,\dots,n+1\}$ such that S contains the number $n+1$ and has a cardinal equal to $m+1$. Let $\tilde{\mathcal{T}}$ be a tableau of shape λ filled with numbers belonging to S such that the numbers in the nodes are in strictly ascending orders along rows and columns in right and down directions. Let γ be the node of $\tilde{\mathcal{T}}$ with the number $n+1$ and μ be the shape of the tableau obtained from $\tilde{\mathcal{T}}$ by removing the node γ. Define the following rational function

$$\tilde{F}_{\tilde{\mathcal{T}}}(u)=\frac{u-c(\gamma)}{u}\prod_{\alpha\in\mu}\frac{(u-c(\alpha))^2}{(u-c(\alpha)+1)(u-c(\alpha)-1)}. \tag{30}$$

The product in the right hand side of (30) depends only on the shape μ and one has

$$\prod_{\alpha\in\mu}\frac{(u-c(\alpha))^2}{(u-c(\alpha)+1)(u-c(\alpha)-1)}=u\prod_{\beta\in\mathcal{E}_-(\mu)}(u-c(\beta))\prod_{\alpha\in\mathcal{E}_+(\mu)}(u-c(\alpha))^{-1}\ ,$$

where $\mathcal{E}_-(\mu)$ (respectively, $\mathcal{E}_+(\mu)$) is the set of removable (respectively, addable) nodes of μ. Therefore the rational function $\tilde{F}_{\tilde{\mathcal{T}}}(u)$ is regular at $u = c(\gamma)$ and moreover

$$\tilde{F}_{\tilde{\mathcal{T}}}(c(\gamma)) = \prod_{\beta \in \mathcal{E}_-(\mu)} (c(\gamma) - c(\beta)) \prod_{\alpha \in \mathcal{E}_+(\mu) \setminus \{\gamma\}} (c(\gamma) - c(\alpha))^{-1} \ . \tag{31}$$

It is known that the right hand side of (31) is equal to

$$\prod_{\alpha \in \lambda} \big(h_\lambda(\alpha)\big)^{-1} \prod_{\alpha \in \mu} h_\mu(\alpha). \tag{32}$$

Define $\tilde{\mathcal{T}}$ to be the tableau of the standard 2-tableau $\mathcal{T}$ which contains the node with number $n+1$. The assertion of the Proposition 4 follows, the only observation one has to make is that the 2-nodes (α, k) with $p_k \neq p_{n+1}$ do not contribute to (29). □

The Theorem 2 follows, by induction on n, from the formula (15), the Proposition 3 and the Proposition 4. □

4. Fusion procedure for the complex reflection group $G(m, 1, n+1)$

We extend the results of the previous Section to the complex reflection groups $G(m, 1, n+1)$ for all positive integers m. We skip the proofs when they are completely similar to the proofs in the preceding Section; we only indicate modifications.

4.1 Definitions

The complex reflection group $G(m, 1, n+1)$ is generated by the elements $s_1, \dots, s_n$ and t with the defining relations (1), (2) and

$$t^m = 1. \tag{33}$$

In particular, $G(1, 1, n+1)$ is isomorphic to the symmetric group S_{n+1} and $G(2, 1, n+1)$ to B_{n+1}.

We extend the definition (4) to the generators $s_1, \dots, s_n$ of $G(m, 1, n+1)$:

$$\mathbf{s}_i(p, p', a, a') := s_i + \frac{\delta_{p,p'}}{a - a'}, \quad i = 1, \dots, n \ . \tag{34}$$

The Jucys–Murphy elements for the group $G(m, 1, n+1)$ are the elements j_i, $i = 1, \dots, n+1$, and $\tilde{j}_i$, $i = 1, \dots, n+1$, of the group ring defined inductively by the following initial conditions and recursions:

$$j_1 = t \ , \quad j_{i+1} = s_i j_i s_i \quad \text{and} \quad \tilde{j}_1 = 0 \ , \quad \tilde{j}_{i+1} = s_i \tilde{j}_i s_i + \frac{1}{m} \sum_{k=0}^{m-1} j_i^k s_i j_i^{m-k}. \tag{35}$$

For $m = 1$, that is, for S_{n+1}, $j_k = 1$, $k = 1, \dots, n+1$; the recursion formula for $\tilde{j}_{i+1}$ reduces to $\tilde{j}_{i+1} = s_i \tilde{j}_i s_i + s_i$.

As for $m=2$, the elements j_i, $i=1,\dots,n+1$, and $\tilde{j}_i$, $i=1,\dots,n+1$, form a maximal commutative set in $\mathbb{C}G(m,1,n+1)$, see [20, 21]; in addition, j_i and $\tilde{j}_i$ commute with all s_k, except s_i and s_{i-1}.

The definitions of a 2-partition, 2-node, standard 2-tableaux and hook length generalize naturally to any m; for example, an m-partition is an m-tuple of partitions and an m-node $\alpha^{(m)}$ is a pair (α,k), $k=1,\dots,m$. For an m-node $\alpha^{(m)}=(\alpha,k)$ of an m-partition $\lambda^{(m)}$ such that the node α lies in the line x and the column y of the k-th diagram, we define $c(\alpha^{(m)}):=c(\alpha)=y-x$. Let $\{\xi_1,\dots,\xi_m\}$ be the set of distinct m-th roots of unity, ordered arbitrarily; we define $p(\alpha^{(m)}):=\xi_k$.

As for $m=2$, we define

$$f_{\nu^{(m)}} := \Big(\prod_{\alpha^{(m)}\in\nu^{(m)}} h_{\nu^{(m)}}(\alpha^{(m)})\Big)^{-1}\ , \tag{36}$$

where $h_{\nu^{(m)}}(\alpha^{(m)})$ is the hook length of $\alpha^{(m)}$ calculated in $\nu^{(m)}$.

The irreducible representations of $G(m,1,n+1)$ are parameterized by the m-partitions of length $n+1$; for a given m-partition $\lambda^{(m)}$, elements of the semi-normal basis of the corresponding representation are indexed by the standard m-tableaux of shape $\lambda^{(m)}$. We denote by $E_{\mathcal{T}}$ the idempotent of the group ring corresponding to a standard m-tableau $\mathcal{T}$.

4.2 Fusion formula for idempotents of $G(m,1,n+1)$

Let $\lambda^{(m)}$ be an m-partition of length $n+1$; fix a standard m-tableau $\mathcal{T}$ of shape $\lambda^{(m)}$. Let $\mathcal{U}$ be the standard m-tableau obtained by removing from $\mathcal{T}$ the m-node containing $n+1$ and let $\mu^{(m)}$ be the shape of $\mathcal{U}$. Set, for $i=1,\dots,n+1$, $p_i:=p(\mathcal{T}|i)$ and $c_i:=c(\mathcal{T}|i)$

The same reasoning as in Subsection 2.2 leads to the formula for $E_{\mathcal{T}}$ (cf (12)):

$$E_{\mathcal{U}}\,\frac{u-c_{n+1}}{u-\tilde{j}_{n+1}}\,\frac{v-p_{n+1}}{v-j_{n+1}}\,\Big|_{\substack{u\,=\,c_{n+1}\\ v\,=\,p_{n+1}}} = E_{\mathcal{T}}\ . \tag{37}$$

Since j_{n+1} takes values in $\{\xi_1,\dots,\xi_m\}$, the rational function $\frac{v-p_{n+1}}{v-j_{n+1}}$ is non-singular for $v=p_{n+1}$ and

$$\frac{v-p_{n+1}}{v-j_{n+1}}\Big|_{v=p_{n+1}} = \frac{1}{m}\sum_{k=0}^{m-1} p_{n+1}^{m-k} j_{n+1}^{k}\ . \tag{38}$$

An analogue of (14) is

$$\mathbf{j}_i(p) := \frac{1}{m}\sum_{k=0}^{m-1} p^{m-k} j_i^{k}\quad \text{for } i=1,\dots,n+1. \tag{39}$$

For $i=1$ we shall write $\mathbf{t}(p):=\frac{1}{m}\sum\limits_{k=0}^{m-1} p^{m-k}t^k$ instead of $\mathbf{j}_1(p)$. Combining (37) and (38), we obtain for the idempotent $E_{\mathcal{T}}$ (cf (15))

$$E_{\mathcal{T}} = E_{\mathcal{U}}\ \mathbf{j}_{n+1}(p_{n+1})\ \frac{u-c_{n+1}}{u-\tilde{j}_{n+1}}\Big|_{u=c_{n+1}}. \tag{40}$$

We generalize the Lemma 1 to an arbitrary positive integer m.

Lemma 5. *For any integer l, $1 \leqslant l \leqslant n$, we have*

(i) $\tilde{j}_{n+1} = s_n s_{n-1} \dots s_l \tilde{j}_l s_l \dots s_{n-1} s_n + \dfrac{1}{m} \sum_{i=l}^{n} s_n \dots s_{i+1} s_i s_{i+1} \dots s_n \sum_{k=0}^{m-1} j_{n+1}^k j_i^{m-k}$.

(ii) $\mathbf{j}_l(p) s_l \dots s_{n-1} s_n \tilde{j}_{n+1} = \mathbf{j}_l(p) \tilde{j}_l s_l \dots s_{n-1} s_n + \dfrac{1}{m} \sum_{i=l}^{n} s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \mathbf{j}_i(p) \sum_{k=0}^{m-1} j_{n+1}^k j_i^{m-k}$;

the product $s_l s_{l+1} \dots s_{i-1}$ in the right-hand-side of (ii) is understood to be equal to 1 if $i = l$.

Proof. The proof is completely similar to the proof of the Lemma 1. $\square$

Let $\phi_1(v,u) := \mathbf{t}(v)$; for $k = 1, \dots, n$ define

$$\begin{aligned} \phi_{k+1}(v_1, \dots, v_k, v, u_1, \dots, u_k, u) &:= \mathbf{s}_k(v, v_k, u, u_k) \phi_k(v_1, \dots, v_{k-1}, v, u_1, \dots, u_{k-1}, u) s_k \\ &= \mathbf{s}_k(v, v_k, u, u_k) \mathbf{s}_{k-1}(v, v_{k-1}, u, u_{k-1}) \dots \mathbf{s}_1(v, v_1, u, u_1) \mathbf{t}(v) s_1 \dots s_{k-1} s_k \ . \end{aligned} \tag{41}$$

The formula (41) reads in the same way for any m; only the definition of $\mathbf{t}(v)$ depends on m.

Define the following rational function with values in the group ring of $G(m,1,n)$:

$$\Phi(v_1, \dots, v_{n+1}, u_1, \dots, u_{n+1}) := \overset{\leftarrow}{\prod_{k=0,\dots,n}} \phi_{k+1}(v_1, \dots, v_k, v_{k+1}, u_1, \dots, u_k, u_{k+1}) \ ; \tag{42}$$

the arrow over $\prod$ indicates that the (non-commuting) factors are taken in the descending order.

Theorem 6. *The idempotent $E_{\mathcal{T}}$ corresponding to the standard m-tableau $\mathcal{T}$ of shape $\lambda^{(m)}$ can be obtained by the following consecutive evaluations*

$$E_{\mathcal{T}} = f_{\lambda^{(m)}} \Phi(v_1, \dots, v_{n+1}, u_1, \dots, u_{n+1}) \Big|_{v_i = p_i, \ i=1,\dots,n+1} \Big|_{u_1 = c_1} \dots \Big|_{u_n = c_n} \Big|_{u_{n+1} = c_{n+1}} \ . \tag{43}$$

Proof. Define

$$F_{\mathcal{T}}(u) := \frac{u - c_{n+1}}{u} \prod_{i=1}^{n} \frac{(u - c_i)^2}{(u - c_i)^2 - \delta_{p_i, p_{n+1}}} \ . \tag{44}$$

Proposition 7. *We have*

$$F_{\mathcal{T}}(u) \phi_{n+1}(p_1, \dots, p_n, p_{n+1}, c_1, \dots, c_n, u) E_{\mathcal{U}} = \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \mathbf{j}_{n+1}(p_{n+1}) E_{\mathcal{U}}. \tag{45}$$

Proof. The proof follows the same lines as the proof of the Proposition 3; actually it is exactly the same until the calculation of $\mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n (u - \tilde{j}_{n+1}) E_{\mathcal{U}}$ just after the formula (25). We give the modified end of the proof.

Here, we rewrite $\mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n (u - \tilde{j}_{n+1}) E_{\mathcal{U}}$ using the Lemma 5, (ii):

$$\Big(\mathbf{j}_l(p_{n+1})(u - \tilde{j}_l) s_l s_{l+1} \dots s_n - \frac{1}{m} \sum_{i=l}^{n} s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \mathbf{j}_i(p_{n+1}) \sum_{k=0}^{m-1} j_{n+1}^k j_i^{m-k} \Big) E_{\mathcal{U}} \ . \tag{46}$$

As $j_i E_{\mathcal{U}} = p_i E_{\mathcal{U}}$ for $i = 1, \dots, n$, the expression (46) is equal to

$$\Big(\mathbf{j}_l(p_{n+1})(u - \tilde{j}_l)s_l s_{l+1} \dots s_n - \frac{1}{m}\sum_{i=l}^{n} s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \sum_{k=0}^{m-1} p_{n+1}^k p_i^{m-k} \mathbf{j}_{n+1}(p_i)\Big) E_{\mathcal{U}} \ . \quad (47)$$

Since $\frac{1}{m}\sum\limits_{k=0}^{m-1} p_{n+1}^k p_i^{m-k} = \delta_{p_i, p_{n+1}}$ we obtain that $\mathbf{j}_l(p_{n+1})s_l s_{l+1} \dots s_n(u - \tilde{j}_{n+1})E_{\mathcal{U}}$ equals

$$\Big(\mathbf{j}_l(p_{n+1})(u - \tilde{j}_l)s_l s_{l+1} \dots s_n - \delta_{p_l, p_{n+1}} s_{l+1} \dots s_{n-1} s_n \mathbf{j}_{n+1}(p_l)\Big) E_{\mathcal{U}} \ . \quad (48)$$

This concludes the proof. □

The analogue of the Proposition 4 holds as well.

Proposition 8. *The rational function $F_{\mathcal{T}}(u)$, defined by (44), is regular at $u = c_{n+1}$ and moreover*

$$F_{\mathcal{T}}(c_{n+1}) = f_{\lambda^{(m)}}(f_{\mu^{(m)}})^{-1}. \quad (49)$$

Proof. The proof is completely similar to the proof of the Proposition 4. □

Similarly to the Theorem 2, the Theorem 6 follows, using induction on n, from the formula (40), the Proposition 7 and the Proposition 8. □

For calculations, it is sometimes useful to write the function Φ in a slightly different form. Namely, let $\tilde{\phi}_1(v, u) := 1$ and define

$$\begin{aligned} \tilde{\phi}_{k+1}(v_1, \dots, v_k, v, u_1, \dots, u_k, u) &:= \mathbf{s}_k(v, v_k, u, u_k)\tilde{\phi}_k(v_1, \dots, v_{k-1}, v, u_1, \dots, u_{k-1}, u)s_k \\ &= \mathbf{s}_k(v, v_k, u, u_k)\mathbf{s}_{k-1}(v, v_{k-1}, u, u_{k-1}) \dots \mathbf{s}_1(v, v_1, u, u_1)s_1 \dots s_{k-1}s_k \ , \end{aligned} \quad (50)$$

for $k = 1, \dots, n$. The elements $\tilde{\phi}_{k+1}(v_1, \dots, v_k, v, u_1, \dots, u_k, u)$ do not involve the generator t and $\Phi(v_1, \dots, v_{n+1}, u_1, \dots, u_{n+1})$, defined in (42), equals

$$\overset{\leftarrow}{\prod_{k=0,\dots,n}} \tilde{\phi}_{k+1}(v_1, \dots, v_{k+1}, u_1, \dots, u_{k+1}) \cdot \mathbf{j}_1(v_1)\mathbf{j}_2(v_2) \dots \mathbf{j}_{n+1}(v_{n+1}) \ . \quad (51)$$

For example, let $m = 2$; choose the order $\{1, -1\}$ on the set of square roots of 1. The primitive idempotent, corresponding to the standard 2-tableau $\big(\boxed{1}\boxed{3}, \boxed{2}\big)$ reads $s_2(1+s_1)s_2\mathbf{j}_1(1)\mathbf{j}_2(-1)\mathbf{j}_3(1)/16$.

References

[1] Cherednik I., *On special bases of irreducible finite-dimensional representations of the degenerate affine Hecke algebra*, Funct. Analysis Appl. 20 (1986), 87–89.

[2] Grime J., *The hook fusion procedure*, Electron. J. of Comb. 12(1), (2005), R26.

[3] Grime J., *The hook fusion procedure for Hecke algebras*, J. Algebra 309, (2007), 744–759.

[4] Guizzi V. and Papi P., *A combinatorial approach to the fusion process for the symmetric group*, Europ. J. Comb. 19 (1998), 835–845.

[5] Isaev A. and Molev A., *Fusion procedure for the Brauer algebra*, Algebra i Analiz, 22:3 (2010), 142–154.

[6] Isaev A., Molev A. and Os'kin A., *On the idempotents of Hecke algebras*, Lett. Math. Phys. 85 (2008), 79–90.

[7] Isaev A., Molev A. and Ogievetsky O., *A new fusion procedure for the Brauer algebra and evaluation homomorphisms*, Int. Math. Res. Not. (2011), doi: 10.1093/imrn/rnr126.

[8] Isaev A., Molev A. and Ogievetsky O., *Idempotents for Birman-Murakami-Wenzl algebras and reflection equation*, arXiv:1111.2502

[9] Jimbo M., Kuniba A., Miwa T. and Okado M., *The $A_n^{(1)}$ face models*, Commun. Math. Phys. 119 (1988), 543–565.

[10] Jones A., *The structure of the Young symmetrizers for spin representations of the symmetric group I*, J. Algebra 205 (1998), 626–660.

[11] Jones A. and Nazarov M., *Affine Sergeev algebra and q-analogues of the Young symmetrizers for projective representations of the symmetric group* Proc. London Math. Soc. 78 (1999), 481–512.

[12] Jucys A., *On the Young operators of the symmetric group*, Lietuvos Fizikos Rinkinys 6 (1966), 163–180.

[13] Molev A., *On the fusion procedure for the symmetric group*, Reports Math. Phys. 61 (2008), 181–188.

[14] Nazarov M., *Young's symmetrizers for projective representations of the symmetric group*, Adv. Math. 127 (1997), 190–257.

[15] Nazarov M., *Yangians and Capelli identities*, in: "Kirillov's Seminar on Representation Theory" (G. I. Olshanski, Ed.) Amer. Math. Soc. Transl. 181, Amer. Math. Soc., Providence, RI, (1998), 139–163.

[16] Nazarov M., *Mixed hook-length formula for degenerate affine Hecke algebras*, Lecture Notes in Math. 1815 (2003), 223–236.

[17] Nazarov M., *A mixed hook-length formula for affine Hecke algebras*, European J. Combinatorics 25 (2004), 1345–1376.

[18] Nazarov M. and Tarasov V., *On irreducibility of tensor products of Yangian modules associated with skew Young diagrams* Duke Math. J. 112 (2002), 343–378.

[19] Ogievetsky O. and Poulain d'Andecy L., *On representations of cyclotomic Hecke algebras*, Mod. Phys. Lett. A, 26, No. 11 (2011), 795-803.

[20] Ogievetsky O. V. and Poulain d'Andecy L., *Cyclotomic Hecke algebras: Jucys-Murphy elements, representations, classical limit*, to appear.

[21] Pushkarev I.A., *On the representation theory of wreath products of finite groups and symmetric groups*, J. Math. Sci. 96 (1999), 3590–3599.

[22] Ram A., *Seminormal representations of Weyl groups and Iwahori-Hecke algebras*, Proc. London Math. Soc. 75 No.3 (1997), 99–133.

[23] Wang W., *Vertex algebras and the class algebras of wreath products*, Proc. London Math. Soc. 88 (2004), 381–404.

[24] Young A., *On quantitative substitutional analysis, V*, Proc. London Math. Soc. (2) **31** (1930), 273–288.