Alternative Classical Mechanics 4

Alejandro A. Torassa

(2014) Buenos Aires, Argentina
atorassa@gmail.com

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This paper presents an alternative classical mechanics which is invariant under transformations between reference frames and which can be applied in any reference frame without the necessity of introducing fictitious forces.

The Inertial Reference Frame

The inertial reference frame Ŝ is a reference frame fixed to a system of particles, whose origin coincides with the center of mass of the system of particles. This system of particles (referred from now on as the free-system) is always free of external and internal forces.

The inertial position $\hat{\mathbf{r}}_a$, the inertial velocity $\hat{\mathbf{v}}_a$ and the inertial acceleration $\hat{\mathbf{a}}_a$ of a particle A relative to the inertial reference frame Ŝ, are as follows:

$$\hat{\mathbf{r}}_a \doteq (\mathbf{r}_a)$$

$$\hat{\mathbf{v}}_a \doteq d(\mathbf{r}_a)/dt$$

$$\hat{\mathbf{a}}_a \doteq d^2(\mathbf{r}_a)/dt^2$$

where $\mathbf{r}_a$ is the position of particle A relative to the inertial reference frame Ŝ.

The New Dynamics

[1] A force is always caused by the interaction between two particles.

[2] The resultant force $\mathbf{F}_a$ acting on a particle A of mass m_a produces an inertial acceleration $\hat{\mathbf{a}}_a$ according to the following equation: $\hat{\mathbf{a}}_a = \mathbf{F}_a/m_a$

[3] In this paper, not all forces obey Newton's third law (in its strong form or in its weak form)

The Definitions

For a system of N particles, the following definitions are applicable:

Mass	$M \doteq \sum_i m_i$
Linear Momentum	$\hat{\mathbf{P}} \doteq \sum_i m_i \hat{\mathbf{v}}_i$
Angular Momentum	$\hat{\mathbf{L}} \doteq \sum_i m_i \hat{\mathbf{r}}_i \times \hat{\mathbf{v}}_i$
Work	$\hat{W} \doteq \sum_i \int_1^2 \mathbf{F}_i \cdot d\hat{\mathbf{r}}_i = \sum_i \Delta\, {}^1\!/_2\, m_i\, (\hat{\mathbf{v}}_i)^2$
Kinetic Energy	$\Delta \hat{K} \doteq \sum_i \Delta\, {}^1\!/_2\, m_i\, (\hat{\mathbf{v}}_i)^2$
Potential Energy	$\Delta \hat{U} \doteq \sum_i - \int_1^2 \mathbf{F}_i \cdot d\hat{\mathbf{r}}_i$
Lagrangian	$\hat{L} \doteq \hat{K} - \hat{U}$

The Principles of Conservation

The linear momentum $\hat{\mathbf{P}}$ of an isolated system of N particles remains constant if the internal forces obey Newton's third law in its weak form.

$$\hat{\mathbf{P}} = \text{constant} \qquad \left[\, d(\hat{\mathbf{P}})/dt = \sum_i m_i \hat{\mathbf{a}}_i = \sum_i \mathbf{F}_i = 0 \,\right]$$

The angular momentum $\hat{\mathbf{L}}$ of an isolated system of N particles remains constant if the internal forces obey Newton's third law in its strong form.

$$\hat{\mathbf{L}} = \text{constant} \qquad \left[\, d(\hat{\mathbf{L}})/dt = \sum_i m_i \hat{\mathbf{r}}_i \times \hat{\mathbf{a}}_i = \sum_i \hat{\mathbf{r}}_i \times \mathbf{F}_i = 0 \,\right]$$

The mechanical energy $\hat{E}$ of a system of N particles remains constant if the system is only subject to conservative forces.

$$\hat{E} \doteq \hat{K} + \hat{U} = \text{constant} \qquad \left[\, \Delta \hat{E} = \Delta \hat{K} + \Delta \hat{U} = 0 \,\right]$$

The Transformations

The inertial position $\hat{\mathbf{r}}_a$, the inertial velocity $\hat{\mathbf{v}}_a$ and the inertial acceleration $\hat{\mathbf{a}}_a$ of a particle A relative to a reference frame S, are given by:

$$\hat{\mathbf{r}}_a \;=\; \mathbf{r}_a - \mathbf{R}$$

$$\hat{\mathbf{v}}_a \;=\; \mathbf{v}_a - \boldsymbol{\omega} \times (\mathbf{r}_a - \mathbf{R}) - \mathbf{V}$$

$$\hat{\mathbf{a}}_a \;=\; \mathbf{a}_a - 2\,\boldsymbol{\omega} \times (\mathbf{v}_a - \mathbf{V}) + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\mathbf{r}_a - \mathbf{R})] - \boldsymbol{\alpha} \times (\mathbf{r}_a - \mathbf{R}) - \mathbf{A}$$

where $\mathbf{r}_a$, $\mathbf{v}_a$ and $\mathbf{a}_a$ are the position, the velocity and the acceleration of particle A relative to the reference frame S. $\mathbf{R}$, $\mathbf{V}$ and $\mathbf{A}$ are the position, the velocity and the acceleration of the center of mass of the free-system relative to the reference frame S. $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are the angular velocity and the angular acceleration of the free-system relative to the reference frame S.

The position $\mathbf{R}$, the velocity $\mathbf{V}$ and the acceleration $\mathbf{A}$ of the center of mass of the free-system relative to the reference frame S, and the angular velocity $\boldsymbol{\omega}$ and the angular acceleration $\boldsymbol{\alpha}$ of the free-system relative to the reference frame S, are as follows:

$$M \;\doteq\; \textstyle\sum_i^N m_i$$

$$\mathbf{R} \;\doteq\; M^{-1} \textstyle\sum_i^N m_i\, \mathbf{r}_i$$

$$\mathbf{V} \;\doteq\; M^{-1} \textstyle\sum_i^N m_i\, \mathbf{v}_i$$

$$\mathbf{A} \;\doteq\; M^{-1} \textstyle\sum_i^N m_i\, \mathbf{a}_i$$

$$\boldsymbol{\omega} \doteq\; \mathbf{I}^{-1} \cdot \mathbf{L}$$

$$\boldsymbol{\alpha} \doteq\; d(\boldsymbol{\omega})/dt$$

$$\mathbf{I} \;\doteq\; \textstyle\sum_i^N m_i\, [|\mathbf{r}_i - \mathbf{R}|^2\, \mathbf{1} - (\mathbf{r}_i - \mathbf{R}) \otimes (\mathbf{r}_i - \mathbf{R})]$$

$$\mathbf{L} \;\doteq\; \textstyle\sum_i^N m_i\, (\mathbf{r}_i - \mathbf{R}) \times (\mathbf{v}_i - \mathbf{V})$$

where M is the mass of the free-system, $\mathbf{I}$ is the inertia tensor of the free-system (relative to $\mathbf{R}$) and $\mathbf{L}$ is the angular momentum of the free-system relative to the reference frame S.

The Equation of Motion

From the third transformation it follows that the acceleration $\mathbf{a}_a$ of a particle A of mass m_a relative to a reference frame S, is given by:

$$\mathbf{a}_a \;=\; \mathbf{F}_a/m_a + 2\,\boldsymbol{\omega}\times(\mathbf{v}_a - \mathbf{V}) - \boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\mathbf{r}_a - \mathbf{R})] + \boldsymbol{\alpha}\times(\mathbf{r}_a - \mathbf{R}) + \mathbf{A}$$

where $\mathbf{F}_a$ is the resultant force acting on particle A.

Observations

The alternative classical mechanics of particles presented in this paper is invariant under transformations between reference frames and can be applied in any reference frame without the necessity of introducing fictitious forces.

This paper considers, on one hand, that not all forces obey Newton's third law (in its strong form or in its weak form) and, on the other hand, that all forces are invariant under transformations between reference frames $(\mathbf{F}'\!=\mathbf{F})$

Additionally, from the above equation it follows that a reference frame S is inertial when $\boldsymbol{\omega} = 0$ and $\mathbf{A} = 0$ but the reference frame S is non-inertial when $\boldsymbol{\omega} \neq 0$ or $\mathbf{A} \neq 0$.

Bibliography

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Appendix

For a system of N particles, the following definitions are also applicable:

Angular Momentum	$\hat{\mathbf{L}}' \doteq \sum_i m_i\,(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_{cm}) \times (\hat{\mathbf{v}}_i - \hat{\mathbf{v}}_{cm})$
Work	$\hat{W}' \doteq \sum_i \int_1^2 \mathbf{F}_i \cdot d(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_{cm}) = \sum_i \Delta\, {}^1\!/_2\, m_i\,(\hat{\mathbf{v}}_i - \hat{\mathbf{v}}_{cm})^2$
Kinetic Energy	$\Delta\,\hat{K}' \doteq \sum_i \Delta\, {}^1\!/_2\, m_i\,(\hat{\mathbf{v}}_i - \hat{\mathbf{v}}_{cm})^2$
Potential Energy	$\Delta\,\hat{U}' \doteq \sum_i - \int_1^2 \mathbf{F}_i \cdot d(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_{cm})$
Lagrangian	$\hat{L}' \doteq \hat{K}' - \hat{U}'$

where $\hat{\mathbf{r}}_{cm}$ and $\hat{\mathbf{v}}_{cm}$ are the inertial position and the inertial velocity of the center of mass of the system of particles. $\sum_i \int_1^2 m_i\,\hat{\mathbf{a}}_i \cdot d(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_{cm}) = \sum_i \int_1^2 m_i\,(\hat{\mathbf{a}}_i - \hat{\mathbf{a}}_{cm}) \cdot d(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_{cm}) = \sum_i \Delta\, {}^1\!/_2\, m_i\,(\hat{\mathbf{v}}_i - \hat{\mathbf{v}}_{cm})^2$

The angular momentum $\hat{\mathbf{L}}'$ of an isolated system of N particles remains constant if the internal forces obey Newton's third law in its strong form.

$$\hat{\mathbf{L}}' = \text{constant}$$

$$d(\hat{\mathbf{L}}')/dt = \sum_i m_i\,(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_{cm}) \times (\hat{\mathbf{a}}_i - \hat{\mathbf{a}}_{cm}) = \sum_i m_i\,(\mathbf{r}_i - \mathbf{r}_{cm}) \times \hat{\mathbf{a}}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i = 0$$

$$\hat{\mathbf{L}}' \doteq \sum_i m_i\,(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_{cm}) \times (\hat{\mathbf{v}}_i - \hat{\mathbf{v}}_{cm}) = \sum_i m_i\,(\mathbf{r}_i - \mathbf{r}_{cm}) \times [\mathbf{v}_i - \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{r}_{cm}) - \mathbf{v}_{cm}]$$

The mechanical energy $\hat{E}'$ of a system of N particles remains constant if the system is only subject to conservative forces.

$$\hat{E}' \doteq \hat{K}' + \hat{U}' = \text{constant}$$

$$\Delta\,\hat{E}' = \Delta\,\hat{K}' + \Delta\,\hat{U}' = 0$$

$$\Delta\,\hat{K}' \doteq \sum_i \Delta\, {}^1\!/_2\, m_i\,(\hat{\mathbf{v}}_i - \hat{\mathbf{v}}_{cm})^2 = \sum_i \Delta\, {}^1\!/_2\, m_i\,[\mathbf{v}_i - \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{r}_{cm}) - \mathbf{v}_{cm}]^2$$

$$\Delta\,\hat{U}' \doteq \sum_i - \int_1^2 \mathbf{F}_i \cdot d(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_{cm}) = \sum_i - \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{r}_{cm})$$

where $\mathbf{r}_{cm}$ and $\mathbf{v}_{cm}$ are the position and the velocity of the center of mass of the system of particles relative to a reference frame S, and $\boldsymbol{\omega}$ is the angular velocity of the free-system relative to the reference frame S. If the system of particles is isolated and if the internal forces obey Newton's third law in its weak form then: $\sum_i - \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{r}_{cm}) = \sum_i - \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i$