Bound state energies and wave functions of spherical quantum dots in presence of a confining potential model

Sameer M. Ikhdair[1,*]

[1]*Physics Department, Near East University, Nicosia, Mersin 10, Turkey*

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Abstract

We obtain the exact energy spectra and corresponding wave functions of the radial Schrödinger equation (RSE) for any (n,l) state in the presence of a combination of psudoharmonic, Coulomb and linear confining potential terms using an exact analytical iteration method. The interaction potential model under consideration is Cornell-modified plus harmonic (CMpH) type which is a correction form to the harmonic, Coulomb and linear confining potential terms. It is used to investigates the energy of electron in spherical quantum dot and the heavy quarkonia (QQ-onia).

Keywords: Schrödinger equation, confining potentials, spherical quantum dots, Cornell-modified potential, pseudoharmonic oscillator

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*E-mail: sikhdair@neu.edu.tr

I. INTRODUCTION

The problem of the inverse-power potential, $1/r^n$, has been used on the level of both classical and quantum mechanics. Some series of inverse power potentials are applicable to the interatomic interaction in molecular physics [1-3]. The interaction in one-electron atoms, muonic, hadronic and Rydberg atoms takes into account inverse-power potentials [4]. Indeed, it has also been used for the magnetic interaction between spin-1/2 particles with one or more deep wells [5]. The analytical exact solutions of this class of inverse-power potentials, $V(r) = Ar^{-4} + Br^{-3} + Cr^{-2} + Dr^{-1}$, $A > 0$, were presented by Barut *et al.* [6] and Özçelik and Şimşek [7] by making an available ansatz for the eigenfunctions. The Laurent series solutions of the Schrödinger equation for power and inverse-power potentials with two coupling constants $V(r) = Ar^2 + Br^{-4}$ and three coupling constants $V(r) = Ar^2 + Br^{-4} + Cr^{-6}$ are obtained [8,9].

The analytic exact iteration method (AEIM) which demands making a trial ansatz for the wave function [7] is general enough to be applicable to a large number of power and inverse-power potentials [10]. Recently, this method is applied to a class of power and inverse-power confining potentials of three coupling constants and containing harmonic oscillator, linear and Coulomb confining terms [11]. This kind of Cornell plus Harmonic (CpH) confining potential of the form $V(r) = ar^2 + br - cr^{-1}$ is mostly used to study individual spherical quantum dots in semiconductors [12] and heavy quarkonia (QQ-nia) [13,14]. So far, such potentials containing quadratic, linear and Coulomb terms have been studied [15,16].

The present work considers the the following confining interaction potential consisting of a sum of pseudoharmonic, linear and Coulombic potential terms:

$$V(r) = V_{\text{har}}(r) + V_{\text{Corn-mod}}(r) = ar^2 + br - \frac{c}{r} - \frac{d}{r^2},\ a > 0, \tag{1}$$

where a, b, c and d are arbitrary constant parameters to be determined later. The above potential includes the well-known funnel or Cornell potential, i.e., a Coulomb plus Linear static potential (CpH), $V_{\text{Corn}}(r) = br - c/r$ [13], and a term $-d/r^2$ is incorporated into the quarkonium potential for the sake of coherence [14]. We will refer to the potential model (1) as a Cornell-modified plus harmonic (CMpH) potential, since the functional form has been improved by the additional $-d/r^2$ piece; besides the contribution from the additional term also alters the value of b and c [14,17]. The authors of Refs. [14,18] did not consider the harmonic or power-law as the results are expected to be similar. The CMpH potential

is plotted in Figure 1 for the values of parameters: $a = 1\ eV.fm^{-2}$, $b = 0.217\ eV.fm^{-1}$, $c = 0.400\ eV.fm$ and $d = 0.010\ eV.fm^2$.

We will apply the AEIM used in [7,11] to obtain the exact energy eigenvalues and wave functions of the radial Schrödinger radial equation (RSE) for the CMpH potential for any arbitrary (n, l) state.

The paper is structured as follows: In Sect. 2, we obtain the exact energy eigenvalues and wave functions of the RSE in three-dimensions (3D) for the confining CMpH potential model by proposing asuitable form for the wave function. In Sect. 3, we apply our results to an electron in spherical quantum dot of InGaAs semiconductor. The relevant conclusions are given in Sect. 4.

II. EXACT SOLUTION OF RSE FOR THE CONFINING POTENTIAL MODEL

The three-dimensional $(3D)$ Schrödinger equation takes the form [19]

$$\left[-\frac{\hbar^2}{2m}\Delta + V(r)\right]\psi(r,\theta,\varphi) = E_{nl}\psi(r,\theta,\varphi), \tag{2}$$

with

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L^2(\theta,\varphi)}{\hbar^2 r^2},$$

where m is the isotropic effective mass and E_{nl} is the total energy of the particle. For any arbitrary state, the complete wave function, $\psi(r,\theta,\varphi)$, can be written as

$$\psi(r,\theta,\varphi) = \sum_{n,l} N_l \psi_{nl}(r) Y_{lm}(\theta,\varphi), \tag{3}$$

where spherical harmonic $Y_{lm}(\theta,\varphi)$ is the eigenfunction of $L^2(\theta,\varphi)$ satisfying

$$L^2(\theta,\varphi)Y_{lm}(\theta,\varphi) = l(l+1)\hbar^2 Y_{lm}(\theta,\varphi), \tag{4}$$

and the radial wave function $\psi_{nl}(r)$ is the solution of the equation

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}\right)\psi_{nl}(r) + \frac{2m}{\hbar^2}\left[E_{nl} - V(r)\right]\psi_{nl}(r) = 0, \tag{5}$$

where r stands for the relative radial coordinates. The radial wave function $\psi_{nl}(r)$ is well-behaved at the boundaries (the finiteness of the solution requires that $\psi_{nl}(0) = \psi_{nl}(r \to \infty) = 0$). Now, the transformation

$$\psi_{nl}(r) = \frac{1}{r}\phi_{nl}(r), \tag{6}$$

reduces Eq. (5) to the simple form

$$\phi''_{nl}(r)+\left[\varepsilon_{n,l}-a_1r^2-b_1r+\frac{c_1}{r}+\frac{d_1-l(l+1)}{r^2}\right]\phi_{nl}(r)=0, \tag{7}$$

where $\phi_{nl}(r)$ is the reduced radial wave function and

$$\varepsilon_{nl}=\frac{2m}{\hbar^2}E_{nl},\ a_1=\frac{2m}{\hbar^2}a,\ b_1=\frac{2m}{\hbar^2}b,\ c_1=\frac{2m}{\hbar^2}c,\ d_1=\frac{2m}{\hbar^2}d. \tag{8}$$

The analytic exact iteration method (AEIM) requires making the following ansatze for the wave function [9],

$$\phi_{nl}(r)=f_n(r)\exp\left[g_l(r)\right], \tag{9}$$

with

$$f_n(r)=\begin{cases}1, & n=0,\\ \prod\limits_{i=1}^{n}\left(r-\alpha_i^{(n)}\right), & n=1,2,\cdots,\end{cases} \tag{10a}$$

$$g_l(r)=-\frac{1}{2}\alpha r^2-\beta r+\delta\ln r,\ \alpha>0,\ \beta>0. \tag{10b}$$

It is clear that $f_n(r)$ are equivalent to the Laguerre polynomials [20]. Substituting Eq. (9) into Eq. (5) we obtain

$$\phi''_{nl}(r)=\left(g''_l(r)+g'^2_l(r)+\frac{f''_n(r)+2g'_l(r)f'_n(r)}{f_n(r)}\right)\phi_{nl}(r). \tag{11}$$

and comparing Eq. (11) and Eq. (7) yields

$$a_1r^2+b_1r-\frac{c_1}{r}+\frac{l(l+1)-d_1}{r^2}-\varepsilon_{nl}=g''_l(r)+g'^2_l(r)+\frac{f''_n(r)+2g'_l(r)f'_n(r)}{f_n(r)}. \tag{12}$$

First of all, for $n=0$, let us take $f_0(r)$ and $g_l(r)$ given in Eq. (10b) to solve Eq. (12),

$$a_1r^2+b_1r-\varepsilon_{0l}-\frac{c_1}{r}+\frac{l(l+1)-d_1}{r^2}=\alpha^2r^2+2\alpha\beta r-\alpha\left[1+2\left(\delta+0\right)\right]+\beta^2-\frac{2\beta(\delta+0)}{r}+\frac{\delta\left(\delta-1\right)}{r^2}. \tag{13}$$

By comparing the corresponding powers of r on both sides of Eq. (13) we find the following corresponding energy and the restrictions on the potential parameters,

$$\alpha=\sqrt{a_1}, \tag{14a}$$

$$\beta=\frac{b_1}{2\sqrt{a_1}},\ a_1>0, \tag{14b}$$

$$c_1=2\beta\left(\delta+0\right), \tag{14c}$$

$$\delta = \frac{1}{2}\left(1 \pm l'\right), \text{ where } l' = \sqrt{\left(2l+1\right)^2 - \frac{8m}{\hbar^2}d} \tag{14d}$$

$$\varepsilon_{0l} = \alpha\left[1 + 2\left(\delta + 0\right)\right] - \beta^2. \tag{14e}$$

Actually, to have well-behaved solutions of the radial wave function at boundaries, namely the origin and the infinity, we need to take δ from Eq. (14d) as

$$\delta = \frac{1}{2}\left(1 + l'\right). \tag{15}$$

Therefore, the lowest (ground) state energy from Eq. (14e) together with Eqs. (14a)-(14c), Eq. (15) and Eq. (8) is given as follows

$$E_{0l} = \sqrt{\frac{\hbar^2 a}{2m}}\left(2 + l'\right) - \frac{2mc^2}{\hbar^2\left(1+l'\right)^2}, \tag{16}$$

where the parameter c of potential (1) should satisfy the following restriction:

$$c = \frac{b}{2\sqrt{\frac{2ma}{\hbar^2}}}\left(1 + \sqrt{\left(2l+1\right)^2 - \frac{8m}{\hbar^2}d}\right). \tag{17}$$

Furthermore, the substitution of α, β and δ from Eqs. (14a), (14b) and (15), respectively, together with the parameters given in Eq. (8) into Eqs. (9) and (10), we finally obtain the following ground state wave function:

$$\psi_{0l}(r) = N_{0l} r^{(-1+l')/2} \exp\left(-\frac{1}{2}\sqrt{\frac{2ma}{\hbar^2}}r^2 - \frac{2mc}{\hbar^2\left(1+l'\right)}r\right), \tag{18}$$

with

$$N_{0l} = \frac{1}{\sqrt{\Gamma(l')D_{-l'}\left(\frac{4mc}{\hbar^2(1+l')}\sqrt{\frac{\hbar}{2\sqrt{2ma}}}\right)}}\left(2\sqrt{\frac{2ma}{\hbar^2}}\right)^{l'/4}\exp\left(-\frac{1}{2}\sqrt{\frac{2m}{\hbar^2 a}}\frac{mc^2}{\hbar^2\left(1+l'\right)^2}\right),$$

where $D_\nu(z)$ are the parabolic cylinder functions [21]. It should be noted that the above solutions are well-behaved at the boundaries, i.e., a regular solution near the origin could be $\phi_{nl}(r \to 0) \to r^{(1+l')/2}$ and asymptotically at infinity as $\phi_{nl}(r \to \infty) \to \exp\left(-\alpha r^2 - \beta r\right) \to 0$. When $b = 0$ $(c = 0)$, the problem turns to become the commoly known pseudoharmonic oscillator (p.h.o.) interaction ($a = m\omega^2/2$), and consequently $\alpha = m\omega$, $\beta = b/\omega$ and $c = (b\delta/m\omega)$ yielding $E_{0l} = \left(2 + l'\right)\frac{\hbar\omega}{2} - \frac{2mc^2}{\hbar^2(1+l')^2}$ and wave function $\psi_{0l}(r) = N_l r^{(-1+l')/2}\exp\left(-\frac{1}{2}\frac{m\omega}{\hbar}r^2 - \frac{2mc}{\hbar^2(1+l')}r\right)$, where

$$N_{0l} = \frac{1}{\sqrt{\Gamma(l')D_{-l'}\left(\frac{4mc}{\hbar^2(1+l')}\sqrt{\frac{\hbar}{2m\omega}}\right)}}\left(\frac{2m\omega}{\hbar}\right)^{l'/4}\exp\left(-\frac{mc^2}{4\hbar^3\omega\left(1+l'\right)^2}\right).$$

The formula (17) is a relationship between parameters of the potential a, b, c and d. Therefore, the solutions (16) and (18) are valid for the potential parameters satisfying the restriction (17). Moreover, the relation between the potential parameters (17) depends on the orbital quantum number l which means that the potential has to be different for different quantum numbers. In applying the AEIM, the obtained solution for any potential is always found to be subjected to certain restrictions on potential parameters as can be traced in other works (see, for example, [7-9,11]).

Secondly, for the first node ($n = 1$), using $f_1(r) = (r - \alpha_1^{(1)})$ and $g_l(r)$ from Eq. (10b) to solve Eq. (12),

$$a_1 r^2 + b_1 r - \varepsilon_{1l} - \frac{c_1}{r} + \frac{l(l+1) - d_1}{r^2} = \alpha^2 r^2 + 2\alpha\beta r$$

$$- \alpha\left[1 + 2\left(\delta + 1\right)\right] + \beta^2 - \frac{2\left[\beta\left(\delta + 1\right) + \alpha\alpha_1^{(1)}\right]}{r} + \frac{\delta\left(\delta - 1\right)}{r^2}. \tag{19}$$

The relations between the potential parameters and the coefficients α, β, δ and $\alpha_1^{(1)}$ are

$$\alpha = \sqrt{a_1},\ \beta = \frac{b_1}{2\sqrt{a_1}},\ \delta = \frac{1}{2}\left(1 + l'\right),\ \varepsilon_{1l} = \alpha\left[1 + 2\left(\delta + 1\right)\right] - \beta^2.$$

$$c_1 - 2\beta\left(\delta + 1\right) = 2\alpha\alpha_1^{(1)},\ \ \left(c_1 - 2\beta\delta\right)\alpha_1^{(1)} = 2\delta, \tag{20}$$

where c_1 and $\alpha_1^{(1)}$ are found from the constraint relations,

$$c = \frac{b}{2\sqrt{\frac{2ma}{\hbar^2}}}\left(2 + l'\right) + \sqrt{\frac{b^2}{\frac{8ma}{\hbar^2}} + \frac{\hbar^2}{m}\sqrt{\frac{\hbar^2 a}{2m}}\left(1 + l'\right)}, \tag{21a}$$

$$\alpha\alpha_1^{(1)2} + \beta\alpha_1^{(1)} - \delta = 0 \ \rightarrow\ \alpha_1^{(1)} = -\frac{b}{4a} + \sqrt{\frac{b^2}{16a^2} + \frac{\left(1 + l'\right)}{2\sqrt{\frac{2ma}{\hbar^2}}}}. \tag{21b}$$

The energy eigenvalue is

$$E_{1l} = \sqrt{\frac{\hbar^2 a}{2m}}\left(4 + l'\right) - \frac{b^2}{4a},$$

$$b = 2\sqrt{\frac{2ma}{\hbar^2}}\frac{\left(2 + l'\right)c}{\left(1 + l'\right)\left(3 + l'\right)}\left[1 + \sqrt{1 + \left(\frac{\hbar^2}{mc^2}\sqrt{\frac{\hbar^2 a}{2m}}\left(1 + l'\right) - 1\right)\frac{\left(1 + l'\right)\left(3 + l'\right)}{\left(2 + l'\right)^2}}\right], \tag{22}$$

and the wave function is

$$\psi_{1l}(r) = N_{1l}\left(r - \alpha_1^{(1)}\right) r^{(-1+l')/2} \exp\left(-\frac{1}{2}\sqrt{\frac{2ma}{\hbar^2}}r^2 - \sqrt{\frac{m}{2\hbar^2 a}}br\right), \tag{23}$$

with

$$N_{1l}=\frac{\left(2\sqrt{\frac{2ma}{\hbar^2}}\right)^{l'/4}\exp\left(-\frac{1}{16}\sqrt{\frac{2m}{a^3}}\hbar^3b^2\right)}{\sqrt{\left(2\sqrt{\frac{2ma}{\hbar^2}}\right)^{-1}\Gamma(l'+2)S_1+\alpha_1^{(1)2}\Gamma(l')S_2-2\left(2\sqrt{\frac{2ma}{\hbar^2}}\right)^{-1/2}\alpha_1^{(1)}\Gamma(l'+1)S_3}},$$

where

$$S_1=D_{-(l'+2)}\left(\sqrt{\frac{\hbar}{2a}\sqrt{\frac{2m}{a}}}b\hbar\right),\ S_2=D_{-l'}\left(\sqrt{\frac{\hbar}{2a}\sqrt{\frac{2m}{a}}}b\hbar\right),\ S_3=D_{-(l'+1)}\left(\sqrt{\frac{\hbar}{2a}\sqrt{\frac{2m}{a}}}b\hbar\right),$$

and $\alpha_1^{(1)}$ is given in Eq. (21b). If there is a p.h.o. interaction, the energy becomes

$$E_{1l}=(4+l')\,\frac{\hbar\omega}{2}-\frac{b^2}{2m\omega^2}, \tag{24}$$

and the wave function

$$\psi_{1l}(r)=N_{1l}\left(r-\alpha_1^{(1)}\right)r^{(-1+l')/2}\exp\left(-\frac{1}{2}m\omega r^2-\frac{b}{\omega}r\right), \tag{25}$$

with

$$N_{1l}=\frac{\left(\frac{2m\omega}{\hbar}\right)^{l'/4}\exp\left(-\frac{1}{4}\frac{\hbar^3b^2}{m\omega^3}\right)}{\sqrt{\frac{\hbar}{2m\omega}\Gamma(l'+2)S_1+\alpha_1^{(1)2}\Gamma(l')S_2-2\alpha_1^{(1)}\sqrt{\frac{\hbar}{2m\omega}}\Gamma(l'+1)S_3}},$$

$$S_1=D_{-(l'+2)}\left(\sqrt{\frac{2\hbar}{m\omega}}\frac{b\hbar}{\omega}\right),\ S_2=D_{-l'}\left(\sqrt{\frac{2\hbar}{m\omega}}\frac{b\hbar}{\omega}\right),\ S_3=D_{-(l'+1)}\left(\sqrt{\frac{2\hbar}{m\omega}}\frac{b\hbar}{\omega}\right),$$

where

$$b=2\frac{m\omega}{\hbar}\frac{(2+l')\,c}{(1+l')\,(3+l')}\left[1+\sqrt{1+\left(\frac{\hbar^3\omega}{2mc^2}\,(1+l')-1\right)\frac{(1+l')\,(3+l')}{(2+l')^2}}\right],$$

and $\alpha_1^{(1)}=\frac{(l'+1)}{2m\omega}$.

Following the analytic iteration procedures for the second node ($n=2$) with $f_2(r)=(r-\alpha_1^{(2)})(r-\alpha_2^{(2)})$ and $g_l(r)$ as defined in Eq. (10b), we obtain

$$a_1r^2+b_1r-\varepsilon_{2,l}-\frac{c_1}{r}+\frac{l(l+1)-d_1}{r^2}=\alpha^2r^2+2\alpha\beta r$$

$$-\alpha\left[1+2\left(\delta+2\right)\right]+\beta^2-\frac{2\left[\beta\left(\delta+2\right)+\alpha\sum_{i=1}^{2}\alpha_i^{(2)}\right]}{r}+\frac{\delta\left(\delta-1\right)}{r^2}, \tag{26}$$

The relations between the potential parameters and the coefficients α, β, δ, $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ are

$$\alpha=\sqrt{a_1},\ \beta=\frac{b_1}{2\sqrt{a_1}},\ \delta=\frac{1}{2}\left(1+l'\right),\ \varepsilon_{2,l}=\alpha\left[1+2\left(\delta+2\right)\right]-\beta^2.$$

$$c_1-2\beta\left(\delta+2\right)=2\alpha\sum_{i=1}^{2}\alpha_i^{(2)},\ \ \left(c_1-2\beta\delta\right)\sum_{i<j}^{2}\alpha_i^{(2)}\alpha_j^{(2)}=2\delta\sum_{i=1}^{2}\alpha_i^{(2)},$$

$$\left[c_1-2\beta\left(\delta+1\right)\right]\sum_{i=1}^{2}\alpha_i^{(2)}=4\alpha\sum_{i<j}^{2}\alpha_i^{(2)}\alpha_j^{(2)}+2\left(2\delta+1\right), \tag{27}$$

The coefficients $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ are found from the constraint relations,

$$\alpha\sum_{i=1}^{2}\alpha_i^{(2)2}+\beta\sum_{i=1}^{2}\alpha_i^{(2)}-\left(2\delta+1\right)=0, \tag{28a}$$

$$\delta\sum_{i=1}^{2}\alpha_i^{(2)2}-\left(\beta\sum_{i=1}^{2}\alpha_i^{(2)}+1\right)\sum_{j<k}^{2}\alpha_j^{(2)}\alpha_k^{(2)}-2\alpha\sum_{j<k}^{2}\alpha_j^{(2)2}\alpha_k^{(2)2}=0. \tag{28b}$$

Hence, the energy eigenvalue is

$$E_{2l}=\sqrt{\frac{\hbar^2a}{2m}}\left(6+l'\right)-\frac{b^2}{4a}, \tag{29}$$

and the associated wave function is

$$\psi_{2l}(r)=N_l\prod_{i=1}^{2}\left(r-\alpha_i^{(2)}\right)r^{(-1+l')/2}\exp\left(-\frac{1}{2}\sqrt{\frac{2ma}{\hbar^2}}r^2-\sqrt{\frac{m}{2\hbar^2a}}br\right), \tag{30}$$

where $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ should satisfy the restriction relations (28a) and (28b).

We apply the present method for the third node ($n=3$) by taking $f_3(r)=(r-\alpha_1^{(3)})(r-\alpha_2^{(3)})(r-\alpha_3^{(3)})$ and $g_l(r)$ as defined in Eq. (10b) to obtain

$$a_1r^2+b_1r-\varepsilon_{3,l}-\frac{c_1}{r}+\frac{l(l+1)-d_1}{r^2}=\alpha^2r^2+2\alpha\beta r$$

$$-\alpha\left[1+2\left(\delta+3\right)\right]+\beta^2-\frac{2\left[\beta\left(\delta+3\right)+\alpha\sum_{i=1}^{3}\alpha_i^{(3)}\right]}{r}+\frac{\delta\left(\delta-1\right)}{r^2}. \tag{31}$$

The relations between the potential parameters and the coefficients α, β, δ, $\alpha_1^{(3)}$, $\alpha_2^{(3)}$ and $\alpha_3^{(3)}$are

$$\alpha=\sqrt{a_1},\ \beta=\frac{b_1}{2\sqrt{a_1}},\ \delta=\frac{1}{2}\left(1+l'\right),\ \varepsilon_{3,l}=\alpha\left[1+2\left(\delta+3\right)\right]-\beta^2.$$

$$c_1 - 2\beta\left(\delta+3\right) = 2\alpha\sum_{i=1}^{3}\alpha_i^{(3)},\ \ \left(c_1 - 2\beta\delta\right)\sum_{i<j<k}^{3}\alpha_i^{(3)}\alpha_j^{(3)}\alpha_k^{(3)} = 2\delta\sum_{i<j}^{3}\alpha_i^{(3)}\alpha_j^{(3)},$$

$$\left[c_1 - 2\beta\left(\delta+2\right)\right]\sum_{i=1}^{3}\alpha_i^{(3)} = 4\alpha\sum_{i<j}^{3}\alpha_i^{(3)}\alpha_j^{(3)} + 3\left(2\delta+2\right). \tag{32}$$

The coefficients $\alpha_1^{(3)}$, $\alpha_2^{(3)}$ and $\alpha_3^{(3)}$ are found from the constraint relation,

$$\alpha\sum_{i=1}^{3}\alpha_i^{(3)2} + \beta\sum_{i=1}^{3}\alpha_i^{(3)} - 3\left(\delta+1\right) = 0, \tag{33}$$

The energy eigenvalue is

$$E_{3l} = \sqrt{\frac{\hbar^2 a}{2m}}\left(8+l'\right) - \frac{b^2}{4a}, \tag{34}$$

and the wave function is

$$\psi_{3l}(r) = N_l\prod_{i=1}^{n=3}\left(r-\alpha_i^{(n)}\right)r^{(-1+l')/2}\exp\left(-\frac{1}{2}\sqrt{\frac{2ma}{\hbar^2}}r^2 - \sqrt{\frac{m}{2\hbar^2 a}}br\right). \tag{35}$$

We can repeat this iteration procedures several times to write the exact energies of the CMpH potential for any n state as

$$E_{nl} = \sqrt{\frac{\hbar^2 a}{2m}}\left(2+2n+l'\right) - \frac{b^2}{4a}, \tag{36}$$

and the wave functions is

$$\psi_{nl}(r) = N_l\prod_{i=1}^{n}\left(r-\alpha_i^{(n)}\right)r^{(-1+l')/2}\exp\left(-\frac{1}{2}\sqrt{\frac{2ma}{\hbar^2}}r^2 - \sqrt{\frac{m}{2\hbar^2 a}}br\right). \tag{37}$$

The relations between the potential parameters and the coefficients α, β, δ, $\alpha_1^{(n)}$, $\alpha_2^{(n)}, \cdots,$ $\alpha_n^{(n)}$ are

$$\alpha = \sqrt{a_1},\ \beta = \frac{b_1}{2\sqrt{a_1}},\ \delta = \frac{1}{2}\left(1+l'\right),\ \varepsilon_{2,l} = \alpha\left[1+2\left(\delta+n\right)\right] - \beta^2,$$

$$c_1 - 2\beta\left(\delta+n\right) = 0,\ \left(n=0\right)$$

$$c_1 - 2\beta\left(\delta+n\right) = 2\alpha\sum_{i=1}^{n}\alpha_i^{(n)},\ n = 1,2,3,\cdots$$

$$\left[c_1 - 2\beta\left(\delta+n-1\right)\right]\sum_{i=1}^{n}\alpha_1^{(n)} = n\left(2\delta+n-1\right),\ \left(n=1\right)$$

$$\left[c_1 - 2\beta\left(\delta+n-1\right)\right]\sum_{i=1}^{n}\alpha_1^{(n)} = 4\alpha\sum_{i<j}^{n}\alpha_i^{(n)}\alpha_j^{(n)} + n\left(2\delta+n-1\right),\ n = 2,3,4,\cdots$$

$$[c_1 - 2\beta(\delta + n - 2)] \sum_{i<j}^{n} \alpha_i^{(n)} \alpha_j^{(n)} = (n-1)(2\delta + n - 2) \sum_{i=1}^{2} \alpha_i^{(2)}, \ \ (n = 2)$$

$$(\ c_1 - 2\beta\delta) \sum_{i<j<k}^{n} \alpha_i^{(n)} \alpha_j^{(n)} \alpha_k^{(n)} = 2\delta \sum_{i<j}^{n} \alpha_i^{(n)} \alpha_j^{(n)}, \ \ (n = 3),$$

$$[\ c_1 - 2\beta(\delta + n - 2)] \sum_{i<j}^{n} \alpha_i^{(n)} \alpha_j^{(n)} = (n-1)(2\delta + n - 2) \sum_{i<j}^{n} \alpha_i^{(n)} \alpha_j^{(n)}$$

$$+ 4\alpha \sum_{i<j<k}^{n} \alpha_i^{(n)} \alpha_j^{(n)} \alpha_k^{(n)}, \ n = 3, 4, 5, \cdots, \tag{38}$$

and so on.

III. RESULTS AND DISCUSSIONS

Now, we consider a special case of potential (1) and an application to our results. For example, when $b = 0$ then leads to $c = 0$, then we have the p.h.o potential, i.e., $V_{ph}(r) = \frac{1}{2}m\omega^2 r^2 - \frac{d}{r^2}$, hence, the energy difference between the ground state and the excited states is

$$\Delta E = E_{1l} - E_{0l} = (4 + l')\frac{\hbar\omega}{2} - (2 + l')\frac{\hbar\omega}{2} = \hbar\omega, \tag{39}$$

which can be used to calculate the values of the potential parameters for the desired system.

We now apply the present results to describe a realistic physical system called indium gallium arsenide (InGaAs) quantum dot, i.e., a piece of this material of a spherical form which is considered as a semiconductor composed of indium, gallium and arsenic [11]. It is used in high-power and high-frequency (say, $\omega \sim 10^{15}\ Hz$) electronics because of its superior electron velocity with respect to the more common semiconductors silicon and gallium arsenide. InGaAs bandgap also makes it the detector material of choice in optical fiber communication at 1300 and 1550 nm. The gallium indium arsenide (GaInAs) is an alternative name for InGaAs. In Fig. 2, we plot the ground state electron energy

$$E_{0l}(\omega) = \left(2 + \sqrt{(2l+1)^2 - \frac{8m}{\hbar^2}d}\right)\frac{\hbar\omega}{2} - \frac{2mc^2}{\hbar^2}\left(1 + \sqrt{(2l+1)^2 - \frac{8m}{\hbar^2}d}\right)^{-2}, \tag{40}$$

versus ω in the interval $2 \times 10^{14} \leq \omega \leq 10 \times 10^{14}\ Hz$ taking the value of $c = 0.001\ eV.fm$ and $d = 0\ eV.fm^2$ for the cases $l = 0$ and $l = 1$, respectively (harmonic, Coulomb and linear combination terms). In Fig. 3, we take instead the value of the parameter $d = 0.1\ eV.fm^2$

(pseudoharmonic, Coulomb and linear combination terms). The effective mass of electron in the InGaAs semiconductor has been chosen as $m = 0.05m_e$ and $\hbar = 6.5821 \times 10^{-16}\ eV.s$. It is seen from Fig. 2 and Fig. 3 how the increase in the value of ω leads to an increase in the energy of electron. The flexibility in the adjustment of the parameter d allows one to fit the spectrum of the desired model properly (cf. Fig. 2 and Fig. 3). The parameter d should satisfy the condition $d \leq (2l+1)^2 \hbar^2/(8m)$. In Fig. 4 we plot the ground state wave function $\psi_{0,l}(r)$ of the CpH potential for the cases $l = 0$ and $l = 1$, respectively, using the values of potential parameter $c = 0.001\ eV.nm$ for an electron with effective mass $m = 0.05$ m_e and frequency $\omega = 10 \times 10^{14}\ Hz$. Further, in Fig. 5 we plot the ground state wave function $\psi_{0,l}(r)$ of the CMpH potential for the cases $l = 0$ and $l = 1$, respectively, using the values of potential parameters $c = 0.001\ eV.nm$ and $d = 0.01\ eV.nm^2$ for an electron with effective mass $m = 0.05\ m_e$ and frequency $\omega = 10 \times 10^{14}\ Hz$. In Figs. 6 and 7, we show electron energy as a function of parameter c in the interval $6 \times 10^{-2} \leq c \leq 10 \times 10^{-2}\ eV.nm$ and $d = 0.01\ eV.nm^2$ for frequency $\omega = 8 \times 10^{14}\ Hz$ and effective mass $m = 0.05\ m_e$ for the cases $l = 0$ and $l = 1$, respectively. From Fig. 5, the increase in c leads in the decrease in the electron energy in the InGaAs semiconductor. In Fig. 8, we plot the first excited state electron energy

$$E_{1l}(\omega) = (4+l')\frac{\hbar\omega}{2} - \frac{2(2+l')^2}{(1+l')^2(3+l')^2}\frac{mc^2}{\hbar^2}$$
$$\times \left[1 + \sqrt{1 + \frac{(1+l')(3+l')}{(2+l')^2}\left(\frac{\hbar^3\omega}{2mc^2}(1+l') - 1\right)}\right]^2, \tag{41}$$

versus ω in the interval $2 \times 10^{14} \leq \omega \leq 10 \times 10^{14}\ Hz$ taking the value of $c = 0.001\ eV.fm$ and $d = 0\ eV.fm^2$ for the cases $l = 0$ and $l = 1$, respectively. In Fig. 9, we take the value of the parameter $d = 0.1\ eV.fm^2$. We remark that the strongly attractive singular part $-d/r^2$ is physically incorporated into the quarkonium Cornell potential as the first perturbative term for the sake of coherence to describe the heavy quarkonia (QQ-nia) (see, for example, [13,14] and the references therein). It also resemles the centrifugal barrier term $l(l+1)/r^2$ in the Schrödinger equation. This attractive term $-d/r^2$ together with the h.o. part ar^2 constitute the so-called p.h.o. when $b = 0$ ($\beta = 0$) in Eq. (14b) leading to $c = 0$ in Eq. (14c).

In Table 1, we calculate the lowest ($n = 0$) energy states ($l = 0, 1$ and 2) from Eq. (36) and from the numerical solution of the radial Schrödinger equation (7) using the values of

parameters given by Ref. [34] using the supersymmetry quantum mechanics (SUSYQM). It is clear that the calculated energy states in the present work are in good agreement with the results obtained numerically and SUSYQM [34]. The accracy of our numerical results is $0.0070\% - 0.0095\%$.

IV. CONCLUSIONS AND OUTLOOK

In this work, we explored the analytical exact solution for the energy eigenvalues and their associated wave functions of a particle in the field of Cornell-modified plus harmonic confining potential. We have used the analytical exact iteration method (AEIM) which required making a trial ansatz for the wave function. The general equation for the energy eigenvalues is given by Eq. (36) with some restrictions on the potential parameters. If one takes $b = 0$ then $c = 0$, hence, the potential (1) turns to the p.h.o. potential with energy eigenvalues:

$$E_{nl} = \sqrt{\frac{\hbar^2 a}{2m}} \left(2 + 2n + \sqrt{(2l+1)^2 - \frac{8m}{\hbar^2} d} \right) \tag{42}$$

and wave functions:

$$\psi_{nl}(r) = N_l \prod_{i=1}^{n} \left(r - \alpha_i^{(n)} \right) r^{(-1+l')/2} \exp\left(-\frac{1}{2} \sqrt{\frac{2ma}{\hbar^2}} r^2 \right). \tag{43}$$

The present results in Eqs. (42) and (43) coincide with Eqs. (15) and (16) of Ref. [22] obtained by the exact polynomial method, Eqs. (72) and (78) of Ref. [23] obtained by the Nikiforov-Uvarov method and Eqs. (30) and (31) of Ref. [24] obtained by the wave function ansatz method after setting $D_0/r_0^2 = a$, $D_0 r_0^2 = -d_0$ and $2D_0 = 0$. The model solved in the present work can be used in modeling the quarkonium [14] perturbed by the field of p.h.o. or electron confined in spherical quantum dots [11]. Finally, our solution to this confining potential is being considered important in many different fields of physics, such as atomic and molecular physics [25,26], particle physics [13,27,28], plasma physics and solid-state physics [29-33].

Acknowledgments

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[1] G.C. Maitland, M. Rigby, E.B. smith and W.A. Wakeham, Intermolecular forces (Oxford Univ. Press, Oxford, 1987).

[2] R.J. Le Roy and W. Lam, Chem. Phys. Lett. 71, 544 (1980); R.J. Le Roy and R.B. Bernstein, J. Chem. Phys. 52, 3869 (1970).

[3] S.M. Ikhdair and R. Sever, J. Molec. Struct. :Theochem 855, 13 (2008).

[4] B.H. Bransden and C.J. Joachain, Physics of atoms and molecules (Longman, London, 1983).

[5] A.O. Barut, J. Math. Phys. 21, 568 (1980).

[6] A.O. Barut, M. Berrondo and G. Garcia-Calderon, J. Math. Phys. 21, 1851 (1980).

[7] S. Özçelik and M. Şimşek, Phys. Lett. A, 152 (1991).

[8] M. Znojil, J. Math. Phys. 30, 23 (1989).

[9] M. Znojil, J. Math. Phys. 31, 108, 1955 (1990).

[10] S.M. Ikhdair and R. Sever, Int. J. Mod. Phys. C 18 (10), 1571 (2007).

[11] H. Hassanabadi and A.A. Rajabi, Phys. Lett. A 373, 679 (2008).

[12] F. Geerinckx, F.M. Peeters and J.T. Devreese, J. Appl. Phys. 68 (7), 3435 (1990).

[13] N.V. Maksimenko and S.M. Kuchin, Russ. Phys. J. 54, 57 (2011); E. Eichten, K. Gottfried, T. Kinoshita, K.D. Lane and T.M. Yan, Phys. Rev. D 21, 203 (1980).

[14] J.-L. Domenech-Garret, M.-A. Sanchis-Lozano, Phys. Lett. B 669, 52 (2008).

[15] B. Szafran and F.M. Peeters, Phys. Rev. B 72, 155316 (2005).

[16] G.A. Farias and F.M. Peeters, Solid State Commun. 100, 711 (1996).

[17] Y. Koma, M. Koma and H. Wittig, arXiv: 0711.2322 [hep-lat].

[18] S. Deoghuria and S. Chakrabarty, Z. Phys. C 53, 293 (1992).

[19] R.L. Liboff, Introductory Quantum Mechanics, 4th edn. (Pearson Education, Inc., San Francisco, CA, 2003).

[20] A. Erdelyi, W. Magnus, F. Oberhettinger and F. Tricomi, Beteman Manuscript Project, Higher transcendental functions (McGraw-Hill, New York, 1953).

[21] I.S. Gradshteyn and I.M. Ryzhik, Tables of Integrals, Series, and Products, 5th ed. (Academic, New York, 1994).

[22] S.M. Ikhdair and R. Sever, J. Molec. Struct. :Theochem 806, 155 (2007).

[23] S.M. Ikhdair and R. Sever, Cent. Eur. J. Phys. 6(3), 685 (2008).

[24] S.M. Ikhdair and R. Sever, Cent. Eur. J. Phys. 6(3), 697 (2008).

[25] E.J. Austin, Mol. Phys. 40, 893 (1980).

[26] J. Killingbeck, J. Phys. A. Math. Gen. 19, 2903 (1986); 13, L393 (1980).

[27] E.R. Vrscay, Int. J. Quantum Chem. 32, 613 (1987); Phys. Rev. A 31, 2054 (1985).

[28] V. Gupta and A. Khare, Phys. Lett. 70, 313 (1979); C. Quigg and J.L. Rosner, Phys. Rep. 56, 167 (1979).

[29] J.E. Avron, Ann. Phys. 131, 73 (1991).

[30] S. Skupsky, Phys. Rev. A 21, 1316 (1980).

[31] R. Cauble, M. blaha and J. Davis, Phys. Rev. A 29, 3280 (1984).

[32] S. Glasberg *et al.*, Phys. rev. B 59, R10425 (1999).

[33] G. Yusa, H. Shatrikman and I. Bar-Joseph, cond-mat/0103505.

[34] R.N. Chaudhury and M. Mondal, Phys. Rev. A 52, 1850 (1995).

TABLE I: Lowest ($n = 0$) energy spectra (for $\hbar = m = 1$).

a	b	c	d	l	Numerical	Present	SUSYQM [34]
$\frac{1}{32}$	1	4	0	0	-7.618	-7.625	-7.625
$\frac{1}{32}$	1	8	0	1	-7.368	-7.375	-7.375
$\frac{1}{32}$	1	12	0	2	-7.120	-7.125	-7.125

FIG. 1: A plot of the CMpH potential [see Eq. (1)] with the selected values of parameters: $a = 1$ $eV.fm^{-2}$, $b = 0.217$ $eV.fm^{-1}$, $c = 0.400$ $eV.fm$ and $d = 0.010$ $eV.fm^2$.

FIG. 3: The ground state electron energy in InGaAs semiconductor versus ω in the field of the CMpH potential with $c = 0.001$ $eV.nm$ and $d = 0.1$ $eV.nm^2$ for the cases $l = 0$ and $l = 1$, respectively.

FIG. 4: Behaviour of the ground state wave function $\psi_{n=0,l=0}(r)$ (dashed line) and $\psi_{n=0,l=1}(r)$ (continuous line) in the field of the CpH potential with the value of $c = 0.001$ $eV.nm$ for an electron with effective mass $m = 0.05$ m_e and frequency $\omega = 10 \times 10^{14}$ Hz in the InGaAs semiconductor.

FIG. 2: The ground state electron energy in InGaAs semiconductor versus ω in the field of CpH potential with $c = 0.001$ $eV.nm$ for cases $l = 0$ and $l = 1$, respectively.

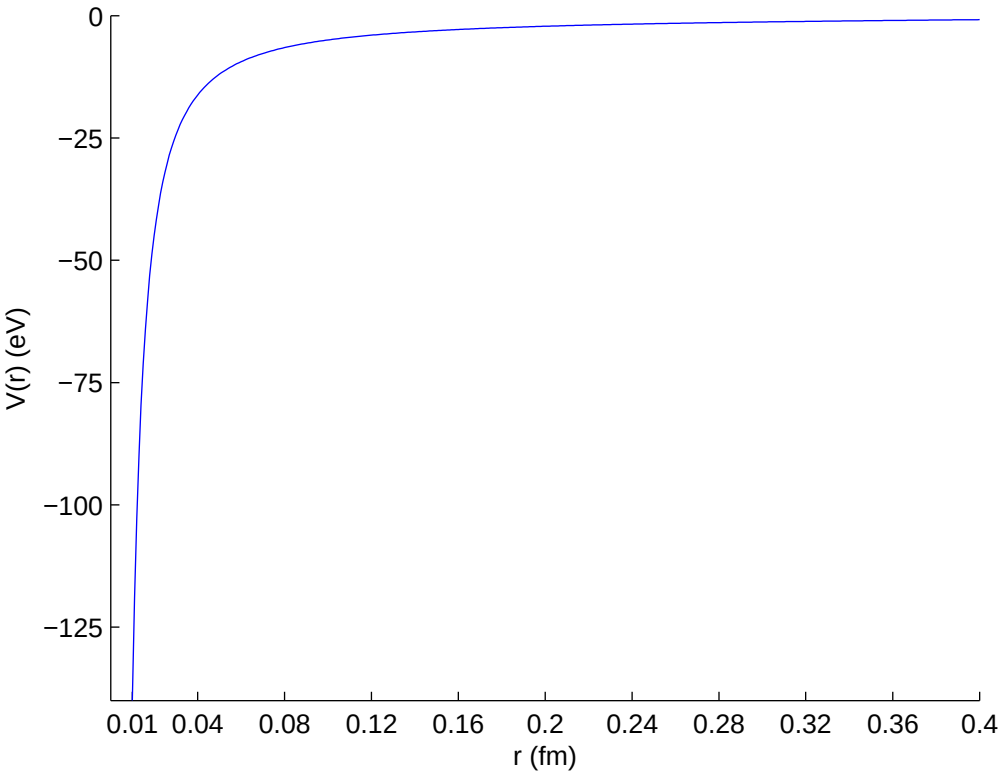
FIG. 5: Behaviour of the ground state wave function $\psi_{n=0,l=0}(r)$ (dashed line) and $\psi_{n=0,l=1}(r)$ (continuous line) of the CMpH potential with the values of $c = 0.001\ eV.nm$ and $d = 0.01\ eV.nm^2$ for an electron with an effective mass $m = 0.05\ m_e$ and frequency $\omega = 10 \times 10^{14}\ Hz$ in the InGaAs semiconductor.

FIG. 6: Ground state energy of electron versus c, for the case $l = 0$, $\omega = 8 \times 10^{14}\ Hz$ and $d = 0.01\ eV.nm^2$.

FIG. 7: Ground state energy of electron versus c, for the case $l = 1$, $\omega = 8 \times 10^{14}\ Hz$ and $d = 0.01\ eV.nm^2$.

FIG. 8: The first excited state electron energy in InGaAs semiconductor versus ω in the field of CpH potential with $c = 0.001\ eV.nm$ for cases $l = 0$ and $l = 1$, respectively.

FIG. 9: The first excited state electron energy in InGaAs semiconductor versus ω in the field of the CMpH potential with $c = 0.001\ eV.nm$ and $d = 0.1\ eV.nm^2$ for the cases $l = 0$ and $l = 1$, respectively.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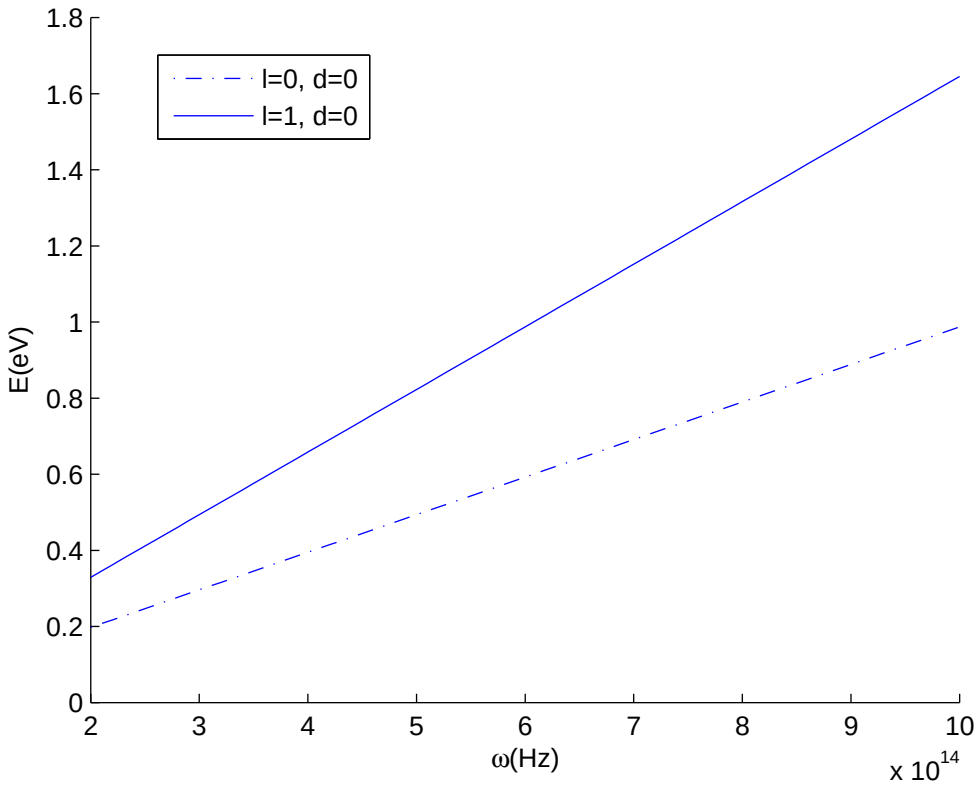
0.24
0.28
0.32
0.36
0.4
r (fm)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ω(Hz)
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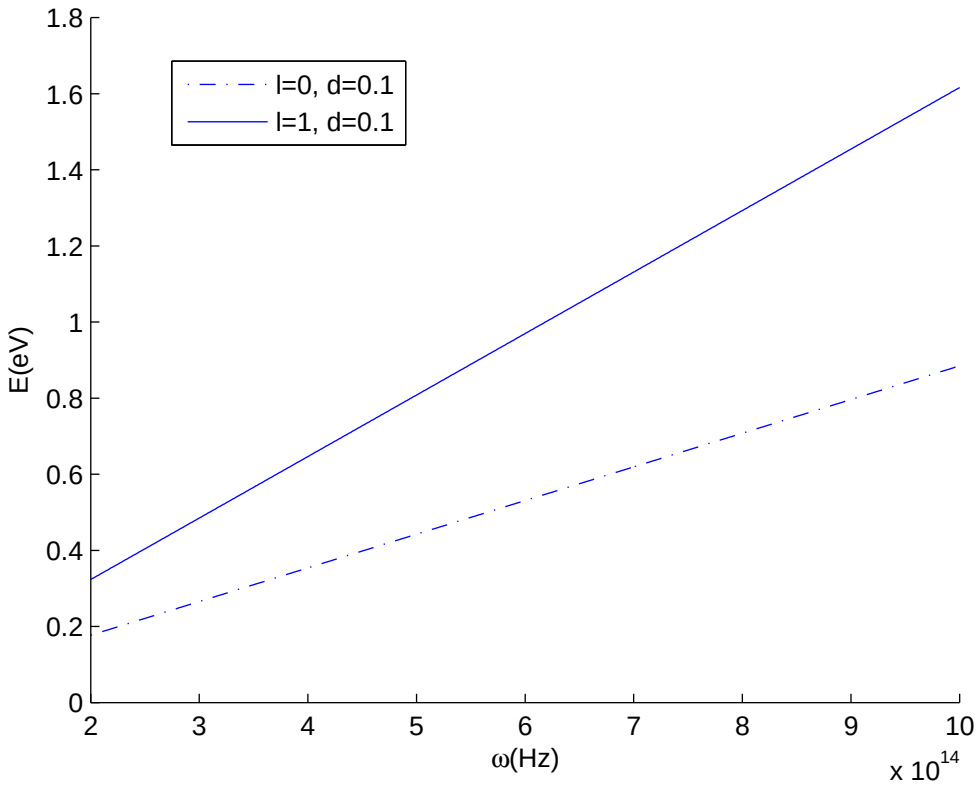
1.4
1.6
1.8
2
3
4
5
6
7
8
9
10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l=1, d=0.1
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ω(Hz)
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1.6
1.8
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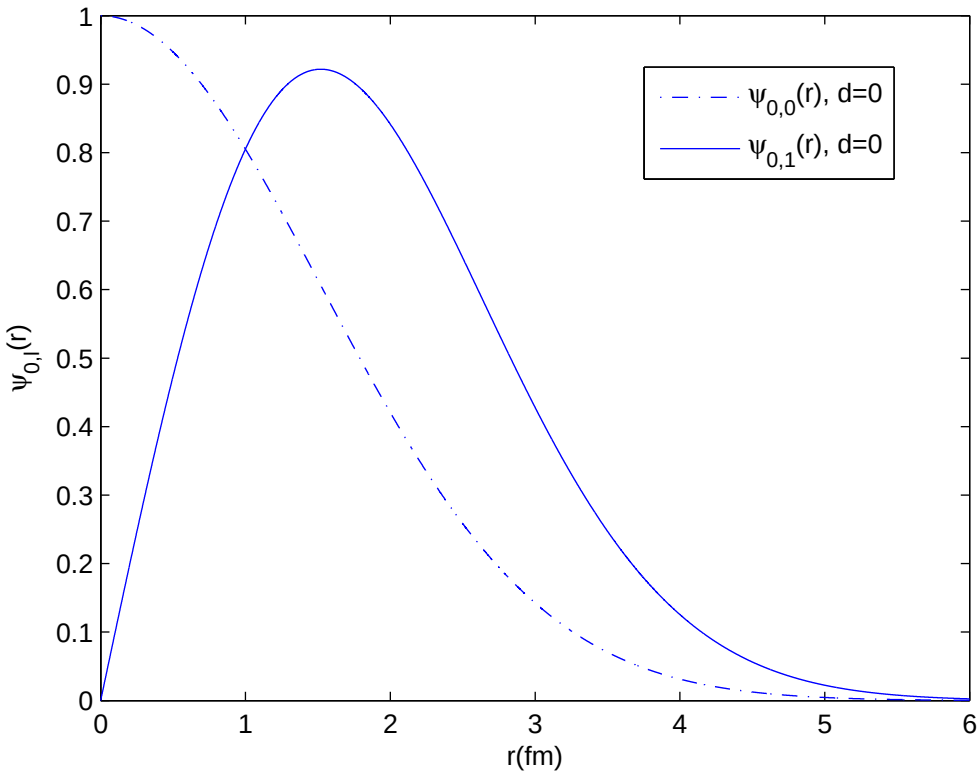
3
4
5
6
7
8
9
10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0.1
0
0
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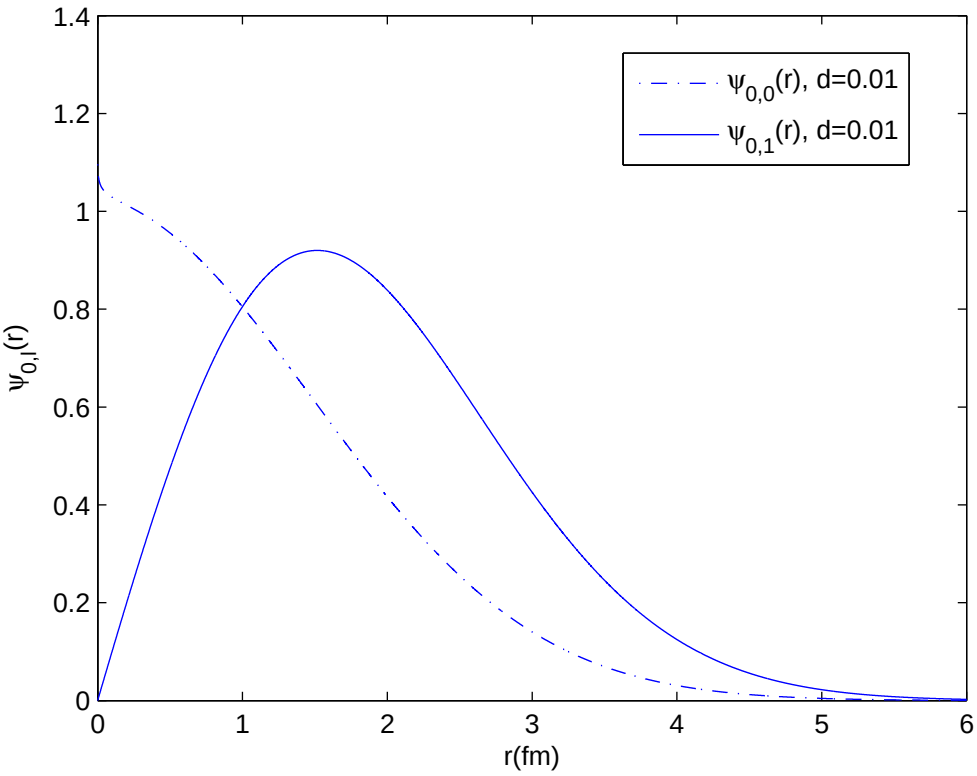
2
3
4
5
6
r(fm)
ψ0,l(r)
ψ0,0(r), d=0
ψ0,1(r), d=0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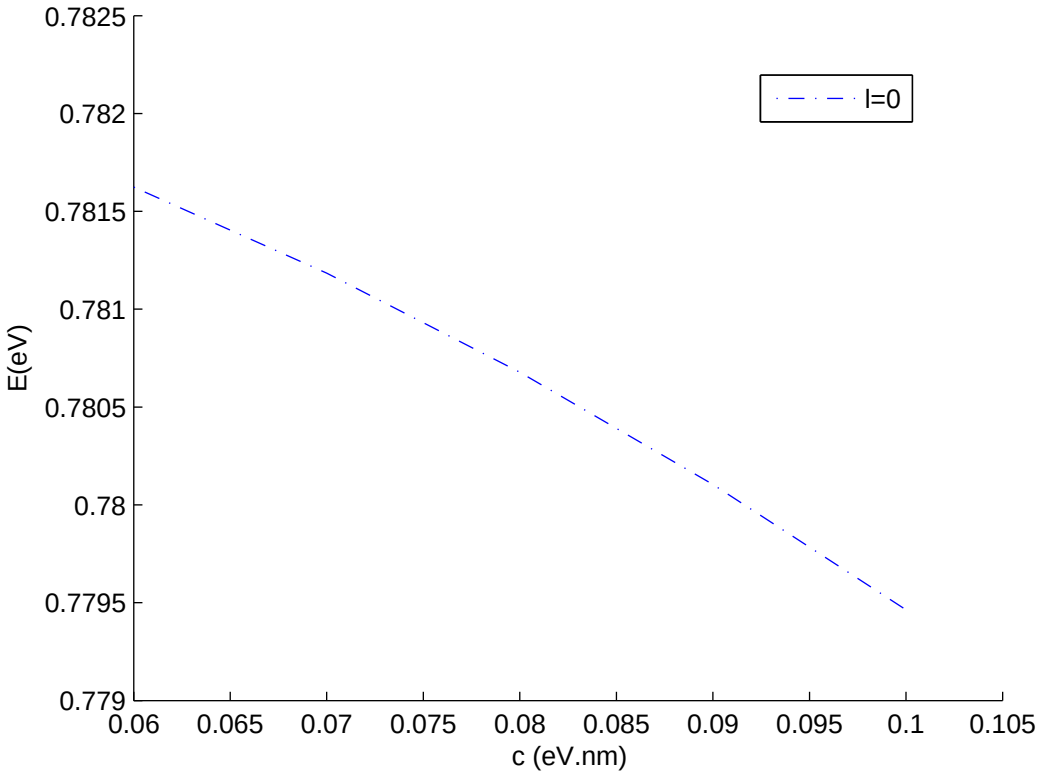
ψ0,l(r)
r(fm)
ψ0,0(r), d=0.01
ψ0,1(r), d=0.01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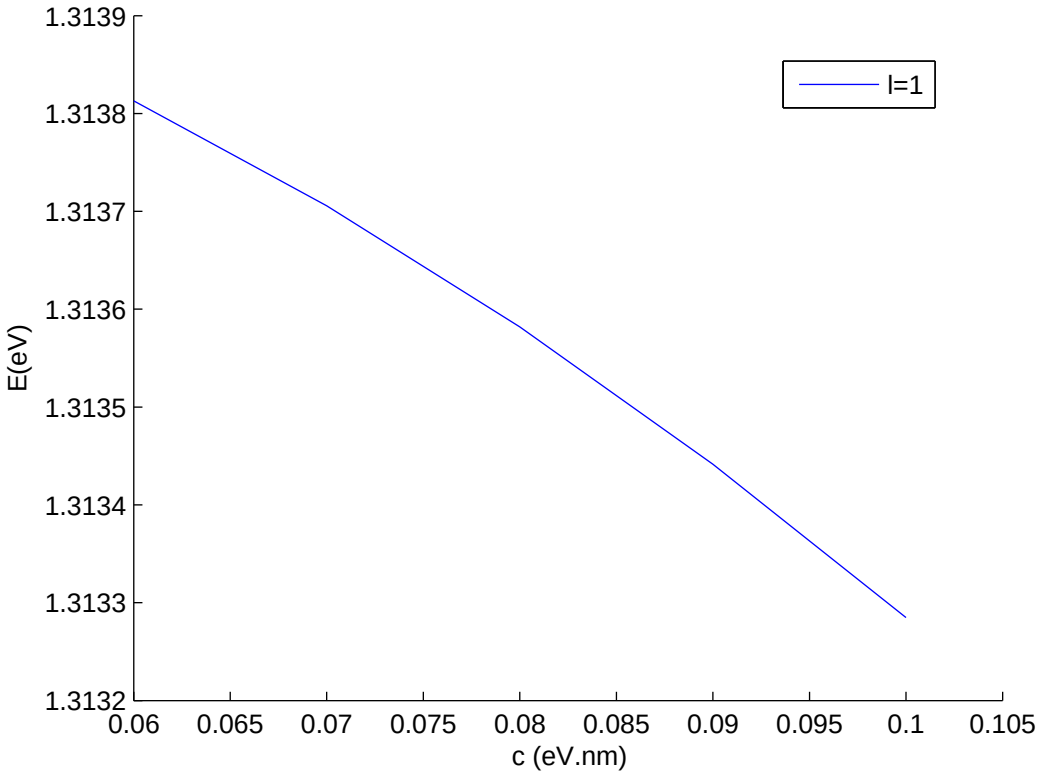
0.09
0.095
0.1
0.105
c (eV.nm)
l=0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0.095
0.1
0.105
c (eV.nm)
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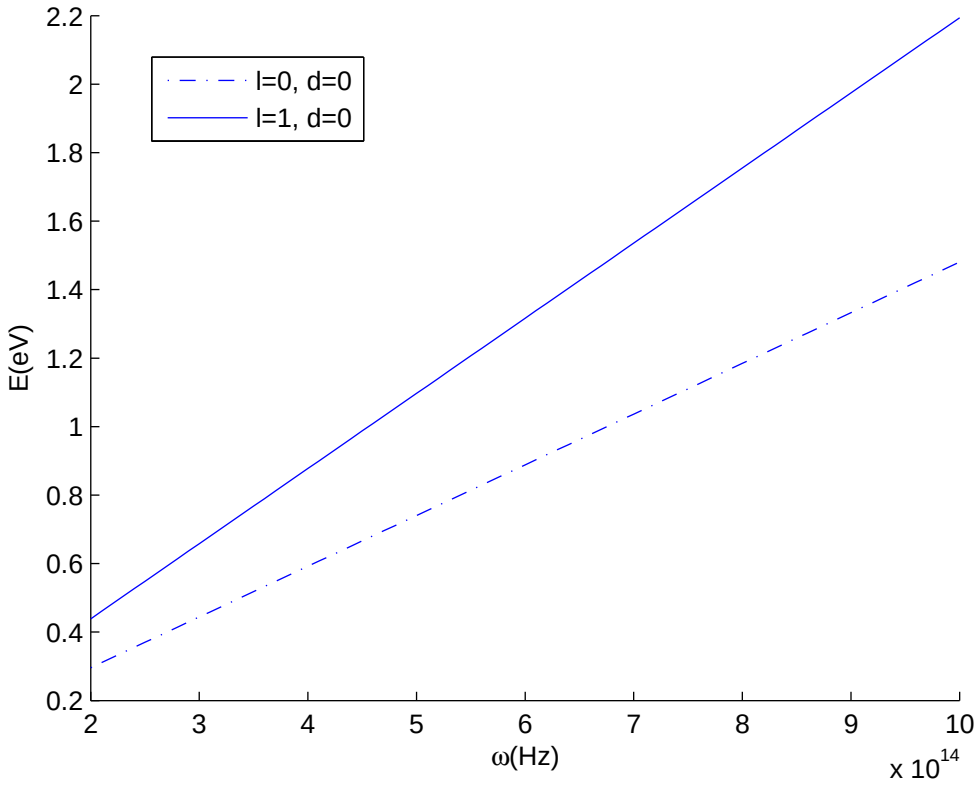
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ω(Hz)
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1.6
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0.8
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0.2
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3
4
5
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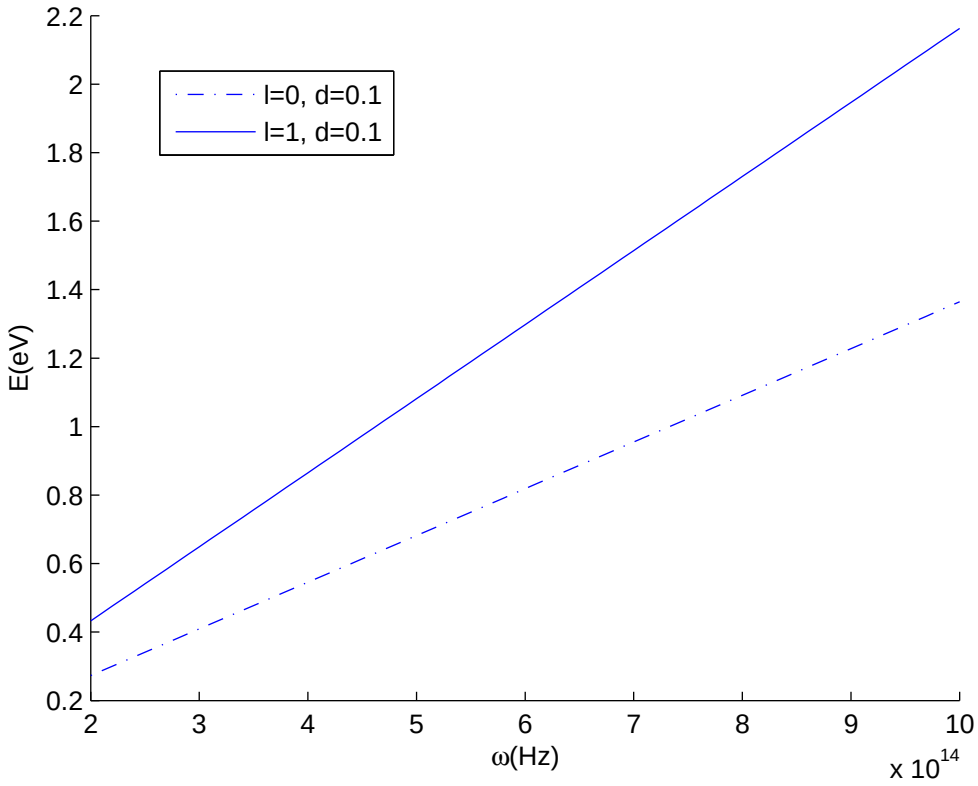
7
8
9
10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l=1, d=0.1
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1.4
1.2
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0.8
0.6
0.4
0.2
2
3
4
5
6
7
8
9
10
ω(Hz)
x 10^14