The Equivalence Principle Applicability Boundaries, QFT in Flat Space and Measurability I.Free Quantum Fields

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Abstract

The present paper continues a study of a quantum field theory in terms of the **measurability** notion introduced in the previous author's works. It should be noted that, without detriment to the consideration and results, we can lift some initial restrictions (limiting conditions) imposed in the above-mentioned papers. Specifically, it is not supposed initially that a theory involves some minimal length. Starting from some maximal momentum, we can use it subsequently together with a specific formula to derive the quantity with a dimension of length that is called the **primary** length. The first part of this paper is devoted to analysis of the applicability limit of Einstein's Equivalence Principle (EP). It is noted that a natural applicability limit of this Principle, associated with the development of quantum-gravitational effects at Planck's scales, is absolute, its more accurate estimation being dependent on the processes under study and on the sizes of the corresponding particles. It is shown that, neglecting the applicability limit of EP, one can obtain senseless results on estimation of the relevant quantities within the scope of the well-known Quantum Field Theory (QFT). Besides, such a neglect may be responsible for ultraviolet divergences in this theory. In the second part of the work the author presents the general principles and mathematical apparatus for framing QFT in terms of the **measurability** notion introduced by the author earlier, considering

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the above-mentioned remark concerning replacement of a minimal length by the **primary** length. Next the author studies QFT in the **measurable** form for free quantum fields at low energies $E \ll E_p$. In such QFT in the general case it is expedient to indicate the energy regions, where EP is valid and where it loses its force, in an effort to find a natural solution of the ultraviolet divergences problem.

1 Introduction

This paper is a continuation of previous works by the author [1]–[6]. The first part is devoted to analysis of the applicability limit of Einstein's Equivalence Principle (EP). It is noted that a natural applicability limit of this Principle, associated with the development of quantum-gravitational effects at Planck's scales, is absolute, its more accurate estimation being dependent on the processes under study and on the sizes of the corresponding particles. It is shown that, neglecting the applicability limit of EP, one can obtain senseless results on estimation of the relevant quantities within the scope of the well-known Quantum Field Theory (QFT), in particular, of the cosmological term λ in General Relativity (GR). Besides, neglect of the applicability limit of EP may be responsible for ultraviolet divergences in OFT.
The idea that all the processes studied in QFT should be considered separately in two different energy ranges

$$\begin{gathered} E \ll E_p \\ and \\ E \approx E_p \end{gathered} \tag{1}$$

is substantiated. Then the results earlier obtained by the author [1]–[6] are used. However now the author lifts some initial restrictions (limiting conditions) imposed in the above-mentioned papers. Specifically, it is not supposed initially that a theory involves some minimal length l_{min}; we start from the maximal momentum $p = p_{max}$,formula (10) in Section 3 (a certain maximal bound for the measured momenta), and then from this formula we can derive the length ℓ and the corresponding time $\tau = \ell/c$. ℓ is called the *primary* length, whereas τ is called the *primary* time. The whole formalism developed in [1]–[6] on condition that ℓ is a minimal length is fully valid for the case when ℓ is the *primary* length. It is important that there is a possibility to lift the formal requirement for involvement of l_{min} in the theory just from the start. The need for replacement of the minimal length l_{min} by the *primary* length ℓ according to the proposed approach is substantiated in the following section, see the paragraph titled **Explanation**.

The principal idea of the above-mentioned works is as follows. Proceeding from the **measurability** notion, initially defined in [2] and also in Section 3 of this paper, we can reformulate quantum theory and gravity, removing from them the abstract infinitesimal variations $dt, dx_i, dp_i, dE, i = 1, ..., 3$ and replacing them by the quantities depending on the existent energies expressed in terms of the quantity ℓ. Within the scope of these terms, at low energies a theory becomes discrete, it is very close to the initial theory formulated in the continuous space-time. Actually, discreteness is revealed at high energies only.

At the present time these theories are defined in the continuous space-time paradigm but are associated with serious problems, in particular with the (ultraviolet and infrared) divergences in QFT.

It is obvious that, when ℓ exists, all variations in a physical system, irrespective of the energies, should be expressed in terms of ℓ. Though with the use of new terms, at low energies a theory becomes discrete, it is very close to the initial theory formulated in the continuous space-time. Actually, discreteness is revealed at high energies only. The main instrument for realization of the idea put forward by the author is the notion of **measurability**, initially defined in [2] and further developed in [5],[6].

The principal results obtained are presented in Section 4, where the initial mathematical apparatus is extended for studies of QFT in terms of the **measurability** notion (or simpler, in the **"measurable"** format) for free fields, with regard to the results of Section 2 concerning the applicability boundary of EP.

2 Equivalence Principle Applicability Boundary and QFT Ultraviolet Behavior in Canonical Theory

The Einstein's Equivalence Principle (EP) is a basic principle not only in the General Relativity (GR) [7]–[9], but also in the fundamental physics as a whole. In the standard formulation it is as follows: ([9],p.68):
¡¡*at every space-time point in an arbitrary gravitational field it is possible to choose* **a locally inertial coordinate system** *such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation¿¿.*
Then in ([9],p.68) ¡¡...There is also a question, how small is **"sufficiently small"**. Roughly speaking, we mean that the region must be small enough so that gravitational field in sensible constant throughout it...¿¿.
However, the statement **"sufficiently small"** is associated with another problem. Indeed, let $\overline{x}$ be a certain point of the space-time manifold $\mathcal{M}$ (i.e. $\overline{x} \in \mathcal{M}$) with the geometry given by the metric $g_{\mu\nu}(\overline{x})$. Next, in accordance with EP, there is some **sufficiently small** region $\mathcal{V}$ of the point $\overline{x}$ so that, within $\mathcal{V}$, we have

$$g_{\mu\nu}(\overline{x}) \equiv \eta_{\mu\nu}(\overline{x}), \tag{2}$$

where $\eta_{\mu\nu}(\overline{x})$ is Minkowskian metric.
In essence, **sufficiently small** $\mathcal{V}$ means that the region $\mathcal{V}^{'}$, for which $\overline{x} \in \mathcal{V}^{'} \subset \mathcal{V}$, satisfies (2) as well. In this way we can construct the sequence

$$... \subset \mathcal{V}^{''} \subset \mathcal{V}^{'} \subset \mathcal{V}. \tag{3}$$

The problem arises, is there any lower limit for the sequence in formula (3)? The answer is positive. Currently, there is no doubt that at very high energies (on the order of Planck's energies $E \approx E_p$), i.e. on Planck's scales, $l \approx l_p$ quantum fluctuations of any metric $g_{\mu\nu}(\overline{x})$ are so high that in this case the geometry determined by $g_{\mu\nu}(\overline{x})$ is replaced by the "geometry" following from **space-time foam** that is defined by great quantum fluctuations of $g_{\mu\nu}(\overline{x})$,i.e. by the characteristic dimensions of the quantum-gravitational

region (for example, [10]–[15]). The above-mentioned geometry is drastically differing from the locally smooth geometry of continuous space-time and EP in it is no longer valid [16]–[23].
From this it follows that the region $\mathcal{V}_{\overline{r},\overline{t}}$ with the characteristic spatial dimension $\overline{r} \approx l_p$ (and hence with the temporal dimension $\overline{t} \approx t_p$) is the lower (approximate) limit for the sequence in (3).
It is difficult to find the exact lower limit for the sequence in formula (3)–it seems to be dependent on the processes under study. Specifically, when the involved particles are considered to be point, their dimensions may be neglected in a definition of the EP applicability limit. When the characteristic spatial dimension of a particle is $\mathbf{r}$, the lower limit of the sequence from formula (3) seems to be given by the region $\mathcal{V}_{\mathbf{r}'}$ containing the above-mentioned particle with the characteristic dimensions $\mathbf{r}' > \mathbf{r}$, i.e. the space EP applicability limit should always be greater than dimensions of the particles considered in this region. By the present time, it is known that spatial dimensions of gauge bosons, quarks, and leptons within the limiting accuracy of the conducted measurements $< 10^{-18}m$. Because of this, the condition $\mathbf{r}' \geq 10^{-18}m$ must be fulfilled. In addition, the radius of interaction of particles $\mathbf{r}_{int}$ must be taken into account in quantum theory. And this fact also imposes a restriction on considering concrete processes in quantum theory. However, the interactions radii of all known processes lie in the energy scales $E \ll E_p$.
Therefore, it is assumed that the Equivalence Principle is valid for the locally smooth space-time and this suggests that all the energies E of the particles in the most general form meet the necessary condition

$$E \ll E_p. \tag{4}$$

Then, if not stipulated otherwise, we can assume that the condition (4) is valid.
The canonical quantum field theory (QFT) [24]– [26] is a local theory considered in continuous space-time with a plane geometry, i.e with the Minkowskian metric $\eta_{\mu\nu}(\overline{x})$ *. In addition, it is assumed that all objects in QFT are point-like. However, as noted above, this assumption will be true to a certain limit: the assumptions that (a) even local space-time geometry is plane and (b) all objects in QFT are point-like have natural applicability boundaries*

directly specifying the EP applicability boundary.
In reality, any interaction introduces some disturbances, introducing an additional local (little) curvature into the initially flat Minkowskian space $\mathcal{M}$. Then the metric $\eta_{\mu\nu}(\overline{x})$ is replaced by the metric $\eta_{\mu\nu}(\overline{x}) + o_{\mu\nu}(\overline{x})$, where the increment $o_{\mu\nu}(\overline{x})$ is small. But, when it is assumed that EP is valid, the increment $o_{\mu\nu}(\overline{x})$ in the local theory has no important role and, in a fairly small neighborhood of the point $\overline{x}$, formula (2)is valid and we have $\eta_{\mu\nu}(\overline{x}) + o_{\mu\nu}(\overline{x}) \equiv \eta_{\mu\nu}(\overline{x})$.
Within the scope of the canonical QFT, the process of passage to more higher energies without a change in the local curvature has no limits [24]–[26], just this fact is the reason for ultraviolet divergences in QFT. But as follows from the previous section, this is not the case. Actually, on passage to the Planck energies $E \approx E_p$ (Planck scales $l \approx l_p$), the space in the Planck neighborhood $\mathcal{V}_{\overline{r},\overline{t}}$ of the point $\overline{x}$ one cannot consider flat even locally and in this case (as noted above) EP is not valid.
Then we introduce the following assumption:

Assumption 2.1
In the canonical QFT in calculations of the quantities it is wrong to sum (or same consider within a single sum) the contributions corresponding to space-time manifolds with locally nonzero or zero curvatures since these contributions are associated with different processes: (1) with the existence of a gravitational field that, in principle, can hardly be excluded; (2) in the absence of a gravitational field.

From the start, we can isolate the case when EP is valid (at sufficiently low energies, specifically satisfying the condition (4)) from the cases when EP becomes invalid (for example, Planck energies $E \approx E_p$).
Let us consider a widely known example when **Assumption 2.1** is not fulfilled leading to the senseless results.
In his well-known lectures [27] at the Cornell University Steven Weinberg considered an example of calculating, within the scope of QFT, the expected value for the vacuum energy density $< \rho >$ that is proportional to the cosmological term λ. To this end, zero-point energies of all normal modes of some field with the mass m are summed up to the wave number cutoff

$\Lambda \gg m$ for the selected normalization $\hbar = c = 1$ (formula (3.5) in [27]):

$$< \rho >= \int_0^{\Lambda} \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{2}\sqrt{k^2+m^2} \simeq \frac{\Lambda^4}{16\pi^2}. \quad (5)$$

Assuming, similar to [27], that GR is valid at all the energy scales up to the Planck's, we have the cutoff $\Lambda \simeq (8\pi G)^{-1/2}$ and hence (formula (3.6) in [27]) leads to the following result:

$$< \rho >\approx 2 \cdot 10^{71} GeV^4, \quad (6)$$

that by 10^{118} orders of magnitude differs from the well-known experimental value for the vacuum energy density

$$< \rho_{exp} >\preceq 10^{-29}\mathrm{g}/cm^3 \approx 10^{-47} GeV^4. \quad (7)$$

Here G is a gravitational constant.
It is clear that in this case **Assumption 2.1** fails as Planck's scales and those close to them at lower energies are included into consideration. By the author's opinion, this is impermissible because for Planck's scales the quantum rather than classical gravity is true and the space even in a small neighborhood of the point is hardly flat. But in formula (5) for the cutoff $\Lambda \simeq (8\pi G)^{-1/2}$ this fact is not included because all calculations in the canonical QFT [26] are valid for the locally flat space and hence (5) in this case leads to senseless results.
Of particular interest is the **inverse problem**: if the experimental value of the vacuum energy density $< \rho_{exp} >$ is known from (7), substituting it into formula (5), we can estimate Λ_{exp} at the upper limit of integration by the above formula

$$< \rho_{exp} >= \int_0^{\Lambda_{exp}} \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{2}\sqrt{k^2+m^2} \simeq 10^{-47} GeV^4. \quad (8)$$

Note that Λ_{exp} may be found in other way. Denoting by Λ_{UV} the quantity $\simeq (8\pi G)^{-1/2}$ from formula (5), corresponding to the cutoff at Planck's scale $\approx 1,6 \cdot 10^{-33} cm$ that is taken as the ultraviolet cutoff, denoting the required quantity $< \rho >$ by $< \rho_{UV} >$, by Λ_{IF} denoting the quantity from the same formula, that corresponds to the cutoff at the scale of a visible part of the Universe $\approx 10^{28} cm$, and the corresponding quantity $< \rho >$ denoting as $< \rho_{IF} >$ (infrared limit), in accordance with [28],[29], we obtain

$$< \rho_{exp} >= \sqrt{< \rho_{UV} >< \rho_{IF} >}. \tag{9}$$

Obviously, Λ_{exp} derived from formulae (8), (9) satisfies the condition (4) and in this case **Assumption 2.1** is fulfilled.

Remark 2.2
In this work we, in fact, consider two extremes:
a)low energies $E \ll E_p$ and
b)very high (essentially maximal) energies $E \approx E_p$.
Then it should be noted that, as all the experimentally involved energies E are low, they satisfy condition a). Specifically, for LHC maximal energies are $\approx 10TeV = 10^4 GeV$, that is by 15 orders of magnitude lower than the Planck energy $\approx 10^{19} GeV$.
Moreover, the characteristic energy scales of all fundamental interactions also satisfy condition a). Indeed, in the case of strong interactions this scale is $\Lambda_{QCD} \sim 200 MeV$; for electroweak interactions this scale is determined by the vacuum average of a Higgs boson and equals $\upsilon \approx 246 GeV$; finally, the scale of the (Grand Unification Theory (GUT)) M_{GUT} lies in the range of $\sim 10^{14} GeV - -10^{16} GeV$. It is obvious that all the above figures satisfy condition a).
Thus, only the expected characteristic energy scale of quantum gravity satisfies condition b).

From **Remark 2.2** it directly follows that even very high energies arising on unification of all the interaction types $M_{GUT} \approx 10^{14} GeV - \sim 10^{16} GeV$,(except of gravitational),satisfy the condition (4).
At the same time, it is clear that the requirement of the Lorentz-invariant QFT, due to the action of Lorentz boost (or same hyperbolic rotations) (formula (3) in [8]), results in however high momenta and energies. But it has been demonstrated that unlimited growth of the momenta and energies is impossible because in this case we fall within the energy region, where the conventional quantum field theory [24]– [26] is invalid.

Note that at the present time there are experimental indications that Lorentz-invariance is violated in QFT on passage to higher energies (for example, [30]). Besides, one should note important recent works associated with EP applicability boundaries and violation in nuclei and atoms at low energies (for example [31]). Proceeding from the above, the requirement for Lorentz-invariance and EP is possible only within the scope of the condition (4).

3 Measurability Notion. A Brief Preliminary Information and Some Important Refinements

In this Section we briefly consider some of the results from [1]–[6] which are essential for subsequent studies. Without detriment to further consideration, in the initial definitions we lift some unnecessary restrictions and make important specifications.
Presently, many researchers are of the opinion that at very high energies (Plank's or trans-Planck's) the ultraviolet cutoff exists that is determined by some maximal momentum.
Therefore, it is further assumed that there is a maximal bound for the measurement momenta $p = p_{max}$ represented as follows:

$$p_{max} \doteq p_{\ell} = \hbar/\ell, \tag{10}$$

where ℓ is some small length and $\tau = \ell/c$ is the corresponding time. Let us call ℓ the *primary* length and τ the *primary* time.
Without loss of generality, we can consider ℓ and τ at Plank's level, i.e. $\ell \propto l_p, \tau = \kappa t_p$, where the numerical constant κ is on the order of 1. Consequently, we have $E_{\ell} \propto E_p$ with the corresponding proportionality factor, where $E_{\ell} \doteq p_{\ell}c$.

Explanation. In the theory under study it is not assumed from the start that there exists some minimal length l_{min} and that ℓ is such. In fact, the minimal length is defined with the use of Heisenberg's Uncertainty Principle (HUP) $\Delta x \cdot \Delta p \geq \frac{1}{2}\hbar$ or of its generalization to high (Planck) energies

– Generalized Uncertainty Principle (GUP) [33]–[41], for example, of the form [33]

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_p^2 \frac{\Delta p}{\hbar}, \tag{11}$$

where α' is a constant on the order of 1. Evidently this formula (11) initially leads to the minimal length $\widetilde{\ell}$ on the order of the Planck length $\widetilde{\ell} \doteq 2\sqrt{\alpha'} l_p$. Besides, other forms of GUP [41] also lead to the minimal length.

But, as is currently known, HUP has been verified and operates well only at low energies $E \ll E_p$. Moreover, there are some serious arguments against GUP as demonstrated in Section IX of review [41]. Because of this, in the present work validity of this principle is not implied from the start. GUP it is given merely as an example. As p_{max} (10) is taken at Planck's level, it is clear that HUP is inapplicable. Taking this into consideration, the existence of a certain minimal length $\widetilde{\ell}$ is not mandatory. So, we start from the *primary* length ℓ and the *primary* time τ. The whole formalism, developed in [1]–[6] on condition that ℓ is the minimal length, is valid for the case when ℓ is the *primary* length but now we can lift the formal requirement for involvement of l_{min} in the theory from the start.

3.1. The **primarily measurable** space-time quantities (variations) are understood as the quantities Δx_i and Δt taking the form

$$\Delta x_i = N_{\Delta x_i} \ell, \Delta t = N_{\Delta t} \tau, \tag{12}$$

where $N_{\Delta x_i}, N_{\Delta t}$ are integer numbers. Further in the text we use both $N_{\Delta x_i}, N_{\Delta t}$ and the equivalent N_{x_i}, N_t.

3.2. Similarly, the **primarily measurable** momenta are considered as a subset of the momenta characterized by the property

$$p_{x_i} \doteq p_{N_{x_i}} = \frac{\hbar}{N_{x_i} \ell}, \tag{13}$$

where N_{x_i} is a nonzero integer number and p_{x_i} is the momentum corresponding to the coordinate x_i.

3.3. Finally, let us define any physical quantity as **primarily or elementary measurable** when its value is consistent with point **3.1**,**3.2** and formulae (12), (13).
Then we consider formula (13) with the addition of the momenta $p_{x_0} \doteq p_{N_0} = \frac{\hbar}{N_{x_0}\ell}$, where N_{x_0} is an integer number corresponding to the time coordinate ($N_{\Delta t}$ in formula (12)).
For convenience, we denote **Primarily Measurable Quantities** satisfying **3.1**–**3.3** in the abbreviated form as **PMQ**. Also, for the **Primarily Measurable Momenta** we use the abbreviation **PMM.**

First, we consider the case of **Low Energies**, i.e. $E \ll E_\ell$ (same $E \ll E_p$. It is obvious that all the nonzero integer numbers N_{x_i}, N_t (or same $N_{x_\mu}; \mu = 0, ..., 3$) from formulae (12),(13) should satisfy the condition $|N_{x_\mu}| \gg 1$. It is clear that all the momenta p_i at **low energies** $E \ll E_p$ meet the condition $p_i = \hbar/(N_i\ell)$, where $|N_i| \gg 1$ but is not necessarily an integer. With regard for smallness of ℓ and for the condition $|N_i| \gg 1$, we can easily show that the difference $1/(N_i\ell) - 1/([N_i]\ell), (\hbar/(N_i\ell) - \hbar/([N_i]\ell))$ is negligible and in this way all momenta in the region of low energies $E \ll E_p$ may be taken as **PMM** with a high accuracy.
It is obviously that the case of **Low Energies** in this section is coincident with the "low energies" condition from **Remark 2.2.**
It is assumed that a theory we are trying to resolve is a deformation of the initial continuous theory.

Remark 3.0
The deformation is understood as an extension of a particular theory by inclusion of one or several additional parameters in such a way that the initial theory appears in the limiting transition [32].

Then it should be noted that **PMQ** is inadequate for studies of the physical processes. In fact, among **PMQ**, we have no quantities capable to give the infinitesimal quantities $dx_\mu, \mu = 0, ..., 3$ in the limiting transition in a continuous theory.
Therefore, it is reasonable to use notion of **Generalized Measurability**

We define any physical quantity at all energy scales as **generalized measurable** *or, for simplicity,* **measurable** *if any of its values may be obtained in terms of* **PMQ** *specified by points* **3.1**–**3.3**.
The **generalized measurable** quantities will be denoted as **GMQ**.
Note that the space-time quantities

$$\frac{\tau}{N_t} = p_{N_t c}\frac{\ell^2}{c\hbar}$$
$$\frac{\ell}{N_i} = p_{N_i}\frac{\ell^2}{\hbar}, 1 = 1, ..., 3, \tag{14}$$

where $p_{N_i}, p_{N_t c}$ are **Primarily Measurable** momenta, up to the fundamental constants, are coincident with $p_{N_i}, p_{N_t c}$ and they may be involved at any stage of the calculations but, evidently, they are not **PMQ**, but they are **GMQ**.
So, in the proposed paradigm at low energies $E \ll E_p$ a set of the **PMM** is discrete, and in every measurement of $\mu = 0, ..., 3$ there is the discrete subset $\mathbf{P}_{\mathbf{x}_\mu} \subset$ **PMM**:

$$\mathbf{P}_{\mathbf{x}_\mu} \doteq \{..., p_{N_{x_\mu}-1}, p_{N_{x_\mu}}, p_{N_{x_\mu}+1}, ...\}. \tag{15}$$

In this case, as compared to the canonical quantum theory, in continuous space-time we have the following substitution:

$$\mathbf{\Delta p}_\mu \mapsto dp_\mu, \mathbf{\Delta p}_{\mathbf{N}_{\mathbf{x}_\mu}} = \mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}} - \mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}+\mathbf{1}} = \mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}(\mathbf{N}_{\mathbf{x}_\mu}+\mathbf{1})};$$
$$\frac{\mathbf{\Delta}}{\mathbf{\Delta p}_\mu} \mapsto \frac{\partial}{\partial \mathbf{p}_\mu}; \frac{\mathbf{\Delta F}(\mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}})}{\mathbf{\Delta p}_\mu} = \frac{\mathbf{F}(\mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}}) - \mathbf{F}(\mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}+\mathbf{1}})}{\mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}} - \mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}+\mathbf{1}}} = \frac{\mathbf{F}(\mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}}) - \mathbf{F}(\mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}+\mathbf{1}})}{\mathbf{p}_{\mathbf{N}_{\mathbf{x}_\mu}(\mathbf{N}_{\mathbf{x}_\mu}+\mathbf{1})}}. \tag{16}$$

And

$$\frac{\ell}{\mathbf{N}_{\mathbf{x}_\mu}} \mapsto dx_\mu;$$
$$\frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}} \mapsto \frac{\partial}{\partial x_\mu}, \frac{\mathbf{\Delta F}(\mathbf{x}_\mu)}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}} = \frac{\mathbf{F}(\mathbf{x}_\mu + \ell/\mathbf{N}_{\mathbf{x}_\mu}) - \mathbf{F}(\mathbf{x}_\mu)}{\ell/\mathbf{N}_{\mathbf{x}_\mu}}. \tag{17}$$

It is clear that for sufficiently high integer values of $|N_{x_\mu}|$, formulae (16),(17) reproduce a continuous paradigm in the momentum space to any preassigned accuracy. However, at low energies $E \ll E_\ell$ a set of **PMM** clearly is not a space. Considering this, the formulae at low energies offer **the Correspondence to Continuous Theory (CCT).**

It is important to make the following remarks in medias res:

Remark 3.1.
In this way any point $\{x_\mu\} \in \mathcal{M} \subset \mathbf{R^4}$ and any set of integer numbers high in absolute values $\{N_{x_\mu}\}$ are correlated with a system of the neighborhoods for this point $(x_\mu \pm \ell/N_{x_\mu})$. It is clear that, with an increase in $|N_{x_\mu}|$, the indicated system converges to the point $\{x_\mu\}$. In this case all the ingredients of the initial (continuous) theory the partial derivatives including are replaced by the corresponding finite differences.

Remark 3.2.
It is further assumed that at low energies $E \ll E_\ell$ (same $E \ll E_p$) all the *observable quantities* **are PMQ**.
Because of this, values of the length ℓ/N_i and of the time ℓ/N_t from formula (14) could not appear in expressions for *observable quantities*, being involved only in intermediate calculations, especially at the summation for replacement of the infinitesimal quantities $dt, dx_i; i = 1, 2, 3$ on passage from a continuous theory to its measurable variant.

Further it is assumed that at **High Energies**, $E \approx E_p$, **PMQ** are *inadequate* to study the theory at these energies. This assumption is quite natural. For example, if GUP (11) is valid and if $\ell = \widetilde{\ell}$, then formula (11) at high energies generates the momenta $\Delta p(N_{\Delta x}, GUP)$ which are not **primarily measurable** [4] –[6].

Remark 3.3
When at low energies $E \ll E_p$ we lift restrictions on integrality of N_{x_μ}, from formulae (16),(17) it directly follows that in this case we have a continuous analog of the well-known theory with the only difference: all the used small quantities become dependent on the existent energies and we can correlate

them. In this way formula (17) may be written as

$$\begin{aligned} dx_\mu \leftrightarrow \frac{\ell}{\mathbf{N_{x_\mu}}} \rightarrow \frac{\ell}{[\mathbf{N_{x_\mu}}]}, \\ \frac{\partial}{\partial x_\mu} \leftrightarrow \frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\mu}}}} \rightarrow \frac{\mathbf{\Delta}}{\mathbf{\Delta_{[N_{x_\mu}]}}} \end{aligned} \quad (18)$$

where $|N_{x_\mu}| \gg 1$ is a sufficiently large number that varies continuously. It is clear that in formula (18) the first arrow corresponds to the continuous theory with a specific selection of values of the infinitesimal quantities dx_μ. As noted above, the difference $\ell/N_{x_\mu} - \ell/[N_{x_\mu}]$ is negligible and hence the second arrow corresponds to passage from the initial continuous theory to a similar discrete theory. Of course, formula (16) may be rewritten in the like manner. In what follows, formula (18) plays a crucial part in derivation of the results and is greatly important for their understanding.
The main target of the author is to form a quantum theory and gravity only in terms of **PMQ**.

Measurable form arbitrary metric and Minkowskian metric

According to the previous works, the **measurable** variants of quantum theory and gravity at low energies $E \ll E_p$ should be formulated in terms of the **measurable** space-time quantities $\ell/N_{\Delta x_\mu}$ or **primary measurable** momenta $p_{N_{\Delta x_\mu}}$.
Let us consider the case of the random metric $g_{\mu\nu} = g_{\mu\nu}(x)$ [7],[8], where $x \in R^4$ is some point of the four-dimensional space R^4 defined in **measurable** terms. Now, any such point $x \doteq \{x_\chi\} \in R^4$ and any set of integer numbers $\{N_{x_\chi}\}$ dependent on the point $\{x_\chi\}$ with the property $|N_{x_\chi}| \gg 1$ may be correlated to the "bundle" with the base R^4 as follows:

$$B_{N_{x_\chi}} \doteq \{x_\chi, \frac{\ell}{N_{x_\chi}}\} \mapsto \{x_\chi\}. \quad (19)$$

It is clear that $\lim\limits_{|N_{x_\chi}| \rightarrow \infty} B_{N_{x_\chi}} = R^4$.
As distinct from the normal one, the "bundle" $B_{N_{x_\chi}}$ is distinguished only by the fact that the mapping in formula (19) is not continuous (smooth)

but discrete in fibers, being continuous in the limit $|N_{x_\chi}| \to \infty$.
Then as a *canonically measurable prototype* of the infinitesimal space-time interval square [7],[8]

$$ds^2(x) = g_{\mu\nu}(x) dx^\mu dx^\nu \tag{20}$$

we take the expression

$$\Delta s^2_{N_{x_\chi}}(x) \doteq g_{\mu\nu}(x, N_{x_\chi}) \frac{\ell^2}{N_{x_\mu} N_{x_\nu}}. \tag{21}$$

Here $g_{\mu\nu}(x, N_{x_\chi})$ – metric $g_{\mu\nu}(x)$ from formula (20) with the property that minimal **measurable** variation of metric $g_{\mu\nu}(x)$ in point x has form

$$\Delta g_{\mu\nu}(x, N_{x_\chi})_\chi = g_{\mu\nu}(x + \ell/N_{x_\chi}, N_{x_\chi}) - g_{\mu\nu}(x, N_{x_\chi}). \tag{22}$$

Let us denote by $\Delta_\chi g_{\mu\nu}(x, N_{x_\chi})$ quantity

$$\Delta_\chi g_{\mu\nu}(x, N_{x_\chi}) = \frac{\Delta g_{\mu\nu}(x, N_{x_\chi})_\chi}{\ell/N_{x_\chi}}. \tag{23}$$

It is obvious that in the case under study the quantity $\Delta g_{\mu\nu}(x, N_{x_\chi})_\chi$ is a **measurable** analog for the infinitesimal increment $dg_{\mu\nu}(x)$ of the χ-th component $(dg_{\mu\nu}(x))_\chi$ in a continuous theory, whereas the quantity $\Delta_\chi g_{\mu\nu}(x, N_{x_\chi})$ is a **measurable** analog of the partial derivative $\partial_\chi g_{\mu\nu}(x)$.
In this manner we obtain the (19)-formula induced bundle over the metric manifold $g_{\mu\nu}(x)$:

$$B_{g, N_{x_\chi}} \doteq g_{\mu\nu}(x, N_{x_\chi}) \mapsto g_{\mu\nu}(x). \tag{24}$$

Referring to formula (14), we can see that (21) may be written in terms of the **primary measurable** momenta $(p_{N_i}, p_{N_t}) \doteq p_{N_\mu}$ as follows:

$$\Delta s^2_{N_{x_\mu}}(x) = \frac{\ell^4}{\hbar^2} g_{\mu\nu}(x, N_{x_\chi}) p_{N_{x_\mu}} p_{N_{x_\nu}}. \tag{25}$$

Considering that $\ell \propto l_P$ (i.e., $\ell = \kappa l_P$), where $\kappa = const$ is on the order of 1, to within the constant $\ell^4/\hbar^2$, we have

$$\Delta s^2_{N_{x_\mu}}(x) = g_{\mu\nu}(x, N_{x_\chi}) p_{N_{x_\mu}} p_{N_{x_\nu}}. \tag{26}$$

As follows from the previous formulae, the **measurable** variant of General Relativity should be defined in the bundle $B_{g,N_{x_\chi}}$.
Analogously, a *canonically measurable prototype* of the **relativistic** infinitesimal space-time interval square

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu. \tag{27}$$

is given by

$$\Delta s^2_{N_{x_\chi}}(x) \doteq \eta_{\mu\nu}(x, N_{x_\chi}) \frac{\ell^2}{N_{x_\mu} N_{x_\nu}}, \tag{28}$$

where $\eta_{\mu\nu}$ is the Minkowskian metric

$$||\eta_{\mu\nu}|| = ||\eta^{\mu\nu}|| = Diag\,(1, -1, -1, -1)\,. \tag{29}$$

Here the integers N_{x_χ} naturally satisfy the condition $|N_{x_\chi}| \gg 1$, components of the **measurable** Minkowskian metric $||\eta_{\mu\nu}(x, N_{x_\chi})||$ are "close" to $||\eta_{\mu\nu}||$,i.e. we have

$$\lim_{(|N_{x_\chi}|)\to\infty} \eta_{\mu\nu}(x, N_{x_\chi}) = \eta_{\mu\nu}. \tag{30}$$

Without loss of generality, we can assume that $\eta_{\mu\nu}(x, N_{x_\chi}) = 0, \mu \neq \nu$.
Thus $||\eta_{\mu\nu}(x, N_{x_\chi})||$ is the diagonal matrix too and $||\eta^{\mu\nu}(x, N_{x_\chi})||$ is its inverse matrix, i.e.

$$||\eta^{\mu\nu}(x, N_{x_\chi})|| \cdot ||\eta_{\mu\nu}(x, N_{x_\chi})|| = 1 \tag{31}$$

Further we assume that the integers N_{x_χ} are sufficiently large in absolute value and, due to formula (30), the metric $||\eta_{\mu\nu}(x, N_{x_\chi})||$, to a high accuracy, is equal to $||\eta_{\mu\nu}||$; then formula (28) is as follows:

$$\Delta s^2_{N_{x_\chi}}(x) \doteq \eta_{\mu\nu} \frac{\ell^2}{N_{x_\mu} N_{x_\nu}}, \tag{32}$$

4 Quantum Field Theory in Measurable Format

In what follows we suppose: $E \ll E_\ell$ (or same $E \ll E_p$). Besides, some integer numbers $N_{x_\chi}, \chi = 0, ..., 3$ are selected on the condition that $|N_{x_\chi}| \gg 1$.

4.1 Main Notations and Theses

Generally, in canonical QFT [24] natural units $\hbar = c = 1$ are used everywhere. Lorentz indices are always denoted by Greek symbols $\mu, \nu, .. = 0, 1, 2, 3$. Then four-vectors for space–time coordinates and particle momenta have the following contravariant components:

$$x = (x^\mu) = (x^0, \vec{x}), \quad x^0 = t \tag{33}$$

$$p = (p^\mu) = (p^0, \vec{p}), \quad p^0 = E = \sqrt{\vec{p}^2 + m^2}. \tag{34}$$

The covariant 4-vector components are related to the contravariant components according to

$$a_\mu \quad = \eta_{\mu\nu}\, a^\nu, \tag{35}$$

were $||\eta_{\mu\nu}||$ is the Minkowskian metric from the formula (29) yielding the 4-dimensional squares (scalar products),

$$a^2 \quad = \eta_{\mu\nu}\, a^\mu a^\nu = a_\mu a^\mu, \quad a \cdot b = a_\mu b^\mu = a^0 b^0 - \vec{a} \cdot \vec{b}. \tag{36}$$

What variations are associated with a **measurable** consideration?
In this case it is assumed that all the above-mentioned vectors should be considered in the **measurable** form. Then $x = (x^\mu)$ is a space–time point, with the coordinates contravariant to the **measurable** components. **Measurability** means that all variations of these coordinates are **GMQ**. On the contrary, the coordinates p^μ of the point p, due to the results from the previous section and according to the condition $E \ll E_p$, are **PMQ**. All variations of these coordinates, because of **Remark 3.3**, may be considered within the scope of **PMQ**.
Formulae (35) and (36) in the **measurable** consideration are retained as the ordinary derivatives in this case are replaced by the covariant ones in line with formula (17), and contravariant components of the **"measurable"** derivatives are as follows:

$$\frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\mu}}}} = \eta_{\mu\nu} \frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x^\nu}}}}, [\frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_0}}}} = \frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x^0}}}}, \quad \frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_i}}}} = -\frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x^i}}}}];$$

$$\Box_{\mathbf{N_{x_\mu}}} = \frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\mu}}}} \frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x^\mu}}}}. \tag{37}$$

Due to the fact that $E \ll E_p$, all the momenta under consideration are given by formula (15); then the quantum mechanical states of spin-s particles, with the momentum $p = (p^0, \vec{p})$ and helicity $\sigma = -s, -s+1, \cdots, +s$, are denoted in the conventional way by Dirac kets $|p_{N_{x_\mu}}, \sigma>$ and all N_{x_μ} are integer numbers on the condition that $|N_{x_\mu}| \gg 1$. They are normalized according to the relativistically invariant convention

$$<p\,\sigma\,|\,p'\sigma'>=<p_{N_{x_\mu}}\,\sigma\,|\,p_{N'_{x_\mu}}\sigma'>= 2p^0\,\delta^3(\vec{p}-\vec{p}')\,\delta_{\sigma\sigma'} = \\ = 2p^0\,\delta^3((N_{x_i})-(N'_{x_i}))\,\delta_{\sigma\sigma'}. \qquad (38)$$

A special state, the zero-particle state or the vacuum, respectively, is denoted by $|0>$. It is normalized to unity

$$<0\,|\,0> \;\; = 1\,. \qquad (39)$$

Considerations of Section 3 point to the fact that the Least Action Principle at low energies $E \ll E_\ell$ is valid in the **measurable** form with substitution of the measurable analogs defined in the foregoing Section for all the components involved in the proof of these arguments. For the canonical (continuous) case we use the notation of Section 3 in [24].
Let φ be a set of all the considered fields $\varphi \doteq (\varphi_1, \varphi_2, ...)$. Then the action S in the continuous case taking the form

$$S = \int \mathcal{L}(\varphi, \partial_\mu \varphi) d^4x \qquad (40)$$

is replaced by the **measurable** action $S_{meas,N}$

$$S_{meas,\{N\}} = \sum \mathcal{L}_{meas,\{N\}}(\varphi, \frac{\mathbf{\Delta}\varphi}{\mathbf{\Delta}_{\mathbf{N_{x_\mu}}}}) \prod \frac{\ell}{N_{x_\mu}}, \qquad (41)$$

where N_{x_μ} – integers with the property $|N_{x_\mu}| \gg 1$,$\mathcal{L}_{meas,N}$–Lagrangian density of the **measurable** fields φ and of their **measurable** analogs for partial derivatives in formula (17) $\frac{\mathbf{\Delta}\varphi}{\mathbf{\Delta}_{\mathbf{N_{x_\mu}}}}$. This means that all variations of these functions are expressed in terms of only **measurable** quantities. In the product $\prod$ the index μ takes the values $\mu = 0, ..., 3$, and $\{N\}$–collection of all N_{x_μ},i.e. $\{N\} \doteq \{N_{x_\mu}\}$. Further, where this causes no confusion, for the

measurable quantities corresponding to the set $\{N\}$ we can equally use both the lower index $\{N\}$ and N.
According to **Remarks 3.1.,3.3.** for the integer numbers N_{x_μ} sufficiently high in absolute value we, to a high accuracy, have

$$S = S_{meas,\{N\}}. \tag{42}$$

Then it is assumed that all the considered functions are **measurable**, i.e. all variations of these functions are expressed in terms of only **measurable** quantities. In this case well-known formula

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{43}$$

is replaced by the expression

$$\frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\mu}}}} \frac{\partial \mathcal{L}_{meas,\{N\}}}{\partial(\frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\mu}}}}\phi)} - \frac{\partial \mathcal{L}_{meas,\{N\}}}{\partial \phi} = 0 \tag{44}$$

The paper [6] presents in detail a measurable form of the Least Action Principle.
It is clear that in all the formulae, similar to formula (41), on passage from QFT in continuous consideration to the **measurable** form of QFT, in accordance with (16) and (17), the substitution is performed

$$\int \mapsto \sum; \partial_\mu \mapsto \frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\mu}}}}; d^4x \mapsto \prod \frac{\ell}{N_{x_\mu}}, ... \tag{45}$$

It is clear that the above-mentioned *discrete* (*almost-continuous*) (QFT), with a cut-off at a certain upper limit of the momenta which are considerably much lower than the Planck, should be ultraviolet-finite. In this case passage to higher energies means going from the momenta $p_N, |N| \gg 1$ to the momenta $p_{N'}, |N| > |N'| \gg 1$ and, vice versa, passage to lower energies is going in the last inequality from the integers N' to the integers N.

4.2 Basic Principles for Construction of QFT Measurable Variant

Based on the foregoing results and considering the energy bounds $E \ll E_p$, QFT in the **measurable** form is close to canonical QFT [24]. At the same time, it is important to take into account the following remark.

Remark 4.0
As distinct from canonical QFT, for the indicated energies in **"measurable"** *QFT a set of momenta and energies is a certain bounded set* **PMQ** rather than space.

Besides, in canonical QFT there are no serious limitations on space-time coordinates for the *observable quantities*. But this is not the case in **"measurable"** QFT. Indeed, according to **Remark 3.2.**, at the indicated low energies *observable quantities* correspond to **PMQ** but at low energies $E \ll E_p$ space-time **PMQ** is nothing else than

$$\{x_\mu\} = N_{x_\mu}\ell, N_{x_\mu} \quad are \quad integer \quad numbers, |N_{x_\mu}| \gg 1 \quad ore \quad N_{x_\mu} = 0. \tag{46}$$

It is clear that this is a really discrete set, whereas the corresponding set of **Primarily Measurable Momenta** or **PMM**, by definition and to a high accuracy, gives all the (continuous) set of momenta at these energies, as it has been shown in Section 3. Moreover, it is obvious that in essence the Minkowskian space-time in terms of the **generalized measurable** quantities (or simply **measurable** quantities is not different from the ordinary Minkowskian space-time in a canonical theory [24]. This conclusion directly follows from the fact that any real number may be approximated by rational numbers to however high accuracy (in the present case by the rational number multiplied by ℓ or τ).
Proceeding from the above, in a **measurable** consideration one can use the Fourier transformation with due regard for the replacement indicated in formula (45). Still, taking into account **Remark 4.0**, we use *only* the Fourier transformation from momentum representation to position representation.

4.3 Free Quantum Fields in Measurable Form

4.3.1 Simple Lagrangians With Scalars Fields

Using formula (37), we can easily obtain, instead of the well-known Lagrangian for a free real scalar field

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu \phi)^2 - \frac{m^2}{2}\,\phi^2 \tag{47}$$

and the corresponding *Klein–Gordon equation* or *KGE*

$$(\Box + m^2)\,\phi = 0 \tag{48}$$

their **measurable** forms

$$\mathcal{L}_{meas,\{N\}} = \frac{1}{2}\,(\frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\mu}}}}\phi_{meas})^2 - \frac{m^2}{2}\,\phi^2_{meas} \tag{49}$$

and

$$(\Box_{N_{x_\mu}} + m^2)\,\phi_{meas} = 0\ . \tag{50}$$

Similarly, using the replacement from formula (45), a solution of the equation (48) in terms of a complete set of the plane waves $e^{\pm ikx}$

$$\phi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k}{2k^0}\,[a(k)\,e^{-ikx} + a^\dagger(k)\,e^{ikx}\,] \tag{51}$$

in the **measurable** form should be written as follows:

$$\begin{aligned}\phi(x, N^*, N_*)_{meas} &= \frac{1}{(2\pi)^{3/2}}\sum_{N_i=N^*}^{N_*}\frac{\Delta^3 k}{2k^0}\,[a(k)\,e^{-ikx} + a^\dagger(k)\,e^{ikx}] = \\ &= \frac{1}{(2\pi)^{3/2}}\sum_{p_{N^*}}^{p_{N_*}}\frac{\Delta^3 k}{2k^0}\,[a(k)\,e^{-ikx} + a^\dagger(k)\,e^{ikx}].\end{aligned} \tag{52}$$

Here, according to Subsection 5.2., $x = \{x_i\}, x_i = N_{x_i}\ell$, N_{x_i}–integers with the property $|N_{x_i}| \gg 1$, $k \doteq \{k_i\}, k_i \doteq p_{N_i} = \hbar/(N_i\ell), \Delta k_i \doteq k_i - k_{i+1} = k_{i(i+1)}, \Delta^3 k \doteq \prod_{i=1}^3 \Delta k_i, k^0 = \sqrt{\vec{k_i}^{\,2} + m^2}$, N_i are integer numbers too, and

$$|N^*| \geq |N_i| \geq |N_*| \gg 1, |N^*| \gg |N_*|. \tag{53}$$

Which conditions should be satisfied by the lower N^* and upper N^* bounds of the summation in formula (52)?
Clearly, in the **measurable** case the function $\phi(x, N^*, N_*)_{meas}$ from this formula is not a complete analog of the function $\phi(x)$ from formula (52). It is only an analog of the function $\phi(x, N^*, N_*)$

$$\phi(x, N^*, N_*) \doteq \frac{1}{(2\pi)^{3/2}} \int\limits_{p_{N^*}}^{p_{N_*}} \frac{\mathrm{d}^3 k}{2k^0} \, [a(k)\, e^{-ikx} \, + \, a^\dagger(k)\, e^{ikx}\,] \tag{54}$$

that seems to be a certain low-energy part of $\phi(x)$. Naturally, it is desirable so that $\phi(x, N^*, N_*)$ should meet the equation KGE to a high accuracy (48) to have

$$(\Box \, + m^2)\, \phi(x, N^*, N_*) = 0\,. \tag{55}$$

As $\phi(x, N^*, N_*)_{meas}$ is very close to $\phi(x, N^*, N_*)$ and, respectively, the operator $\Box_{N_{x\mu}}$ – to the operator $\Box$, the only initial constraint imposed on the integers N^*, N^* is the condition that the function $\phi(x, N^*, N_*)_{meas}$ to a high accuracy satisfies the equation

$$(\Box_{N_{x\mu}} \, + m^2)\, \phi(x, N^*, N_*)_{meas} \approx 0\,. \tag{56}$$

Then, similar to canonical QFT, in the **measurable** pattern for the momentum representation we have

$$\begin{aligned} a^\dagger(k)\,|0\!>=|k\!> \\ a(k)\,|k'\!>= 2k^0\,\delta^3((N_i) - (N'_i\,)\,|0\!> \ . \end{aligned} \tag{57}$$

Similar formulae are easily derived for the complex scalar field $\phi^\dagger \neq \phi$ having two degrees of freedom and describing spinless particles, which carry the charge ± 1 and can be interpreted as particles or antiparticles.
The corresponding Lagrangian of canonical QFT [24]

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \, \phi^\dagger \phi \tag{58}$$

is replaced by

$$\mathcal{L}_{meas,\{N\}} = (\frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x\mu}}}}\phi)^\dagger (\frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x^\mu}}}}\phi) - m^2 \, \phi^\dagger \phi, \tag{59}$$

where $\frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\mu}}}}$ and $\frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x^\mu}}}}$ are related by formula (37).
Consequently, a **measurable** analog of the Fourier expansion in canonical QFT with the annihilation and creation operators a, $a^\dagger$ for the particle states $|+,k>$ and b, $b^\dagger$ and for the antiparticle states $|-,k>$,

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 k}{2k^0} \left[a(k)\, e^{-ikx} \; + \; b^\dagger(k)\, e^{ikx} \right] \tag{60}$$

has the form

$$\begin{aligned} \phi(x, N^*, N_*)_{meas} &= \frac{1}{(2\pi)^{3/2}} \sum_{N_i = N^*}^{N_*} \frac{\Delta^3 k}{2k^0} \left[a(k)\, e^{-ikx} \; + \; b^\dagger(k)\, e^{ikx} \right] = \\ &= \frac{1}{(2\pi)^{3/2}} \sum_{p_{N^*}}^{p_{N_*}} \frac{\Delta^3 k}{2k^0} \left[a(k)\, e^{-ikx} \; + \; b^\dagger(k)\, e^{ikx} \right], \end{aligned} \tag{61}$$

where all the basic notations are taken from formulae (52),(53). And, in analogy with formula (57), we have

$$\begin{aligned} a^\dagger(k)\,|0> &= \, |+, k>, b^\dagger(k)\,|0> = \, |-, k>, \\ a(k)\,|+, k'> &= \, 2k^0\, \delta^3((N_i) - (N'_i))\,|0>, \\ b(k)\,|-, k'> &= \, 2k^0\, \delta^3((N_i) - (N'_i))\,|0> \; . \end{aligned} \tag{62}$$

In the proposed paradigm for propagators in momentum and position representations with a **measurable** picture we have the same formalism as of the wave functions.
Specifically, in canonical QFT the propagators in the momentum and position representations $D(k)$ and $D(x-y)$ are related by the Fourier transformation

$$D(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, D(k)\, e^{-ik(x-y)}, \tag{63}$$

and the Green function $D(x-y)$ is a solution of the inhomogeneous field equation

$$(\Box + m^2)\, D(x-y) = -\delta^4(x-y)\,. \tag{64}$$

Similar to formula (54), at low energies $E \ll E_p$ in the canonical case of the well-known QFT for $D(x-y)$ one can introduce the cutoff $D(x-y, N^*, N_*)$:

$$D(x-y, N^*, N_*) \doteq \int\limits_{p_{N^*}}^{p_{N_*}} \frac{\mathrm{d}^4 k}{(2\pi)^4}\, D(k)\, e^{-ik(x-y)}, \tag{65}$$

where N^*, N_*–sets of the numbers from formula (53) with the added corresponding bound for p_0. $D(x-y, N^*, N_*)$ – low-energy term dependent on N^*, N_* in the full propagator $D(x-y)$. In analogy with wave functions, on the number sets N^*, N_* the following constraint is imposed:

$$(\Box + m^2)\, D(x-y, N^*, N_*) \approx -\delta^4(x-y)\,. \tag{66}$$

As in a **measurable** consideration at low momenta in the domain of $E \ll E_\ell$ the approximate equality $D(x-y, N^*, N_*)_{meas} \approx D(x-y, N^*, N_*)$ should be the case, formula (65) is replaced by

$$D(x-y, N^*, N_*)_{meas} \doteq \sum_{p_{N^*}}^{p_{N_*}} \frac{\Delta^4 k}{(2\pi)^4}\, D(k)_{meas}\, e^{-ik(x-y)}, \tag{67}$$

where $\Delta^4 k$ is defined in the same way as $\Delta^3 k$ in formula (52) with the addition of the zero-index coordinate. As a result, formula (66) in the case under study is transformed to the following expression:

$$(\Box_{N_{x_\mu}} + m^2)\, D(x-y, N^*, N_*)_{meas} \approx -\delta^4(x-y)\,. \tag{68}$$

As the wave functions describe free particles without space–time limitations, with regard to the foregoing formula, in a **measurable** consideration for the *propagator* in the momentum space, similar to the canonical case, we have

$$(k^2 - m^2)\, D(k)_{meas} = 1\,. \tag{69}$$

A solution for (69) is of the form

$$i\, D(k)_{meas} = \frac{i}{k^2 - m^2 + i\epsilon}. \tag{70}$$

However, now k takes a discrete, restricted set of values that is represented by formulae (52),(53) and expanded for the four-dimensional case in formula (67).
The Feynman diagram associated with a scalar propagator in this case is of the standard form

$$i\, D(k)_{meas} \qquad \underset{k}{\bullet\text{- - -¿- - -}\bullet} \qquad .$$

Of particular importance are the following remarks.
Remark 4.1
It should be noted that, when coordinates of the space-time points x and y are the coordinates of some *observable quantities*, in the **measurable** picture they satisfy the condition (46). Moreover, as energies are low, coordinates of the point $x - y$ should satisfy the same condition when $x \neq y$.

Remark 4.2
From **Remark 3.3** it follows directly that the functions $\phi(x, N^*, N_*)_{meas}$ and $\phi(x, N^*, N_*)$ from formulae (52) and (54), respectively, are close to each other and in the majority of calculations we can assume that $\phi(x, N^*, N_*) \approx \phi(x, N^*, N_*)_{meas}$. The only cardinal distinction of $\phi(x, N^*, N_*)_{meas}$ from $\phi(x, N^*, N_*)$ is the fact that, if the number N^* is finite, the first of the mentioned functions in its Fourier series expansion is determined by the finite number of the terms, whereas the second – by the infinite number. Similarly close are the Green functions $D(x-y, N^*, N_*)$ and $D(x-y, N^*, N_*)_{meas}$ from formulae (65) and (67), respectively.

Remark 4.3
If we assume that $|N^*|$ are large enough (and the corresponding momenta p_N^* are thus small) so that, without any detriment for the performed calculations, we can consider $|N^*| = \infty$, (and, consequently, $p_N^* = 0$).
Then the above-mentioned functions $\phi(x, N^*, N_*)_{meas}, (\phi(x, N^*, N_*)), D(x-y, N^*, N_*)_{meas}, (D(x-y, N^*, N_*)), ...$ should be dependent on the single parameter N_*

$$\begin{aligned} \phi(x, \infty, N_*)_{meas} &\equiv \phi(x, N_*)_{meas}, \\ D(x-y, \infty, N_*)_{meas} &\equiv D(x-y, N_*)_{meas}, ... \end{aligned} \tag{71}$$

It is clear that $|N_*|$–integer cutoff parameter at the upper bound of momenta: the lower the parameter, the large the momenta $p_{|N_*|}$, the closer the quantities $\phi(x,N_*)_{meas}, \phi(x,N_*), D(x-y,N_*)_{meas}, D(x-y,N_*),...$ to the initial ones $\phi(x), D(x-y),....$ But, considering that (see Section 2) the Einstein Equivalence Principle (EP) should have the applicability boundary, it is assumed:

4.3.1 The parameter $|N_*|$ has the *minimum* $|\widetilde{N}_*|$ determined by linear dimensions of the *minimal* neighborhood $|\widetilde{N}_*|\ell$, for which (EP) remains valid in the process under study. The fact that such a neighborhood should be the case, with $|N_*| \gg 1$, was noted in Section 2.

4.3.2 Vector and Dirac Fields

It is obvious that for vector and Dirac fields the whole formalism from **4.3.1** remains valid.
In particular, let $A_\mu(x)$ is vector field which describes a particle with spin 1. If this particle has the mass m, then the corresponding Lagrangian for a free system ('massive photon') in a canonical theory [24] is of the form

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}\, A_\mu A^\mu \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{72}$$

yielding from (43) (with $\phi \to A_\nu$) the *Proca equation*

$$\left[(\Box + m^2)\, \eta^{\mu\nu} - \partial^\mu \partial^\nu\right] A_\nu = 0\,. \tag{73}$$

And, due to (45), formulae (72),(73) are replaced in a **measurable** picture by the formulae

$$\mathcal{L}_{meas} = -\frac{1}{4}\, F_{\mu\nu,meas}F^{\mu\nu}_{meas} - \frac{m^2}{2}\, A_\mu A^\mu \quad \text{with}$$
$$F_{\mu\nu,meas} = \frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\mu}}}} A_\nu - \frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\nu}}}} A_\mu, \tag{74}$$

and

$$\left[(\Box_{N_{x_\mu}} + m^2)\, \eta^{\mu\nu} - \frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x_\mu}}}} \frac{\mathbf{\Delta}}{\mathbf{\Delta_{N_{x^\nu}}}}\right] A_\nu = 0\,. \tag{75}$$

Correspondingly, the field A_μ that can be represented for canonical QFT as a Fourier expansion

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}} \sum_\lambda \int \frac{\mathrm{d}^3 k}{2k^0} \left[a_\lambda(k)\, \epsilon_\mu^{(\lambda)}(k)\, e^{-ikx} \;+\; a_\lambda^\dagger(k)\, \epsilon_\mu^{(\lambda)}(k)^*\, e^{ikx} \right], \tag{76}$$

similar to formula (52), may be written in the **measurable** form as follows:

$$A_\mu(x, N^*, N_*)_{meas} =$$

$$= \frac{1}{(2\pi)^{3/2}} \sum_\lambda \sum_{p_{N^*}}^{p_{N_*}} \frac{\Delta^3 k}{2k^0} \; \left[a_\lambda(k)\, \epsilon_\mu^{(\lambda)}(k)\, e^{-ikx} \;+\; a_\lambda^\dagger(k)\, \epsilon_\mu^{(\lambda)}(k)^*\, e^{ikx} \right], \tag{77}$$

where in the last formula all the basic notations are taken from formulae (52),(53) and helicity $\lambda = \pm 1, 0$ for massive particles, $\lambda = \pm 1$ for particles with zero mass.

Similar to the scalar case, the *Feynman propagator* of a vector field in the measurable form, $D(x-y, N^*, N_*)_{meas}$, is a solution of the inhomogeneous field equation

$$\left[(\Box_{N_{x\mu}} + m^2)\, \eta^{\mu\rho} - \frac{\boldsymbol{\Delta}}{\boldsymbol{\Delta}_{\mathbf{N_x}^\mu}} \frac{\boldsymbol{\Delta}}{\boldsymbol{\Delta}_{\mathbf{N_x}^\rho}} \right] D_{\rho\nu}(x-y, N^*, N_*)_{meas} \approx \eta^\mu{}_\nu\, \delta^4(x-y) \tag{78}$$

by the respective Fourier transformation

$$D(x-y, N^*, N_*)_{meas} \doteq \sum_{p_{N^*}}^{p_{N_*}} \frac{\Delta^4 k}{(2\pi)^4}\, D(k)\, e^{-ik(x-y)}, \tag{79}$$

that naturally agrees with formula (67).

Repeating all steps from **4.3.1**, we can derive the Feynman propagator of a massive vector field in a **measurable consideration** in the standard form

$$i\, D_{\rho\nu}(k)_{meas} = \frac{i}{k^2 - m^2 + i\epsilon} \left(-\eta_{\nu\rho} + \frac{k_\nu k_\rho}{m^2} \right), \tag{80}$$

where, as noted in **4.3.1**, k takes a discrete, restricted set of values that is determined by the numbers N^* and N_*.
The measurable case for a massless vector field and Dirac fields is treated in the same way. Of course, in the process all the foregoing remarks **Remark 4.1–Remark 4.3** remain valid.
In conclusion the procedure in terms of **measurability** notion is shown for gauge theories at low energies.

4.4 Measurability for Gauge Theories at Low Energies

In this section we use the formalism from [24],[25].
It is easily seen that at low energies $E \ll E_p$ for the gauge theories written in the **measurable** form all formulae of the canonical (continuous) theory are valid when using the corresponding substitution in accordance with formulae (16),(17),(45).
Indeed, let $\mathbf{G}$ – gauge group and $\{N\} \doteq \{N_{x_\mu}\}$, similar to formulae from the preceding section,– fixed set of the integers $|N_{x_\mu}| \gg 1$ sufficiently large in the absolute value.
As $\mathbf{G}$ - group of the local internal symmetries of a physical system and the definition of **measurability** refer only to the space-time indexes, we can get the following correspondences:

$$\begin{aligned} \mathbf{W}'_\mu = U\,\mathbf{W}_\mu\,U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \mapsto \mathbf{W}'_{\mu,\{N\}} \doteq \\ \doteq U\,\mathbf{W}_{\mu,\{N\}},U^{-1} - \frac{i}{g}(\frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}}U)U^{-1}, \\ D_\mu = \partial_\mu - ig\,\mathbf{W}_\mu \mapsto D_{\mu,\{N\}} \doteq \\ \doteq \frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}} - ig\,\mathbf{W}_{\mu,\{N\}}, \\ \mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{W}_\nu - \partial_\nu \mathbf{W}_\mu - i\,g\,[\mathbf{W}_\mu, \mathbf{W}_\nu] \mapsto \mathbf{F}_{\mu\nu,\{N\}} \doteq \\ \doteq \frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}}\mathbf{W}_{\nu,\{N\}} - \frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\nu}}}\mathbf{W}_{\mu,\{N\}} - i\,g\,[\mathbf{W}_{\mu,\{N\}}, \mathbf{W}_{\mu,\{N\}}]. \end{aligned} \tag{81}$$

And, similarly, we have

$$\overline{\Psi}\,(i\gamma^{\mu}D_{\mu}-m)\Psi \mapsto \overline{\Psi}_{\{N\}}\,(i\gamma^{\mu}D_{\mu,\{N\}}-m)\Psi_{\{N\}}. \tag{82}$$

Here g is a coupling constant,$\mathbf{W}_{\mu}$ – space-time components of gauge fields, $\Psi,\overline{\Psi}$–corresponding material fields (in this case fermion),D_{μ}–covariant derivative and U - element of the gauge group $\mathbf{G}$.
Passage in formulae (81),(82) from the left- to the right-hand side is associated with the transition from the canonical (continuous) consideration to the representation in terms of **measurable** quantities for the fixed set $\{N\} \doteq \{N_{x_{\mu}}\}$. It is clear that in this case all the transformable quantities in the right-hand sides of these formulae should depend on $\{N\}$,that is indicated by the additional lower index $\{N\}$. In a similar way, **the "measurable"** metric $g_{\mu\nu}(x, N_{x_{\chi}}) \equiv g_{\mu\nu}(x,\{N\})$ from formula (21) is dependent on $\{N\}$.
However, considering that the energies are low and the numbers $|N_{x_{\mu}}| \gg 1$ are sufficiently high, the above-mentioned relationship is very weak.
As follows from formulae (81),(82) and from the paragraph preceding these formulae, if $\mathcal{L}$ – gauge-invariant Lagrangian associated with the left-hand sides of these formulae, the corresponding Lagrangian given in terms of **measurable** quantities $\mathcal{L}_{meas,\{N\}}$ is also gauge-invariant by $\mathbf{G}$ and we have

$$\mathcal{L} \approx \mathcal{L}_{meas,\{N\}}. \tag{83}$$

Besides, from the above formulae it follows that all the known relations for the gauge theory with the group $\mathbf{G}$ are valid, to a high accuracy, at low energies for a **measurable** variant of this theory on replacement of all basic quantities in the initial theory by the corresponding quantities with the additional lower index $\{N\}$.
Specifically, the "gauge" analog *Bianchi identity*

$$D_{\rho}\mathbf{F}_{\mu\nu} + D_{\mu}\mathbf{F}_{\nu\rho} + D_{\nu}\mathbf{F}_{\rho\mu} = 0 \tag{84}$$

in the **measurable** form is replaced, to a high accuracy, by the identity

$$D_{\rho,\{N\}}\mathbf{F}_{\mu\nu,\{N\}} + D_{\mu,\{N\}}\mathbf{F}_{\nu\rho,\{N\}} + D_{\nu,\{N\}}\mathbf{F}_{\rho\mu,\{N\}} = 0. \tag{85}$$

Obviously, this accuracy is the higher the greater the absolute values of the numbers from the set $\{N\}$.
Similar to the canonical case, formula (84) is equivalent to the *Jacoby identity*

$$\sum_{cyclic\ permutations} [D_{\rho}, [D_{\mu}, D_{\nu}]] = 0, \tag{86}$$

in the **measurable** consideration formula (85) to a high accuracy is equivalent to the **measurable** form of *Jacoby identity*

$$\sum_{cyclic\ permutations} [D_{\rho,\{N\}}, [D_{\mu,\{N\}}, D_{\nu,\{N\}}]] = 0. \tag{87}$$

5 Conclusion

In the proposed approach we have considered QFT in the **measurable** form at the energies $E \ll E_p$ (or same $E \ll E_l$). In the process the upper bounds for a sum of the contributions made by the particles during calculations of physical quantities are determined by the EP applicability boundaries which in specific cases should be much lower than E_p.
This paper presents only a single case of free quantum fields. To use the potentialities of this approach in full, we should study thoroughly the case of interacting fields in terms of the **measurability** notion and the EP applicability boundaries dependence for specific processes. Within the scope of the UV divergence problem in canonical QFT [24]– [26], most important for this consideration are loop Feynman diagrams.
Initially, we have chosen some maximal bound for the measurement momenta $p = p_{max}$ at Plank's level and the corresponding *primary* length ℓ from formula (10). Of course, all the results should not be dependent on this choice.

References

[1] Shalyt-Margolin, A.E. Minimal Length and the Existence of Some Infinitesimal Quantities in Quantum Theory and Gravity. *Adv. High Energy*

Phys., **2014** (2014), 8.
http://dx.doi.org/10.1155/2014/195157

[2] Shalyt-Margolin,Alexander. Minimal Length, Measurability and Gravity. *Entropy* **2016**, *18(3)*, 80.
http://dx.doi.org/10.3390/e18030080

[3] A.E. Shalyt-Margolin, Uncertainty Principle at All Energies Scales and Measurability Conception for Quantum Theory and Gravity,*Nonlinear Phenomena in Complex Systems,* **19(2)** (2016),166–181.

[4] Shalyt-Margolin,Alexander E. Minimal Length, Minimal Inverse Temperature, Measurability and Black Hole. *Electronic Journal of Theoretical Physics* **2018**, *14(37)*, 35-54.

[5] Shalyt-Margolin,Alexander. Minimal Quantities and Measurable Variant of Gravity in the General Form.*Advanced Studies in Theoretical Physics* **2018**, *12(2)*, 57 - 78.
https://doi.org/10.12988/astp.2018.71265

[6] Shalyt-Margolin,Alexander. Minimal Quantities and Measurability. Gravity in Measurable Format and Natural Transition to High Energies.*Nonlinear Phenomena in Complex Systems* **2018**, *21(2)*, 138 - 163.

[7] Wald,R.M. *General Relativity*,University of Chicago Press, Chicago, Ill,USA,1984.
http://dx.doi.org/10.7208/chicago/9780226870373.001.0001

[8] Akhmedov,Emil T. *Lectures on General Theory of Relativity*, arXiv:1601.04996 [gr-qc].

[9] Weinberg,S. *Gravitation and Cosmology.Principles and Applications of General Theory of Relativity.*1972

[10] Wheeler,J. A."Geons".*Phys.Rev.* **1955**, *97*, 511.

[11] Wheeler,J. A.*Geometrodynamics* (Academic Press, New York and London, 1962).

[12] Misner,C. W., Thorne,K. S. and Wheeler,J. A. *Gravitation* Freeman, San Francisco, (1973).

[13] Hawking,S. W.,Space-time foam.*Nuclear Phys.B* **1978**,*114*, 349

[14] Y. J. Ng, Selected topics in Planck-scale physics, Mod.Phys.Lett.A., vol.18,pp.1073–1098,2003.

[15] Scardigli,Fabio, Black Hole Entropy: a spacetime foam approach, *Class.Quant.Grav.*,**14** (1997),1781–1793.

[16] Garattini,Remo,A Spacetime Foam approach to the cosmological constant and entropy. *Int. J. Mod. Phys. D.* **4** (2002) 635–652.

[17] Garattini,Remo, A Spacetime Foam Approach to the Schwarzschild-de Sitter Entropy. *Entropy* **2** (2000) 26–38.

[18] Garattini,Remo,Entropy and the cosmological constant: a spacetime-foam approach. *Nucl.Phys.Proc.Suppl.* **88** (2000) 297–300.

[19] Garattini,Remo, Entropy from the foam. *Phys.Lett.B* **1999**, *459*, 461–467.

[20] Scardigli,Fabio. Generalized Uncertainty Principle in Quantum Gravity from Micro-Black Hole Gedanken Experiment. *Phys.Lett.B.* **1999**, *452*, 39–44.

[21] Scardigli,Fabio. Gravity coupling from micro-black holes. *Nucl.Phys.Proc.Suppl.* **2000**, *88*, 291–294.

[22] Garay,Luis J. Thermal properties of spacetime foam. *Phys.Rev. D* **1998**, *58*, 124015.

[23] Garay,Luis J. Spacetime foam as a quantum thermal bath. *Phys.Rev.Lett.* **1998**, *80*, 2508–2511.

[24] Rayder,Lewis.H.*Quantum Field Theory*,University of Kent and Canterbuty,.

[25] M.E. Peskin, D.V. Schroeder, *An Introduction to Quantum Field Theory*,Addison-Wesley Publishing Company, 1995.

[26] Steven Weinberg, *The Quantum Theory of Fields*,Vol.1,2. Cambridge University Press, 1995.

[27] Weinberg,S.The cosmological constant problem.*Rev. Mod. Phys.* **1989**,*61*,1.

[28] Padmanabhan, T., 2005, Darker side of the universe . and the crying need for some bright ideas! Proceedings of the 29th International Cosmic Ray Conference, Pune, India, 47-62.

[29] Padmanabhan,T.,2006, Dark Energy: Mystery of the Mil-lennium. Paris 2005, Albert Einstein's century, AIP Con-ference Proceedings 861, American Institute of Physics, New York, 858-866

[30] Kostelecky, V.A., Russell, N. Data tables for Lorentz and CPT violation. *Rev. Mod. Phys.* **2011**, *83(1)*, 11–31.

[31] Flambaum V.V. Enhanced violation of the Lorentz invariance and Einstein's equivalence principle in nuclei and atoms. *Phys. Rev. Lett.* **2016**, *117*,072501.

[32] Faddeev,L. Mathematical View on Evolution of Physics. *Priroda* **1989**,*5*, 11

[33] Adler R. J. and Santiago,D. I. On gravity and the uncertainty principle. *Mod. Phys. Lett. A* **1999**, *14*, 1371–1378.
http://dx.doi.org/10.1142/s0217732399001462

[34] Maggiore,M. Black Hole Complementarity and the Physical Origin of the Stretched Horizon. *Phys. Rev. D*, **1994** *49*, 2918–2921.
http://dx.doi.org/10.1103/physrevd.49.2918

[35] Maggiore,M. Generalized Uncertainty Principle in Quantum Gravity. *Phys. Lett. B*, **1993** *304*, 65–69.
http://dx.doi.org/10.1016/0370-2693(93)91401-8

[36] Maggiore,M. The algebraic structure of the generalized uncertainty principle.**1993** *319*, 83–86.
http://dx.doi.org/10.1016/0370-2693(93)90785-g

[37] Witten,E. Reflections on the fate of spacetime. *Phys. Today* **1996**, *49*, 24–28. http://dx.doi.org/10.1063/1.881493

[38] Amati,D., Ciafaloni M. and Veneziano, G. A. Can spacetime be probed below the string size? *Phys. Lett. B* **1989**, *216*, 41–47.
http://dx.doi.org/10.1016/0370-2693(89)91366-x

[39] Capozziello,S., Lambiase G., and Scarpetta,G. The Generalized Uncertainty Principle from Quantum Geometry. *Int. J. Theor. Phys.* **2000**, *39*, 15–22.
http://dx.doi.org/10.1023/a:1003634814685

[40] Kempf,A., Mangano,G. and Mann,R.B. Hilbert space representation of the minimal length uncertainty relation.*Phys. Rev. D.* **1995**, *52*, 1108–1118. http://dx.doi.org/10.1103/physrevd.52.1108

[41] Abdel Nasser Tawfik, Abdel Magied Diab.Generalized Uncertainty Principle: Approaches and Applications.*Int. J. Mod. Phys. D* **2014**, *23*,1430025.