On a property of the n-dimensional cube

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Abstract. We show that in any subset of vertices of the n-dimensional cube which contains at least $2^{n-1}+1$ vertices ($n \geq 4$), there are four vertices that induce a claw, or there are eight vertices that induce the cycle of length eight.

We consider finite graphs $G = (V, E)$ with vertex set V and edge set E. The graphs contain no multiple edges or loops. The n-dimensional cube is denoted by Q_n, and a claw is the complete bipartite graph $K_{1,3}$. Moreover, the vertex of a degree three in a claw is called a claw-center. Non-defined terms and concepts can be found in [1].

The main result of the paper is the following:

Theorem 1. *Let $n \geq 4$ and let $V' \subseteq V(Q_n)$. If $|V'| \geq 2^{n-1}+1$, then at least one of the following two conditions holds:*

(a) there are four vertices in V' that induce a claw;

(b) there are eight vertices in V' that induce a simple cycle.

Proof. Our proof is by induction on n. Suppose that $n = 4$. Clearly, without loss of generality, we can assume that $|V'| = 9$. Consider the following partition of the vertices of Q_4:

$$V_1 = \{(0, \alpha_2, \alpha_3, \alpha_4) : \alpha_i \in \{0, 1\}, 2 \leq i \leq 4\}, V_2 = \{(1, \alpha_2, \alpha_3, \alpha_4) : \alpha_i \in \{0, 1\}, 2 \leq i \leq 4\}.$$

Clearly, the subgraphs of Q_4 induced by V_1 and V_2 are isomorphic to Q_3. Define:

$$V_1' = V_1 \cap V', V_2' = V_2 \cap V'.$$

We shall assume that $|V_1'| \geq |V_2'|$. We shall complete the proof of the base of induction by considering the following cases:

Case 1: $|V_1'| = 8$ and $|V_2'| = 1$. Clearly, any vertex from V_1' is a claw-center.

Case 2: $|V_1'| = 7$ and $|V_2'| = 2$. It is not hard to see that V_1' contains a claw-center.

Case 3: $|V_1'| = 6$ and $|V_2'| = 3$. Again, it is a matter of direct verification that V' contains a claw-center.

Case 4: $|V_1'| = 5$ and $|V_2'| = 4$. Consider the subgraph G_1 of Q_4 induced by V_1'. Clearly, if G_1 contains a vertex of a degree three, then this vertex is a claw-center. Therefore, without loss of generality, we can assume that any vertex in G_1 has a degree at most two. It is not hard to see that this implies that G_1 contains no isolated vertex. Moreover, since $|V_1'| = 5$, we can conclude that G_1 is a connected graph, and, consequently, it is the path of length four.

Now, let a_1, a_2, a_3 be the internal vertices of G_1, and let b_1, b_2 be the end-vertices of G_1. Clearly, we can assume that neither of a_1, a_2, a_3 has a neighbour in V_2'. Since $|V_2| = 8$ and $|V_2'| = 4$, we have that there are five possibilities for V_2'. We invite the reader to check that in four of these cases one can find a claw-center in V_2', and in the final case V' has a vertex z such that $V' \backslash \{z\}$ induces a simple cycle.

Now, let us assume that the statement is true for $n-1$, and a subset V' of the vertices of Q_n satisfies the inequality $|V'| \geq 2^{n-1}+1$. Consider the following partition of the vertices of Q_n:

$$V_1 = \{(0, \alpha_2, ..., \alpha_n) : \alpha_i \in \{0, 1\}, 2 \leq i \leq n\}, V_2 = \{(1, \alpha_2, ..., \alpha_n) : \alpha_i \in \{0, 1\}, 2 \leq i \leq n\}.$$

Clearly, the subgraphs of Q_n induced by V_1 and V_2 are isomorphic to Q_{n-1}. Moreover, it is not hard to see that at least one of the following two inequalities is true: $|V_1 \cap V'| \geq 2^{n-2}+1$ and $|V_2 \cap V'| \geq 2^{n-2}+1$. Thus the proof follows from the induction hypothesis. $\square$

For the case of $n = 3$ we have:

Proposition 1. *Let $V' \subseteq V(Q_3)$ and let $|V'| \geq 6$. Then at least one of the following two conditions holds:*

- *there are four vertices in V' that induce a claw;*
- *there are six vertices in V' that induce a simple cycle.*

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References

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