UTILITY MAXIMIZATION WITH ADDICTIVE CONSUMPTION HABIT FORMATION IN INCOMPLETE SEMIMARTINGALE MARKETS

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Abstract. This paper studies the problem of continuous time expected utility maximization of consumption together with addictive habit formation in general incomplete semimartingale financial markets. Introducing an auxiliary state processes and a modified dual space, we embed our original problem into an auxiliary time-separable utility maximization problem with the shadow random endowment. We establish existence and uniqueness of the optimal solution using convex duality on the product space $\mathbb{L}^0_+(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}})$ by defining the primal value function as depending on both the initial wealth and initial standard of living. We also provide market independent sufficient conditions on both stochastic discounting processes of the habit formation process and on the utility function for our original problem to be well posed and to modify the convex duality approach when the auxiliary dual process is not necessarily integrable.

1. Introduction

During the past decades, the assumption of time-additivity of von Neumann-Morgenstern preferences on consumption plan has been challenged due to its lack of consistency with many observed empirical evidences. For instance, the celebrated magnitude of the equity premium (*Mehra and Prescott* [24]) can not be reconciled with the time separable preference $\mathbb{E}[\int_0^T U(t,c_t)dt]$ when the instantaneous utility function U is only derived from the consumption rate. As an alternative modeling tool, habit formation has attracted a lot of attention and has been actively investigated in recent years. This new way to compare consumption stream is defined by $\mathbb{E}[\int_0^T U(t,c_t,Z_t)dt]$, where the accumulative process Z_t, called the standard of living, describes the consumption history impact. The habit forming preference is not only more prominently capable to explain many empirical facts (*Constantinides* [5]), but also can intuitively reflect consumers' rationality from the psychological perspective. In contrast to the traditional time additive utilities, the concept of habit formation characterizes the non-neglectable effect of past consumption patterns on current and future economic decisions. It specifies that the utility of consumption at time t depends also negatively on the history of consumption up to time t. In particular, an increase in consumption today increases current utility but depresses all future utilities through the induced increase in future standards of living.

The study of habit formation in modern economics dates back to *Hicks* [14], *Ryder and Heal* [13]. More recently, the utility maximization problem with consumption habits in continuous time

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has been studied by *Constantinides* [5] to explain the equity premium puzzle. In complete Itô processes markets, *Detemple and Zapatero* [10] and [11] employed martingale methods to the general nonlinear habit formation utility optimization problem $\mathbb{E}[\int_0^T U(t, c_t, Z_t)dt]$ and established some recursive stochastic differential equations for the consumption rate process c_t. They derived a closed form solution for the optimal consumption under preferences of the type $\mathbb{E}[\int_0^T U(t, c_t - Z_t)dt]$ when $U : [0,\infty) \times (0,\infty) \to \mathbb{R}$, i.e., the habit is assumed to be linear and addictive. Later, *Schroder and Skiadas* [28] made an insightful observation that to solve the optimal portfolio selection with utilities incorporating linear habit formation $\mathbb{E}[\int_0^T U(t, c_t - Z_t)dt]$ in the complete market is equivalent to solving the time additive utility maximization $\mathbb{E}[\int_0^T U(t, c'_t)dt]$ in the isomorphic complete market without habit formation. This isomorphism is given by the relationship that the optimal policies $c'^*_t = c^*_t - Z^*_t$ holds true. They also gave the construction of the isomorphic market based on the original market under some appropriate assumptions. *Detemple and Karatzas* [9] further considered the linear non-addictive habits $\mathbb{E}[\int_0^T U(t, c_t - Z_t)dt]$, where instead they define $U : [0,\infty) \times (-\infty,\infty) \to \mathbb{R}$. Their consumption c_t is required to be non-negative but is allowed to fall below the *"the standard of living"* index Z_t that aggregates past consumption. They provided a constructive proof for the existence of an optimal consumption, however, the market completeness is still a key assumption. *Egglezos and Karatzas* [12] exploited the interplay between stochastic partial differential equations and the utility maximization with linear addictive habit formation by taking advantage of the first order condition in the non-Markovian complete market, therefore obtaining some stochastic feedback formulae for the optimal portfolio and consumption policies. Although significant progress has been made, it is still an open problem to investigate the existence of optimal consumption policy under utility maximization for the habit-forming investor when the financial market ceases to be complete, which consists one of the primary motivations of our present work.

In the current article, we consider instead the general incomplete semimartingale framework, and allow all the driving factors of habit formation index to be unbounded optional processes in the given probabilistic setting. For the utility maximization problem with addictive consumption habits, we will routinely assume the lower bound constraint on consumption rates called *"the standard of living"*, which is to say the marginal utility from the difference of consumption c_t and the *"habit formation process"* Z_t is infinite at zero. In essence, this assumption requires the optimal consumption rate c_t shall never fall below the current standard of living Z_t. The main challenge in our problem lies in the fact that the intermediate utility function $U(t, c_t - Z_t)$ depends both on the current consumption rate and its past path integral due to the habit formation requirement. To overcome this intrinsic path-dependent complexity, we propose to define the closely related auxiliary processes $\tilde{c}_t = c_t - Z_t$ in the spirit of Market Isomorphism result by *Schroder and Skiadas* [28], and reduce our original path-dependent state constraint problem into a more natural time separable utility maximization problem on the auxiliary product space. Using the properly modified dual domain as well as treating the variables of the optimization problem both as the initial capital and initial habit, we are able to embed our time separable auxiliary optimization problem into an abstract utility maximization problem with the shadow random endowment, and build the conjugate duality following the idea appeared in *Hugonnier and Kramkov* [16] for the optimal investment problem with non-traded contingent claims.

More precisely speaking, the time non-additive nature of the primal optimization problem prevents us from applying Legendre transform to define the conjugate utility function merely on the current state of the well-known supermartingale deflator process $Y_t \in \mathcal{Y}$. As a consequence, the critical lower semi-continuity property with respect to Y_t, which is a key step to show the existence of the dual optimizer, may fail. On the contrary, our innovative transformation of path-dependent optimization problem into the auxiliary optimization problem enables us to derive the auxiliary dual problem in a very simple non path-dependent formulation. After which, we resort to work on the carefully chosen auxiliary dual domain as a set of processes Γ_t instead of Y_t. The negative side although exists that the extra exogenous random term, i.e. w_t and $\tilde{w}_t$ (see their definitions in (3.10)), appears simultaneously due to the special structure of habit formation process Z_t. Despite of this drawback, we can still successfully embed this problem into the auxiliary utility maximization problem by treating the extra random term as some shadow random endowment source in the abstract space. On the other hand, we are facing the issue to apply the classical convex duality results to the auxiliary processes $\tilde{c}_t$ and Γ_t due to the fact that the dual domain may not be a subset of $\mathbb{L}^1_+(\Omega\times[0,T],\mathcal{O},\bar{\mathbb{P}})$, because the auxiliary dual process Γ_t is defined via the unbounded stochastic discounting factors α_t and δ_t. To this end, we are interested in revising some classical proofs based on space $\mathbb{L}^1$, and provide the market independent sufficient conditions on habit formation discounting factors α_t and δ_t, see Assumption (3.3) and (3.4), to guarantee the well-posedness of the Primal optimization problem, as well as the Asymptotic Elasticity conditions on utility functions U both at $x \to 0$ and $x \to \infty$, i.e., $AE_0[U] < \infty$ and $AE_\infty[U] < 1$ (see Assumption (2.9) and (2.10)), for the validity of several key assertions of our main results to hold true. To the best of our knowledge, our paper is the first one which aims to solve the utility maximization problem with consumption habit formation in continuous time framework in the general incomplete semimartingale financial markets. However, we also refer the readers to the very recent work by *Muraviev*, [25] with additive habit formation in the discrete time incomplete markets with random endowment.

We should also stress the present paper is our first step to study the utility maximization problem with general nonlinear habit formation $\mathbb{E}[\int_0^T U(t,c_t,Z_t)dt]$ in incomplete semimartingale markets, in the sense that the investor's preference depends nonlinearly on both the current consumption rate process c_t and his past consumption path accumulative index Z_t. This generalized nonlinear habit formation problem includes the non-addictive linear habits considered earlier by *Detemple and Karatzas* [9]. We intend to provide similar convex duality conclusions as well as some specific characterizations of the optimal consumption structures in the future research. Another main motivation behind this work is the role it plays as a necessary step for the existence and uniqueness for equilibrium in continuous-time incomplete markets, together with internal/external habit formation or other time non-separable preferences, see *Detemple and Zapatero* [10] and *Bank and Riedel* [1], [2] for examples in complete markets.

The convex duality approach plays an important role in the treatment of general utility maximization problems in the framework of incomplete markets. To list a very small subset of the existing literature in optimal investment and consumption problems, we refer to *Karatzas, Lehoczky, Shreve, and Xu* [17], *Kramkov and Schachermayer* [21], [22], *Cvitanic, Schachermayer and Wang*

[6], *Karatzas and Žitković* [18], *Hugonnier and Kramkov* [16], *Žitković* [29], [30], *Kauppila* [19] and *Larsen and Žitković* [23].

The remainder of this paper is organized in the following way: Section 2 describes the financial market and introduces the definition of consumption habit formation as well as poses the utility maximization problem. In Section 3, we introduce some functional set-up on the product space, and define the auxiliary process domain, we embed our original problem into an auxiliary abstract utility maximization problem without habit formation, however, with the shadow random endowment. Section 4 is devoted to the definition of the dual problem for the auxiliary optimization problem and the formulation of the main theorems. Finally, Section 5 contains the proofs of our main results.

2. Market Model

2.1. **The Financial Market Model.** We consider a financial market with $d \in \mathbb{N}$ risky assets modeled by a d-dimensional semimartingale

$$S = (S_t^{(1)}, \dots, S_t^{(d)})_{t\in[0,T]} \tag{2.1}$$

on a given filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$, where the filtration $\mathbf{F}$ satisfies the usual conditions and the maturity time is given by T. To simplify our notation, we take $\mathcal{F} = \mathcal{F}_T$.

We make the standard assumption that there exists one riskless bond $S_t^{(0)} \equiv 1, \forall t \in [0,T]$, which amounts to consider $S_t^{(0)}$ as the numéraire asset.

The portfolio process $H = (H_t^{(1)}, \dots, H_t^{(d)})_{t\in[0,T]}$ is a predictable S-integrable process representing the number of shares of each risky asset held by the investor at time $t \in [0,T]$. The accumulated gains/losses process of the investor under his trading strategy H by time t is given by:

$$X_t^H = (H\cdot S)_t = \sum_{k=1}^{d} \int_0^t H_u^{(k)} dS_u^{(k)}, \quad t \in [0,T], \tag{2.2}$$

2.2. **Admissible Portfolios and Consumption Habits.** The portfolio process $(H_t)_{t\in[0,T]}$ is called **admissible** if the gains/losses process X_t^H is bounded below, which is to say, there exists a constant bound $a \in \mathbb{R}$ such that $X_t^H \geq a$, $a.s.$ for all $t \in [0,T]$.

Now, given the initial wealth $x > 0$, the agent will also choose an intermediate consumption plan during the whole investment process, and we denote the consumption rate process by c_t. The resulting self-financing **wealth process** $(W_t^{x,H,c})_{t\in[0,T]}$ is given by

$$W_t^{x,H,c} \triangleq x + (H\cdot S)_t - \int_0^t c_s ds, \quad t \in [0,T]. \tag{2.3}$$

Apart from the wealth process, the associated index process of consumption history as $Z_\cdot \equiv Z(\cdot; c)$ is defined in the following way:

$$dZ_t = (\delta_t c_t - \alpha_t Z_t)dt,$$
$$Z_0 = z,$$

where the stochastic discounting factors α_t and δ_t are assumed to be nonnegative optional processes and the given real number $z \geq 0$ is called *"initial habit"*.

Equivalently, we can write it as:

$$Z_t = ze^{-\int_0^t \alpha_v dv} + \int_0^t \delta_s e^{-\int_s^t \alpha_v dv} c_s ds, \tag{2.4}$$

which is called *"the standard of living"* process and represents the *"Habit Formation"* of the investor, an index as exponentially weighted average of agent's past consumption integral. Here, these stochastic discounting factors α_t and δ_t measure, respectively, the persistence of the initial habits level and the intensity of consumption history.

Throughout this paper, we make the assumption that the consumption habit is addictive, i.e. $c_t \geq Z_t,\ \ \forall t \in [0,T]$, which is to say, the investor's current consumption rate shall never fall below his *"the standard of living"* process. A consumption process $(c_t)_{t\in[0,T]}$ is defined to be (x,z)-**financeable** if there exists an admissible portfolio process $(H_t)_{t\in[0,T]}$ such that $W_t^{x,H,c} \geq 0,\ \forall t \in [0,T]$, a.s. and the addictive habit formation constraint $c_t \geq Z_t,\ \forall t \in [0,T]$ a.s. holds. The class of all (x,z)-financeable consumption rate processes will be denoted by $\mathcal{A}(x,z)$, for $x>0,\ z\geq 0$.

2.3. **Absence of Arbitrage.** A probability measure $\mathbb{Q}$ is called an **equivalent local martingale measure** if it is equivalent to $\mathbb{P}$ and if X_t^H is a local martingale under $\mathbb{Q}$. We denote by $\mathcal{M}$ the family of equivalent local martingale measures and in order to rule out the arbitrage opportunities in the market, we assume that

$$\mathcal{M} \neq \emptyset. \tag{2.5}$$

We refer the readers to *Delbaen and Schachermayer* [7] and [8] for a detailed discussion on the topic of arbitrage.

Define the RCLL process $Y^{\mathbb{Q}}$ by

$$Y_t^{\mathbb{Q}} = \mathbb{E}\Big[\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{F}_t\Big]$$

for the $\mathbb{Q} \in \mathcal{M}$, then $Y^{\mathbb{Q}}$ is called a equivalent local martingale measure density and we shall always identify the equivalent local martingale measure $\mathbb{Q}$ with its density process $Y^{\mathbb{Q}}$, and hence denote $\mathcal{M}$ also as the set of all equivalent local martingale density processes.

The celebrated Optional Decomposition Theorem, see *Kramkov* [20], enables us to characterize the (x,z)-financeble consumption process in terms of linear inequalities with respect to $Y_t^{\mathbb{Q}} \in \mathcal{M}$, called **Budget Constraint**, and this serves as an important ingredient in the treatment of our utility maximization problem via convex duality approach.

Proposition 2.1. *The process $(c_t)_{t\in[0,T]}$ is (x,z)- financeable if and only if $c_t \geq Z_t,\ \forall t \in [0,T]$ and*

$$\mathbb{E}\Big[\int_0^T c_t Y_t^{\mathbb{Q}} dt\Big] \ \leq\ x, \quad \forall Y_t^{\mathbb{Q}} \in \mathcal{M}. \tag{2.6}$$

2.4. **The Utility Function.** The individual investor's preference is represented by a utility function $U : [0,T] \times (0,\infty) \to \mathbb{R}$, such that, for every $x > 0$, $U(\cdot, x)$ is continuous on $[0,T]$, and for every $t \in [0,T]$, the function $U(t,\cdot)$ is strictly concave, strictly increasing, continuously differentiable and satisfies the Inada conditions:

$$U'(t,0) \triangleq \lim_{x\to 0} U'(t,x) = \infty, \qquad U'(t,\infty) \triangleq \lim_{x\to\infty} U'(t,x) = 0. \tag{2.7}$$

where $U'(t,x) \triangleq \frac{\partial}{\partial x} U(t,x)$.

According to these assumptions, the inverse $I(t,\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ of the function $U'(t,\cdot)$ exists for every $t \in [0,T]$, and is continuous and strictly decreasing with:

$$I(t,0) \triangleq \lim_{x\to 0} I(t,x) = \infty, \qquad I(t,\infty) \triangleq \lim_{x\to\infty} I(t,x) = 0. \tag{2.8}$$

The convex conjugate of the agents' utility function, also known as the Legendre-Fenchel transform, is defined as follows:

$$V(t,y) \triangleq \sup_{x>0}\{U(t,x) - xy\}, \qquad y > 0.$$

Under the Inada conditions (2.7), the conjugate of $V(t,\cdot)$ is a continuously differentiable, strictly decreasing and strictly convex function satisfying $V'(t,0) = -\infty$, $V'(t,\infty) = 0$ and $V(t,0) = U(t,\infty)$, $V(t,\infty) = U(t,0)$, see, for example, *Karatzas, Lehoczky, Shreve, and Xu* [17] for reference.

Follow *Kramkov and Schachermayer* [21], see also *Karatzas and Žitković* [18], we shall make additional assumptions on U for future purposes:

Assumption 2.1.
Utility functions U satisfies the Reasonable Asymptotic Elasticity condition that

$$AE_\infty[U] = \limsup_{x\to\infty}\Big(\sup_{t\in[0,T]} \frac{x\,U'(t,x)}{U(t,x)}\Big) < 1, \tag{2.9}$$

and

$$AE_0[U] = \limsup_{x\to 0}\Big(\sup_{t\in[0,T]} \frac{x\,U'(t,x)}{|U(t,x)|}\Big) < \infty. \tag{2.10}$$

Moreover, in order to get some inequalities uniformly in time t, we shall assume

$$\lim_{x\to\infty}\Big(\inf_{t\in[0,T]} U(t,x)\Big) > 0, \tag{2.11}$$

and

$$\lim_{x\to 0}\Big(\sup_{t\in[0,T]} U(t,x)\Big) < 0. \tag{2.12}$$

Remark 2.1. Many well known Utility functions satisfy Reasonable Asymptotic Elasticity Assumptions (2.9) and (2.10), for example, the discounted log utility function $U(t,x) = e^{-\beta t}\log(x)$ or discounted power utility function $U(t,x) = e^{-\beta t}\frac{x^p}{p}$ ($p < 1$ and $p \neq 0$), for a constant $\beta > 0$.

Remark 2.2. The utility function $U(t,x)$ satisfies Reasonable Asymptotic Elasticity Assumptions (2.9) and (2.10) if and only if its affine transform $a+bU(t,x)$ satisfies Reasonable Asymptotic Elasticity Assumptions (2.9) and (2.10) for arbitrary constants $a,b>0$. Hence, the adjoint Assumption (2.11) and Assumption (2.12) are not restrictive.

The next technical result gives the equivalent characterization of the Reasonable Asymptotic Elasticity condition $AE_\infty[U]$, which follows the similar proof of Lemma 6.3 of *Kramkov and Schachermayer* [21], see also Proposition 3.7 of *Karatzas and Žitković* [18].

Lemma 2.1. *Let* $U(t,x)$ *be a utility function satisfying* (2.9) *and* (2.11). *In each of the subsequent assertions, the infimum of* $\gamma>0$ *for which these assertions hold true equals the Reasonable Asymptotic Elasticity* $AE_\infty[U]$.

(i) *There is* $x_0>0$ *for all* $t\in[0,T]$ *s.t.*

$$U(t,\lambda x)<\lambda^\gamma U(t,x)\quad \textit{for } \lambda>1, x\geq x_0.$$

(ii) *There is* $x_0>0$ *for all* $t\in[0,T]$ *s.t.*

$$U'(t,x)<\gamma\frac{U(t,x)}{x}\quad \textit{for } x\geq x_0.$$

(iii) *There is* $y_0>0$ *for all* $t\in[0,T]$ *s.t.*

$$V(t,\mu y)<\mu^{-\frac{\gamma}{1-\gamma}}V(t,y)\quad \textit{for } 0<\mu<1, 0<y\leq y_0.$$

(iv) *There is* $y_0>0$ *for all* $t\in[0,T]$ *s.t.*

$$-V'(t,y)<\Big(\frac{\gamma}{1-\gamma}\Big)\frac{V(t,y)}{y}\quad \textit{for } 0<y\leq y_0.$$

Corollary 2.1. *Under Assumptions* (2.10) *and* (2.12), *we have* $AE_0[U]<\infty$ *if and only if* $AE_\infty[V]<1$, *where we define*

$$AE_\infty[V]=\limsup_{y\to\infty}\Big(\sup_{t\in[0,T]}\frac{yV'(t,y)}{V(t,y)}\Big)<1,$$

and hence similarly, we have each of the following assertions, the infimum of $\gamma>0$ *for which these assertions hold true equals the Reasonable Asymptotic Elasticity* $AE_\infty[V]$.

(i) *There is* $y_0>0$ *for all* $t\in[0,T]$ *s.t.*

$$V(t,\lambda y)>\lambda^\gamma V(t,y)\quad \textit{for } \lambda>1, y\geq y_0.$$

(ii) *There is* $y_0>0$ *for all* $t\in[0,T]$ *s.t.*

$$V'(t,y)>\gamma\frac{V(t,y)}{y}\quad \textit{for } y\geq y_0.$$

(iii) *There is* $x_0>0$ *for all* $t\in[0,T]$ *s.t.*

$$U(t,\mu x)>\mu^{-\frac{\gamma}{1-\gamma}}U(t,x)\quad \textit{for } 0<\mu<1, 0<x\leq x_0.$$

(iv) *There is* $x_0>0$ *for all* $t\in[0,T]$ *s.t.*

$$-U'(t,x)>\Big(\frac{\gamma}{1-\gamma}\Big)\frac{U(t,x)}{x}\quad \textit{for } 0<x\leq x_0.$$

3. A New Characterization of Financeable Consumption Processes

3.1. **Some Functional Set Up.** In the spirit of *Bouchard and Pham* [3] who treats the wealth dependent problem (see also *Žitković* [30] on consumption and endowment with stochastic clock), let $\mathcal{O}$ denotes the σ-algebra of optional sets relative to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and we define the product measure $d\bar{\mathbb{P}} = dt \times d\mathbb{P}$ be the finite measure on the product space $(\Omega \times [0,T], \mathcal{O})$:

$$\bar{\mathbb{P}}[A] = \mathbb{E}^{\mathbb{P}}\Big[\int_0^T \mathbf{1}_A(t,\omega)dt\Big], \quad \text{for } A \in \mathcal{O}. \tag{3.1}$$

We denote by $\mathbb{L}^0(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}})$ ($\mathbb{L}^0$ for short) the set of all random variables on the product space $\Omega \times [0,T]$ under the product measure $\bar{\mathbb{P}}$ with respect to the optional σ-algebra $\mathcal{O}$. And from now on, we shall always identify the optional stochastic process $(Y_t)_{t\in[0,T]}$ with the random variable $Y \in \mathbb{L}^0(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}})$. We also define the positive orthant $\mathbb{L}^0_+(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}})$ ($\mathbb{L}^0_+$ for short) the set of elements $Y = Y(t,\omega)$ of $\mathbb{L}^0$ such that:

$$Y \geq 0, \quad \bar{\mathbb{P}}\ a.s..$$

For any $Y^1, Y^2 \in \mathbb{L}^0_+$, we shall say that

$$Y^1 \equiv Y^2 \ \text{ if } \ Y^1 = Y^2, \ \ \bar{\mathbb{P}}\ a.s..$$

Endow $\mathbb{L}^0_+$ with the bilinear form valued in $[0,\infty]$ as:

$$\big\langle X, Y \big\rangle = \mathbb{E}\Big[\int_0^T X_t Y_t dt\Big], \ \text{ for all } X, Y \in \mathbb{L}^0_+.$$

We also define a partial ordering on $\mathbb{L}^0_+$ for convenience:

$$Y^1 \preceq (\prec) Y^2 \Longleftrightarrow Y^1 \leq (<) Y^2, \quad \bar{\mathbb{P}}\ a.s..$$

3.2. **Path-dependence Reduction by Auxiliary Processes.** At this point, we are able to define the set of all (x,z)-financeable consumption rate processes as a set of random variables on the product space $(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}})$ and the **Budget Constraint** Proposition 2.1 states that:

$$\begin{aligned}\mathcal{A}(x,z) \triangleq \Big\{ & c \in \mathbb{L}^0_+ : c_t \geq Z_t \text{ and } W_t = x + \\ & (H \cdot S)_t - \int_0^t c_s ds \geq 0, \forall t \in [0,T] \text{ and } H \text{ is admissible}\Big\} \\ = \Big\{ & c \in \mathbb{L}^0_+ : c_t \geq Z_t, \ \forall t \in [0,T] \text{ and } \big\langle c, Y \big\rangle \leq x, \ \forall Y \in \mathcal{M}\Big\}.\end{aligned}$$

where process Z_t is defined by (2.4). However, the family $\mathcal{A}(x,z)$ may be empty for some values $x > 0, z \geq 0$. We shall restrict ourselves to the *effective domain* $\bar{\mathcal{H}}$ which is defined as the union of the *interior* of set such that $\mathcal{A}(x,z)$ is not empty and the one side boundary $\{x > 0, z = 0\}$:

$$\bar{\mathcal{H}} \triangleq int\Big\{(x,z) \in (0,\infty) \times [0,\infty) : \mathcal{A}(x,z) \neq \varnothing\Big\} \cup (0,\infty) \times \{0\}. \tag{3.2}$$

We want the effective domain $\bar{\mathcal{H}}$ to include the special case of zero initial habit by $z = 0$.

Before we state the next result, we shall first impose some additional conditions on the stochastic discounting factors α_t and δ_t, which are essential for the well-posedness of our primal utility optimization problem :

Assumption 3.1.
We assume the nonnegative optional processes α_t and δ_t satisfy:

$$\sup_{Y\in\mathcal{M}} \mathbb{E}\Big[\int_0^T e^{\int_0^t(\delta_v-\alpha_v)dv} Y_t dt\Big] < \infty. \tag{3.3}$$

and there exists a constant $\bar{x} > 0$ such that

$$\mathbb{E}\Big[\int_0^T U(t, \bar{x}e^{-\int_0^t \alpha_v dv})dt\Big] > -\infty. \tag{3.4}$$

Remark 3.1. If stochastic discounting processes α_t and δ_t are assumed to be bounded, Assumptions (3.3) and (3.4) will be satisfied, and are redundant.

Remark 3.2. Assumption (3.3) is the well known super-hedging property of the random variable $\int_0^T e^{\int_0^t(\delta_v-\alpha_v)dv}dt$ in our original financial market. This assumption is basically equivalent to the statement that for all $z \geq 0$, there exists a $x > 0$, such that $\bar{\mathcal{A}}(x, z) \neq \emptyset$, as we will see below.

On the other hand, we make Assumption (3.4) to guarantee the existence of some $(x, z) \in \bar{\mathcal{H}}$ such that the value function $u(x, z) > -\infty$, which is always taken as granted in the utility maximization problem with pure investment or consumption without habit formation. The acceptance of this convention in the classical problem lies in the fact that there exists some strict positive constants in the corresponding admissible space of wealth or consumption processes. However, this convention will be violated, due to the addictive habits constraint $c_t > Z_t$ and the fact that stochastic discounting factors are unbounded, hence our auxiliary dual process Γ_t (see its definition in (3.8)) is not necessary in $\mathbb{L}^1$. It is interesting to note, however, in the future we will see the process $\tilde{w}_t \triangleq e^{-\int_0^t \alpha_v dv}$ somehow plays the same role as the constant 1 to be a universal strictly positive element in the corresponding admissible space by rescaling. And we remark here that one can also take $\tilde{w}_t \triangleq e^{-\int_0^t \alpha_v dv}$ as the abstract numeráre.

Lemma 3.1. *Under Assumption* (3.3)*, the effective domain $\bar{\mathcal{H}}$ can be rewritten explicitly as:*

$$\bar{\mathcal{H}} = \Big\{(x, z) \in (0, \infty) \times [0, \infty) : x > z \sup_{Y\in\mathcal{M}} \mathbb{E}\Big[\int_0^T e^{\int_0^t(\delta_v-\alpha_v)dv} Y_t dt\Big]\Big\}. \tag{3.5}$$

By choosing $(x, z) \in \bar{\mathcal{H}}$, we can now define our **Primal Utility Maximization Problem** as:

$$u(x, z) \triangleq \sup_{c\in\mathcal{A}(x,z)} \mathbb{E}\Big[\int_0^T U(t, c_t - Z_t)dt\Big], \quad (x, z) \in \bar{\mathcal{H}}. \tag{3.6}$$

Now, for fixed $(x, z) \in \bar{\mathcal{H}}$, and each (x, z)-financeable consumption rate process, we want to generalize the Market Isomorphism idea by *Schroder and Skiadas* [28] in order to reduce the path dependency. We are ready to introduce the auxiliary process $\tilde{c}_t = c_t - Z_t$, and define the auxiliary set of $\mathcal{A}(x, z)$ as:

$$\bar{\mathcal{A}}(x,z) \triangleq \Big\{\tilde{c} \in \mathbb{L}^0_+ : \ \tilde{c}_t = c_t - Z_t, \ \forall t \in [0,T], \quad c \in \mathcal{A}(x,z)\Big\}. \tag{3.7}$$

For each fixed $(x, z) \in \bar{\mathcal{H}}$, it is clear that there is one to one correspondence between sets $\mathcal{A}(x, z)$ and $\bar{\mathcal{A}}(x, z)$, and hence we have $\bar{\mathcal{A}}(x, z) \neq \emptyset$ for $(x, z) \in \bar{\mathcal{H}}$.

Let's turn our attention to the set $\mathcal{M}$ of equivalent local martingale measures, and for each $Y \in \mathcal{M}$, according to Assumption (3.3) we can define the auxiliary optional process with respect to Y_t as:

$$\Gamma_t \triangleq Y_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s(\delta_v - \alpha_v)dv} Y_s ds \Big| \mathcal{F}_t\Big], \quad \forall t \in [0,T]. \tag{3.8}$$

Let's denote the set of all these auxiliary optional processes as:

$$\widetilde{\mathcal{M}} = \Big\{\Gamma \in \mathbb{L}^0_+ : \Gamma_t = Y_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s(\delta_v - \alpha_v)dv} Y_s ds \Big| \mathcal{F}_t\Big], \quad \forall t \in [0,T], \quad Y \in \mathcal{M}\Big\}. \tag{3.9}$$

We remark again here that since stochastic discounting processes δ_t and α_t are unbounded, under Assumption (3.3), the auxiliary dual process Γ is well defined, but it is not necessarily in $\mathbb{L}^1$.

The following important equalities serve as critical ingredients in embedding our original utility maximization problem into an auxiliary abstract optimization problem on the product space, for which we are able to apply the convex duality approach:

Proposition 3.1. *Under Assumption* (3.3)*, for each nonnegative optional process c_t such that $c_t \geq Z_t$ with Z_t defined by* (2.4) *for fixed initial standard of living $z \geq 0$ and the nonnegative optional process Y_t, we have the following equalities with respect to their corresponding auxiliary processes $\tilde{c}_t = c_t - Z_t$ and Γ_t which is defined by* (3.8)*, that:*

$$\begin{aligned} \big\langle c, Y\big\rangle &= \big\langle \tilde{c}, \Gamma\big\rangle + z\big\langle w, Y\big\rangle \\ &= \big\langle \tilde{c}, \Gamma\big\rangle + z\big\langle \tilde{w}, \Gamma\big\rangle, \end{aligned} \tag{3.10}$$

where we define these extra exogenous random processes $w, \ \tilde{w} \in \mathbb{L}^0_+$ as

$$w_t \triangleq e^{\int_0^t(\delta_v - \alpha_v)dv} \quad \textit{and} \quad \tilde{w}_t \triangleq e^{\int_0^t(-\alpha_v)dv} \ \ \textit{for all } t \in [0,T]. \tag{3.11}$$

Remark 3.3. These extra random processes w_t and $\tilde{w}_t$ in (3.11) defined by stochastic discounting factors α_t and δ_t will play the role of shadow random endowment rate processes in the future formulation of the dual optimization problem. In an attempt to analyze this special structure, we will naturally adopt some classical convex duality framework with respect to market random endowment source, and try to prove some similar results.

Based on previous Propositions 2.1 and 3.1, under Assumptions (3.3) and (3.4), clearly we will have the alternative budget constraint characterization of the consumption rate process c_t as:

Proposition 3.2. *For any given pair $(x,z)\in\bar{\mathcal{H}}$, we call the consumption rate process c is (x,z)-financeable if and only if $c_t\geq Z_t,\ \forall t\in[0,T]$ and*

$$\big\langle c-Z,\Gamma\big\rangle\leq x-z\big\langle\tilde{w},\Gamma\big\rangle,\ \textit{for all}\ \Gamma\in\widetilde{\mathcal{M}}.$$

Proposition 3.2 provides us the alternative definition of set $\bar{\mathcal{A}}(x,z)$ for $(x,z)\in\bar{\mathcal{H}}$ as:

$$\bar{\mathcal{A}}(x,z)=\Big\{\tilde{c}\in\mathbb{L}^0_+:\ \big\langle\tilde{c},\Gamma\big\rangle\leq x-z\big\langle\tilde{w},\Gamma\big\rangle,\ \forall\Gamma\in\widetilde{\mathcal{M}}\Big\}. \tag{3.12}$$

We see that the path-dependent addictive habits constraint on c_t such that $c_t\geq Z_t$ eventually turns to be a natural constraint that $\tilde{c}\in\mathbb{L}^0_+$, and (3.12) states that the auxiliary set $\bar{\mathcal{A}}(x,z)$ is solid, convex and closed in measure $\bar{\mathbb{P}}$ although $\mathcal{A}(x,z)$ does not hold all these properties. Hence this path-dependence reduction from c_t to $\tilde{c}_t$ is crucial to enable us to work with convex duality approach.

3.3. Embedding into an Abstract Utility Maximization Problem with Shadow Random Endowments.

In order to accommodate to the classical convex duality approach with the random endowment in the next section, due to some technical reasons, we need to first enlarge the domain of the set $\bar{\mathcal{H}}$ to $\mathcal{H}$ and enlarge the corresponding auxiliary set $\bar{\mathcal{A}}(x,z)$ to $\widetilde{\mathcal{A}}(x,z)$ defined as:

$$\widetilde{\mathcal{A}}(x,z)\triangleq\Big\{\tilde{c}\in\mathbb{L}^0_+:\big\langle\tilde{c},\Gamma\big\rangle\leq x-z\big\langle\tilde{w},\Gamma\big\rangle,\ \forall\Gamma\in\widetilde{\mathcal{M}}\Big\}, \tag{3.13}$$

where now $(x,z)\in\mathbb{R}^2$, and is restricted in the enlarged domain $\mathcal{H}$:

$$\mathcal{H}\triangleq int\Big\{(x,z)\in\mathbb{R}^2:\widetilde{\mathcal{A}}(x,z)\neq\emptyset\Big\}.$$

Under Assumption (3.3) and Proposition 3.1, we have the following equivalent characterization of $\widetilde{\mathcal{A}}(x,z)$:

Lemma 3.2.

$$\begin{aligned}\mathcal{H}=&\Big\{(x,z)\in\mathbb{R}^2:x>z\big\langle\tilde{w},\Gamma\big\rangle,\ \textit{for all}\ \Gamma\in\widetilde{\mathcal{M}}\Big\}\\=&\Big\{(x,z)\in\mathbb{R}^2:x>\bar{p}z,z\geq0\Big\}\cup\Big\{(x,z)\in\mathbb{R}^2:x>\underline{p}z,z<0\Big\}.\end{aligned} \tag{3.14}$$

where

$$\bar{p}\triangleq\sup_{Y\in\mathcal{M}}\big\langle w,Y\big\rangle=\sup_{\Gamma\in\widetilde{\mathcal{M}}}\big\langle\tilde{w},\Gamma\big\rangle, \tag{3.15}$$

and

$$\underline{p}\triangleq\inf_{Y\in\mathcal{M}}\big\langle w,Y\big\rangle=\inf_{\Gamma\in\widetilde{\mathcal{M}}}\big\langle\tilde{w},\Gamma\big\rangle. \tag{3.16}$$

where $\bar{p},\underline{p}<\infty$ and $\mathcal{H}$ is a well defined convex cone in $\mathbb{R}^2$. Moreover

$$\begin{aligned}cl\mathcal{H}=&\Big\{(x,z)\in\mathbb{R}^2:\widetilde{\mathcal{A}}(x,z)\neq\emptyset\Big\}\\=&\Big\{(x,z)\in\mathbb{R}^2:x\geq z\big\langle\tilde{w},\Gamma\big\rangle,\ \textit{for all}\ \Gamma\in\widetilde{\mathcal{M}}\Big\}\end{aligned} \tag{3.17}$$

where $cl\mathcal{H}$ denotes the closure of the set $\mathcal{H}$ in $\mathbb{R}^2$.

We will now define the **Auxiliary Primal Utility Maximization Problem** based on the abstract auxiliary domain $\widetilde{\mathcal{A}}(x,z)$ as:

$$\tilde{u}(x,z) \triangleq \sup_{\tilde{c}\in\widetilde{\mathcal{A}}(x,z)} \mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)dt\Big],\ (x,z)\in\mathcal{H}. \tag{3.18}$$

By definitions of $\bar{\mathcal{A}}(x,z)$ for $(x,z)\in\bar{\mathcal{H}}$ and $\widetilde{\mathcal{A}}(x,z)$ for $(x,z)\in\mathcal{H}$, we successfully embedded our original utility maximization problem (3.6) with consumption habit formation into the auxiliary abstract utility maximization problem (3.18) without habit formation, however, with some shadow random endowments. More precisely, the following equivalence can be guaranteed that for any $(x,z)\in\bar{\mathcal{H}}\subset\mathcal{H}$:

$$\bar{\mathcal{A}}(x,z) = \widetilde{\mathcal{A}}(x,z), \tag{3.19}$$

and the two value functions coincide

$$u(x,z) = \tilde{u}(x,z), \tag{3.20}$$

in addition, the immediate byproduct consequence states that c_t^* is the optimal solution for $u(x,z)$ if and only if $\tilde{c}_t^* = c_t^* - Z_t^* \geq 0$ for all $t\in[0,T]$ is the optimal solution for $\tilde{u}(x,z)$, when $(x,z)\in\bar{\mathcal{H}}$.

4. The Dual Optimization Problem and Main Results

Inspired by the idea in *Hugonnier and Kramkov* [16] for optimal investment with random endowment, we concentrate now on the construction of the dual problem by firstly introducing the set $\mathcal{R}$, which is the *relative interior* of the polar cone of $-\mathcal{H}$:

$$\mathcal{R} \triangleq ri\Big\{(y,r)\in\mathbb{R}^2 : xy - zr \geq 0 \text{ for all } (x,z)\in\mathcal{H}\Big\}. \tag{4.1}$$

To exclude the easy case, let's make the following assumption on stochastic discounting processes α_t and δ_t:

Assumption 4.1.
The random variable defined by

$$\mathcal{E} \triangleq \int_0^T w_t dt = \int_0^T e^{\int_0^t (\delta_v - \alpha_v)dv} dt \tag{4.2}$$

is not replicable under our original financial market.

Remark 4.1. Our Assumption (4.2) above prevents the foundation of the nice work by *Schroder and Skiadas* [28] which states that there exists an equivalence between the primal utility maximization problem with habit formation in the original market and the utility maximization problem in the Isomorphic market without consumption habits. And our work generally extends their conclusion and provides the existence and uniqueness of the optimal solution in the incomplete market when the Market Isomorphism does not hold.

Remark 4.2. We remark here that even if $\mathcal{E} \triangleq \int_0^T w_t dt$ is replicable in the original incomplete market such that $\bar{p} = \underline{p}$, the market isomorphism relation by *Schroder and Skiadas* [28] may still not hold. In this case, however, the original utility maximization problem becomes easier since we do not need to take care of the exogenous term $\tilde{w}_t$ and the primal value function $u(x)$ becomes one dimensional function only depending on the initial wealth. We can therefore embed our original problem into the framework by *Kramkov and Schachermayer* [21], [22] to build the corresponding one dimensional conjugate duality relation and provide the existence and uniqueness of the optimal consumption strategy.

Lemma 4.1. *By Assumption* (4.2)*, we know that $\mathcal{R}$ is an open convex cone in $\mathbb{R}^2$, and can be rewritten as:*

$$\mathcal{R} = \Big\{(y,r) \in \mathbb{R}^2 : y > 0, \ \textit{and} \ \ \underline{p}y < r < \bar{p}y\Big\}, \tag{4.3}$$

where $\bar{p}$ and $\underline{p}$ are defined by (3.15) *and* (3.16)*, and $\bar{p} < \underline{p}$.*

Following the framework of *Hugonnier and Kramkov* [16], for an arbitrary pair $(y,r) \in \mathcal{R}$, we denote by $\widetilde{\mathcal{Y}}(y,r)$ the set of nonnegative processes as a proper extension of the auxiliary set $\widetilde{\mathcal{M}}$ in the way that:

$$\widetilde{\mathcal{Y}}(y,r) \triangleq \Big\{\Gamma \in \mathbb{L}^0_+ : \ \big\langle \tilde{c}, \Gamma \big\rangle \leq xy - zr, \ \text{for all } \tilde{c} \in \widetilde{\mathcal{A}}(x,z), \ \text{and } (x,z) \in \mathcal{H}\Big\}. \tag{4.4}$$

Based on previous efforts, we are ready to establish the **Auxiliary Dual Utility Maximization Problem** to (3.18) defined as:

$$\tilde{v}(y,r) \triangleq \inf_{\Gamma \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\Big[\int_0^T V(t,\Gamma_t)dt\Big], \quad (y,r) \in \mathcal{R}. \tag{4.5}$$

The following theorems constitute our main results. And we provide their proofs through a number of auxiliary results in the next section.

Theorem 4.1. *Assume conditions* (2.5)*,* (2.7)*,* (3.3)*,* (3.4)*,* (4.2)*. Assume also that* (2.11)*,* (2.12) *and* (2.10)*, (i.e., $AE_0[U] < \infty$) hold true together with*

$$\tilde{u}(x,z) < \infty \qquad \textit{for some } (x,z) \in \mathcal{H}. \tag{4.6}$$

we will have:

(i) *The function $\tilde{u}$ is $(-\infty,\infty)$-valued on $\mathcal{H}$ and $\tilde{v}(y,r)$ is $(-\infty,\infty]$-valued on $\mathcal{R}$. And for each $(y,r) \in \mathcal{R}$ there exists a constant $s = s(y,r) > 0$ such that $\tilde{v}(sy,sr) < \infty$. Moreover, we* *have the conjugate duality of value functions $\tilde{u}$ and $\tilde{v}$:*

$$\tilde{u}(x,z) = \inf_{(y,r)\in\mathcal{R}} \{\tilde{v}(y,r) + xy - zr\}, \qquad (x,z) \in \mathcal{H}$$
$$\tilde{v}(y,r) = \sup_{(x,z)\in\mathcal{H}} \{\tilde{u}(x,z) - xy + zr\}, \qquad (y,r) \in \mathcal{R}.$$

(ii) *The solution $\Gamma^*(y,r)$ to the optimization problem (4.5) exists and is unique (in the sense of $\equiv$ in $\mathbb{L}^0_+$) for all $(y,r) \in \mathcal{R}$ such that $\tilde{v}(y,r) < \infty$.*

Theorem 4.2. *We now assume in addition to conditions of Theorem 4.1 that Assumption (2.9) holds, (i.e., $AE_\infty[U] < 1$). Then in addition to assertions of Theorem 4.1, we also have:*

(i) *The value function $\tilde{v}(y,r)$ is $(-\infty,\infty)$-valued on $(y,r) \in \mathcal{R}$ and $\tilde{v}$ is continuously differentiable on $\mathcal{L}$. And the optimal solution $\Gamma^*_t(y,r) > 0$, $\mathbb{P}$-a.s. for all $t \in [0,T]$.*

(ii) *The solution $\tilde{c}^*(x,z)$ to optimization problem (3.18) exists and is unique (in the sense of $\equiv$ in $\mathbb{L}^0_+$) for any $(x,z) \in \mathcal{H}$. And the optimal solution $\tilde{c}^*_t(x,z) > 0$, $\mathbb{P}$-a.s. for all $t \in [0,T]$.*

(iii) *The superdifferential of $\tilde{u}$ maps $\mathcal{H}$ into $\mathcal{R}$, i.e.,*

$$\partial\tilde{u}(x,z) \subset \mathcal{R}, \qquad (x,z) \in \mathcal{H}. \tag{4.7}$$

Moreover, if $(y,r) \in \partial\tilde{u}(x,z)$, then $\tilde{c}^*(x,z)$ and $\Gamma^*(y,r)$ are related by:

$$\begin{aligned} &\Gamma^*_t(y,r) = U'(t,\tilde{c}^*_t(x,z)) \quad \text{or} \quad \tilde{c}^*_t(x,z) = I(t,\Gamma^*_t(y,r)), \\ &\langle \Gamma^*(y,r), \tilde{c}^*(x,z) \rangle = xy - zr. \end{aligned} \tag{4.8}$$

(iv) *If we restrict the choice of $(x,z) \in \hat{\mathcal{H}} \subset \mathcal{H}$, the solution $c^*_t(x,z)$ to our primal utility optimization problem (3.6) exists and is unique, moreover,*

$$\tilde{c}^*_t(x,z) = c^*_t(x,z) - Z^*_t(x,z). \tag{4.9}$$

5. Proofs of Main Results

5.1. **The proof of Theorem** 4.1**.** The following Proposition will serve as the key step to build some future Bipolar relationships:

Proposition 5.1. *Assume all assumptions of Theorem 4.1 hold true. Then the families $\left(\widetilde{\mathcal{A}}(x,z)\right)_{(x,z)\in\mathcal{H}}$ and $\left(\widetilde{\mathcal{Y}}(y,r)\right)_{(y,r)\in\mathcal{R}}$ have the following properties:*

(i) *For any $(x,z) \in \mathcal{H}$, the set $\widetilde{\mathcal{A}}(x,z)$ contains a strictly positive random variable on the product space. A nonnegative random variable $\tilde{c}$ belongs to $\widetilde{\mathcal{A}}(x,z)$ if and only if*

$$\big\langle \tilde{c}, \Gamma \big\rangle \leq xy - zr \quad \text{for all } (y,r) \in \mathcal{R} \text{ and } \Gamma \in \widetilde{\mathcal{Y}}(y,r). \tag{5.1}$$

(ii) *For any $(y,r) \in \mathcal{R}$, the set $\widetilde{\mathcal{Y}}(y,r)$ contains a strictly positive random variable on the product space. A nonnegative random variable Γ belongs to $\widetilde{\mathcal{Y}}(y,r)$ if and only if*

$$\big\langle \tilde{c}, \Gamma \big\rangle \leq xy - zr \text{ for all } (x,z) \in \mathcal{H} \text{ and } \tilde{c} \in \widetilde{\mathcal{A}}(x,z). \tag{5.2}$$

In order to prove Proposition 5.1, for any $p > 0$, we denote by $\mathcal{M}(p)$ the subset of $\mathcal{M}$ that consists of measure densities $Y \in \mathcal{M}$ such that $\big\langle w, Y \big\rangle = p$. Then for any density process $Y \in \mathcal{M}(p)$, define the auxiliary set as

$$\widetilde{\mathcal{M}}(p) \triangleq \Big\{ \Gamma \in \mathbb{L}^0_+ : \Gamma_t = Y_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v) dv} Y_s ds \Big| \mathcal{F}_t \Big], \quad \forall t \in [0,T], \ Y \in \mathcal{M}(p) \Big\}. \tag{5.3}$$

We have $\big\langle \tilde{w}, \Gamma \big\rangle = \big\langle w, Y \big\rangle = p$.

Define $\mathcal{P}$ as the open interval $\mathcal{P} = (\underline{p}, \bar{p})$, where $\underline{p}, \bar{p}$ are defined in (3.15) and (3.16). We have the following result.

Lemma 5.1. *Assume that conditions of Proposition* 5.1 *hold true and let $p > 0$. Then the set $\widetilde{\mathcal{M}}(p)$ is not empty if and only if $p \in \mathcal{P} = (\underline{p}, \bar{p})$, where $\underline{p}, \bar{p}$ are defined in* (3.15) *and* (3.16) . *In particular,*

$$\bigcup_{p \in \mathcal{P}} \widetilde{\mathcal{M}}(p) = \widetilde{\mathcal{M}}. \tag{5.4}$$

where the set $\widetilde{\mathcal{M}}$ is defined by (3.9).

Proof. The proof reduces to verifying that $\mathcal{P} = \mathcal{P}'$, where we define

$$\mathcal{P}' \triangleq \{ p > 0 : \ \widetilde{\mathcal{M}}(p) \neq \emptyset \}.$$

Similar to the proof of Lemma 8 of *Hugonnier and Kramkov* [16], one direction inclusion that $\mathcal{P} \subseteq \mathcal{P}'$ is obvious.

For the inverse, let $p \in \mathcal{P}'$, $(x,z) \in cl\mathcal{H}$, $\Gamma \in \widetilde{\mathcal{M}}(p)$, and we first claim there exists a $\tilde{c} \in \widetilde{\mathcal{A}}(x,z)$ such that

$$\bar{\mathbb{P}}[\tilde{c} \succ 0] > 0,$$

so we get

$$0 < \big\langle \tilde{c}, \Gamma \big\rangle \leq x - zp.$$

As (x,z) is an arbitrary element of $cl\mathcal{H}$, we have $p \in \mathcal{P}$.

As for the above claim, according to Theorem 2.11 of *Schachermayer* [27], Assumption (4.2) guarantees that for all $Y \in \mathcal{M}$, we have

$$\underline{p} < \langle w, Y \rangle < \bar{p},$$

which is

$$\underline{p} < \langle \tilde{w}, \Gamma \rangle < \bar{p},$$

for all the $\Gamma \in \widetilde{\mathcal{M}}$. Then by the definition of $cl\mathcal{H}$ in Lemma 3.2, we observe that for any $(x, z) \in cl\mathcal{H}$, we will have

$$x - z\big\langle \tilde{w}, \Gamma \big\rangle > 0,$$

for all the $\Gamma \in \widetilde{\mathcal{M}}$, and the claim holds by the definition of $\widetilde{\mathcal{A}}(x, z)$. $\square$

Lemma 5.2. *Assume that conditions of Proposition* 5.1 *hold true and let* $p \in \mathcal{P} = (\underline{p}, \bar{p})$, *we have then* $\widetilde{\mathcal{M}}(p) \subseteq \widetilde{\mathcal{Y}}(1, p)$.

Proof. The conclusion can be directly derived in light of the definition of $\widetilde{\mathcal{A}}(x, z)$ and $\widetilde{\mathcal{Y}}(1, p)$. $\square$

Lemma 5.3. *Assume that conditions of Proposition* 5.1 *hold true. For any* $(x, z) \in \mathcal{H}$, *a nonnegative random variable* $\tilde{c}$ *belongs to* $\tilde{\mathcal{A}}(x, z)$ *if and only if*

$$\big\langle \tilde{c}, \Gamma \big\rangle \leq x - zp \quad \textit{for all } p \in \mathcal{P} \textit{ and } \Gamma \in \widetilde{\mathcal{M}}(p). \tag{5.5}$$

Proof. If $\tilde{c} \in \widetilde{\mathcal{A}}(x, z)$, the definition of $\widetilde{\mathcal{A}}(x, z)$ and the fact $\widetilde{\mathcal{M}}(p) \subset \widetilde{\mathcal{M}}$ guarantee the validity of (5.5).

On the other hand, for any $\tilde{c} \in \mathbb{L}^0_+$ such that (5.5) holds true, we will have:

$$\begin{aligned}
\sup_{\Gamma \in \widetilde{\mathcal{M}}} \big\langle \tilde{c} + z\tilde{w}, \Gamma \big\rangle &= \sup_{p \in \mathcal{P}} \sup_{\Gamma \in \widetilde{\mathcal{M}}(p)} \big\langle \tilde{c} + z\tilde{w}, \Gamma \big\rangle \\
&= \sup_{p \in \mathcal{P}} \sup_{\Gamma \in \widetilde{\mathcal{M}}(p)} \Big(\big\langle \tilde{c}, \Gamma \big\rangle + zp \Big) \leq x.
\end{aligned}$$

The claim holds according to the definition of $\widetilde{\mathcal{A}}(x, z)$. $\square$

PROOF OF PROPOSITION 5.1.

For the validity of assertion (i), consider $(x, z) \in \mathcal{H}$, there exists a $\lambda > 0$ such that $(x - \lambda, z) \in \mathcal{H}$ since $\mathcal{H}$ is an open set.

Let $\tilde{c} \in \widetilde{\mathcal{A}}(x - \lambda, z)$, we will have for any $\Gamma \in \widetilde{\mathcal{M}}$, and $\tilde{w}_t = e^{-\int_0^t \alpha_v dv} \succ 0$,

$$\big\langle \tilde{c}, \Gamma \big\rangle \leq x - \lambda - z\big\langle \tilde{w}, \Gamma \big\rangle. \tag{5.6}$$

By Assumption (3.3) and Proposition 3.1, we define $\rho_t \triangleq \frac{\lambda}{\bar{p}} \tilde{w}_t > 0$ for all $t \in [0, T]$, then for all $\Gamma \in \widetilde{\mathcal{M}}$:

$$\begin{aligned}
\big\langle \rho, \Gamma \big\rangle \leq \big\langle \tilde{c} + \rho, \Gamma \big\rangle &\leq x - \lambda - z\big\langle \tilde{w}, \Gamma \big\rangle + \frac{\lambda}{\bar{p}} \big\langle \tilde{w}, \Gamma \big\rangle \\
&\leq x - \lambda - z\big\langle \tilde{w}, \Gamma \big\rangle + \lambda \leq x - z\big\langle \tilde{w}, \Gamma \big\rangle.
\end{aligned}$$

Hence, we have shown the existence of a strictly positive element $\rho_t \succ 0 \in \widetilde{\mathcal{A}}(x,z)$ by the definition of $\widetilde{\mathcal{A}}(x,z)$.

If (5.1) holds for some $\tilde{c} \in \mathbb{L}^0_+$. The density process $\Gamma \in \widetilde{\mathcal{M}}(p)$ belongs to $\widetilde{\mathcal{Y}}(1,p)$ for all $p \in \mathcal{P}$ by Lemma 5.2, and hence (5.5) holds. Lemma 5.3 then implies that $\tilde{c} \in \widetilde{\mathcal{A}}(x,z)$.Conversely, suppose now $\tilde{c} \in \widetilde{\mathcal{A}}(x,z)$, the definition of sets $\widetilde{\mathcal{Y}}(y,r)$, $(y,r) \in \mathcal{R}$ implies (5.1) and we complete the proof of assertion (i).

For the proof the assertion (ii), notice

$$k\widetilde{\mathcal{Y}}(y,r) = \widetilde{\mathcal{Y}}(ky,kr) \text{ for all } k>0, \ (y,r) \in \mathcal{R}.$$

therefore we just need to consider $(y,r) = (1,p)$ for some $p \in \mathcal{P}$. Lemma 5.2 implies $\Gamma \in \widetilde{\mathcal{M}}(p) \subseteq \widetilde{\mathcal{Y}}(1,p)$, and the existence of $Y \succ 0 \in \mathcal{M}(p)$ takes care of the existence $\Gamma \succ 0 \in \widetilde{\mathcal{M}}(p)$, $\bar{\mathbb{P}}$-a.s.

The second part is a direct consequence of the definition of $\widetilde{\mathcal{Y}}(y,r)$. $\square$

For the proof of Theorem 4.1, we will also need the following lemmas:

Lemma 5.4. *Under assumptions of Theorem* 4.1*, the value function $\tilde{u}$ is $(-\infty,\infty)$-valued on $\mathcal{H}$.*

Proof. First, by Lemma 2.1, the assumption $AE_0[U] < \infty$ implies that for any positive constant $s>0$, the existence of $s_1>0$ and $s_2>0$ such that for all $t \in [0,T]$:

$$U(t,x/s) \geq s_1 U(t,x) + s_2, \quad x>0, \tag{5.7}$$

According to Assumption (3.4) and the proof of Proposition 5.1, for each fixed pair $(x,z) \in \mathcal{H}$, there exists $\lambda = \lambda(x,z) > 0$ such that $\frac{\lambda}{\bar{p}}\tilde{w}_t \in \widetilde{\mathcal{A}}(x,z)$, therefore we deduce that $\bar{x}\tilde{w}_t \in \widetilde{\mathcal{A}}(\frac{\bar{x}\bar{p}}{\lambda}x, \frac{\bar{x}\bar{p}}{\lambda}z)$, and

$$\tilde{u}(\frac{\bar{x}\bar{p}}{\lambda}x, \frac{\bar{x}\bar{p}}{\lambda}z) = \sup_{\tilde{c}\in\widetilde{\mathcal{A}}(\frac{\bar{x}\bar{p}}{\lambda}x, \frac{\bar{x}\bar{p}}{\lambda}z)} \mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)dt\Big] \geq \mathbb{E}\Big[\int_0^T U(t,\bar{x}\tilde{w}_t)dt\Big] > -\infty,$$

hence, for any $(x,z) \in \mathcal{H}$, we get the existence of a constant $s(x,z)>0$, such that $\tilde{u}(sx,sz) > -\infty$, with $s(x,z) = \frac{\bar{x}\bar{p}}{\lambda}$.

Since, for any constant $s>0$,

$$\widetilde{\mathcal{A}}(x,z) = \widetilde{\mathcal{A}}(sx,sz)/s,$$

we derive $\tilde{u}(x,z) > -\infty$ if $\tilde{u}(sx,sz) > -\infty$ holds for a constant $s = s(x,z) > 0$, follow the result above, we conclude that $\tilde{u}(x,z) > -\infty$ in the whole domain $\mathcal{H}$.

Now, since the set $\mathcal{H}$ is open and $\tilde{u}(x,z) < \infty$ for some $(x,z) \in \mathcal{H}$ by assumption (4.6), we deduce that $\tilde{u}$ is finitely valued on $\mathcal{H}$ by the concavity of $\tilde{u}$ on $\mathcal{H}$. And the proof is complete. $\square$

Before we state the next lemma, let's introduce a special concept of compactness which was originally defined in *Žitković* [31].

Definition 5.1. *A convex subset C of a topological vector space X is said to be convexly compact if for any non-empty set A and any family $\{F_a\}_{a\in A}$ of closed, convex subsets of C, the condition*

$$\forall D \in Fin(A), \bigcap_{a\in D} F_a \neq \emptyset \implies \bigcap_{a\in A} F_a \neq \emptyset$$

where the set $Fin(A)$ consists of all non-empty finite subsets of A for an arbitrary non-empty set A.

Without the restriction that the sets $\{F_a\}_{a\in A}$ must be convex, this definition would be equivalent to compactness in the original sense. Thus any convex and compact set is convexly compact and Definition 5.1 extends the concept of compactness.

Žitković [31] furthermore derived an easy characterization on the space of non-negative, measurable functions.

Theorem 5.1. *A closed and convex subset C of $\mathbb{L}^0_+$ is convexly compact if and only if it is bounded in finite measure.*

Based on the above theorem, we have the following lemma on the convexly compactness of sets $\widetilde{\mathcal{A}}(x,z)$ and $\widetilde{\mathcal{Y}}(y,r)$:

Lemma 5.5. *For each pair $(x,z)\in\mathcal{H}$ and $(y,r)\in\mathcal{R}$, the sets $\widetilde{\mathcal{A}}(x,z)$ and $\widetilde{\mathcal{Y}}(y,r)$ are convex, solid and closed in the topology of convergence in measure $\bar{\mathbb{P}}$. Moreover, they are both bounded in $\mathbb{L}^0_+(\Omega\times[0,T],\mathcal{O},\bar{\mathbb{P}})$, hence they are both convexly compact.*

Proof. For $(y,r)\in\mathcal{R}$, we now define two auxiliary sets as

$$\begin{aligned} \mathfrak{H}(y,r) &\triangleq \Big\{(x,z)\in\mathcal{H}:\ xy-zr\leq 1\Big\} \\ \mathfrak{A}(k) &\triangleq \bigcup_{(x,z)\in k\mathfrak{H}(y,r)} \widetilde{\mathcal{A}}(x,z), \end{aligned} \tag{5.8}$$

and denote by $\widetilde{\mathfrak{A}}(k)$ the closure of $\mathfrak{A}(k)$ with respect to convergence in measure $\bar{\mathbb{P}}$.

From Proposition 5.1, we deduce that

$$\Gamma\in\widetilde{\mathcal{Y}}(y,r) \ \Leftrightarrow\ \big\langle \tilde{c},\Gamma\big\rangle \leq 1,\ \ \forall \tilde{c}\in\widetilde{\mathfrak{A}}(1)$$

Hence, sets $\widetilde{\mathcal{Y}}(y,r)$ and $\widetilde{\mathfrak{A}}(1)$ satisfy

$$\widetilde{\mathcal{Y}}(y,r)=\widetilde{\mathfrak{A}}(1)^{\circ}.$$

At the same time, by its definition, we have $\widetilde{\mathfrak{A}}(1)$ itself is closed, convex and solid, by the Bipolar theorem in *Brannath and Schachermayer* [4], we have $\widetilde{\mathfrak{A}}(1)=\widetilde{\mathfrak{A}}(1)^{\circ\circ}$, and hence we have the following Bipolar relationship:

$$\begin{aligned} \widetilde{\mathfrak{A}}(1) &= \widetilde{\mathcal{Y}}(y,r)^{\circ} \\ \widetilde{\mathcal{Y}}(y,r) &= \widetilde{\mathfrak{A}}(1)^{\circ}. \end{aligned} \tag{5.9}$$

The Bipolar theorem on $\mathbb{L}^0_+$ gives the convexity, solidness and closure in measure $\bar{\mathbb{P}}$.

Similarly, for $(x,z) \in \mathcal{H}$, now define the set:

$$\begin{aligned} &\mathfrak{R}(x,z) \triangleq \{(y,r) \in \mathcal{R} : xy - zr \leq 1\}, \\ &\mathfrak{Y}(k) \triangleq \bigcup_{(y,r)\in k\mathfrak{R}(x,z)} \widetilde{\mathcal{Y}}(y,r), \end{aligned} \tag{5.10}$$

and denote by $\widetilde{\mathfrak{Y}}(k)$ the closure of $\mathfrak{Y}(k)$ with respect to convergence in measure $\bar{\mathbb{P}}$.

Now, again Proposition 5.1 implies

$$\tilde{c} \in \widetilde{\mathcal{A}}(x,z) \Leftrightarrow \big\langle \tilde{c}, \Gamma \big\rangle \leq 1, \ \forall \Gamma \in \widetilde{\mathfrak{Y}}$$

and the Bipolar relationship:

$$\begin{aligned} \widetilde{\mathfrak{Y}}(1) &= \widetilde{\mathcal{A}}(x,z)^{\circ} \\ \widetilde{\mathcal{A}}(x,z) &= \widetilde{\mathfrak{Y}}(1)^{\circ}. \end{aligned} \tag{5.11}$$

Hence, we also have $\widetilde{\mathcal{A}}(x,z)$ is convex, solid and closed in the topology of convergence in measure $\bar{\mathbb{P}}$.

Moreover, thanks to the existence of $0 \prec \Gamma \in \widetilde{\mathcal{M}}(p)$ which is also in $\widetilde{\mathcal{Y}}(1,p)$, we deduce the set $\widetilde{\mathcal{A}}(x,z)$ is bounded in measure $\bar{\mathbb{P}}$ by Proposition 5.1 part (i).

Similarly, as we have derived $0 \prec \rho_t = \frac{\lambda}{\bar{p}}\tilde{w}_t \in \widetilde{\mathcal{A}}(x,z)$, due to Proposition 5.1 part (ii), we get the set $\widetilde{\mathcal{Y}}(y,r)$ is also bounded in measure $\bar{\mathbb{P}}$. And therefore both of them are convexly compact in $\mathbb{L}^0_+$. ☐

Lemma 5.6. *Under assumptions of theorem* 4.1*, we have for each fixed* $(y,r) \in \mathcal{R}$

$$\sup_{\Gamma \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\Big[\int_0^T V^-(t,\Gamma_t)dt\Big] < \infty.$$

Proof. Assumption (3.4) admits the existence of $\bar{x}\tilde{w}_t \in \mathbb{L}^0_+$ such that $\mathbb{E}\Big[\int_0^T U(t,\bar{x}\tilde{w}_t)dt\Big] > -\infty$, and moreover, by the proof of Proposition 5.1, we also know for each fixed $(y,r) \in \mathcal{R}$, find the fixed pair $(x,z) \in \widetilde{\mathfrak{H}}(y,r)$, there exists a constant $\lambda(x,z) > 0$ such that $\tilde{w} \in \widetilde{\mathfrak{A}}(\frac{\bar{p}}{\lambda})$, where $\bar{p}$ is defined by (3.15). Taking into account the inequality $U(t,x) \leq V(t,y) + xy$, we have for any $\Gamma \in \widetilde{\mathcal{Y}}(y,r)$ and $y_0(t) \triangleq \inf\{y > 0 : \ V(t,y) < 0\}$

$$\begin{aligned} &\mathbb{E}\Big[\int_0^T V^-(t,\Gamma_t)dt\Big] \leq -\mathbb{E}\Big[\int_0^T V(t,\Gamma_t 1_{\{\Gamma_t \geq y_0(t)\}} + y_0(t)1_{\{\Gamma_t < y_0(t)\}})dt\Big] \\ \leq &- \mathbb{E}\Big[\int_0^T U(t,\bar{x}\tilde{w}_t)dt\Big] + \bar{x}\mathbb{E}\Big[\int_0^T \tilde{w}_t\Gamma_t dt\Big] + \bar{x}\mathbb{E}\Big[\int_0^T \tilde{w}_t(y_0(t) - \Gamma_t)1_{\{\Gamma_t < y_0(t)\}}dt\Big] \\ \leq &- \mathbb{E}\Big[\int_0^T U(t,\bar{x}\tilde{w}_t)dt\Big] + \bar{x}\frac{\bar{p}}{\lambda} + \bar{x}\int_0^T y_0(t)dt. \end{aligned}$$

which is finitely valued and independent of the initial choice of Γ since we have $\tilde{w}_t \triangleq e^{\int_0^t(-\alpha_v)dv} \leq 1$ for $t \in [0,T]$ and $\sup_{t\in[0,T]} y_0(t) < \infty$ by Assumption (2.12), and thus our conclusion holds true. ☐

Lemma 5.7. *Under assumptions of theorem 4.1, we have for any $(y,r)\in\mathcal{R}$, $\Big(V^{-}(\cdot,\Gamma.)\Big)$ is uniformly integrable for all $\Gamma\in\widetilde{\mathcal{Y}}(y,r)$.*

Proof. By Corollary 2.1, the assumption $AE_0[U]<\infty$ is equivalent to the following assertions:

$$\exists y_0>0,\ \text{and}\ \mu\in(1,2),\quad \forall x\geq y_0,\quad V(t,2y)\geq \mu V(t,y) \tag{5.12}$$

Let $y_0>0$ and $\mu\in(1,2)$ be the constants in the above (5.12). Take $\gamma=\log_2\mu\in(0,1)$, we define the auxiliary function $\widetilde{V}(t,y):[0,\infty)\times(0,\infty)\to\mathbb{R}$ by

$$\widetilde{V}(t,y)\triangleq\begin{cases} -\frac{2y_0}{\gamma}V'(t,2y_0)-V(t,y), & y\geq 2y_0,\\ -V(t,2y_0)-\frac{2y_0}{\gamma}V'(t,2y_0)(\frac{y}{2y_0})^{\gamma}, & y<2y_0.\end{cases} \tag{5.13}$$

For each fixed $t>0$, $\widetilde{V}(t,y)$ is a nonnegative, concave, and nondecreasing function which agrees with $-V(t,y)$ up to a constant for large enough values of y and satisfies

$$\widetilde{V}(t,2y)\leq\mu\widetilde{V}(t,y),\ \text{for all}\ y>0. \tag{5.14}$$

Lemma 5.6 asserts

$$\sup_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\Big[\int_0^T V^{-}(t,\Gamma_t)dt\Big]<\infty$$

and hence in light of the fact that V^{-} and $\widetilde{V}$ differ only by a constant in a neighborhood of ∞, we will get

$$\sup_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\Big[\int_0^T \widetilde{V}(t,\Gamma_t)dt\Big]<\infty. \tag{5.15}$$

The validity of uniform integrability of the sequence $\Big(V^{-}(\cdot,\Gamma^n_\cdot)\Big)_{n\geq1}$ for $\Gamma^n\in\widetilde{\mathcal{Y}}(y,r)$, is therefore equivalent to the uniform integrability of $(\widetilde{V}(\cdot,\Gamma^n_\cdot))_{n\geq1}$.

To this end, we argue by contradiction. Suppose this sequence is not uniformly integrable, then by Rosenthal's subsequence splitting lemma, we can find a subsequence $(f^n)_{n\geq1}$, a constant $\varepsilon>0$ and a disjoint sequence $(A^n)_{n\geq1}$ of $(\Omega\times[0,T],\mathcal{O})$ with

$$A^n\in\mathcal{O},\quad A^i\cap A^j=\varnothing\quad \text{if}\ i\neq j,$$

such that

$$\mathbb{E}\Big[\int_0^T\widetilde{V}(t,f^n_t)1_{A^n}dt\Big]\geq\varepsilon,\quad\text{for}\ n\geq1$$

We define the sequence of random variables $(h^n)_{n\geq1}$

$$h^n_t=\sum_{k=1}^n f^k_t 1_{A^k}.$$

For any $\tilde{c}\in\widetilde{\mathfrak{A}}(1)$,

$$\big\langle\tilde{c},h^n\big\rangle\leq\sum_{k=1}^n\big\langle\tilde{c},f^k\big\rangle\leq n.$$

Hence $\frac{h^n}{n} \in \widetilde{\mathcal{Y}}(y,r)$.

One the other hand,

$$\mathbb{E}\Big[\int_0^T \widetilde{V}(t,h_t^n)dt\Big] \geq \sum_{k=1}^n \mathbb{E}\Big[\int_0^T \widetilde{V}(t,f_t^k)1_{A^k}dt\Big] \geq \varepsilon n,$$

and therefore by taking $n=2^m$, via iteration, it produces

$$\mu^m \sup_{\Gamma_t \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\Big[\int_0^T \widetilde{V}(t,\Gamma_t)dt\Big] \geq \mu^m \mathbb{E}\Big[\int_0^T \widetilde{V}(t,\frac{h_t^{2^m}}{2^m})dt\Big]$$

$$\geq \mathbb{E}\Big[\int_0^T \widetilde{V}(t,h_t^{2^m})dt\Big] \geq 2^m \epsilon,$$

since $\mu \in (1,2)$, this contradicts (5.15) for m large enough, therefore the conclusion holds true. □

Lemma 5.8. *For any pair $(y,r) \in \mathcal{R}$, the optimal solution Γ^* to the optimization problem* (4.5) *exists and is unique such that $\tilde{v}(y,r) < \infty$.*

Proof. Now fix $(y,r) \in \mathcal{R}$, let $(\Gamma^n)_{n\geq 1}$ be a sequence in $\widetilde{\mathcal{Y}}(y,r)$ such that

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_0^T V(t,\Gamma_t^n)dt\Big] = \tilde{v}(y,r).$$

There exists a sequence of forward convex combinations $f^n \in conv(\Gamma^n,\Gamma^{n+1},\ldots)$ which converges almost surely to a random variable Γ^* with values in $[0,\infty]$. Since the set $\widetilde{\mathcal{Y}}(y,r)$ is closed and bounded in measure $\bar{\mathbb{P}}$ in $\mathbb{L}_+^0$ by Lemma 5.5, we deduce that Γ^* is almost surely finitely valued, moreover, Γ^* belongs to $\widetilde{\mathcal{Y}}(y,r)$. We claim that Γ^* is the optimal solution to (4.5), that is

$$\mathbb{E}\Big[\int_0^T V(t,\Gamma_t^*)dt\Big] = \tilde{v}(y,r).$$

The concavity of V produces

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_0^T V(t,f_t^n)dt\Big] = \tilde{v}(y,r),$$

and Fatou's lemma implies

$$\liminf_{n\to\infty} \mathbb{E}\Big[\int_0^T V^+(t,f_t^n)dt\Big] \geq \mathbb{E}\Big[\int_0^T V^+(t,\Gamma_t^*)dt\Big].$$

The optimality of Γ_t^* will follow if we can show

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_0^T V^-(t,f_t^n)dt\Big] = \mathbb{E}\Big[\int_0^T V^-(t,\Gamma_t^*)dt\Big], \tag{5.16}$$

but the validity of (5.16) is a consequence of Lemma 5.7. □

For the proof of conjugate duality relations between value functions $\tilde{u}(x,z)$ and $\tilde{v}(y,r)$, by the exact same proof, we can generalize Lemma 11 of *Hugonnier and Kramkov* [16] in the following way:

Lemma 5.9. *If $\mathcal{G}\subseteq\mathbb{L}^0_+$ is convex and contains a strictly positive random variable. Then*

$$\sup_{g\in\mathcal{G}}\mathbb{E}\Big[\int_0^T U(t,xg_t)dt\Big]=\sup_{g\in cl\mathcal{G}}\mathbb{E}\Big[\int_0^T U(t,xg_t)dt\Big],\quad x>0$$

where $cl\mathcal{G}$ denotes the closure of $\mathcal{G}$ with respect to convergence in probability $\bar{\mathbb{P}}$.

Lemma 5.10. *For $\tilde{w}_t\triangleq e^{\int_0^t(-\alpha_v)dv}$, we have the following result:*

$$\mathbb{E}\Big[\int_0^T V^-(t,U'(t,\tilde{w}_t))dt\Big]<\infty \tag{5.17}$$

Proof. Similar to the proof of Lemma 5.6, recall the Assumption that $\mathbb{E}\Big[\int_0^T U(t,\bar{x}\tilde{w}_t)dt\Big]>-\infty$, taking into account the inequality $U(t,x)<V(t,y)+xy$, we have for any $y_0(t)\triangleq\inf\{y>0:V(t,y)<0\}$

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T V^-(t,U'(t,\tilde{w}_t))dt\Big]\leq-\mathbb{E}\Big[\int_0^T V(t,U'(t,\tilde{w}_t)1_{\{U'(t,\tilde{w}_t)\geq y_0(t)\}}+y_0(t)1_{\{U'(t,\tilde{w}_t)<y_0(t)\}})dt\Big]\\
\leq&-\mathbb{E}\Big[\int_0^T U(t,\bar{x}\tilde{w}_t)dt\Big]+\bar{x}\mathbb{E}\Big[\int_0^T\tilde{w}_tU'(t,\tilde{w}_t)dt\Big]+\bar{x}\mathbb{E}\Big[\int_0^T\tilde{w}_t(y_0(t)-U'(t,\tilde{w}_t))1_{\{U'(t,\tilde{w}_t)<y_0(t)\}}dt\Big]\\
\leq&-\mathbb{E}\Big[\int_0^T U(t,\bar{x}\tilde{w}_t)dt\Big]+\bar{x}\mathbb{E}\Big[\int_0^T\tilde{w}_tU'(t,\tilde{w}_t)dt\Big]+\bar{x}\int_0^T y_0(t)dt.
\end{aligned}\tag{5.18}$$

We already know the first term and the third term are bounded, as for the second term, we have two different cases:

1. If we have $\bar{x}\leq 1$, then we can rewrite the second term as

$$\mathbb{E}\Big[\int_0^T\tilde{w}_tU'(t,\tilde{w}_t)dt\Big]=\mathbb{E}\Big[\int_0^T\tilde{w}_tU'(t,\tilde{w}_t)1_{\{\tilde{w}_t\leq x_0\}}dt\Big]+\mathbb{E}\Big[\int_0^T\tilde{w}_tU'(t,\tilde{w}_t)1_{\{\tilde{w}>x_0\}}dt\Big],$$

where x_0 is the uniform constant in Corollary 2.1 such that for all $t\in[0,T]$,

$$xU'(t,x)<\Big(\frac{\gamma}{1-\gamma}\Big)\Big(-U(t,x)\Big)\ \text{ for } 0<x\leq x_0. \tag{5.19}$$

Again, use the fact that $\tilde{w}\preceq 1$, we have

$$\mathbb{E}\Big[\int_0^T\tilde{w}_tU'(t,\tilde{w}_t)1_{\{\tilde{w}>x_0\}}dt\Big]<\infty,$$

and we also have

$$\mathbb{E}\Big[\int_0^T\tilde{w}_tU'(t,\tilde{w}_t)1_{\{\tilde{w}_t\leq x_0\}}dt\Big]\leq-\Big(\frac{\gamma}{1-\gamma}\Big)\mathbb{E}\Big[\int_0^T U(t,\tilde{w}_t)dt\Big]\leq-\Big(\frac{\gamma}{1-\gamma}\Big)\mathbb{E}\Big[\int_0^T U(t,\bar{x}\tilde{w}_t)dt\Big]<\infty$$

by using the inequality (5.19), the increasing property of $U(t,x)$ with respect to x and the Assumption (3.4).

2. If we have $\bar{x} > 1$, then we rewrite the second term as:

$$\mathbb{E}\Big[\int_0^T \tilde{w}_t U'(t, \tilde{w}_t) dt\Big] = \mathbb{E}\Big[\int_0^T \tilde{w}_t U'(t, \tilde{w}_t) 1_{\{\bar{x}\tilde{w}_t \leq x_0\}} dt\Big] + \mathbb{E}\Big[\int_0^T \tilde{w}_t U'(t, \tilde{w}_t) 1_{\{\bar{x}\tilde{w} > x_0\}} dt\Big],$$

where x_0 is the uniform constant in Corollary 2.1 such that for all $t \in [0, T]$, the inequality (5.19) holds and moreover,

$$U(t, \frac{1}{\bar{x}} x) > (\frac{1}{\bar{x}})^{-\frac{\gamma}{1-\gamma}} U(t, x) \quad \text{for } 0 < x \leq x_0, \tag{5.20}$$

holds for all $t \in [0, T]$.
Then, again, the second term is bounded since $\bar{x}\tilde{w} \preceq \bar{x}$, and for the first term, we have

$$\begin{aligned}\mathbb{E}\Big[\int_0^T \tilde{w}_t U'(t, \tilde{w}_t) 1_{\{\bar{x}\tilde{w}_t \leq x_0\}} dt\Big] &\leq -\Big(\frac{\gamma}{1-\gamma}\Big) \mathbb{E}\Big[\int_0^T U(t, \tilde{w}_t) 1_{\{\bar{x}\tilde{w}_t \leq x_0\}} dt\Big] \\ &\leq -\Big(\frac{\gamma}{1-\gamma}\Big)(\frac{1}{\bar{x}})^{-\frac{\gamma}{1-\gamma}} \mathbb{E}\Big[\int_0^T U(t, \bar{x}\tilde{w}_t) dt\Big] < \infty\end{aligned}$$

by the inequality (5.19) and (5.20) and the Assumption (3.4).
Hence we proved the second term in (5.18) is also finite, and we can therefore conclude that result (5.17) holds true. □

We should now emphasize the fact that because these stochastic discounting factors α_t and δ_t are only assumed to be nonnegative and optional, the auxiliary dual domain $\widetilde{\mathcal{Y}}(y, r)$ is then not necessary a subset of $\mathbb{L}^1$, which fundamentally differs from the usual observations that the dual domain of pure investment or consumption optimization problem without habit formation is primarily a subset of $\mathbb{L}^1$. As a consequence, we have to revise the usual Minimax theorem based on $\mathbb{L}^1$ to derive the important conjugate duality relationship. Fortunately, the following Minimax theorem by *Kauppila* [19] can serve as a substitute tool on the space $\mathbb{L}^0_+$ without any priori assumption on the integrability of the dual process.

Theorem 5.2 (Minimax Theorem)**.** *Let A be a nonempty convex subset of a topological space, and B a nonempty, closed, convex, and convexly compact subset of a topological vector space. Let $H : A \times B \to \mathbb{R}$ be convex on A, and concave and upper-semicontinuous on B. Then*

$$\sup_B \inf_A H = \inf_A \sup_B H.$$

Lemma 5.11. *Under assumptions of Theorem 4.1, the conjugate duality relations hold:*

$$\begin{aligned}\tilde{u}(x, z) &= \inf_{(y,r)\in\mathcal{R}} \{\tilde{v}(y, r) + xy - zr\}, \qquad (x, z) \in \mathcal{H} \\ \tilde{v}(y, r) &= \sup_{(x,z)\in\mathcal{H}} \{\tilde{u}(x, z) - xy + zr\}, \qquad (y, r) \in \mathcal{R}.\end{aligned} \tag{5.21}$$

Proof. For $n > 0$, we define $\mathcal{S}_n$ as a subset in $\mathbb{L}^0_+(\Omega \times [0, T], \mathcal{O}, \bar{\mathbb{P}})$ as

$$\mathcal{S}_n = \{\tilde{c} \in \mathbb{L}^0_+ : 0 \preceq \tilde{c} \preceq n\tilde{w}\}.$$

It is clear that sets $\mathcal{S}_n$ are closed, convex, and bounded in probability, and hence convexly compact in $\mathbb{L}^0_+$.

We start to show the functional

$$\tilde{c} \mapsto \mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}_t) - \tilde{c}_t\Gamma_t\Big)dt\Big]$$

is upper-semicontinuous on $\mathcal{S}_n$ in the topology of convergence in measure $\bar{\mathbb{P}}$, for all $\Gamma \in \widetilde{\mathcal{Y}}(y,r)$ and $(y,r) \in \mathcal{R}$:

In fact, by passing if necessary to a subsequence denoted by $(\tilde{c}^m)_{m\geq 1}$ converges almost surely to $\tilde{c} \in \mathcal{S}_n$, Fatou's lemma deduces both

$$\liminf_{m\to\infty}\mathbb{E}\Big[\int_0^T U(t,\tilde{c}^m_t)^- dt\Big] \geq \mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)^- dt\Big], \tag{5.22}$$

and

$$\liminf_{m\to\infty}\mathbb{E}\Big[\int_0^T \tilde{c}^m_t\Gamma_t dt\Big] \geq \mathbb{E}\Big[\int_0^T \tilde{c}_t\Gamma_t dt\Big]. \tag{5.23}$$

Moreover, on $\mathcal{S}_n$, it is clear that $\mathbb{E}\Big[\int_0^T U(t,\tilde{c}^m_t)^+ dt\Big]$ is uniformly integrable, and hence

$$\lim_{m\to\infty}\mathbb{E}\Big[\int_0^T U(t,\tilde{c}^m_t)^+ dt\Big] = \mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)^+ dt\Big]. \tag{5.24}$$

Now, together with (5.22) and (5.23), we have

$$\limsup_{m\to\infty}\mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}^m_t) - \tilde{c}^m_t\Gamma_t\Big)dt\Big] \leq \mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}_t) - \tilde{c}_t\Gamma_t\Big)dt\Big].$$

Noting that, by Lemma 5.5, $\widetilde{\mathcal{Y}}(y,r)$ is a closed convex subset of $\mathbb{L}^0_+$, we may use the above Minimax Theorem 5.2 to get the following equality, for n fixed:

$$\sup_{\tilde{c}\in\mathcal{S}_n}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}_t) - \tilde{c}_t\Gamma_t\Big)dt\Big] = \inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\sup_{\tilde{c}\in\mathcal{S}_n}\mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}_t) - \tilde{c}_t\Gamma_t\Big)dt\Big].$$

Recall now the Bipolar relationship (5.9), and from the definition, we have

$$\bigcup_{(x,z)\in\mathcal{H}} \widetilde{\mathcal{A}}(x,z) = \bigcup_{k>0}\widetilde{\mathfrak{A}}(k). \tag{5.25}$$

As a preparation of the following proof, we define the auxiliary set

$$\mathfrak{A}'(k) \triangleq \Big\{\tilde{c} \in \widetilde{\mathfrak{A}}(k) : \sup_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\langle\tilde{c},\Gamma\rangle = k\Big\}$$

and clearly, we also have

$$\bigcup_{k>0}\widetilde{\mathfrak{A}}(k) = \bigcup_{(x,z)\in\mathcal{H}}\widetilde{\mathcal{A}}(x,z) = \bigcup_{k>0}\mathfrak{A}'(k). \tag{5.26}$$

We show first that

$$
\begin{aligned}
&\lim_{n\to\infty}\sup_{\tilde{c}\in\mathcal{S}_n}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\Big[\int_0^T\Big(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\Big)dt\Big]\\
&=\sup_{k>0}\sup_{\tilde{c}\in\mathfrak{A}'(k)}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\Big[\int_0^T\Big(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\Big)dt\Big].
\end{aligned}\tag{5.27}
$$

The direction of inequality "$\geq$" holds by

$$
\begin{aligned}
&\lim_{n\to\infty}\sup_{\tilde{c}\in\mathcal{S}_n}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\Big[\int_0^T\Big(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\Big)dt\Big]\\
\geq&\lim_{n\to\infty}\sup_{\tilde{c}\in\mathfrak{A}'(k)\cap\mathcal{S}_n}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\Big[\int_0^T\Big(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\Big)dt\Big]\\
=&\sup_{\tilde{c}\in\mathfrak{A}'(k)}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\Big[\int_0^T\Big(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\Big)dt\Big],\quad\forall k>0,
\end{aligned}
$$

while the other direction "$\leq$" is obvious since for any $(x,z)\in\mathcal{H}$, we have $n\tilde{w}\in\mathfrak{A}'(n\bar{p})$, and hence $\mathcal{S}_n\subset\mathfrak{A}'(n\bar{p})$.

To show the next step, we need to prepare some finiteness results as below:

From definitions in Lemma 5.5 and by Lemma 5.9, we know

$$
\sup_{\tilde{c}\in\widetilde{\mathfrak{A}}(k)}\mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)dt\Big]=\sup_{\tilde{c}\in\mathfrak{A}(k)}\mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)dt\Big]=\sup_{(x,z)\in k\mathfrak{H}(y,r)}\tilde{u}(x,z),\quad k>0.\tag{5.28}
$$

and we claim that

$$
\sup_{(x,z)\in k\mathfrak{H}(y,r)}\tilde{u}(x,z)<\infty,\quad k>0.\tag{5.29}
$$

To prove (5.29), recall that the set $\mathcal{R}$ is open, the set $\mathfrak{H}(y,r)$ is bounded and (5.29) follows from the concavity of $\tilde{u}$ and $\tilde{u}(x,z)<\infty$ for all $(x,z)\in\mathcal{H}$.

Now, by (5.26), (5.27), (5.28), (5.29) and the definition of domain $\mathcal{H}$, we have further equalities:

$$
\begin{aligned}
&\sup_{k>0}\sup_{\tilde{c}\in\mathfrak{A}'(k)}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\Big[\int_0^T\Big(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\Big)dt\Big]\\
=&\sup_{k>0}\Big\{\sup_{\tilde{c}\in\mathfrak{A}'(k)}\mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)dt\Big]-k\Big\}\\
=&\sup_{k>0}\Big\{\sup_{\tilde{c}\in\widetilde{\mathfrak{A}}(k)}\mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)dt\Big]-k\Big\}\\
=&\sup_{k>0}\Big\{\sup_{(x,z)\in k\mathfrak{H}(y,r)}\tilde{u}(x,z)-k\Big\}\\
=&\sup_{(x,z)\in\mathcal{H}}\{\tilde{u}(x,z)-xy+zr\}.
\end{aligned}
$$

On the other hand,

$$
\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\sup_{\tilde{c}\in\mathcal{S}_n}\mathbb{E}\Big[\int_0^T\Big(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\Big)dt\Big]=\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\Big[\int_0^T V^n(t,\Gamma_t,\omega)dt\Big]\triangleq\tilde{v}^n(y,r),
$$

where we define $V^n(t,y,\omega)$ according to the definition of set $\mathcal{S}_n$ as

$$V^n(t,y,\omega)=\sup_{0<x\leq n\tilde{w}}\Big[U(t,x)-xy\Big].$$

Consequently, it is sufficient to show that

$$\lim_{n\to\infty}\tilde{v}^n(y,r)=\lim_{n\to\infty}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\Big[\int_0^T V^n(t,\Gamma_t,\omega)dt\Big]=\tilde{v}(y,r),\quad (y,r)\in\mathcal{R}.$$

Evidently, $\tilde{v}^n(y,r)\leq\tilde{v}(y,r)$, for $n\geq 1$. Let $(\Gamma^n)_{n\geq 1}$ be a sequence in $\widetilde{\mathcal{Y}}(y,r)$ such that

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_0^T V^n(t,\Gamma_t^n,\omega)dt\Big]=\lim_{n\to\infty}\tilde{v}^n(y,r).$$

Then we can find a sequence $h^n\in conv(\Gamma^n,\Gamma^{n+1},\ldots)$, $n\geq 1$, converging almost surely to a variable Γ. We have $\Gamma\in\widetilde{\mathcal{Y}}(y,r)$, because the set $\widetilde{\mathcal{Y}}(y,r)$ is closed under convergence in probability.

Now, we claim the sequence of processes $(V^n(\cdot,h^n_\cdot,\omega)^-),n\geq 1$ is uniformly integrable, and in fact, we can rewrite

$$\Big(V^n(t,h^n_t,\omega)\Big)^-=\Big(V^n(t,h^n_t,\omega)\Big)^-1_{\{h^n_t\leq U'(t,\tilde{w}_t)\}}+\Big(V^n(t,h^n_t,\omega)\Big)^-1_{\{h^n_t>U'(t,\tilde{w}_t)\}},\tag{5.30}$$

and since $V^n(t,y,\omega)=V(t,y)$ for $y\geq U'(t,\tilde{w}_t)\geq U'(t,n\tilde{w}_t)$ by the definition. The argument from Lemma 5.7 asserts the uniform integrability of the sequence of processes $\Big(V^n(\cdot,h^n_\cdot,\omega)\Big)^-1_{\{h^n_\cdot>U'(\cdot,\tilde{w}_\cdot)\}},n\geq 1$.

On the other hand, by the monotonicity of $(V^n)^-$, we have for all $n>1$,

$$\Big(V^n(t,h^n_t,\omega)\Big)^-1_{\{h^n_t\leq U'(t,\tilde{w}_t)\}}\leq\Big(V^1(t,h^n_t,\omega)\Big)^-1_{\{h^n_t\leq U'(t,\tilde{w}_t)\}}\leq\Big(V(t,U'(t,\tilde{w}_t))\Big)^-\tag{5.31}$$

and by Lemma 5.10 the right hand side is integrable in the product space, and hence we conclude the sequence $\Big(V^n(\cdot,h^n_\cdot,\omega)\Big)^-1_{\{h^n_\cdot\leq U'(\cdot,\tilde{w}_\cdot)\}},n\geq 1$ is also uniformly integrable, and hence our claim holds true. Moreover, we will have the following inequalities:

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_0^T V^n(t,\Gamma_t^n,\omega)dt\Big]\geq\liminf_{n\to\infty}\mathbb{E}\Big[\int_0^T V^n(t,h_t^n,\omega)dt\Big]\geq\mathbb{E}\Big[\int_0^T V(t,\Gamma_t)dt\Big]\geq\tilde{v}(y,r).$$

which proves:

$$\tilde{v}(y,r)=\sup_{(x,z)\in\mathcal{H}}\{\tilde{u}(x,z)-xy+zr\}.\tag{5.32}$$

For the other equality (5.21), define the function $f(x,z)$ from $\mathbb{R}^2$ to $\bar{\mathbb{R}}$ as

$$f(x,z)\triangleq\begin{cases} cl(-\tilde{u}(x,z)) & (x,z)\in cl\mathcal{H},\\ \infty, & \text{otherwise.}\end{cases}\tag{5.33}$$

where $cl(-\tilde{u}(x,z))$ is the lower semicontinuous hull of function $-u(x,z)$. Then f is a proper, convex and lower-semicontinuous function on $\mathbb{R}$ and notice $int(dom(f))=\mathcal{H}$. By Corollary 12.2.2 in *Rockafella* [26], its Fenchel-Legendre transform is defined by

$$\tilde{f}(y,r)=\sup_{(x,z)\in\mathbb{R}^2}(-xy+zr-f(x,z))=\sup_{(x,z)\in\mathcal{H}}(-xy+zr+\tilde{u}(x,z)),\quad (y,r)\in\mathbb{R}^2.$$

Observe that if $(y,r)\in\mathcal{R}$, we have $\tilde{f}(y,r)=\tilde{v}(y,r)$ by (5.32), and if $(y,r)\notin cl\mathcal{R}$, we have by the increasing property of $\tilde{u}(x,z)$ that

$$\tilde{f}(y,r)\geq s(-x_0y+z_0r)+\tilde{u}(x_0,z_0)$$

for any $s>1$ and fixed $(x_0,z_0)\in\mathcal{H}$. We can therefore conclude that $\tilde{f}(y,r)=\infty$ for $(y,r)\notin cl\mathcal{R}$ since $-x_0y+z_0r>0$ by the definition of $\mathcal{R}$. We can thus apply Theorem 12.2 in *Rockafella* [26] to derive that

$$f(x,z)=\sup_{(y,r)\in\mathbb{R}^2}(-xy+zr-\tilde{f}(y,r)),\quad \forall (x,z)\in\mathbb{R}^2,$$

Again, by Corollary 12.2.2 in *Rockafella* [26] and the fact that $int(dom(\tilde{f}))=int(dom(\tilde{v}))\subseteq\mathcal{R}$, we further have

$$f(x,z)=\sup_{(y,r)\in\mathcal{R}}(-xy+zr-\tilde{v}(y,r))=-\inf_{(y,r)\in\mathcal{R}}(\tilde{v}(y,r)+xy-zr),\quad \forall (x,z)\in\mathbb{R}^2,.$$

In particular, we deduce that relation

$$\tilde{u}(x,z)=\inf_{(y,r)\in\mathcal{R}}\{\tilde{v}(y,r)+xy-zr\},\quad \forall (x,z)\in\mathcal{H},.$$

$\square$

PROOF OF THEOREM 4.1.

It is now sufficient to show the conjugate value function $\tilde{v}$ is $(-\infty,\infty]$-valued on $\mathcal{R}$.

Now, according to the definition of Legendre transform, we have

$$U(t,x)\leq V(t,y)+xy$$

by integration, it is easy to see for any $\tilde{c}\in\widetilde{\mathcal{A}}(x,z)$ and $\Gamma\in\widetilde{\mathcal{Y}}(y,r)$, we have

$$\mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)dt\Big]\leq\mathbb{E}\Big[\int_0^T V(t,\Gamma_t)dt\Big]+\mathbb{E}\Big[\int_0^T \tilde{c}_t\Gamma_t dt\Big],$$

from which Proposition 5.1 deduces that

$$\tilde{u}(x,z)\leq\tilde{v}(y,r)+xy-zr,$$

and hence we obtain for all $(y,r)\in\mathcal{R}$, we have $\tilde{v}(y,r)>-\infty$ by Lemma 5.4.

On the other hand, thanks to conjugate duality (5.21) and Bipolar relationship (5.9), follow the proofs in Lemma 5.5 and Lemma 5.11, we also have for each fixed $(y,r)\in\mathcal{R}$

$$\sup_{(x,z)\in k\mathfrak{H}(y,r)}\tilde{u}(x,z)=\inf_{s>0}\{\tilde{v}(sy,sr)+ks\}.$$

The finiteness result (5.29) for all $k>0$ in the proof of Lemma 5.11 guarantees the existence of a constant $s(y,r)>0$, such that $\tilde{v}(sy,sr)<\infty$.

$\square$

5.2. **The Proof of Theorem** 4.2**.** Let's move on to the proof of Theorem 4.2, to this end, we will need some further lemmas and priori results.

Lemma 5.12. *Under assumptions of Theorem* 4.2*, we have* $\tilde{v}(y,r)$ *is* $(-\infty,\infty)$*-valued on* $\mathcal{R}$*.*

Proof. Similar to the proof of Lemma 5.4, under the additional Assumption (2.9), we can show $\tilde{v}(y,r)<\infty$ if $\tilde{v}(sy,sr)<\infty$ for a constant $s=s(y,r)>0$. And we have shown that Theorem 4.1 asserts the existence of $s=s(y,r)>0$. □

To obtain the existence and uniqueness of our auxiliary primal Utility Maximization problem (3.18), we resort to a further auxiliary optimization problem of the auxiliary dual Utility Minimization problem (4.5), and make advantage of the Bipolar results built in Lemma 5.5.

Lemma 5.13. *Define the auxiliary optimization problem to the auxiliary dual Utility Minimization problem* (4.5) *as:*

$$\hat{v}(k)=\inf_{\Gamma\in\widetilde{\mathfrak{Y}}(k)}\mathbb{E}\Big[\int_0^T V(t,\Gamma_t)dt\Big], \tag{5.29}$$

where $\widetilde{\mathfrak{Y}}(k)$ *is defined in Lemma* 5.5 *as the bipolar set of* $\widetilde{\mathcal{A}}(x,z)$ *on the product space for any* $(x,z)\in\mathcal{H}$*.*

Then, for all $k>0$*, under hypothesis of Theorem* 4.2*, the value function* $\hat{v}(k)<\infty$ *for all* $k>0$*, and the optimal solution* $\hat{\Gamma}(k)$ *exists and is unique and* $\hat{\Gamma}_t(k)>0$ *for all* $t\in[0,T]$*. Moreover, for each* $k>0$*, and any* $\Gamma\in\widetilde{\mathfrak{Y}}(k)$*, we have*

$$\mathbb{E}\Big[\int_0^T(\Gamma_t-\hat{\Gamma}_t(k))I(t,\hat{\Gamma}_t(k))dt\Big]\le 0.$$

Proof. According to the definition in Lemma 5.5, it is easy to see

$$\hat{v}(k)=\inf_{\Gamma\in\widetilde{\mathfrak{Y}}(k)}\mathbb{E}\Big[\int_0^T V(t,\Gamma_t)dt\Big]\le\inf_{\Gamma\in\mathfrak{Y}(k)}\mathbb{E}\Big[\int_0^T V(t,\Gamma_t)dt\Big]=\inf_{(y,r)\in k\mathfrak{R}(x,z)}\tilde{v}(y,r)<\infty, k>0.$$

by Lemma 5.12.

Taking into account the Bipolar relationship (5.11), we have $\widetilde{\mathfrak{Y}}(k)$ is convexly compact in $\mathbb{L}^0_+$, the existence and uniqueness of optimal solution $\hat{\Gamma}(k)$ will follow the similar proof of Theorem 4.1.

Now, for $k>0$, $\epsilon\in(0,1)$ and define $\Gamma^\epsilon_t=(1-\epsilon)\hat{\Gamma}_t(k)+\epsilon\Gamma_t$, for all $t\in[0,T]$, the optimality of $\hat{\Gamma}(k)$ implies

$$\begin{aligned}0\le&\frac{1}{\epsilon}\mathbb{E}\Big[\int_0^T\Big(V(t,\Gamma^\epsilon_t)-V(t,\hat{\Gamma}_t(k))\Big)dt\Big]\\ \le&\frac{1}{\epsilon}\mathbb{E}\Big[\int_0^T\Big(\hat{\Gamma}_t(k)-\Gamma^\epsilon_t\Big)I(t,\Gamma^\epsilon_t)dt\Big]\\ =&\mathbb{E}\Big[\int_0^T\Big(\hat{\Gamma}_t(k)-\Gamma_t\Big)I(t,\Gamma^\epsilon_t)dt\Big].\end{aligned} \tag{5.30}$$

We claim the family $\Big\{\Big((\Gamma_t-\hat{\Gamma}_t(k))I(t,\Gamma_t^{\epsilon})\Big)^-, \epsilon\in(0,1)\Big\}$ is uniformly integrable with respect to $\bar{\mathbb{P}}$, since first

$$\Big((\Gamma_t-\hat{\Gamma}_t(k))I(t,\Gamma_t^{\epsilon})\Big)^- \leq \hat{\Gamma}_t(k)I(t,\Gamma_t^{\epsilon}) \leq \hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k)),\ \forall t\in[0,T].$$

We fix $\epsilon_0<1$ and observe that for $\epsilon<\epsilon_0$, we have for each $t\in[0,T]$,

$$\begin{aligned}\Big|\hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k))\Big| \leq & \Big|\hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k))\Big|1_{\{\hat{\Gamma}_t(k)\leq y_1\}} + \Big|\hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k))\Big|1_{\{\hat{\Gamma}_t(k)\geq \frac{y_2}{1-\epsilon_0}\}}\\ &+\Big|\hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k))\Big|1_{\{y_1<\hat{\Gamma}_t(k)<\frac{y_2}{1-\epsilon_0}\}}.\end{aligned}$$

Now fix $\epsilon_0<1$ and observe that for $\epsilon<\epsilon_0$, recall by Lemma 2.1 and Corollary 2.1, assumptions on Reasonable Asymptotic Elasticity $AE_0[U]<\infty$ and $AE_\infty[U]<1$ imply for fixed $\mu>0$, the existence of constants $C_1>0$, $C_2>0$, $y_1>0$ and $y_2>0$ such that

$$\begin{aligned} -V'(t,\mu y) &< C_1\frac{V(t,y)}{y} \quad \text{for } 0<y\leq y_1,\\ -V'(t,y) &< C_2\frac{-V(t,y)}{y} \quad \text{for } y_2\leq y.\end{aligned} \tag{5.31}$$

Hence, the first term is dominated by

$$\Big|\hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k))\Big|1_{\{\hat{\Gamma}_t(k)\leq y_1\}} \leq \frac{1}{1-\epsilon_0}C_1V(t,\hat{\Gamma}_t(k)),$$

and the send term is dominated by

$$\Big|\hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k))\Big|1_{\{\hat{\Gamma}_t(k)\geq \frac{y_2}{1-\epsilon_0}\}} \leq \frac{-1}{1-\epsilon_0}C_2V(t,(1-\epsilon)\hat{\Gamma}_t(k)) \leq \frac{-1}{1-\epsilon_0}C_2V(t,\hat{\Gamma}_t(k)).$$

These two terms are both in $\mathbb{L}^1$ by the finiteness of $\hat{v}(k)$. On the other hand, the third remaining term $\Big|\hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k))\Big|1_{\{y_1<\hat{\Gamma}_t(k)<\frac{y_2}{1-\epsilon_0}\}}$ is dominated by $k\hat{\Gamma}_t(k)1_{\{y_1<\hat{\Gamma}_t(k)<\frac{y_2}{1-\epsilon_0}\}}$ for a constant $k>0$, and it is obviously integrable as well.

Now we can let $\epsilon\to 0$ and apply Dominated convergence theorem and Fatou's lemma to obtain the stated inequality.

To show the optimal solution $\hat{\Gamma}_t(k)>0$ for all $t\in[0,T]$, it is enough to rewrite the inequality (5.30) as

$$0\geq \mathbb{E}\Big[\int_0^T\Big(\Gamma_t-\hat{\Gamma}_t(k)\Big)I(t,\Gamma_t^{\epsilon})\mathbf{1}_{\{\hat{\Gamma}_t>0\}}dt\Big]+\mathbb{E}\Big[\int_0^T\Big(\Gamma_t-\hat{\Gamma}_t(k)\Big)I(t,\Gamma_t^{\epsilon})\mathbf{1}_{\{\hat{\Gamma}_t=0\}}dt\Big].$$

Now suppose $\bar{\mathbb{P}}\{\hat{\Gamma}_t(k)=0\}>0$, then by the uniform integrability of $\Big\{\Big((\Gamma_t-\hat{\Gamma}_t(k))I(t,\Gamma_t^{\epsilon})\Big)^-, \epsilon\in (0,1)\Big\}$, let ϵ converges to 0, the second term of (5.32) goes to ∞, since $I(t,0)=\infty$, and $\Gamma_t>0$ for all $t\in[0,T]$, and we obtain the contradiction. Hence the conclusion holds.

□

Lemma 5.14. *Under Assumptions of Theorem* 4.2*, the auxiliary dual value function* $\hat{v}(k)$ *is continuously differentiable on* $(0,\infty)$*, and*

$$-k\hat{v}'(k)=\mathbb{E}\Big[\int_0^T\hat{\Gamma}_t(k)I(t,\hat{\Gamma}_t(k))dt\Big]. \tag{5.32}$$

Proof. In order to show $\hat{v}(k)$ is continuously differentiable, notice the convexity property, it is enough to justify that its derivative exists on $(0,\infty)$. Now fix $k>0$, and define the function

$$h(s)\triangleq\mathbb{E}\Big[\int_0^T V(t,\frac{s}{k}\hat{\Gamma}_t(k))dt\Big].$$

This function is convex and by optimality of $\hat{\Gamma}(k)$ of problem (5.29), we have $h(s)\geq\hat{v}(s)$ for all $s>0$ and $h(k)=\hat{v}(k)$. Again, by convexity, we obtain

$$\Delta^-h(k)\leq\Delta^-\hat{v}(k)\leq\Delta^+\hat{v}(k)\leq\Delta^+h(k),$$

where Δ^+ and Δ^- denote right- and left-derivatives, respectively. Now

$$\begin{aligned}\Delta^+h(k)=&\lim_{\epsilon\to0}\frac{h(k+\epsilon)-h(k)}{\epsilon}=\lim_{\epsilon\to0}\frac{1}{\epsilon}\mathbb{E}\Big[\int_0^T\Big(V(t,\frac{k+\epsilon}{k}\hat{\Gamma}_t(k))-V(t,\hat{\Gamma}_t(k))\Big)dt\Big]\\ \leq&\liminf_{\epsilon\to0}\Big(-\frac{1}{k\epsilon}\Big)\mathbb{E}\Big[\int_0^T\epsilon\hat{\Gamma}_t(k)I(t,\frac{k+\epsilon}{k}\hat{\Gamma}_t(k))dt\Big]\\ =&-\frac{1}{k}\mathbb{E}\Big[\int_0^T\hat{\Gamma}_t(k)I(t,\hat{\Gamma}_t(k))dt\Big],\end{aligned}$$

by the Monotone Convergence Theorem. Similarly, we get

$$\Delta^-h(k)\geq\limsup_{\epsilon\to0}\mathbb{E}\Big[-\int_0^T\hat{\Gamma}_t(k)I(t,\frac{k-\epsilon}{k}\hat{\Gamma}_t(k))dt\Big].$$

We can follow the same reasoning as in Lemma 5.13 to show the family $\{(\hat{\Gamma}_t(k)I(t,\frac{k-\epsilon}{k}\hat{\Gamma}_t(k))),\epsilon\in(0,1)\}$ is uniformly integrable, and Dominated Convergence Theorem deduces

$$\Delta^-h(k)\geq-\frac{1}{k}\mathbb{E}\Big[\int_0^T\hat{\Gamma}_t(k)I(t,\hat{\Gamma}_t(k))dt\Big]$$

which completes the proof. □

Lemma 5.15. *The auxiliary dual value function* $\hat{v}(\cdot)$ *has the asymptotic property:*

$$-\hat{v}'(0)=\infty,\qquad-\hat{v}'(\infty)=0. \tag{5.33}$$

Proof. We first show $-\hat{v}'(0)=\infty$, and to this end, we can first derive the result that

$$\hat{v}(0+)\geq\int_0^T V(t,0+)dt. \tag{5.34}$$

To prove the validity of (5.34), we observe that for any $k>0$, by the definition we have

$$\hat{v}(k)=\mathbb{E}\Big[\int_0^T V(t,\hat{\Gamma}_t(k))dt\Big]=\mathbb{E}\Big[\int_0^T V^+(t,\hat{\Gamma}_t(k))dt\Big]-\mathbb{E}\Big[\int_0^T V^-(t,\hat{\Gamma}_t(k))dt\Big],$$

hence, by Fatou's Lemma, firstly, we have

$$\lim_{k\to 0}\mathbb{E}\Big[\int_0^T V^+(t,\hat{\Gamma}_t(k))dt\Big]\geq \mathbb{E}\Big[\int_0^T V^+(t,0+)dt\Big]$$

and on the other hand, similar to the proof of Lemma 5.6, we can show that

$$\mathbb{E}\Big[\int_0^T V^-(t,\hat{\Gamma}_t(1))dt\Big]<\infty, \tag{5.35}$$

and therefore, by the Monotonicity of function $V^-(t,\cdot)$ and Dominated Convergence Theorem, we can easily derive that

$$\lim_{k\to 0}\mathbb{E}\Big[\int_0^T V^-(t,\hat{\Gamma}_t(k))dt\Big]\geq \mathbb{E}\Big[\int_0^T V^-(t,0+)dt\Big],$$

which together with (5.35) implies that (5.34) holds true.

Therefore, if $\int_0^T V(t,0+)dt=\infty$, then we have $\hat{v}(0+)=\infty$, and by convexity, we have $\hat{v}'(0+)=-\infty$.

In the case $\int_0^T V(t,0+)dt<\infty$, we then have

$$-\hat{v}(0+)\geq \lim_{k\to 0}\frac{\hat{v}(0)-\hat{v}(k)}{k}\geq \lim_{k\to 0}\frac{\int_0^T V(t,0+)dt-\mathbb{E}\Big[\int_0^T V(t,\hat{\Gamma}_t(k))dt\Big]}{k}.$$

and hence we have

$$\begin{aligned}-\hat{v}(0+)&\geq \lim_{k\to 0}\frac{\mathbb{E}\Big[\int_0^T V(t,0+)dt\Big]-\mathbb{E}\Big[\int_0^T V(t,\hat{\Gamma}_t(k))dt\Big]}{k}\\&\geq \lim_{k\to 0}\mathbb{E}\Big[\int_0^T \hat{\Gamma}_t(1)I(t,k\hat{\Gamma}_t(1))dt\Big]=\infty\end{aligned}$$

by the Monotone Convergence Theorem.

We can now turn to show that $-\hat{v}'(\infty)=0$, and since the function $-\hat{v}$ is concave and increasing, there is a finite positive limit

$$-\hat{v}'(\infty)\triangleq \lim_{k\to\infty}-\hat{v}'(y).$$

By the definition of Legendre Transform, we clearly have for any $y>0$,

$$-V(t,y)\leq -U(t,x)+xy,\text{for all } x>0,$$

and then for any $\epsilon>0$, we always have:

$$\begin{aligned}0\leq -\hat{v}'(\infty)=\lim_{k\to\infty}\frac{-\hat{v}(k)}{k}&=\lim_{k\to\infty}\frac{\mathbb{E}\Big[\int_0^T -V(t,\hat{\Gamma}_t(k))dt\Big]}{k}\\&\leq \lim_{k\to\infty}\frac{\mathbb{E}\Big[\int_0^T -U(t,\epsilon\tilde{w}_t)dt\Big]}{k}+\lim_{k\to\infty}\frac{\big\langle \epsilon\tilde{w},\hat{\Gamma}(k)\big\rangle}{k}.\end{aligned}$$

Now, recall that for each fixed $(x,z)\in\mathcal{H}$, there exists a constant $\lambda(x,z)>0$ such that we have $\tilde{w}_t\in\widetilde{\mathcal{A}}(\frac{\bar{p}}{\lambda}x,\frac{\bar{p}}{\lambda}z)$, and by the definition of $\widetilde{\mathfrak{Y}}(k)$, we can see the second term above has

$$\lim_{k\to\infty}\frac{\big\langle \epsilon\tilde{w},\hat{\Gamma}(k)\big\rangle}{k}\leq \lim_{k\to\infty}\frac{\epsilon\frac{\bar{p}}{\lambda}k}{k}=\epsilon\frac{\bar{p}}{\lambda}.$$

As for the first term, we claim that $\mathbb{E}\Big[\int_0^T -U(t,\epsilon\tilde{w}_t)dt\Big] < \infty$ for each fixed ϵ small enough, without loss of generality, we just need to consider that $\epsilon < \bar{x}$, and then we will apply Corollary 2.1 again, and since there exists a constant x_0 such that for all $t \in [0,T]$,

$$U(t,\frac{\epsilon}{\bar{x}}x) > (\frac{\epsilon}{\bar{x}})^{-\frac{\gamma}{1-\gamma}}U(t,x) \ \text{ for } 0 < x \leq x_0,$$

we will have

$$\begin{aligned}\mathbb{E}\Big[\int_0^T -U(t,\epsilon\tilde{w}_t)dt\Big] &= \mathbb{E}\Big[\int_0^T -U(t,\epsilon\tilde{w}_t)1_{\{\bar{x}\tilde{w}_t > x_0\}}dt\Big] + \mathbb{E}\Big[\int_0^T -U(t,\epsilon\tilde{w}_t)1_{\{\bar{x}\tilde{w}_t \leq x_0\}}dt\Big]\\ &\leq \mathbb{E}\Big[\int_0^T -U(t,\epsilon\tilde{w}_t)1_{\{\bar{x}\tilde{w}_t > x_0\}}dt\Big] + (\frac{\epsilon}{\bar{x}})^{-\frac{\gamma}{1-\gamma}}\mathbb{E}\Big[\int_0^T -U(t,\bar{x}\tilde{w}_t)dt\Big] < \infty\end{aligned}$$

by the fact that $\tilde{w} \preceq 1$ and the Assumption (3.4).

Hence, we conclude that

$$0 \leq -\hat{v}'(\infty) = \lim_{k\to\infty}\frac{-\hat{v}(k)}{k} \leq \epsilon\frac{\bar{p}}{\lambda},$$

and consequently, we have $-\hat{v}'(\infty) = 0$ by letting ϵ goes to 0. □

Lemma 5.16. *Under assumptions of Theorem 4.2, for any $(x,z) \in \mathcal{H}$, suppose k satisfies $1 = -\hat{v}'(k)$ where $\hat{v}(k)$ is the value function of the auxiliary dual optimization problem* (5.29)*, then $\tilde{c}^*_t(x,z) \triangleq I(t,\hat{\Gamma}_t(k))$ is the unique (in the sense of $\equiv$ in $\mathbb{L}^0_+$) optimal solution to problem* (3.18)*, moreover we have $\tilde{c}^*_t(x,z) > 0$, $\mathbb{P}$-a.s. for all $t \in [0,T]$.*

Proof. Lemma 5.14 asserts

$$\langle \tilde{c}^*(x,z), \hat{\Gamma}(k)\rangle = -k\hat{v}'(k) = k.$$

And for any $\Gamma \in \widetilde{\mathfrak{Y}}(k)$, by Lemma 5.13, we have

$$\langle \tilde{c}^*(x,z), \Gamma(k)\rangle \leq \langle \tilde{c}^*(x,z), \hat{\Gamma}(k)\rangle = k.$$

Hence, we get first $\tilde{c}^*_t(x,z) \in \widetilde{\mathcal{A}}(x,z)$ by the Bipolar relationship (5.11).

Now, for any $\tilde{c} \in \widetilde{\mathcal{A}}(x,z)$, we have

$$\begin{aligned}\langle \tilde{c}, \hat{\Gamma}(k)\rangle &\leq k,\\ U(t,\tilde{c}_t) &\leq V(t,\hat{\Gamma}_t(k)) + \tilde{c}_t\hat{\Gamma}_t(k), \ \forall t \in [0,T].\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)dt\Big] &\leq \hat{v}(k) + k = \mathbb{E}\Big[\int_0^T \Big(V(t,\hat{\Gamma}_t(k)) + \hat{\Gamma}_t I(t,\hat{\Gamma}_t(k))\Big)dt\Big]\\ &= \mathbb{E}\Big[\int_0^T U(t,I(\hat{\Gamma}_t(k)))dt\Big] = \mathbb{E}\Big[\int_0^T U(t,\tilde{c}^*_t)dt\Big],\end{aligned} \tag{5.36}$$

shows the optimality of $\tilde{c}^*$. The uniqueness of the optimal solution follows from the strict concavity of the function U.

Moreover, under assumptions of Theorem 4.2, for any pair $(x,z) \in \mathcal{H}$, by the fact that $\widetilde{\mathfrak{Y}}(k)$ is convexly compact and $\hat{\Gamma}_t(k)$ is bounded in probability, we actually have the optimal solution

$\tilde{c}_t^*(x,z) > 0$, $\mathbb{P}$-a.s. for all $t \in [0,T]$ since $\hat{\Gamma}_t(k)$ is bounded in probability if and only if $\hat{\Gamma}_t(k)$ is finite $\bar{\mathbb{P}}$-a.s. and by definition, we know $I(t,x) > 0$ for $x < \infty$.

$\square$

Let $(x,z) \in cl\mathcal{H}$, the proof of Lemma 5.1 states there exists $\tilde{c} \in \widetilde{\mathcal{A}}(x,z)$ such that $\bar{\mathbb{P}}[\tilde{c} \succ 0] > 0$. Similar to the proof of Lemma 12 of *Hugonnier and Kramkov* [16], we will have:

Lemma 5.17. *Assume that the assumptions of Proposition of* 5.1 *hold true. Let the sequences* $(y^n, r^n) \in \mathcal{R}$ *and* $\Gamma^n \in \widetilde{\mathcal{Y}}(y^n, r^n)$, $n \geq 1$, *converges to* $(y,r) \in \mathbb{R}^2$ *and* $\Gamma \in \mathbb{L}^0_+$, *respectively. If* Γ *is a strictly positive random variable, then* $(y,r) \in \mathcal{R}$ *and* $\Gamma \in \widetilde{\mathcal{Y}}(y,r)$.

PROOF OF THEOREM 4.2.

We first show the dual value function $\tilde{v}(y,z)$ is continuously differentiable on $\mathcal{R}$. Theorem 4.1.1 and 4.1.2 in *Hiriart-Urruty and Lemaréchal* [15] gives the equivalence between the above statement and the fact that the value function $\tilde{u}(x,z)$ is strictly concave on $\mathcal{H}$. Since U is a strictly concave function, to show the value function is strictly concave is equivalent to show for any two distinct points $(x_i, z_i) \in \mathcal{H}$, $i = 1,2$, the optimal consumption policies are different:

$$\bar{\mathbb{P}}[\tilde{c}^*(x_1,z_1) \neq \tilde{c}^*(x_2,z_2)] > 0,$$

which is equivalent to Assumption (4.2).

The attempt of the left proof to Theorem 4.2 reduces to show the assertion (ii) hold, and recall $\hat{\Gamma}(k)$ is the optimal solution of the auxiliary dual problem (5.29), such that

$$\hat{\Gamma}_t(k) = U'(t, \tilde{c}_t^*(x,z)), \quad \forall t \in [0,T], \quad k = \langle \tilde{c}^*(x,z), \hat{\Gamma}(k) \rangle.$$

By the definition that $\widetilde{\mathfrak{Y}}(k)$ is closed with respect to convergence in measure $\bar{\mathbb{P}}$, there exists a sequence $(y^n, r^n) \in k\mathfrak{R}(x,z)$ such that $\Gamma^n \in \widetilde{\mathcal{Y}}(y^n, r^n)$ and Γ^n converges to $\hat{\Gamma}(k)$ $\bar{\mathcal{P}}$-a.s. by passing to a subsequence if necessary, and since set $k\mathfrak{R}(x,z)$ is bounded, there exists a further subsequence (y^n, r^n) converges to $(y,r) \in \mathbb{R}^2$. By passing to this further subsequence, as we have shown $\bar{\mathbb{P}}[\hat{\Gamma}(k) \succ 0] = 1$, we will have $(y,r) \in k\mathfrak{R}(x,z)$ such that $\hat{\Gamma}(k) \in \widetilde{\mathcal{Y}}(y,r)$ due to Lemma 5.17. Moreover, for this pair $(y,r) \in \mathcal{R}$, by Fatou's Lemma and Proposition 5.1, we have the equality that

$$xy - zr = k = \langle \tilde{c}^*(x,z), \hat{\Gamma}(k) \rangle. \tag{5.37}$$

And we have the corresponding optimizer $\Gamma_t^*(y,r)$ of (4.5) verifies

$$\Gamma_t^*(y,r) = \hat{\Gamma}_t(k) = U'(t, \tilde{c}^*(x,z)), \tag{5.38}$$

because on one hand, we have $\hat{\Gamma}(k) \in \widetilde{\mathcal{Y}}(y,r)$, hence

$$\mathbb{E}\int_0^T V(t, \Gamma_t^*(y,r)) = \inf_{\Gamma \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\int_0^T V(t, \Gamma_t(y,r)) \leq \mathbb{E}\int_0^T V(t, \hat{\Gamma}_t(k)),$$

and on the other hand, we have

$$\mathbb{E}\int_0^T V(t,\hat{\Gamma}_t(y,r)) = \inf_{\Gamma\in\widetilde{\mathfrak{Y}}(k)} \mathbb{E}\int_0^T V(t,\Gamma_t(y,r)) \leq \inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\int_0^T V(t,\Gamma_t(y,r)) = \mathbb{E}\int_0^T V(t,\Gamma_t^*(y,r)),$$

By the equality

$$U(t,\tilde{c}_t^*(x,z)) = V(t,\hat{\Gamma}_t(k)) + \tilde{c}_t^*(x,z)\hat{\Gamma}_t(k),$$

we can conclude $(y,r)\in\partial\tilde{u}(x,z)$ by Theorem 23.5 of *Rockafellar* [26], since we have

$$\tilde{u}(x,z) = \tilde{v}(y,z) + xy - zr \tag{5.39}$$

In particular, we get

$$\partial\tilde{u}(x,z)\cap\mathcal{R}\neq\varnothing. \tag{5.40}$$

Similar to the proof of Theorem 2 in *Hugonnier and Kramkov* [16], we can actually show

$$\partial\tilde{u}(x,z)\subset\mathcal{R}.$$

For any $(y,r)\in\partial\tilde{u}(x,z)$, we can find a sequence $(y^n,r^n)\in\partial\tilde{u}(x,z)\cap\mathcal{R}$ converging to (y,r) by (5.40) and the fact that $\partial\tilde{u}(x,z)$ is closed and convex. Since $U'(\cdot,\tilde{c}_\cdot^*(x,z))$ is strictly positive and we know $U'(\cdot,\tilde{c}_\cdot^*(x,z))\in\widetilde{\mathcal{Y}}(y,r)$. Lemma 5.17 now infers $(y,r)\in\mathcal{R}$.

Conversely, for any $(y,r)\in\partial\tilde{u}(x,z)$, then

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \Big|V(t,\Gamma_t^*(y,r)) + \tilde{c}_t^*(x,z)\Gamma_t^*(y,r) - U(t,\tilde{c}_t^*(x,z))\Big|dt\Big]\\
=&\mathbb{E}\Big[\Big(\int_0^T V(t,\Gamma_t^*(y,r)) + \tilde{c}_t^*(x,z)\Gamma_t^*(y,r) - U(t,\tilde{c}_t^*(x,z))dt\Big)\Big]\\
\leq&\tilde{v}(y,r) + xy - zr - \tilde{u}(x,z) = 0,
\end{aligned}$$

which infers (5.37) and (5.38). □

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Appendix A.

A.1. **Proof of Lemma** 3.1.

Proof. It is enough to show for all $(x,z)\in(0,\infty)\times[0,\infty)$, $x\geq z\sup_{Y\in\mathcal{M}}\mathbb{E}\Big[\int_0^T e^{\int_0^t(\delta_v-\alpha_v)dv}Y_t dt\Big]$ if and only if $\bar{\mathcal{A}}(x,z)\neq\emptyset$.

On one hand, if $(x,z)\in(0,\infty)\times[0,\infty)$ and $\bar{\mathcal{A}}(x,z)\neq\emptyset$, by definition, there exists $c\in\mathbb{L}_+^0$ such that $c_t\geq Z_t$ for all $t\in[0,T]$ and $\langle c,Y\rangle\leq x,\forall Y\in\mathcal{M}$. We now claim that we should always have $c_t\geq\bar{c}_t$ for all $t\in[0,T]$ where $\bar{c}_t\equiv Z(\bar{c})_t$ is the subsistent consumption plan which equals its standard of living process. To this end, we first recall by the definition of Z_t that $dZ_t = (\delta_t c_t - \alpha_t Z_t)dt$ with $Z_0 = z \geq 0$, and the constraint that $c_t \geq Z_t$ implies

$$dZ_t \geq (\delta_t Z_t - \alpha_t Z_t)dt, \quad Z_0 = z, \tag{A.1}$$

also, we should have $\bar{c}_t$ satisfies

$$d\bar{c}_t = (\delta_t \bar{c}_t - \alpha_t \bar{c}_t)dt, \quad \bar{c}_0 = z. \tag{A.2}$$

and we can solve $\bar{c}_t = ze^{\int_0^t (\delta_v - \alpha_v)dv}$ for $t \in [0, T]$.

By the simple subtraction of $(A.1)$ and $(A.2)$, one can get

$$d(Z_t - \bar{c}_t) \geq (\delta_t - \alpha_t)(Z_t - \bar{c}_t)dt, \quad Z_0 - \bar{c}_0 = 0,$$

from which we can derive that

$$e^{\int_0^t (\delta_s - \alpha_s)ds}(Z_t - \bar{c}_t) \geq 0, \quad \forall t \in [0, T]. \tag{A.3}$$

Hence, we will conclude that $c_t \geq ze^{\int_0^t (\delta_v - \alpha_v)dv}$ for all $t \in [0, T]$, which gives $x \geq z \sup_{Y \in \mathcal{M}} \mathbb{E}\Big[\int_0^T e^{\int_0^t (\delta_v - \alpha_v)dv} Y_t dt\Big]$ by the consumption Budget Constraint condition (2.6).

One the other hand, if $(x, z) \in (0, \infty) \times [0, \infty)$ and $x \geq z \sup_{Y \in \mathcal{M}} \mathbb{E}\Big[\int_0^T e^{\int_0^t (\delta_v - \alpha_v)dv} Y_t dt\Big]$, we can obviously always construct $\bar{c}_t = ze^{\int_0^t (\delta_v - \alpha_v)dv}$ such that $\bar{c}_t \equiv Z(\bar{c})_t$ for all $t \in [0, T]$ and $\big\langle c, Y \big\rangle \leq x, \forall Y \in \mathcal{M}$, and hence $\bar{\mathcal{A}}(x, z) \neq \emptyset$. The proof is complete.

□

A.2. **Proof of Proposition** 3.1**.**

Proof. By the definition, Z_t solves the ODE: $dZ_t = (\delta_t c_t - \alpha_t Z_t)dt$ with $Z_0 = z$, for each $t \in [0, T]$. If we set $\tilde{c}_t = c_t - Z_t$, we can rewrite c_t in terms of $\tilde{c}_t$ as:

$$c_t = ze^{\int_0^t (\delta_v - \alpha_v)dv} + \tilde{c}_t + \int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v)dv} \tilde{c}_s ds,$$

and hence we will have the following chain equivalence by Tonelli's theorem:

$$\begin{aligned}
\big\langle c, Y \big\rangle =& z\mathbb{E}\Big[\int_0^T e^{\int_0^t (\delta_v - \alpha_v)dv} Y_t dt\Big] + \mathbb{E}\Big[\int_0^T \Big(\tilde{c}_t + \int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v)dv} \tilde{c}_s ds\Big) Y_t dt\Big] \\
=& z\big\langle w, Y \big\rangle + \mathbb{E}\Big[\int_0^T \tilde{c}_t Y_t dt + \int_0^T \delta_s \tilde{c}_s \Big(\int_s^T e^{\int_s^t (\delta_v - \alpha_v)dv} Y_t dt\Big) ds\Big] \\
=& z\big\langle w, Y \big\rangle + \mathbb{E}\Big[\int_0^T \tilde{c}_t Y_t dt + \int_0^T \delta_t \tilde{c}_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v)dv} Y_s ds \Big| \mathcal{F}_t\Big] dt\Big] \\
=& z\big\langle w, Y \big\rangle + \big\langle \tilde{c}, \Gamma \big\rangle,
\end{aligned}$$

which gives the first equality. Similarly, we just observe that:

$$
\begin{aligned}
\big\langle \tilde{w}, \Gamma \big\rangle =&\mathbb{E}\Big[\int_0^T e^{\int_0^t(-\alpha_v)dv} Y_t dt\Big] + \mathbb{E}\Big[\int_0^T e^{\int_0^t(-\alpha_v)dv}\delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s(\delta_v-\alpha_v)dv}Y_s ds\Big|\mathcal{F}_t\Big]dt\Big]\\
=&\mathbb{E}\Big[\int_0^T e^{\int_0^t(-\alpha_v)dv} Y_t dt\Big] + \mathbb{E}\Big[\int_0^T e^{\int_0^s(\delta_v-\alpha_v)dv}Y_s\Big(\int_0^s \delta_t e^{-\int_0^t\delta_v dv}dt\Big)ds\Big]\\
=&\mathbb{E}\Big[\int_0^T e^{\int_0^t(-\alpha_v)dv} Y_t dt\Big] - \mathbb{E}\Big[\int_0^T e^{\int_0^s(\delta_v-\alpha_v)dv}Y_s\Big(e^{-\int_0^s\delta_s ds}-1\Big)ds\Big]\\
=&\mathbb{E}\Big[\int_0^T e^{\int_0^t(-\alpha_v)dv} Y_t dt\Big] - \mathbb{E}\Big[\int_0^T e^{\int_0^t(-\alpha_v)dv} Y_t dt\Big] + \mathbb{E}\Big[\int_0^T e^{\int_0^t(\delta_v-\alpha_v)dv}Y_t dt\Big]\\
=&\big\langle w, Y\big\rangle,
\end{aligned}
$$

which gives the second equality. □

A.3. **Proof of Lemma** 3.2.

Proof. Again, it is just enough to show $\Big\{(x,z)\in\mathbb{R}^2 : x\geq z\big\langle\tilde{w},\Gamma\big\rangle,\ \text{for all } \Gamma_t\in\widetilde{\mathcal{M}}\Big\}$ is equivalent to $\Big\{(x,z)\in\mathbb{R}^2 : \widetilde{\mathcal{A}}(x,z)\neq\emptyset\Big\}$.

On one hand, if $(x,z)\in\Big\{(x,z)\in\mathbb{R}^2 : \widetilde{\mathcal{A}}(x,z)\neq\emptyset\Big\}$, there exists $\tilde{c}\in\mathbb{L}^0_+$ such that $\langle\tilde{c},\Gamma\rangle\leq x-z\langle\tilde{w},\Gamma\rangle$ for all $\Gamma\in\widetilde{\mathcal{M}}$, clearly, we get $x\geq z\big\langle\tilde{w},\Gamma\big\rangle$, for all $\Gamma\in\widetilde{\mathcal{M}}$.

On the other hand, if $(x,z)\in\Big\{(x,z)\in\mathbb{R}^2 : x\geq z\big\langle\tilde{w},\Gamma\big\rangle,\ \text{for all } \Gamma_t\in\widetilde{\mathcal{M}}\Big\}$, it is trivial to construct $\tilde{c}_t\equiv 0\in\widetilde{\mathcal{A}}(x,z)$ for all $t\in[0,T]$, therefore, we have $(x,z)\in\Big\{(x,z)\in\mathbb{R}^2 : \widetilde{\mathcal{A}}(x,z)\neq\emptyset\Big\}$, which completes the proof.

□

A.4. **Proof of Lemma** 4.1.

Proof. Since $\underline{p}<\bar{p}$ by Assumption (4.2), and by Lemma 3.2 the set $cl\mathcal{H}=\{(x,z)\in\mathbb{R}^2 : x\geq \bar{p}z, z\geq 0\}\cup\{(x,z)\in\mathbb{R}^2 : x\geq \underline{p}z, z<0\}$ does not contain any lines passing through the origin. By the properties of polars of convex sets (See *Rockafella* [26], Corollary 14.6.1), $\mathcal{R}$ is an open convex cone in the first orthant of $\mathbb{R}^2$. Moreover, by the inequality constraint $xy-zr\geq 0$ for all $(x,z)\in\mathcal{H}$ and the definition of $\mathcal{H}$, it is obvious that (4.3) holds. □

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