Quantification of Entanglement of teleportation in Arbitrary Dimensions

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We study bipartite entangled states in arbitrary dimensions and obtain different bounds for the teleportation fidelity. In addition, we establish various relations between teleportation fidelity and the entanglement measures depending upon Schmidt rank of the states. These relations and bounds help us to determine the amount of entanglement required for teleportation. We call this amount of entanglement required for teleportation as "Entanglement of Teleportation". These bounds are used to determine the teleportation fidelity as well as the entanglement required for teleportation of states modeled by a two qutrit mixed system as well as two qubit open quantum systems.

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I. INTRODUCTION

Entanglement lies at the heart of quantum mechanics [1]. For a long time it was considered synonymous with quantum correlations and plays a pivotal role in various information processing tasks, including, among others, quantum teleportation [2], super dense coding [3], remote state preparation [4], secret sharing [5], and quantum cryptography [6].

In quantum teleportation, using entangled states as resource, it is possible to transfer quantum information from an unknown qubit to another one placed at a distance. Thus, one of the party, say, Alice makes a two qubit measurement on her qubit and the unknown state in Bell basis, and sends the measurement results through a classical channel to the second party, say, Bob (who is located away from Alice). Accordingly, Bob makes appropriate unitary transformations to obtain the desired state. Thus the ability of teleporting an unknown state depends on the nature of entanglement of the resource state and is called teleportation fidelity.

The situation is very straight forward when we have an unknown qubit to send with the help of a pure entangled state as a resource. However, it is more involved when we have mixed entangled states as a medium of teleportation. For a general two qubit density matrix $\rho = \frac{1}{4}[I \otimes I + \sum_i r_i(\sigma_i \otimes I) + \sum_j s_j(I \otimes \sigma_j) + \sum_{i,j} t_{ij}(\sigma_i \otimes \sigma_j)]$, the teleportation fidelity is a function of the eigenvalues of correlation matrix $T = [t_{ij}]$. Similarly, when we go from qubits to higher dimensional bipartite states the teleportation fidelity is expressed in terms of the singlet fraction of the state. The relation between optimal teleportation fidelity $F(\rho)$ and maximal singlet fraction $f(\rho)$ in a $d \otimes d$ system, if one performs quantum teleportation with the state ρ, is [7]

$$F(\rho) = \frac{df(\rho) + 1}{d + 1}. \quad (1)$$

Here the singlet fraction is defined as, $f(\rho) = max_{|\psi\rangle}\langle\psi|\rho|\psi\rangle$, and $|\psi\rangle$ is a maximally entangled state in $d \otimes d$. If $f(\rho) > \frac{1}{d}$ then the parties can perform quantum teleportation with the average fidelity of the teleported qubit exceeding the classical limit $\frac{2}{d+1}$.

In bipartite two qubit states it is known that the total amount of entanglement present in the resource state is useful for teleportation. Here we try to answer the question; how much entanglement is necessary for teleporting an unknown state when we have a bipartite state in arbitrary dimensions. To answer this question one has to quantify entanglement and find out for what range of entanglement the state can be used as a resource. In other words, one needs to establish a relationship between the amount of entanglement and teleportation fidelity. In the literature, there exist different kinds of entanglement measures, expressed in terms of Schmidt numbers, suitable for quantification of the amount of entanglement present in the system.

Schmidt decomposition [8] is a very good tool to describe composite systems. If $|\Psi\rangle$ is a pure state of composite systems A and B then,

$$|\Psi\rangle = \sum_{i=1}^{d_A} \sqrt{\lambda_i}|i_A\rangle|i_B\rangle, \quad (2)$$

represents the Schmidt decomposition of $|\Psi\rangle$, where $|i_A\rangle(i_A = 1, 2, ..., d_A) \in \mathcal{H_A}$ and $|i_B\rangle(i_B = 1, 2, ..., d_B) \in \mathcal{H_B}$ are orthonormal bases for A and B respectively, and $d_A \leq d_B$. Here λ_i are the Schmidt numbers or coefficients, non-negative real numbers satisfying the relation $\sum_i \lambda_i = 1$.

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We quantify the amount of entanglement present in the resource state to find out the bounds within which these states can be useful for teleportation. In other words, we obtain relations connecting entanglement measures with teleportation fidelity via singlet fraction. Our results are obtained for arbitrary dimensional bipartite states with at most three non vanishing Schmidt coefficients. We implement our results to detect mixed states useful for teleportation.

The plan of the paper is as follows. In section 2, we establish a relation between singlet fraction and different types of entanglement measures for arbitrary dimensional pure two qudit system with a maximum of three Schmidt coefficients. This relation is the key to our work. Then we study the bounds of teleportation fidelity and entanglement measures for two special cases, i) arbitrary dimensional pure bipartite state with two Schmidt coefficients, and ii) arbitrary dimensional pure bipartite state with three Schmidt coefficients. These results are used in section 3, to arbitrary dimensional mixed bipartite systems with Schmidt coefficients two and three. In section 4, we apply our results on examples of mixed states, in particular, two qutrit mixed state with Schmidt rank two and two qubit mixed states generated dynamically by an open system model. Finally, we conclude in section 4.

II. RELATION BETWEEN SINGLET FRACTION AND DIFFERENT ENTANGLEMENT MEASURES FOR PURE TWO QUDIT SYSTEM WITH THREE SCHMIDT COEFFICIENTS

In this section we obtain an explicit relation that will connect the entanglement monotones with the singlet fraction for a pure two qudit system of arbitrary dimension. However, it is very difficult to obtain an analytical expression which relates the entanglement monotones and singlet fraction with all non zero Schmidt coefficients. Nevertheless, we obtain results in $d \otimes d$ systems with two and three non zero Schmidt coefficients.

Let us consider a bipartite $d \otimes d$ system in which three Schmidt coefficients are non zero. Without any loss of generality we assume that the first three Schmidt coefficients are non zero. Any pure two qudit system with three non zero Schmidt coefficients λ_1, λ_2 and λ_3 can be written in Schmidt decomposition form as,

$$|\Psi^d\rangle = \sqrt{\lambda_1}|00\rangle + \sqrt{\lambda_2}|11\rangle + \sqrt{\lambda_3}|22\rangle, \qquad (3)$$

with the Schmidt coefficients summing to one, i.e., $\lambda_1 + \lambda_2 + \lambda_3 = 1$. To quantify the amount of entanglement in $|\Psi^d\rangle$ we consider two different entanglement measures $E^{(d,2)}(|\Psi^d\rangle)$ and $E^{(d,3)}(|\Psi^d\rangle)$ which can be defined as [9],

$$E^{(d,2)}(|\Psi^d\rangle) = \sqrt{\frac{2d}{d-1}(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)}, \qquad (4)$$

$$E^{(d,3)}(|\Psi^d\rangle) = \Big(\frac{6d^2}{(d-1)(d-2)}\Big)^{\frac{1}{3}}(\lambda_1\lambda_2\lambda_3)^{\frac{1}{3}}. \qquad (5)$$

Here $E^{(d,2)}(|\Psi^d\rangle)$ and $E^{(d,3)}(|\Psi^d\rangle)$ denote entanglement measure for a $d \otimes d$ dimensional pure system defined by taking the sum of the product of the Schmidt coefficients taken two or three at a time, respectively. We note that for a Schmidt rank two state, $E^{(d,3)}(|\Psi^d\rangle) = 0$ but $E^{(d,2)}(|\Psi^d\rangle) \neq 0$.

The singlet fraction for the state $|\Psi^d\rangle$ is defined as

$$f(|\Psi^d\rangle) = \max_{|\Phi\rangle} |\langle\Phi|\Psi^d\rangle|^2, \qquad (6)$$

where the maximum is taken over all maximally entangled states $|\Phi\rangle$ in $d \otimes d$ systems. The singlet fraction $f(|\Psi^d\rangle)$ for pure state $|\Psi^d\rangle$ can also be expressed in terms of Schmidt coefficients [10] as

$$f(|\Psi^d\rangle) = \frac{1}{d}\Big(\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3}\Big)^2. \qquad (7)$$

Expanding the the right hand side part of Eq. (7) and using $\lambda_1 + \lambda_2 + \lambda_3 = 1$, we get

$$\sqrt{\lambda_1\lambda_2} + \sqrt{\lambda_2\lambda_3} + \sqrt{\lambda_1\lambda_3} = \frac{df(|\Psi^d\rangle) - 1}{2}. \qquad (8)$$

Also, we have the following identity

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = (\sqrt{\lambda_1\lambda_2} + \sqrt{\lambda_2\lambda_3} + \sqrt{\lambda_1\lambda_3})^2 - 2\sqrt{\lambda_1\lambda_2\lambda_3}(\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3}). \qquad (9)$$

Using (4), (5), (7), (8) and (9) we have

$$\begin{aligned}(E^{(d,2)}(|\Psi^d\rangle))^2 &= \frac{d^3}{2(d-1)}(f(|\Psi^d\rangle) - \frac{1}{d})^2 \\ &- \frac{4}{d-1}\sqrt{\frac{d(d-1)(d-2)}{6}}(E^{(d,3)}(|\Psi^d\rangle))^{\frac{3}{2}} \\ &\times \sqrt{f(|\Psi^d\rangle)}.\end{aligned} \qquad (10)$$

This establishes the required relationship between the entanglement measures $E^{(d,2)}(|\Psi^d\rangle)$ and $E^{(d,3)}(|\psi^d\rangle)$ with the singlet fraction $f(|\Psi^d\rangle)$ for a pure two qudit system $|\Psi^d\rangle$ with three non vanishing Schmidt coefficients.

Next, we will consider separately the cases of states of Schmidt ranks two and three, respectively. For purpose of clarity, in the discussions to follow, we modify the notation of the entanglement measures discussed above, as $E_j^{d,i}$, where d stands for the $d \otimes d$ dimensional system, j indicates the Schmidt rank of the state under consideration and i is the number of coefficients taken at a time.

A. States with Schmidt Rank Two

When one of the Schmidt coefficients (say, λ_3) is zero, i.e., $E_2^{(d,3)}(|\Psi^d\rangle) = 0$, from Eq. (10), we have

$$E_2^{(d,2)}(|\Psi^d\rangle) = \sqrt{\frac{d^3}{2(d-1)}}(f_2(|\Psi^d\rangle) - \frac{1}{d}), \quad (11)$$

where, $f_2(|\Psi^d\rangle)$ denotes the singlet fraction of Schmidt rank two state, and $f_2(|\Psi^d\rangle) > \frac{1}{d}$. If $F_2(|\Psi^d\rangle)$ denotes the teleportation fidelity of Schmidt rank two states, then $E_2^{(d,2)}(|\Psi^d\rangle)$ can be expressed in terms of $F_2(|\Psi^d\rangle)$ as

$$E_2^{(d,2)}(|\Psi^d\rangle) = \sqrt{\frac{d^3}{2(d-1)}}\left[\frac{(d+1)F_2(|\Psi^d\rangle) - 2}{d}\right]. \quad (12)$$

This establishes the relation between the entanglement monotone and teleportation fidelity of Schmidt rank two states. If the state $|\Psi^d\rangle$ has Schmidt number two and useful for teleportation, then we have [12]

$$\frac{1}{d} < f_2(|\Psi^d\rangle) \le \frac{2}{d}. \quad (13)$$

Eq. (13) can be recasted in terms of teleportation fidelity as

$$\frac{2}{d+1} < F_2(|\Psi^d\rangle) \le \frac{3}{d+1}. \quad (14)$$

Using Eq. (14), $E_2^{(d,2)}(|\Psi^d\rangle)$ can be seen to be bounded as

$$0 < E_2^{(d,2)}(|\Psi^d\rangle) \le \sqrt{\frac{d}{2(d-1)}}. \quad (15)$$

When the amount of entanglement lies in the above range we can use the state for teleportation. This quantifies the entanglement required for teleportation for a pure qudit state with two non vanishing Schmidt coefficients.

B. States with Schmidt Rank Three

Next we take up sates where none of the three Schmidt coefficients are zero, i.e., $E_3^{(d,3)}(|\Psi^d\rangle) \neq 0$.

(i) Using the well known result of arithmetic mean (AM) being greater than or equal to geometric mean (GM) on three real quantities $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$ and $\sqrt{\lambda_3}$, we have

$$\frac{\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3}}{3} \ge \left(\sqrt{\lambda_1}\sqrt{\lambda_2}\sqrt{\lambda_3}\right)^{\frac{1}{3}}. \quad (16)$$

Using Eqs. (5), and (7), we get

$$f_3(|\Psi^d\rangle) \ge \frac{9}{d}\Big(\frac{(d-1)(d-2)}{6d^2}\Big)^{\frac{1}{3}} E_3^{(d,3)}(|\Psi^d\rangle). \quad (17)$$

In terms of teleportation fidelity, the above equation can be written as

$$F_3(|\Psi^d\rangle) \ \ge \ \frac{9}{d+1}\left[\Big(\frac{(d-1)(d-2)}{6d^2}\Big)^{\frac{1}{3}} E_3^{(d,3)}(|\Psi^d\rangle)\right] + \frac{1}{d+1}. \quad (18)$$

If the state $|\Psi^d\rangle$ is not useful for teleportation then we know that $f_3(|\Psi^d\rangle) < \frac{1}{d}$. From this, it can be seen that Schmidt rank three states are not useful for teleportation in $d \otimes d$ systems when

$$0 < E_3^{(d,3)}(|\Psi^d\rangle) \le \frac{1}{9}\Big(\frac{6d^2}{(d-1)(d-2)}\Big)^{\frac{1}{3}}. \quad (19)$$

(ii) Once again using $AM \ge GM$ on three real quantities $\sqrt{\lambda_1\lambda_2}$, $\sqrt{\lambda_1\lambda_3}$ and $\sqrt{\lambda_2\lambda_3}$, we have

$$\frac{\sqrt{\lambda_1\lambda_2} + \sqrt{\lambda_1\lambda_3} + \sqrt{\lambda_1\lambda_3}}{3} \ge \left(\sqrt{\lambda_1\lambda_2}\sqrt{\lambda_1\lambda_3}\sqrt{\lambda_2\lambda_3}\right)^{\frac{1}{3}}. \quad (20)$$

Using Eqs. (5), and (8), we have

$$f_3(|\Psi^d\rangle) \ge \frac{6}{d}\left[\Big(\frac{(d-1)(d-2)}{6d^2}\Big)^{\frac{1}{3}} E_3^{(d,3)}(|\Psi^d\rangle)\right] + \frac{1}{d}. \quad (21)$$

Since, the singlet fraction $f_3(|\Psi^d\rangle)$ attains its maximum value unity at $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{d}$, we have

$$\frac{6}{d}\left[\Big(\frac{(d-1)(d-2)}{6d^2}\Big)^{\frac{1}{3}} E_3^{(d,3)}(|\Psi^d\rangle)\right] + \frac{1}{d} \le f_3(|\Psi^d\rangle) \le 1. \quad (22)$$

In terms of teleportation fidelity $F_3(|\Psi^d\rangle)$, the above inequality can be expressed as

$$\frac{2}{d+1} + \frac{6}{d+1}\Big(\frac{(d-1)(d-2)}{6d^2}\Big)^{\frac{1}{3}} E_3^{(d,3)}(|\Psi^d\rangle) \le F_3(|\Psi^d\rangle) \le 1. \quad (23)$$

Hence, pure entangled states with $\frac{1}{9}(\frac{6d^2}{(d-1)(d-2)})^{\frac{1}{3}} < E_3^{(d,3)}(|\Psi^d\rangle) \le 1$ and teleportation fidelity $F_3(|\Psi^d\rangle) > \frac{2}{d+1}$ are Schmidt rank three states useful for teleportation.

To summarize, the classification of entanglement, for rank three states, in terms of teleportation fidelity, i.e., the "Entanglement of Teleportation", is:

(a). $0 < F_3 \le \frac{2}{d+1}$: state $|\Psi^d\rangle$ not useful for teleportation and $0 < E_3^{(d,3)}(|\Psi^d\rangle) \le \frac{1}{9}(\frac{6d^2}{(d-1)(d-2)})^{\frac{1}{3}}$;

(b). $\frac{2}{d+1} < F_3 < 1$: state useful for teleportation and $\frac{1}{9}(\frac{6d^2}{(d-1)(d-2)})^{\frac{1}{3}} < E_3^{(d,3)}(|\Psi^d\rangle) \le 1$.

Thus if the pure qudit state $|\Psi^d\rangle$ of Schmidt rank three has entanglement $E_3^{(d,3)}(|\Psi^d\rangle) > \frac{1}{9}(\frac{6d^2}{(d-1)(d-2)})^{\frac{1}{3}}$, then it is useful for teleportation, otherwise not.

III. BOUNDS ON ENTANGLEMENT MEASURES FOR MIXED TWO QUDIT SYSTEMS USEFUL FOR TELEPORTATION

In this section we would like to answer the following questions : (i) What is the minimum amount of entanglement needed to perform teleportation when Schmidt rank two mixed state is used as a resource in a $d \otimes d$ system? (ii) What is the minimum amount of entanglement needed to perform teleportation when Schmidt rank three mixed state is used as resource in a $d \otimes d$ system?

Let us consider a mixed qudit state described by the density operator $\rho = \sum_{i=1}^{n} p_i \rho_i$, where $\sum_{i=1}^{n} p_i = 1$ and ρ_i $(= |\psi_i\rangle\langle\psi_i|)$ are composite pure states. The singlet fraction $f(\rho)$ of the state ρ is defined as

$$f(\rho) = \max_U \langle\psi^+|U^\dagger \otimes \mathcal{I} \rho U \otimes \mathcal{I}|\psi^+\rangle, \tag{24}$$

where U is the unitary matrix, $\mathcal{I}$ is the identity matrix and $|\psi^+\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |kk\rangle$ represents a pure maximally entangled state.

The entanglement measure $E^{(d,2)}(|\Psi^d\rangle)$ and $E^{(d,3)}(|\Psi^d\rangle)$ given in Eqs. (4) and (5) for pure states can also be defined for a mixed state ρ as

$$E^{(d,2)}(\rho) = \min \sum_{i=1}^{n} p_i E^{(d,2)}(\rho_i), \tag{25}$$

and

$$E^{(d,3)}(\rho) = \min \sum_{i=1}^{n} p_i E^{(d,3)}(\rho_i). \tag{26}$$

Here the minimum is taken over all pure state decompositions of ρ. Now one may ask a question that, like entanglement measures, does the singlet fraction $f(\rho)$ also have the property [11]

$$f(\rho) = \min \sum p_i f(\rho_i), \tag{27}$$

where the minimum is taken over all decomposition of ρ. Unfortunately, the answer is no.

A. Two qudit mixed state of Schmidt rank two

It can be easily shown that the maximum amount of entanglement contained in a two qudit mixed state of Schmidt rank two is $\sqrt{\frac{d}{2(d-1)}}$, i.e.,

$$0 < E_2^{(d,2)}(\rho) \le \sqrt{\frac{d}{2(d-1)}}. \tag{28}$$

From Eq. (10), $E_2^{(d,2)}(\rho_i)$ for any Schmidt rank two bipartite pure qudit state ρ_i whose $f_2(\rho_i) < \frac{1}{d}$, i.e., for states not useful for teleportation, is

$$E_2^{(d,2)}(\rho_i) = \sqrt{\frac{d^3}{2(d-1)}}\Big(\frac{1}{d} - f_2(\rho_i)\Big). \tag{29}$$

Using Eqs. (25) and (29), we have

$$\begin{aligned} E_2^{(d,2)}(\rho) &= \min \sum_i p_i \sqrt{\frac{d^3}{2(d-1)}}(\frac{1}{d} - f_2(\rho_i)) \\ &\le \sum_i p_i \sqrt{\frac{d^3}{2(d-1)}}(\frac{1}{d} - f_2(\rho_i)) \\ &= \sqrt{\frac{d}{2(d-1)}} - \sqrt{\frac{d^3}{2(d-1)}} \sum_i p_i f_2(\rho_i). \end{aligned} \tag{30}$$

Hence, if the mixed state ρ of Schmidt rank two in a $d \otimes d$ system is not useful for teleportation then

$$0 < E_2^{(d,2)}(\rho) \le \sqrt{\frac{d}{2(d-1)}} - \sqrt{\frac{d^3}{2(d-1)}} \sum_i p_i f_2(\rho_i). \tag{31}$$

But a Schmidt rank two pure bipartite qudit state ρ_i can be useful for teleportation if $f_2(\rho_i) > \frac{1}{d}$. Thus, we can say, making use of Eqs. (10), and (28), that mixed states of Schmidt rank two are useful for teleportation only when

$$\sqrt{\frac{d^3}{2(d-1)}} \sum_i p_i f_2(\rho_i) - \sqrt{\frac{d}{2(d-1)}} \le E_2^{(d,2)}(\rho) \le \sqrt{\frac{d}{2(d-1)}}. \tag{32}$$

B. Two qudit mixed state of Schmidt rank three

Using Eq. (26) and the results for pure two qudit states of Schmidt rank three, for two qudit mixed states of Schmidt rank three we have:

(i) if $0 < E_3^{(d,3)}(\rho) \leq \frac{1}{9}(\frac{6d^2}{(d-1)(d-2)})^{\frac{1}{3}}$ then the mixed state ρ of Schmidt rank three is not useful for teleportation.

(ii) If $\frac{1}{9}(\frac{6d^2}{(d-1)(d-2)})^{\frac{1}{3}} < E_3^{(d,3)}(\rho) \leq 1$ then the mixed state ρ of Schmidt rank three is useful for teleportation.

IV. ILLUSTRATIONS AND APPLICATIONS IN OPEN QUANTUM SYSTEMS

In this section we provide examples of qubit and qutrit mixed states as applications of our results. This paves the way for detecting states which are useful for teleportation as well as to quantify the amount of entanglement required for teleportation, in realistic settings.

A. Two qutrit mixed states with Schmidt rank two

We consider a two qutrit mixed state of Schmidt rank two given by

$$\rho_f = \frac{5p}{p+2}\rho_c + \frac{2(1-2p)}{p+2}|\phi\rangle\langle\phi|; 0 \leq p \leq \frac{1}{2}, \quad (33)$$

where, $\rho_c = \frac{1}{2}(|\chi_0\rangle\langle\chi_0| + |\chi_1\rangle\langle\chi_1|)$. This decomposition for state ρ_f is optimal. Here, $|\chi_0\rangle$ and $|\chi_1\rangle$ are of the form $|\chi_0\rangle = \sqrt{\frac{3}{5}}|\psi\rangle + \sqrt{\frac{2}{5}}|\phi\rangle$ and $|\chi_1\rangle = \sqrt{\frac{3}{5}}|\psi\rangle - \sqrt{\frac{2}{5}}|\phi\rangle$, respectively, and the states $|\psi\rangle, |\phi\rangle$ are given by, $|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle - e^{\frac{i\pi}{3}}|22\rangle)$ and $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Also, p is the classical probability of mixing.

We check whether the bounds on $E_2^{(3,2)}(\rho_i)$ works for the above density matrix. For $3 \otimes 3$ dimension, the bound set by Eq. (32) becomes

$$F_p \leq E_2^{(3,2)}(\rho_f) \leq \frac{\sqrt{3}}{2}, \quad (34)$$

where, $F_p = \frac{3\sqrt{3}}{2}\sum_i p_i f_2(\rho_i) - \frac{\sqrt{3}}{2}$. If we calculate the lower bound F_p in Eq. (34) for the state [Eq. (33)], then $F_p = \frac{(1-\sqrt{3})+2(2\sqrt{3}-1)p}{p+2}$, a function of p. The graphical plot of this bound with respect to the parameter p is shown in Fig. (1). From Eqs. (34), and (15), it can be seen that the state (in Eq. (33)) is useful for teleportation for $0 \leq p \leq \frac{1}{2}$.

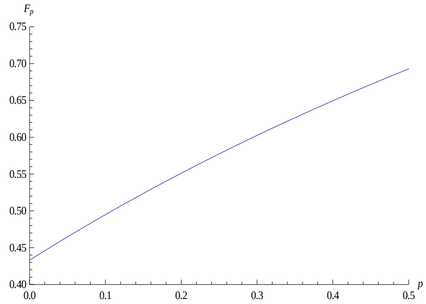

FIG. 1: The function F_p plotted against mixing parameter p

B. States generated as a result of Two-Qubit Interaction with a Squeezed Thermal Bath

Open quantum systems is the systematic study of the evolution of the system of interest, such as a qubit, under the influence of its environment, also called the bath or the reservoir. This results in decoherence and dissipation. Consider the Hamiltonian $H = H_S + H_R + H_{SR}$; where S stands for the system of interest, R for reservoir and SR for the system-reservoir interaction. Depending upon the system-reservoir interaction, open systems can be classified into two broad categories, viz., dissipative or QND (quantum non-demolition). In case of QND dephasing occurs without damping the system, i.e., where $[H_S, H_{SR}] = 0$ while decoherence along with dissipation occurs in dissipative systems, i.e., $[H_S, H_{SR}] \neq 0$. [15–17].

1. States generated as a result of Two-Qubit Open System Interacting with a Squeezed Thermal Bath via a Dissipative Interaction

Here we study the dynamics of the bound [Eq. (32)] for a two-qubit open system interacting with a squeezed thermal bath, modeled as a $3-D$ electromagnetic field (EMF), as well as its specialization to a vacuum bath, where the bath squeezing (r) and temperature (T) are set to zero, and undergoing a dissipative interaction [18]. The model Hamiltonian is

$$\begin{aligned} H &= H_S + H_R + H_{SR} \\ &= \sum_{n=1}^{2}\hbar\omega_n S_n^z + \sum_{\vec{k}_s}\hbar\omega_k(b^\dagger_{\vec{k}_s})b_{\vec{k}_s} + \frac{1}{2}) \\ &\quad - i\hbar\sum_{\vec{k}_s}\sum_{n=1}^{2}[\vec{\mu}_n.\vec{g}_{\vec{k}_s}(\vec{r}_n)(S_n^+ + S_n^-)b_{\vec{k}_s} - h.c.] \end{aligned} \quad (35)$$

Here $\vec{\mu}_n$ are the transition dipole moments, dependent on the different atomic positions $\vec{r}_n$ and S_n^+ $(= \frac{1}{2}|e_n\rangle\langle g_n|)$, and $S_n^-(= \frac{1}{2}|g_n\rangle\langle e_n|)$ are the dipole raising and lowering operators satisfying the usual commutation relations. $S_n^z(= \frac{1}{2}(|e_n\rangle\langle e_n| - |g_n\rangle\langle g_n|))$ is the energy operator of nth atom and $b^{\dagger}_{\vec{k}_s}$, $b_{\vec{k}_s}$ are the creation and anihilation operators of the field mode $\vec{k}_s$ with the wave vector $\vec{k}$ and polarization index $s = 1, 2$. A key feature of the model is that the system-reservoir (S-R) coupling constant $\vec{g}_{\vec{k}_s}(\vec{r}_n)$ is dependent on the position of the qubit r_n and is

$$\vec{g}_{\vec{k}_s}(\vec{r}_n) = (\frac{\omega_k}{2\epsilon\hbar V})^{\frac{1}{2}}\vec{e}_{\vec{k}_s}e^{i\vec{k}.r_n}, \tag{36}$$

where V is the normalization volume and $\vec{e}_{\vec{k}_s}$ is the unit polarization vector of the field. The position dependence of the coupling leads to interesting dynamical consequences and allows the entire dynamics to be classified into two categories, that is, the independent regime, where the interqubit distance is far enough for each qubit to locally interact with an independent bath or the collective regime, where the qubits are close enough for them to interact with the bath collectively. Asuming an initial system-reservoir separable state, with the system in a separable, and the bath in a squeezed thermal state, with time the qubits develop correlations between themselves via a channel setup by the bath. A master equation for the reduced dynamics of the two qubit system is obtained by tracing out the environment (bath), using the usual Born-Markov and rotating wave approximation (RWA). This can be then solved to obtain the dynamics of the reduced density matrix, whose details are presented in [18], for the general case of a squeezed thermal bath at finite temperature as well as for a vacuum reservoir.

Let the reduced two-qubit density matrix of the system be $\rho_f(t)$. Its spectral decomposition corresponding to its eigenvalues ($\lambda_i(t)$) is,

$$\rho_f(t) = \sum_i \lambda_i(t)\rho_i(t). \tag{37}$$

Here $\rho_i(t) = |\psi_i(t)\rangle\langle\psi_i(t)|$, $|\psi_i(t)\rangle$ being the eigenvectors corresponding to the eigenvalues $\lambda_i(t)$ ($\sum_i \lambda_i(t) = 1$). For a two qubit state the Eq. (32) becomes

$$2f - 1 \leq E_2^{(2,2)}(\rho_f(t)) \leq 1 \tag{38}$$

where, $f = \sum_i \lambda_i(t) f_2(\rho_i(t))$ and is the singlet fraction of the mixed state $\rho_f(t)$. Again we can easily say that for two-qubit state $E_2^{(2,2)}$ is nothing but concurrence C. It is clear from Eq. (38) that if $f > \frac{1}{2}$ then the states $\rho_f(t)$ will be useful for teleportation, which is a known fact. If we look at the Figs. (2), and (3), for the case of a vacuum bath ($T = 0, r = 0$), concurrence C (or $E_2^{2,2}$) is seen to decrease with time of evolution t, with a predominantly oscillatory behavior in the collective regime (marked by the inter qubit distance $r_{12} < 1$). The singlet fraction f also shows similar behavior. From these two figures, it is clear that when and where C and f become zero and $\frac{1}{2}$, respectively, the system comes out of the lower bound of Eq. (38).

For the case of a squeezed thermal bath, as the system evolves with time t, concurrence C and f exhibit damped behavior, as seen in Figs. (4) and (5). If we increase the inter-qubit distance r_{12}, then the concurrence C for the system suddenly falls to zero (i.e., sudden death of entanglement in the system). Thus, the system can be used as a resource for teleportation purpose in the range $0 \leq r_{12} < r_d$. Here we define a new term r_d, such that at $r_{12} = r_d$ concurrence C of the system becomes zero. Obviously this r_d will be different for different parameter (T, r) settings. The Figs. (5) depict the abrupt decrease of concurrence C and singlet fraction f as r_{12} increases and the system comes out of the lower bound of Eq. (38).

2. States generated as a result of Two-Qubit Open System Interacting with a Squeezed Thermal Bath via Quantum Nondemolition Interaction

Now we take up the Hamiltonian, describing a QND interaction of two qubits with the bath as

$$\begin{aligned} H &= H_S + H_R + H_{SR} \\ &= \sum_{n=1}^{2} \hbar\varepsilon_n J_z^n + \sum_k \hbar\omega_k b_k^{\dagger} b_k \\ &+ \sum_{n,k} \hbar J_z^n (g_k^n b_k^{\dagger} + g_k^{n*} b_k). \end{aligned} \tag{39}$$

Here H_S, H_R and H_{SR} stand for the Hamiltonians of the system, reservoir and system-reservoir interaction, respectively. $b_k^{\dagger}$, b_k denote the creation and annihilation operators for the reservoir oscillator of frequency ω_k, g_k^n stands for the coupling constant (again assumed to be position dependent) for the interaction of the oscillator field with the qubit system and are taken to be

$$g_k^n = g_k e^{-ik.r_n}, \tag{40}$$

where r_n is the qubit position. Since $[H_S, H_{SR}] = 0$, the Hamiltonian [Eq. 39)] is of QND type. In the parlance of quantum information theory, the noise generated is called the phase damping noise. The position dependence of the coupling constant once more allows for the dynamical classification into the independent and collective regimes. In order to obtain the reduced dynamics of the system , we trace over the reservoir variables, the details of which can be found in [19].

Now we study the behavior of concurrence C (actually $E_2^{2,2}$) and singlet fraction f as the two-qubit system evolves with time t both for collective and localized decoherence model. It can be noticed from Figs. (6), and (7) that the value of concurrence C is higher and lasts longer in the case of collective decoherence model than in the case of localized decoherence model. As expected, the the singlet fraction f

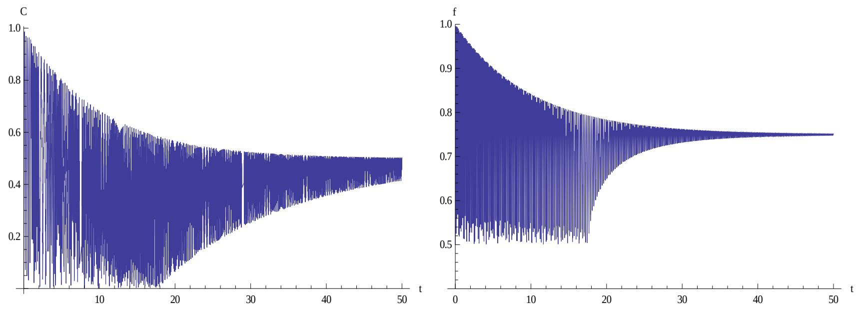

FIG. 2: Plot of (i) concurrence C (or $E_2^{(2,2)}$) and (ii) singlet fraction f with respect to the time of evolution t, respectively. Here we consider the case of a vacuum bath ($T = r = 0$) and the collective decoherence model ($r_{12} = 0.05$).

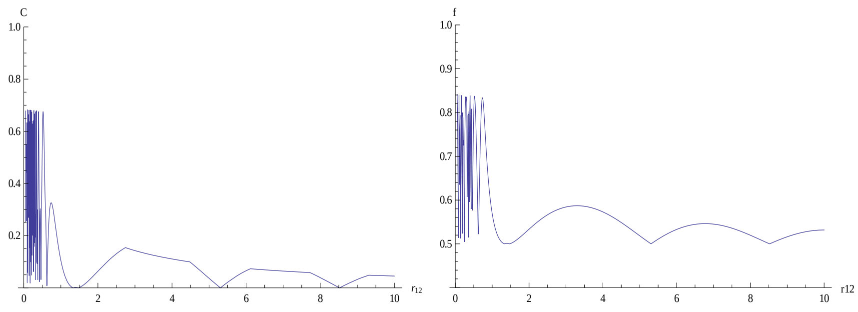

FIG. 3: Plot of (i) concurrence C (or $E_2^{(2,2)}$) and (ii) f with respect to the inter-qubit distance r_{12}, respectively. Here we consider the case of vacuum bath ($T = r = 0$) and system is at time $t = 10$.

shows similar kind of behavior with time t. When C becomes zero, f becomes equal to $\frac{1}{2}$, i.e., the system at this particular time t cannot be useful for teleportation, otherwise it is useful. Hence the system comes out of the lower bound of Eq. (38), when concurrence C vanishes.

V. CONCLUSION

We have made a study of entanglement of teleportation for arbitrary $d \otimes d$ dimensional states having Schmidt rank upto three. The connections are established by developing relations between entanglement measures and singlet fraction. This enables a classification of entanglement as a function of teleportation fidelity, the "Entanglement of Teleportation". These results are then extended to mixed two qudit states, which we illustrate on specific examples of a two qutrit mixed state of Schmidt rank two as well as two qubit states dynamically generated by interaction with an appropriate reservoir, for both pure dephasing as well as dissipative interactions.

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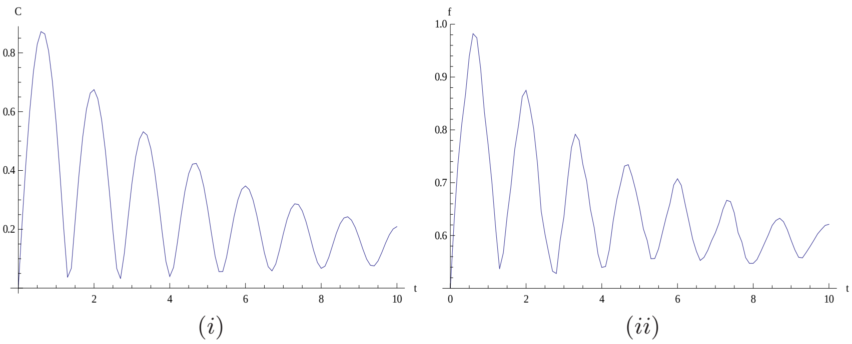


FIG. 4: Plot of (i) concurrence C (or $E_2^{(2,2)}$) and (ii) singlet fraction f with respect to the time of evolution t, respectively, for a squeezed thermal bath ($T = 1, r = 0.1$) in the collective regime ($r_{12} = 0.05$).

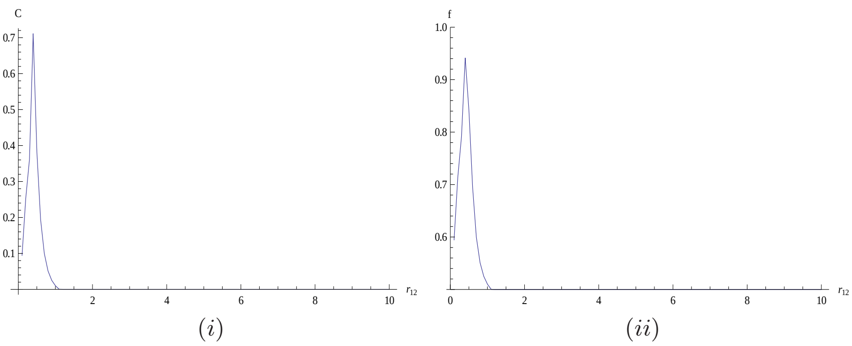


FIG. 5: Plot of (i) concurrence C (or $E_2^{(2,2)}$) and (ii) f with respect to the inter-qubit distance r_{12}, respectively, for a squeezed thermal bath ($T = 1, r = 0.1$) and time of evolution $t = 1$.

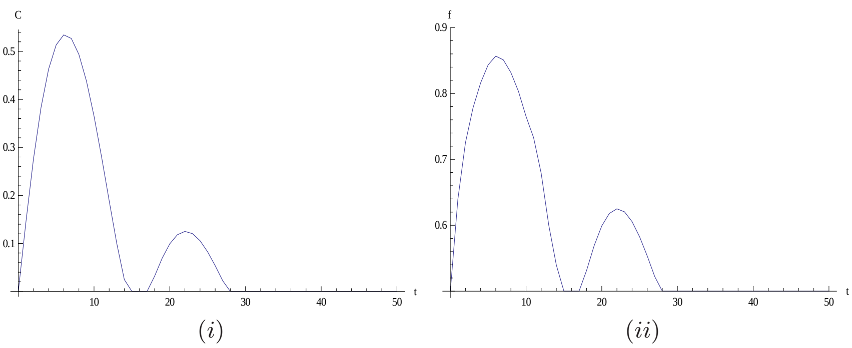
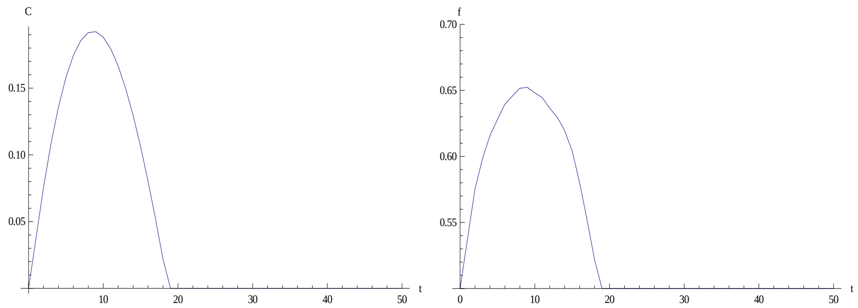


FIG. 6: Plot of (i) concurrence C (or $E_2^{(2,2)}$) and (ii) f as a function of the time of evolution t. Here we consider the case of QND interaction ($T = 5, r = 0.1$), in the collective decoherence regime ($r_{12} = 0.05$).



(i) (ii)

FIG. 7: Plot of (i) concurrence C (or $E_2^{(2,2)}$) and (ii) f as a function of the time of evolution t, for the case of QND interaction ($T = 5, r = 0.1$), in the independent decoherence regime ($r_{12} = 1.1$).

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