DRINFELD-JIMBO QUANTUM LIE ALGEBRA

Oleg Ogievetsky *

Centre de Physique Théorique
Luminy, 13288 Marseille, France
oleg@cpt.univ-mrs.fr

Todor Popov

Institute for Nuclear Research and Nuclear Energy
Bulgarian Academy of Sciences
Sofia, BG-1784, Bulgaria
tpopov@inrne.bas.bg

Abstract

Quantum Lie algebras related to multi-parametric Drinfeld–Jimbo R-matrices of type $GL(m|n)$ are classified.

1 Introduction

Quantum groups are examples of non-commutative manifolds having a rich differential geometry. Woronowicz developed a general theory of the differential calculus on a quantum group, the so called bicovariant differential calculus. Bicovariant bimodules are objects analogous to tensor bundles over Lie groups [16]. The vector space dual to the space of left-invariant differential forms (i.e., the space of the left-invariant vector fields) is endowed with a bilinear operation playing the role of the Lie bracket (for a friendly introduction to the subject see, e.g., [1, 2]). This vector space is a quantum analogue of a Lie algebra. A quantum Lie algebra can be defined axiomatically as a triple (V, σ, C) consisting of a vector space V, a braiding $\sigma : V \otimes V \to V \otimes V$ and a "quantum Lie bracket" $C : V \otimes V \to V$ on it satisfying certain compatibility identities, see below.

Given a braiding σ, it is natural to try to describe quantum Lie algebras compatible with σ. An important class of braidings arises as quantizations

*On leave of absence from P.N. Lebedev Physical Institute, Theoretical Department, Leninsky prospekt 53, 119991 Moscow, Russia

[8, 11] of classical r-matrices corresponding to Belavin–Drinfeld triples [3]. The problem of a description of quantum Lie algebras compatible with the Cremmer–Gervais R-matrix [6] has been addressed in our previous work [15] with the help of a suitable rime Ansatz [14]. The Cremmer–Gervais R-matrix corresponds to a maximal Belavin–Drinfeld triple for the defining fundamental representation of the quantum group of type GL. In the defining representation this braiding is of Hecke type which, in particular, implies that the BRST operator for a quantum Lie algebra with this braiding is finite [9, 10]. In this note we give a complete list of quantum Lie algebras compatible with the Drinfeld-Jimbo R-matrix [7, 12] (which corresponds to the empty Belavin–Drinfeld triple) in the defining fundamental representation on a vector super-space.

2 Bicovariant calculus

We give here a short extract from the differential calculus on quantum groups to motivate the notion of the quantum Lie algebra and related objects used in the sequel.

Let $\mathcal{A}$ be the Hopf algebra of functions on a quantum group. We denote by Δ, ϵ and S the coproduct, the counit and the antipode of $\mathcal{A}$. We use the Sweedler notation for the coproduct

$$\Delta(a) = a_{(1)} \otimes a_{(2)} \ .$$

Woronowicz [16] introduced the notion of the bicovariant bimodule Γ over $\mathcal{A}$. It is an $\mathcal{A}$-bimodule endowed with two coactions

$$\Delta_L : \Gamma \to \mathcal{A} \otimes \Gamma \quad , \quad \Delta_R : \Gamma \to \Gamma \otimes \mathcal{A}$$

satisfying certain compatibility conditions. One supposes that Γ is a free left (and right) $\mathcal{A}$-module admitting a basis ω^i formed by *left-invariant forms*, that is,

$$\Delta_L(\omega^i) = 1 \otimes \omega^i \ . \tag{1}$$

Elements $r^i_j \in \mathcal{A}$ are defined by

$$\Delta_R(\omega^i) = \omega^j \otimes r^i_j \ . \tag{2}$$

Denote the Hopf algebra dual to $\mathcal{A}$ by $\mathcal{A}'$; it has the coproduct Δ', counit ϵ' and antipode S'. There exists a set of elements $f^i_j \in \mathcal{A}'$ relating the left and the right actions

$$\omega^i b = (f^i_j * b)\omega^j := b_{(1)} f^i_j(b_{(2)})\omega^j \ . \tag{3}$$

The consistency implies

$$\Delta' f^i_j = f^i_k \otimes f^k_j \ , \ \epsilon'(f^i_j) = \delta^i_j \ , \ S'(f^i_k) f^k_j = \delta^i_j = f^i_k S'(f^k_j) \ , \tag{4}$$

$$\Delta r^i_j = r^k_j \otimes r^i_k \ , \ \epsilon(r^i_j) = \delta^i_j \ , \ S(r^k_j) r^i_k = \delta^i_j = r^k_j S(r^i_k) \ . \tag{5}$$

The differential (on functions) in the Woronowicz calculus is the map $d : \mathcal{A} \to \Gamma$, given by

$$da = (\chi_i * a)\omega^i \qquad \forall\, a \in \mathcal{A} \;, \tag{6}$$

where the elements $\chi_i \in \mathcal{A}'$ form a basis of the free (left) $\mathcal{A}$-module of the *left-invariant vector fields*.

The dual $\mathcal{A}'$ of a (finite dimensional) Hopf algebra $\mathcal{A}$ is again a Hopf algebra. The Leibniz rule implies the coproduct

$$\Delta'\chi_i = \chi_j \otimes f^j_i + 1 \otimes \chi_i \;. \tag{7}$$

and the duality induces $\epsilon'(\chi_i) = 0$, hence $S'(\chi_j) = -\chi_i S'(f^i_j)$.

The elements χ_i and f^i_j satisfy the quadratic-linear relations

$$\begin{array}{ll} \chi_i\chi_j - \chi_k\chi_l\sigma^{kl}_{ij} = \chi_k C^k_{ij} \;, & \sigma^{ab}_{kl} f^k_i f^l_j = f^a_k f^b_l \sigma^{kl}_{ij} \;, \\ \chi_k f^a_l \sigma^{kl}_{ij} + f^a_l C^l_{ij} = C^a_{kl} f^k_i f^l_j + f^a_i \chi_j \;, & \chi_i f^a_j = f^a_k \chi_l \sigma^{kl}_{ij} \;. \end{array} \tag{8}$$

Here

$$\sigma^{ij}_{kl} = f^i_l(r^j_k) \;\text{ and }\; C^k_{ij} = \chi_j(r^k_i) \;. \tag{9}$$

The compatibility leads to the following relations between the *braiding* σ and *structure constants* C:

$$\sigma^{ab}_{lm}\sigma^{mc}_{nk}\sigma^{ln}_{ij} = \sigma^{bc}_{lm}\sigma^{al}_{in}\sigma^{nm}_{jk} \qquad \text{(braid relation)} \;, \tag{10}$$

$$C^b_{sk}C^s_{ij} = C^b_{is}C^s_{jk} + C^b_{sl}C^s_{ir}\sigma^{rl}_{jk} \qquad \text{(braided Jacobi identity)} \;, \tag{11}$$

$$\sigma^{ab}_{sk}C^s_{ij} = C^b_{sl}\sigma^{as}_{ir}\sigma^{rl}_{jk} \;, \tag{12}$$

$$\sigma^{ab}_{sl}C^s_{ir}\sigma^{rl}_{jk} + \sigma^{ab}_{il}C^l_{jk} = C^a_{rl}\sigma^{lb}_{sk}\sigma^{rs}_{ij} + C^b_{sk}\sigma^{as}_{ij} \;. \tag{13}$$

Denote by $\mathcal{W}$ the algebra with the generators χ_i and f^i_j and the defining relations (8). The formulas (4) and (7) equip $\mathcal{W}$ with a Hopf algebra structure. For further details see the original work [16].

The relations (10)-(13) can be conveniently written [4, 5] as a single braid relation. Let us make a convention that the small indices $i, j, \ldots, k$ run over a set $\mathcal{I} = \{1, \ldots, \dim V\}$ and the capital indices $I, J, \ldots, K$ run over the set $\mathcal{I}_0 := 0 \cup \mathcal{I}$.

Lemma 1. *Let $\hat{\mathsf{R}}^{IJ}_{KL}$ be a matrix whose non-vanishing components are*

$$\hat{\mathsf{R}}^{ij}_{kl} = \sigma^{ij}_{kl} \;, \quad \hat{\mathsf{R}}^{0j}_{kl} = C^j_{kl} \;, \quad \hat{\mathsf{R}}^{0A}_{B0} = \delta^A_B \;, \quad \hat{\mathsf{R}}^{A0}_{0B} = \delta^A_B \;. \tag{14}$$

Then the system (10)-(13) is equivalent to the braid relation

$$\hat{\mathsf{R}}_{12}\hat{\mathsf{R}}_{23}\hat{\mathsf{R}}_{12} = \hat{\mathsf{R}}_{23}\hat{\mathsf{R}}_{12}\hat{\mathsf{R}}_{23} \;, \tag{15}$$

where $\hat{\mathsf{R}}_{12} := \hat{\mathsf{R}} \otimes \mathrm{Id}$ and $\hat{\mathsf{R}}_{23} := \mathrm{Id} \otimes \hat{\mathsf{R}}$. Let T^I_J be the matrix with elements $T^i_j = f^i_j, T^0_j = \chi_j, T^0_0 = 1, T^i_0 = 0$,

$$T^I_J = \begin{pmatrix} 1 & \chi_j \\ 0 & f^i_j \end{pmatrix} \;. \tag{16}$$

Then the relations (8) of the algebra $\mathcal{W}$ take the concise form

$$\hat{\mathsf{R}}^{AB}_{KL} T^K_I T^L_J = T^A_K T^B_L \hat{\mathsf{R}}^{KL}_{IJ} \ . \tag{17}$$

The Hopf structure of $\mathcal{W}$ (see eqs. (4) and (7)) reads

$$\begin{aligned} \Delta' T^I_J = T^I_K \otimes T^K_J \ , \qquad \epsilon'(T^I_J) = \delta^I_J \ , \\ S'(T^I_K) T^K_J = \delta^I_J = T^I_K S'(T^K_J) \ . \end{aligned} \tag{18}$$

The complexity of the relations (8) of the algebra $\mathcal{W}$ is hidden into the single matrix $\hat{\mathsf{R}}$ and the matrix of generators T^I_J.

Recall that the (right) adjoint representation of a Hopf algebra $\mathcal{H}$ on itself is defined by $ad_x(y) := S(x_{(1)}) y x_{(2)}$. The image of the space V in $\mathcal{W}$ is stable with respect to the adjoint action and one has

$$ad_{f^i_j}(\chi_a) = \chi_b \sigma^{ib}_{aj} \ , \ ad_{\chi_i}(\chi_a) = \chi_b C^b_{ai} \ .$$

In this representation the defining relations (8) turn into the (numerical) relations (10)-(13).

3 Quantum Lie algebra

The notion of a quantum Lie algebra formalizes the properties of the subalgebra of "vector fields", generated by χ_i, in $\mathcal{W}$. We give the precise definitions. Let V be the vector space with the basis $\{\chi_i\}_{i=1,\dots,\dim V}$. The space V is endowed with a braiding operator, that is, the operator $\sigma : V \otimes V \to V \otimes V$ which satisfies

$$\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23} \ .$$

We assume that σ is semi-simple and has an eigenvalue 1. Denote by $P_{(1)}$ the projector of σ corresponding to the eigenvalue 1.

Definition 1. A quantum Lie algebra is a triple (V, σ, C) where C is the "bracket" $C : V \otimes V \to V$ such that the following conditions hold:

i) braided symmetry: the bracket is in the kernel of the projector $P_{(1)}$

$$C P_{(1)} = 0 \ , \tag{19}$$

ii) braided Jacobi identity

$$C(C \otimes \mathrm{id}) = C(\mathrm{id} \otimes C) + C(C \otimes \mathrm{id})\sigma_{23} \ , \tag{20}$$

iii) additional, linear in C, identities

$$\begin{aligned} \sigma(C \otimes \mathrm{id}) &= (\mathrm{id} \otimes C)\sigma_{12}\sigma_{23} \ , \\ \sigma(C \otimes \mathrm{id})\sigma_{23} + \sigma(\mathrm{id} \otimes C) &= (C \otimes \mathrm{id})\sigma_{23}\sigma_{12} + (\mathrm{id} \otimes C)\sigma_{12} \ . \end{aligned}$$

Definition 2. The universal enveloping algebra $U_{\sigma,C}(V)$ of the quantum Lie algebra (V,σ,C) is the associative algebra with the generators χ_i, $i=1,\dots,\dim V$, and the defining relations

$$\chi_i\chi_j - \chi_k\chi_l\sigma^{kl}_{ij} = \chi_k C^k_{ij} \ . \tag{21}$$

Notation. We shall often write the compatibility conditions (12) and (13) between σ and C in the form

$$E_{\sigma,C}(i,j,k;a,b) := \sigma^{ab}_{sk}C^s_{ij} - C^b_{s\,l}\sigma^{as}_{ir}\sigma^{rl}_{jk} \ , \tag{22}$$

$$F_{\sigma,C}(i,j,k;a,b) := \sigma^{ab}_{sl}C^s_{ir}\sigma^{rl}_{jk} + \sigma^{ab}_{il}C^l_{jk} - C^a_{rl}\sigma^{lb}_{sk}\sigma^{rs}_{ij} - C^b_{sk}\sigma^{as}_{ij} \ . \tag{23}$$

4 Ice condition

The Boltzmann weights of the 6-vertex model in 2D statistical mechanics are subject to a restriction known as "ice" condition. Namely, the ice condition for the entries of an R-matrix $\hat{R}$ can be different from zero only if the set of the upper and the set of the lower indices coincide

$$\hat{R}^{ij}_{kl} \neq 0 \qquad \Rightarrow \qquad \{i,j\} \equiv \{k,l\} \ . \tag{24}$$

An ice matrix $\hat{R} \in \mathrm{End}(V\otimes V)$ has the form

$$\hat{R}^{kl}_{ij} = a_{ij}\delta^l_i\delta^k_j + b_{ij}\delta^k_i\delta^l_j \ .$$

If the set $\mathcal{I}$ of indices cannot be split into a disjoint union of two subsets $\mathcal{I}'$ and $\mathcal{I}''$ such that $b_{ij} = 0 = b_{ji}$ whenever $i \in \mathcal{I}'$ and $j \in \mathcal{I}''$ then we say that the matrix $\hat{R}$ is indecomposable. Recall that an operator $\hat{R} \in \mathrm{End}(V\otimes V)$ is called skew-invertible if there exists an operator $\Psi \in \mathrm{End}(V\otimes V)$ such that $\Psi^{iu}_{jv}\hat{R}^{vk}_{ul} = \delta^i_l\delta^k_j$. The characteristic function $\theta_{i>j}$ is defined to be 1 when $i>j$ and zero otherwise (similarly for $\theta_{i<j}$).

Lemma 2. *(See [13] for the proof.) Let $\hat{R}$ be an ice solution of the braid equation. Assume that $\hat{R}$ is invertible, skew-invertible and indecomposable. Then, up to a reordering of the basis and rescaling, $\hat{R}$ is the standard multi-parametric Drinfeld-Jimbo R-matrix [7, 12] with a_{ij} and b_{ij} given by*

$$a_{ij} = (-1)^{\hat{i}\delta_{ij}} q^{(1-2\hat{i})\,\delta_{ij}} p_{ij} q^{\theta_{i<j}-\theta_{i>j}} \ , \quad b_{ij} = (q-q^{-1})\theta_{i<j} \ . \tag{25}$$

Here $q \in \mathbb{C}^*$ and $\hat{i} \in \{0,1\}$; the parameters p_{ij} satisfy $p_{ij}p_{ji} = 1$ and $p_{ii} = 1$.

5 Drinfeld-Jimbo quantum Lie algebra

We shall say that $\hat{i}$ is the "parity" of the basis vector χ_i. Thus, $a_{ii} = q$ if $\hat{i} = 0$ and $a_{ii} = -q^{-1}$ if $\hat{i} = 1$.

Recall that the braiding operator σ of a quantum Lie algebra must have an eigenvalue 1. We thus have to rescale the standard R-matrix whose eigenvalues are q and $(-q^{-1})$; there are two possibilities:

$$\sigma = q^{-1}\hat{R} \qquad \text{or} \qquad \sigma = -q\hat{R} \ .$$

The second possibility can be reduced to the first one. Namely, the replacement of q by $(-\tilde{q}^{-1})$ in (25) leads to the standard R-matrix with the parities $\hat{i}'$, parameters $\tilde{q} = -q^{-1}$ and $\tilde{p}_{ij}$ where $\hat{i}' = 1 - \hat{i}$ and $\tilde{p}_{ij} = -p_{ij}\tilde{q}^{2(\theta_{i>j}-\theta_{i<j})}$. Thus we have to investigate the quantum Lie algebras with the braiding operator $\sigma = q^{-1}\hat{R}$ or, explicitly,

$$\begin{gathered} \sigma^{kl}_{ij} = A_{ij}\delta^l_i\delta^k_j + B_{ij}\delta^k_i\delta^l_j \ , \\ A_{ij} = (-1)^{\hat{i}\delta_{ij}} q^{-2\hat{i}\,\delta_{ij}} p_{ij} q^{-2\theta_{i>j}} \quad , \quad B_{ij} = 1 - q^{-2\theta_{i<j}} \ . \end{gathered} \tag{26}$$

The R-matrix is called unitary if it squares to the identity operator. The matrix σ is unitary iff $q^2 = 1$ and not semi-simple iff $q^2 = -1$.

Theorem 3. *Let σ be a standard Drinfeld-Jimbo R-matrix (26). Assume that σ is non-unitary and semi-simple, that is, $q^4 \neq 1$. The non-trivial quantum Lie algebra with the braiding operator σ exists only if the generator χ_1 is even and $p_{1j} = 1$ for $j > 1$. It is then unique (up to a global rescaling of C^k_{ij}) and the structure constants C^k_{ij} are given by*

$$C^k_{ij} = c(\delta^1_i\delta^k_j - \delta^1_j\delta^k_i) \ . \tag{27}$$

Explicitly, the relations (21) for this universal enveloping algebra $U_{\sigma,C}(V)$ read

$$\left\{ \begin{array}{rcll} \chi_1\chi_j - q^2\chi_j\chi_1 & = & q^2 c\chi_j & \text{if} \quad 1 < j \ , \\ \chi_i\chi_j - p_{ij}q^2\chi_j\chi_i & = & 0 & \text{if} \quad 1 < i < j \ , \\ \chi_i^2 & = & 0 & \text{if} \quad \hat{i} = 1 \ . \end{array} \right.$$

We start with the following lemma.

Lemma 4. *For the operator σ given by (26) with $q^4 \neq 1$, the relations (19), (12) and (13) imply*

$$C^k_{ji} = -p_{ji}C^k_{ij} \qquad \textit{and} \qquad C^k_{jj} = 0 \ . \tag{28}$$

Proof. The relation (19) is equivalent to $C^k_{ab}(\sigma^{ab}_{ij} + q^{-2}\delta^a_i\delta^b_j) = 0$ for the Hecke matrix σ with the eigenvalues 1 and $(-q^{-2})$. For $i \neq j$, this immediately yields

$$C^k_{ji} = -p_{ji}C^k_{ij} \quad , \quad i \neq j \ .$$

For $i=j$ and $\hat{j}=0$ the relation (19) reads $(1+q^{-2})C^k_{jj}=0$ so, by the semi-simplicity of σ, we obtain

$$C^k_{jj}=0 \quad , \quad \hat{j}=0 \ . \tag{29}$$

For $i=j$ and $\hat{j}=1$ the relation (19) does not impose any constraint on C^k_{jj} and we have to use eqs. (13) and (12). Consider the equation $E_{\sigma,C}(j,j;j,j,j)=0$ with $\hat{j}=1$. The summation indices get fixed by the ice condition and we obtain

$$\sigma^{jj}_{jj}C^j_{jj}=C^j_{jj}\sigma^{jj}_{jj}\sigma^{jj}_{jj} \text{ (no summation)} \Rightarrow C^j_{jj}\left(1-(-1)^{\hat{j}}q^{-2\hat{j}}\right)=0 \Rightarrow C^j_{jj}=0$$

by the semi-simplicity condition. It is left to show that C^k_{jj} for $\hat{j}=1$ and $j\neq k$. Choose the following equations from the system (12)-(13):

$$\begin{aligned} E_{\sigma,C}(j,j,j;j,k) &= 0 \ , && (30)\\ E_{\sigma,C}(j,j,k;k,k) &= 0 \ , && (31)\\ F_{\sigma,C}(j,j,j;j,k) &= 0 \ . && (32)\end{aligned}$$

Again the ice condition "freezes" all summations and we obtain

$$\begin{aligned} (\sigma^{jk}_{kj}-\sigma^{jj}_{jj}\sigma^{jj}_{jj})C^k_{jj} &= 0 \ ,\\ (\sigma^{kk}_{kk}-\sigma^{kj}_{jk}\sigma^{kj}_{jk})C^k_{jj} &= 0 \ , \qquad \text{(no summation!)}\\ (\sigma^{jj}_{jj}\sigma^{jk}_{kj}+\sigma^{jk}_{jk}-\sigma^{jj}_{jj})C^k_{jj} &= 0 \ . \end{aligned} \tag{33}$$

Substituting the values of the matrix elements of σ we obtain (recall that $\hat{j}=1$)

$$\begin{aligned} \left[p_{kj}q^{-2\theta_{k>j}}-q^{-4}\right]C^k_{jj} &= 0 \ ,\\ \left[(-1)^{\hat{k}}q^{-2\hat{k}}-p^2_{jk}q^{-4\theta_{j>k}}\right]C^k_{jj} &= 0 \ ,\\ \left[-q^{-2}p_{kj}q^{-2\theta_{k>j}}+1-q^{-2\theta_{j<k}}+q^{-2}\right]C^k_{jj} &= 0 \ . \end{aligned} \tag{34}$$

If $C^k_{jj}\neq 0$ then each square bracket in (34) vanishes. It is straightforward to see that this contradicts to the restriction $q^4\neq 1$. $\square$

Proof of Theorem 3. We first prove that

$$C^k_{ij}=0 \quad , \quad i\neq j\neq k\neq i \ .$$

Consider the subsystem

$$\begin{aligned} E_{\sigma,C}(i,j,k;k,k)=0 \quad &\Leftrightarrow \quad A_{kk}C^k_{ij}=A_{jk}A_{ik}C^k_{ij},\\ E_{\sigma,C}(i,j,j;j,k)=0 \quad &\Leftrightarrow \quad A_{kj}C^k_{ij}=A_{jj}A_{ij}C^k_{ij}, \quad \text{(no summation!)}\\ E_{\sigma,C}(i,j,i;i,k)=0 \quad &\Leftrightarrow \quad A_{ki}C^k_{ij}=A_{ji}A_{ii}C^k_{ij} \end{aligned} \tag{35}$$

with three different indices $i\neq j\neq k\neq i$. Since $A_{ij}A_{ji}=q^{-2}$ the system (35) has a non-zero solution C^k_{ij} iff

$$A_{ii}A_{jj}A_{kk}=q^{-2} \quad \text{or} \quad (-1)^{\hat{i}+\hat{j}+\hat{k}}q^{-2(\hat{i}+\hat{j}+\hat{k})}=q^{-2} \ .$$

This equation may have a solution only if $q^4 = -1$ and $\hat{\imath} = \hat{\jmath} = \hat{k} = 1$. But then the equation $F_{\sigma,C}(i,i,j;i,k) = 0$ implies that $C^k_{ij} = 0$.

Next we consider the equation $E_{\sigma,C}(k,i,j;i,k) = 0$ with $i \neq j \neq k \neq i$; by the ice condition, this equation reduces to $0 = \sigma^{ij}_{ij}\sigma^{ik}_{ki}C^k_{kj}$. Since $\sigma^{ik}_{ki} \neq 0$ and $\sigma^{ij}_{ij} \neq 0$ iff $i < j$ we conclude that $C^k_{kj} = 0$ iff $i < j$. If $j > 1$ we can always find $i\colon i < j$. Thus the possibly non-zero structure constants C^k_{kj} are only C^k_{k1}; the structure constants C^k_{1k} may also be different from 0, $C^k_{1k} = -p_{1k}C^k_{k1}$ by Lemma 4. As we have seen, all other structure constants vanish.

The constants C^k_{1k} are subject to further constraints resulting from the equation $E_{\sigma,C}(1,j,k;j,k) = 0$ with $1 < j < k$,

$$\sigma^{jk}_{jk}C^j_{1j} = C^k_{1k}\sigma^{j1}_{1j}\sigma^{jk}_{jk} \ \Rightarrow \ C^j_{1j} = p_{1j}C^k_{1k} \ \text{ for } \ j,k \ \text{ such that } \ 1 < j < k \tag{36}$$

(no summation). The relation $E_{\sigma,C}(i,j,j;j,j) = 0$ reads

$$\sigma^{jj}_{jj}C^j_{ij} = C^j_{ij}\sigma^{ji}_{ij}\sigma^{jj}_{jj} \qquad \text{(no summation)} ,$$

which implies for $i = 1$ that

$$(1 - p_{1j})C^j_{1j} = 0 \ .$$

Thus if $p_{1j} \neq 1$ for some j then $C^j_{1j} = 0$ and, by (36), $C^k_{1k} = 0$ for all k and there is no non-trivial quantum Lie algebra. Therefore $p_{1j} = 1$ and then, by (36), the constants C^k_{1k} for all $k \neq 1$ are equal, $C^k_{1k} = c$. Now consider the equation $E_{\sigma,C}(1,j,1;1,j) = 0$,

$$C^j_{1j}\sigma^{1j}_{j1}(1 - \sigma^{11}_{11}) = 0 \qquad \text{(no summation)} ,$$

So, if χ_1 is odd then all structure constants vanish. This is our final result:

$$p_{1j} = 1 \ \ , \ \ C^k_{ij} = c(\delta^k_j\delta^1_i - \delta^k_i\delta^1_j) \ \ , \ \ \chi_1 \ \text{ is even} \ . \tag{37}$$

It is straightforward to check that (37) defines the quantum Lie algebra. The proof is finished. □

The braid relation is stable under three canonical operations (and their compositions): $\sigma \mapsto \sigma^T$ (transposition), $\sigma \mapsto \sigma_{21} := P\sigma P$, where $P \in \mathrm{End}(V \otimes V)$ is the flip, $P(u \otimes v) = v \otimes u$, and $\sigma \mapsto \sigma^{-1}$. However for the standard R-matrix σ, the matrices σ^T, σ_{21} and σ^{-1} are again standard (modulo a base change and a redefinition of parameters) and we do not need to consider them separately. We conclude that Theorem 3 gives all quantum Lie algebras compatible with the standard Drinfeld-Jimbo R-matrices.

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