Quantum Financial Economics of Games of Strategy and Financial Decisions

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Abstract

A quantum financial approach to finite games of strategy is addressed, with an extension of Nash's theorem to the quantum financial setting, allowing for an entanglement of games of strategy with two-period financial allocation problems that are expressed in terms of: the consumption plans' optimization problem in pure exchange economies and the finite-state securities market optimization problem, thus addressing, within the financial setting, the interplay between companies' business games and financial agents' behavior.

A complete set of quantum Arrow-Debreu prices, resulting from the game of strategy's quantum Nash equilibrium, is shown to hold, even in the absence of securities' market completeness, such that Pareto optimal results are obtained without having to assume the completeness condition that the rank of the securities' payoff matrix is equal to the number of alternative lottery states.

Keywords: Quantum Financial Economics, Finite Games, Quantum Nash Equilibrium, Quantum Arrow-Debreu Prices, Securities Markets

1 Introduction

Underlying economic theory, the notion of game constitutes a fundamental conceptual unit with operativity in approaching the systemic behavior and dynamics of economic systems, an argument that was established by von Neumann and Morgenstern in [12]. Each (formal) game structure attempts to formalize complex decisional problems that demand a calculatory adaptiveness on the part of the players, a main point which is present both in standard game theory [10, 12] as well as in the quantum expansions [9, 11].

Financial economics has approached, foundationally, the decisional contexts for the problem of allocation of resources, from an exposure to games comprised of pure lotteries, that is, a game against nature where nature “chooses” each state with a certain probability [1, 5, 8].

However, this constitutes an oversimplification of financial systems and their interplay with economic systems. Since companies play games of strategy and, ultimately, a securities market for those companies’ shares is about an exposure to such strategic contexts.

Thus, to better understand the consequences of such an interplay between business games and financial systems’ resources and wealth allocation problems [1, 5, 8], we change, in the present work, the fundamental building block of the *pure lottery game*, to make the lottery entangled with a quantum game of strategy, and address how the financial economics of quantum games of strategy may impact on the structure and conclusions of the two basic financial decision problems: **the management and optimization of consumption plans in pure exchange economies** and the **securities’ market optimization problem** [1, 5], for securities that lead to exposure, on the part of the financial agents, to each player’s position in the quantum game.

In **section 2.**, we address the quantum financial economics of finite games of strategy and provide for an illustrative example of a game between two companies, showing how one may interpret both the disentangled quantum mixed strategies equilibrium as well as the quantum entangled solution.

In **section 3.**, we draw upon the formal work of **section 2.** and introduce a quantum game of strategy-dependent lottery, leading to a quantum entanglement between a pure exchange economy for consumption claims on the lottery results and the underlying game of strategy. In this way, it its shown how one can operationalize the traditional two-period exchange economy results of standard financial theory within a setting in which the lottery is entangled with a strategic choice problem.

From these results, the securities market portfolio problem is addressed, being shown that, when each security offers an exposure to a game position, Pareto optimality for the financial agents’ consumption problem is guaranteed, even in incomplete financial markets, as long as the financial prices reflect the quantum game present value of that exposure.

In **section 4.**, we conclude with some final remarks regarding quantum game theory and quantum financial economics.

2 Quantum Financial Economics of Finite Games of Strategy

2.1 The Formalism and Equilibrium Problem

Finite games of strategy, within the framework of noncooperative quantum game theory [10], can be approached from finite chain categories, where, by finite chain category, it is understood a category $\mathcal{C}(n; N)$ that is generated by n objects and N morphic chains, called primitive chains, linking the objects in a specific order, such that there is a single labelling. $\mathcal{C}(n; N)$ is, thus, generated by N primitive chains of the form:

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_1 ... x_{n-1} \xrightarrow{f_n} x_n \tag{1}$$

A finite chain category is interpreted as a finite game category as follows: to each morphism in a chain $x_{i-1} \xrightarrow{f_i} x_i$ there corresponds a strategy played by a player that occupies the position i, in this way, a chain corresponds to a sequence of strategic choices available to the players.

A quantum formal theory, for a finite game category $\mathcal{C}(n; N)$, is defined as a formal structure such that each morphic fundament f_i of the morphic relation $x_{i-1} \xrightarrow{f_i} x_i$ is a tuple of the form:

$$f_i := \left(\mathcal{H}_i, \mathcal{P}_i, \hat{P}_{f_i}\right) \tag{2}$$

where $\mathcal{H}_i$ is the i-th player's Hilbert space, $\mathcal{P}_i$ is a complete set of projectors onto a basis that spans the Hilbert space, and $\hat{P}_{f_i} \in \mathcal{P}_i$. This structure is interpreted as follows: from the strategic Hilbert space $\mathcal{H}_i$, given the pure strategies' projectors $\mathcal{P}_i$, the player chooses to play $\hat{P}_{f_i}$.

From the morphic fundament definition (2), an assumption has to be made on composition in the finite category, we assume the following tensor product composition operation:

$$f_j \circ f_i = f_{ji} \tag{3}$$

$$f_{ji} = \left(\mathcal{H}_{ji} = \mathcal{H}_j \otimes \mathcal{H}_i, \mathcal{P}_{ji} = \mathcal{P}_j \otimes \mathcal{P}_i, \hat{P}_{f_{ji}} = \hat{P}_{f_j} \otimes \hat{P}_{f_i}\right) \tag{4}$$

From this definition of composition, a morphism for a *game choice path* can be introduced as:

$$x_0 \xrightarrow{f_{n...21}} x_n \tag{5}$$

$$f_{n...21} = \left(\mathcal{H}_G = \bigotimes_{i=n}^{1} \mathcal{H}_i, \mathcal{P}_G = \bigotimes_{i=n}^{1} \mathcal{P}_i, \hat{P}_{f_{n...21}} = \bigotimes_{i=n}^{1} \hat{P}_{f_i}\right) \tag{6}$$

in this way, the choices along the chain of players are completely "encoded" in the tensor product projectors $\hat{P}_{f_{n...21}}$. Given the above definitions, there are $N = \prod_{i=1}^{n} \dim\left(\mathcal{H}_i\right)$ such morphisms, a number that coincides with the number of primitive chains in the category $\mathcal{C}(n; N)$.

Each projector can be addressed as a strategic marker of a game path, and leads to the matrix form of an Arrow-Debreu security [1, 5], therefore, we call it *game Arrow-Debreu projector*.

While, in traditional financial economics, the Arrow-Debreu securities pay one unit of numeraire per state of nature, in the present game setting, they pay one unit of payoff per game path at the beginning of the game, however this analogy may be taken it must be addressed with some care, since these are not securities, rather, they represent, projectively, strategic choice chains in the game, so that the price of a projector $\hat{P}_{f_{n...21}}$ (the Arrow-Debreu price) is the price of a strategic choice and, therefore, the result of the strategic evaluation of the game by the different players.

Now, let $|\Psi\rangle$ be a ket vector in the game's Hilbert space $\mathcal{H}_G$, such that:

$$|\Psi\rangle = \sum_{f_{n\ldots21}} \psi\left(f_{n\ldots21}\right)\left|f_{n\ldots21}\right\rangle \tag{7}$$

where $\psi\left(f_{n\ldots21}\right)$ is the Arrow-Debreu price amplitude, with the condition:

$$\sum_{f_{n\ldots21}} \left|\psi\left(f_{n\ldots21}\right)\right|^2 = D \tag{8}$$

for $D > 0$, then, the $\left|\psi\left(f_{n\ldots21}\right)\right|^2$ correspond to the Arrow-Debreu prices for the game path $f_{n\ldots21}$ and D is the discount factor in riskless borrowing, defining an economic scale for temporal connections between one unit of payoff now and one unit of payoff at the end of the game, such that one unit of payoff now can be "capitalized" to the end of the game (when the decision takes place) through a multiplication by $\frac{1}{D}$, while one unit of payoff at the end of the game can be discounted to the beginning of the game through multiplication by D.

In this case, the unit operator $\hat{1} = \sum_{f_{n\ldots21}} \hat{P}\left(f_{n\ldots21}\right)$ has a similar profile as that of a bond in standard financial economics, with $\left\langle \Psi \left| \hat{1} \right| \Psi \right\rangle = D$, on the other hand, the general payoff system, for each player, can be addressed from an operator expansion:

$$\hat{\pi}_i = \sum_{f_{n\ldots21}} \pi_i\left(f_{n\ldots21}\right) \hat{P}_{f_{n\ldots21}} \tag{9}$$

where each weight $\pi_i\left(f_{n\ldots21}\right)$ corresponds to quantities associated with each *Arrow-Debreu projector* that can be interpreted as similar to the quantities of each Arrow-Debreu security for a general asset. Multiplying each weight by the corresponding Arrow-Debreu price, one obtains the payoff value for each alternative such that the total payoff for the player at the end of the game is given by:

$$\left\langle \Psi \left| \hat{\pi}_i \right| \Psi \right\rangle = \sum_{f_{n\ldots21}} \pi_i\left(f_{n\ldots21}\right) \frac{\left|\psi\left(f_{n\ldots21}\right)\right|^2}{D} \tag{10}$$

We can discount the total payoff in (10) to the beginning of the game using the discount factor D, leading to the present value payoff for the player:

$$PV_i = D \left\langle \Psi \left| \hat{\pi}_i \right| \Psi \right\rangle = D \sum_{f_{n\ldots21}} \pi_i\left(f_{n\ldots21}\right) \frac{\left|\psi\left(f_{n\ldots21}\right)\right|^2}{D} \tag{11}$$

In the above equation, the $\pi_i\left(f_{n\ldots21}\right)$ represent quantities, while the ratio $\frac{\left|\psi\left(f_{n\ldots21}\right)\right|^2}{D}$ represents the future value (at the decision moment) of the quantum Arrow-Debreu prices (capitalized quantum Arrow-Debreu prices). Introducing the ket $|Q\rangle \in \mathcal{H}_G$, such that:

$$|Q\rangle = \frac{1}{\sqrt{D}}|\Psi\rangle \tag{12}$$

then, $|Q\rangle$ is a normalized ket for which the price amplitudes are expressed in terms of their future value. Replacing in (11), we have:

$$PV_i = D \langle Q | \hat{\pi}_i | Q \rangle \tag{13}$$

In the quantum game setting, the capitalized Arrow-Debreu price amplitudes $\langle f_{n...21} | Q \rangle$ become quantum strategic configurations, resulting from the strategic cognition of the players with respect to the game. Given $|Q\rangle$, each player's strategic valuation of each pure strategy can be obtained by introducing the projector chains:

$$\hat{C}_{f_i} = \sum_{f_{n...i+1}, f_{i-1...1}} \hat{P}_{f_{n...i+1}} \otimes \hat{P}_{f_i} \otimes \hat{P}_{f_{i-1...1}} \tag{14}$$

with $\sum_{f_i} \hat{C}_{f_i} = \hat{1}$. For each alternative choice of the player i, the chain sums over all of the other choice paths for the rest of the players, such chains are called coarse-grained chains in the decoherent histories approach to quantum mechanics [2, 4, 6], following this approach, one may introduce the pricing functional from the expression for the decoherence functional [2, 4, 6]:

$$\mathcal{D}(f_i, g_i : |Q\rangle) = \left\langle Q \left| \hat{C}^{\dagger}_{f_i} C_{g_i} \right| Q \right\rangle \tag{15}$$

we, then, have, for each player:

$$\mathcal{D}(f_i, g_i : |Q\rangle) = 0, \, \forall f_i \neq g_i \tag{16}$$

this is the usual quantum mechanics' condition for an aditivity of measure (also known as decoherence condition), which, in the present case, means that the capitalized prices for two different alternative choices of player i are additive. Then, we can work with the pricing functional $\mathcal{D}(f_i, f_i : |Q\rangle)$ as giving, for each player an Arrow-Debreu capitalized price associated with the pure strategy, expressed by f_i. Given that the condition (16) is satisfied, each player's quantum Arrow-Debreu pricing matrix, defined analogously to the decoherence matrix from the decoherent histories approach, is a diagonal matrix and can be expanded as a linear combination of the projectors for each player's pure strategies as follows:

$$\mathbf{D}_i(|Q\rangle) = \sum_{f_i} \mathcal{D}(f_i, f_i : |Q\rangle) \hat{P}_{f_i} \tag{17}$$

which has the mathematical expression of a mixed strategy. In particular, $\mathbf{D}_i(|Q\rangle)$ can be regarded as points in a simplex whose vertices are the pure strategies' projectors. Gathering all of the pricing matrices $\mathbf{D}_i(|Q\rangle)$ we obtain the n-tuple $\mathfrak{D}(|Q\rangle) = (\mathbf{D}_1(|Q\rangle), \mathbf{D}_2(|Q\rangle), ..., \mathbf{D}_n(|Q\rangle))$. Introducing $pv_i[\mathfrak{D}(|Q\rangle)] := PV_i = \langle Q | \hat{\pi}_i | Q \rangle$, Nash's equilibrium definition [10], follows, within the quantum financial setting, as:

$$pv_i[\mathfrak{D}(|Q\rangle)] = \max_{\text{all } |Q'\rangle} \left\{ pv_i \left[\mathfrak{D}\left(|Q'\rangle ; \mathbf{D}_i \left(\hat{U} |Q'\rangle \right) \right) \right] \right\} \tag{18}$$

where by $\mathfrak{D}\left(\left|Q'\right\rangle;\mathbf{D}_i\left(\hat{U}\left|Q'\right\rangle\right)\right)$ it is understood that, given the feasible quantum price strategy $\left|Q'\right\rangle$, the i-th player performs a unitary rotation $\hat{U}$ leading to a substitution of $\mathbf{D}_i\left(\left|Q'\right\rangle\right)$ by $\mathbf{D}_i\left(\hat{U}\left|Q'\right\rangle\right)$ leaving all of the rest of the elements in the n-tuple unchanged. Thus, each player chooses from all of the possible quantum computations, the one that maximizes the present value payoff function with all the other strategies held fixed, which is in agreement with Nash [10].

The equilibrium price ket $\left|Q\right\rangle$ is such that each player's present value payoff is optimal, given all of the other players' strategies. The following theorem can now be proven:

Theorem 1. (Quantum Equilibrium Theorem) - Every quantum finite game has a ket of capitalized Arrow-Debreu price amplitudes, unique up to a capitalized Arrow-Debreu price-conserving unitary transformation, that is an equilibrium solution for the game.

Proof. In the present proof, we assume all of the above definitions and game formalism context. Let, then, $\left|Q\right\rangle \in \mathcal{H}_G$ be a ket of capitalized Arrow-Debreu price amplitudes and $\mathfrak{D}\left(\left|Q\right\rangle\right)$ be the corresponding n-tuple of pricing matrices, define also $\left|Q:f_i\right\rangle \in \mathcal{H}_G$ to be such that $\mathbf{D}_j\left(\left|Q:f_i\right\rangle\right) = \mathbf{D}_j\left(\left|Q\right\rangle\right)$ for every $j \neq i$ and $\mathbf{D}_i\left(\left|Q:f_i\right\rangle\right) = \hat{P}_{f_i}$. Following Nash's proof in [10], we introduce the set of continuous functions of $\mathfrak{D}\left(\left|Q\right\rangle\right)$ defined by:

$$\varphi_{if_i}\left[\mathfrak{D}\left(\left|Q\right\rangle\right)\right] := \max\left(0, pv_i\left[\mathfrak{D}\left(\left|Q:f_i\right\rangle\right)\right] - pv_i\left[\mathfrak{D}\left(\left|Q\right\rangle\right)\right]\right) \tag{19}$$

Let, now, $\hat{U}$ be a unitary transformation such that $\hat{U}\left|Q\right\rangle = \left|X\right\rangle$ and, for each, $\mathbf{D}_i\left(\left|Q\right\rangle\right) \in \mathfrak{D}\left(\left|Q\right\rangle\right)$ it holds that:

$$\mathbf{D}_i\left(\left|X\right\rangle\right) = \frac{1}{1+\sum_{f_i}\varphi_{if_i}\left[\mathfrak{D}\left(\left|Q\right\rangle\right)\right]}\left(\mathbf{D}_i\left(\left|Q\right\rangle\right) + \sum_{f_i}\varphi_{if_i}\left[\mathfrak{D}\left(\left|Q\right\rangle\right)\right]\hat{P}_{f_i}\right) \tag{20}$$

then, the unitary transformation leads to a pricing matrix transformation that coincides with the transformation addressed by Nash in [10], such that its fixed points are the equilibrium points. Given the geometric coincidence between Nash's formulation and the structure of the space of pricing matrices for the quantum game, it follows that Nash's theorem applies and every game has an n-tuple of equilibrium pricing matrices.

For any $\mathfrak{D}\left(\left|Q\right\rangle\right)$ which is an equilibrium, there is a family of kets defined by:

$$\mathcal{E}\left(\mathfrak{D}\left(\left|Q\right\rangle\right)\right) = \left\{\left|X\right\rangle \in \mathcal{H}_G : \mathfrak{D}\left(\left|X\right\rangle\right) = \mathfrak{D}\left(\left|Q\right\rangle\right)\right\} \tag{21}$$

all of the kets in $\mathcal{E}\left(\mathfrak{D}\left(\left|Q\right\rangle\right)\right)$ are related to each other by equilibrium preserving unitary transformations, which necessarily conserve the capitalized equilibrium quantum Arrow-Debreu prices. □

In the above theorem, and proof, the probabilistic interpretation is not invoked for the capitalized prices $\frac{\left|\psi\left(f_{n\ldots21}\right)\right|^2}{D}$, which have a mathematical structure, with respect to the payoff operators, similar to probability measures. Within an evolutionary finance framework, one can, however, offer a probabilistic interpretation by working with a notion of *fitness* as a selectibility of a strategic solution in an adaptation problem [3], such that the probability of a strategy being selected is proportional to its adaptive value.

In the present case, the capitalized quantum Arrow-Debreu prices constitute a strategic valuation measure of a game path at the end of the decision frame (future) and, therefore, they become a measure of the adaptive value of a game alternative for the system of players.

The game paths with higher assigned prices are the most desired by the players' system as good adaptive solutions, therefore, these should be more probable of being selected by the system of players, hence, the probabilities of selection of each game path should numerically coincide with the capitalized quantum Arrow-Debreu prices. This is a numerical coincidence, not a conceptual one, since the capitalized prices are expressed in units of payoff, while the probabilities are pure numbers. Under the evolutionary interpretation, the present value of the decision, therefore, leads to an expectation on the future decisional moment discounted to the present.

We now provide for an example from quantum corporate finance.

2.2 A Two-Company Game Example

In order to illustrate the formalism introduced so far, let us assume that two companies, labelled A and B, are evaluating a new investment opportunity, having the choice of implementing the project with one of two alternative technologies, labelled 0 and 1.

Both companies are expected to announce simultaneously, the new investment, making public to the markets the technology chosen, and neither company knows beforehand what the other will choose, so that each decides without knowledge of the other's decision.

If both companies choose the same technology, then, company A expects to obtain a project's return index (PRI) of 2, while it obtains a PRI of 1.5 if the companies do not choose the same technology. Company B, on the other hand, expects to obtain a PRI of: 2.5 if the companies do not choose the same technology; 1.4 if both companies choose technology 0 and 2 if both companies choose technology 1.

There are four primitive chains for this game, with a freedom to choose who to place first in the chain. In the present case, for simplicity, we follow the labels and write these chains as $0 \xrightarrow{A_s} 1 \xrightarrow{B_s} 2$, where A_s means A chooses technology s, with $s = 0, 1$, the same holding for B_s. The game's choice paths' morphisms are, then, expressed as $0 \xrightarrow{B_s A_r} 2$, with $B_s A_r = \left(\mathcal{H}_G, \mathcal{P}_G, \hat{P}_{B_s A_r}\right)$ and $\mathcal{P}_G = \left\{\hat{P}_{B_s A_r} = |B_s A_r\rangle \langle B_s A_r| : r, s = 0, 1\right\}$.

Given the description of the game, the payoff operators are, in turn, given by:

$$\hat{\pi}_A = 2\left(\hat{P}_{B_0A_0} + \hat{P}_{B_1A_1}\right) + 1.5\left(\hat{P}_{B_1A_0} + \hat{P}_{B_0A_1}\right) \tag{22}$$

$$\hat{\pi}_B = 1.4\hat{P}_{B_0A_0} + 2\hat{P}_{B_1A_1} + 2.5\left(\hat{P}_{B_1A_0} + \hat{P}_{B_0A_1}\right) \tag{23}$$

Up to a price-conserving unitary transformation, the quantum Nash equilibrium is, in this case, given by:

$$|Q\rangle = \sqrt{\frac{5}{32}}\left(|B_0A_0\rangle + |B_1A_0\rangle\right) + \sqrt{\frac{11}{32}}\left(|B_0A_1\rangle + |B_1A_1\rangle\right) \tag{24}$$

which is separable in A and B's strategies:

$$|Q\rangle = \left(\frac{1}{\sqrt{2}}|B_0\rangle + \frac{1}{\sqrt{2}}|B_1\rangle\right) \otimes \left(\sqrt{\frac{5}{16}}|A_0\rangle + \sqrt{\frac{11}{16}}|A_1\rangle\right) \tag{25}$$

and the resulting pricing matrices reflect this tensor product structure:

$$\mathbf{D}_A = \frac{5}{16}\hat{P}_{A_0} + \frac{11}{16}\hat{P}_{A_1},\ \mathbf{D}_B = \frac{1}{2}\hat{P}_{A_0} + \frac{1}{2}\hat{P}_{A_1} \tag{26}$$

Assuming that the IRP are calculated for the beginning of the game, which forms part of the "year 0" at which the project is evaluated, we have $D = 1$ and, thus, assuming the quantum Arrow-Debreu prices to be expressed in monetary units: $PV_A = \$1.75$ while $PV_B = \$2.15625$. Under the evolutionary framework, introduced in the previous subsection, A chooses technology 0 with probability equal to $\frac{5}{16}$ and chooses technology 1 with probability $\frac{11}{16}$, while B plays a fair coin game for 0 and 1, these probabilities are conserved for any quantum Arrow-Debreu price conserving transformation of $|Q\rangle$ and the choice of one player is probabilistically independent from the choice of the other player.

If, on the other hand, *company* A were to announce first its decision, then, B could always wait for A and choose its strategy so as to maximize its payoff, a resulting general solution, for $D = 1$, is an entangled ket:

$$|Q\rangle = \psi_A(0)|B_1A_0\rangle + \psi_A(1)|B_0A_1\rangle \tag{27}$$

such that $PV_A = \langle Q|\hat{\pi}_A|Q\rangle = 1.5$ and $PV_B = \langle Q|\hat{\pi}_B|Q\rangle = 2.5$, which is the result of either the path B_1A_0 or the path B_0A_1, being played by the companies, it does not matter what the value of the quantum Arrow-Debreu prices are, the result is always the same for each company.

3 Quantum Games with Exchange Economies

We now expand the formalism of the previous section to include exchange economies, first a pure exchange economy comprised of a lottery that is a single bet game on the result of an underlying quantum game of strategy, and, second, a securities economy.

3.1 Single bet game

In the present section it is useful to introduce Greek letter indexes ranging in $1, 2, ...N$, keeping the Latin lettered indexes for denoting adaptive agents (both agents in the exchange economy as well as game players), thus, one assumes to order the morphic fundaments $f_{n...21}$ and write, according to that order, f_α with the index $\alpha = 1, 2, ..., N$. Assuming this notation, in order to introduce an exchange economy structure, we, first, expand the strategic game of **subsection 2.1** with a lottery, by adding an addicional x_{n+1} object, such that the morphism (5) is expanded to:

$$x_0 \xrightarrow{f_{n...21}} x_n \xrightarrow{\omega} x_{n+1} \tag{28}$$

where ω is an index ranging as $\omega = 1, 2, ...N$. This is called the single bet game, since there is only one lottery type. The ket (12) is, accordingly, replaced by:

$$|Q\rangle = \frac{1}{\sqrt{D}} \sum_{\omega\alpha} \psi\left(f_\alpha, \omega\right) |f_\alpha \omega\rangle \tag{29}$$

The price amplitudes $\psi\left(f_\alpha, \omega\right)$ are now pricing, simultaneously, the game path and the lottery.

Introducing the lottery economy operator $\hat{e}$, such that:

$$\hat{e}\,|Q\rangle = |Q\rangle \tag{30}$$

$$\hat{e}\,|f_\alpha \omega_k\rangle = \begin{cases} |f_\alpha \omega\rangle\,, & \omega = \alpha \\ 0, & \omega \neq \alpha \end{cases} \tag{31}$$

then, it follows that the lottery result is entangled with the game, that is, in the quantum game equilibrium, we have the entangled kets as solutions to the above eigenvalue equation:

$$|Q\rangle = \frac{1}{\sqrt{D}} \sum_{\omega} \psi\left(f_\omega, \omega\right) |f_\omega \omega\rangle \tag{32}$$

thus, the quantum state for the game is no longer a pure state, but, instead a statistical mixture density operator, resulting from tracing out the lottery economy, such that, from the systemic position of the game, we have the density operators for the game and the lottery:

$$\hat{\rho}_{Game} = Tr_{Lottery}\left(|Q\rangle\langle Q|\right) = \sum_{\omega} \frac{\left|\psi\left(f_\omega, \omega\right)\right|^2}{D} \hat{P}_{f_\omega} \tag{33}$$

$$\hat{\rho}_{Lottery} = Tr_{Game}\left(|Q\rangle\langle Q|\right) = \sum_{\omega} \frac{\left|\psi\left(f_\omega, \omega\right)\right|^2}{D} \hat{P}_{\omega} \tag{34}$$

in both cases, the capitalized quantum Arrow-Debreu prices $\frac{|\psi(f_\omega,\omega)|^2}{D}$ are assumed from now on to be, always, the Nash Equilibrium prices for the underlying quantum game of strategy.

Now, let us introduce a two-period pure exchange economy with a single perishable good in both periods, such that the agents, in the economy, choose a consumption at time 0 (beginning of the game) and state contingent claims on lottery-dependent consumption for the end of the game, taken, for simplicity, to be time 1.

The exchange econonomy's agents and the players are, in this case, assumed to be different entities, while the players are addressing the game of strategy, the agents are addressing the lottery, through a consumption allocation problem, such that, c_{i0} is the i-th agent's consumption at time 0, and $c_{i\omega}$ is the i-th agent's consumption at time 1. A utility operator on consumption for each agent is introduced such that, for $i = 1, 2, ..., I$ (I being the number of agents in the exchange economy):

$$\hat{u}_i \left| f_\alpha \omega_k \right\rangle = u_i \left(c_{i0}, c_{i\omega} \right) \left| f_\alpha \omega \right\rangle \tag{35}$$

For two different agents, their utility operators are assumed to commute $[\hat{u}_i, \hat{u}_j] = \delta_{ij}$ and the utility functions u_i are assumed to be increasing and strictly concave functions of the consumption plan $(c_{i0}, c_{i\omega})$. We now have a problem of allocation of state contingent consumption among agents, where, without loss of generality, the single consumption good is used as the numeraire for the exchange economy [5].

Taking C_0 to be the aggregate time-0 consumption available and C_ω the aggregate consumption in the ω-th lottery result at time 1, the feasibility conditions are given by [5]:

$$C_0 = \sum_{i=1}^{I} c_{i0},\ C_\omega = \sum_{i=1}^{I} c_{i\omega} \tag{36}$$

Assuming that each agent knows that the lottery and the strategic game are entangled and assigns a subjective probability to each alternative ω, then, each agent has a subjectively assigned density operator for the lottery:

$$\hat{\rho}^i_{Lottery} = \sum_\omega bel_i \left(f_\omega, \omega \right) \hat{P}_\omega \tag{37}$$

where $bel_i \left(f_\omega, \omega \right)$ is a subjective statistical weight that the agent assigns to the lottery, without any further information available upon the underlying game's payoff system. The (subjective) expected utility is:

$$\left\langle \hat{u}_i \right\rangle_{Bel} = Tr \left(\hat{\rho}^i_{Lottery} \hat{u}_i \right) \tag{38}$$

Then, the following optimization problem, then, ensues for the Pareto optimal allocation:

$$\begin{array}{lc} & \max_{\left\{ (c_{i0}, c_{i\omega})_{i=1}^{I}, \omega = 1,2,...,N \right\}} \sum_{i=1}^{I} \lambda_i \left\langle \hat{u}_i \right\rangle_{Bel} \\ s.t. & \sum_{i=1}^{I} = C_\omega,\ \omega = 1, 2, ..., N \\ & \sum_{i=1}^{I} c_{i0} = C_0 \end{array} \tag{39}$$

Forming the Lagrangian for the optimization problem we have:

$$\max_{\left\{(c_{i0},c_{i\omega})_{i=1}^{I},\omega=1,2,...,N\right\}} L =$$

$$= \sum_{i=1}^{I} \lambda_i \left\langle \hat{u}_i \right\rangle_{Bel} + \phi_0 \left[C_0 - \sum_{i=1}^{I} c_{i0} \right] + \tag{40}$$

$$+ \sum_{\omega=1}^{N} \phi_\omega \left[C_\omega - \sum_{i=1}^{I} c_{i\omega} \right]$$

The first-order conditions lead to:

$$\lambda_i \textstyle\sum_{\omega=1}^{N} bel_i \left(f_\omega, \omega\right) \frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i0}} = \phi_0, \quad i = 1, 2, ..., I \tag{41}$$

$$\lambda_i bel_i \left(f_\omega, \omega\right) \frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i\omega}} = \phi_\omega, \quad \omega = 1, 2, ..., N, \; i = 1, 2, ..., I \tag{42}$$

$$\sum_{i=1}^{I} c_{i\omega} = C_\omega, \; \forall \omega = 1, 2, ..., N \tag{43}$$

$$\sum_{i=1}^{I} c_{i0} = C_0 \tag{44}$$

Since the utility functions are assumed to be increasing and strictly concave and the weights $\{\lambda_i\}_{i=1}^{I}$ are assumed to be strictly positive, the first order conditions are necessary and sufficient for a global maximum [5].

Replacing (42) in (41) for each agent, the marginal rates of substitution between present consumption and future lottery-state contingent consumption are equal across individuals:

$$\frac{bel_i \left(f_\omega, \omega\right) \frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i\omega}}}{\sum_{\omega=1}^{N} bel_i \left(f_\omega, \omega\right) \frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i0}}} = \frac{\phi_\omega}{\phi_0} \tag{45}$$

for $\omega = 1, 2, ..., N$ and $i = 1, 2, ..., I$.

Following Huang and Litzenberger in [5], the above optimization problem can be solved for the cases where, taking $\phi_0 = 1$, ϕ_ω become the lottery-state contingent Arrow-Debreu prices. In the present case, since the contingency does not come from a pure lottery, but, instead from a lottery that is entangled with a finite quantum game of strategy, we may let ϕ_ω equal the quantum Arrow-Debreu price for the game, that is:

$$\phi_\omega = \left| \psi \left(f_\omega, \omega\right) \right|^2 \tag{46}$$

If we replace (46) in (45), and let $\phi_0 = 1$, we obtain:

$$\frac{bel_i \left(f_\omega, \omega\right) \frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i\omega}}}{\sum_{\omega=1}^{N} bel_i \left(f_\omega, \omega\right) \frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i0}}} = \left| \psi \left(f_\omega, \omega\right) \right|^2 \tag{47}$$

going further, and assuming that each agent has enough information to form a financially sustained rational equilibrium prediction about the game, that is, provided each agent, in the transaction economy, has enough information on the game of strategy to compute the quantum Nash equilibrium for the game and, therefore, the probabilities for the lottery, then:

$$bel_i\left(f_\omega,\omega\right)=\frac{\left|\psi\left(f_\omega,\omega\right)\right|^2}{D} \tag{48}$$

Replacing this last result in (47), and given the positivity of the quantum Arrow-Debreu prices, the following result holds:

$$\frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i\omega}}=D\sum_{\omega=1}^{N}\frac{\left|\psi\left(f_\omega,\omega\right)\right|^2}{D}\cdot\frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i0}} \tag{49}$$

establishing the relation between the marginal utilities and the quantum game of strategy solution.

3.2 Securities market game

Assuming the single-lottery framework of equations (30) to (32), we can introduce a transaction market for m securities with payoff operators:

$$\hat{x}_j\left|f_\alpha\omega_k\right\rangle=\begin{cases}x_j\left(\omega\right)\left|f_\alpha\omega\right\rangle, & \omega=\alpha\\ 0, & \omega\neq\alpha\end{cases} \tag{50}$$

for $j=1,2...,m$. Each operator is financially consistent with the lottery economy, in the sense that $[\hat{x}_j,\hat{e}]=0$ having the same entangled eigenstates described by equation (32). Letting s_{ij} and S_j denote, respectively, the number of securities held by the agent i, in equilibrium, and the price for security j at time 0, the relevant optimization problem for each agent's portfolio, given the previous subsection's framework, can be formulated as follows [5]:

$$\begin{array}{c}\max_{\{c_{i0},s_{ij};j=1,2,\ldots,m\}}\left\langle\hat{u}_i\right\rangle_{Bel}\\ s.t.\quad c_{i0}+\sum_j s_{ij}S_j=e_{i0}+\sum_{j=1}^m w_{ij}S_j\end{array} \tag{51}$$

where e_{i0} is an endowment of time 0 consumption and w_{ij} is a time 0 endowment of shares of security j. Given the previously assumed properties of the utility functions, the necessary and sufficient conditions for the i-th agent's portfolio problem are given by:

$$\sum_{\omega=1}^{m}\frac{bel_i\left(f_\omega,\omega\right)\frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i\omega}}}{\sum_{\omega=1}^m bel_i\left(f_\omega,\omega\right)\frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i0}}}x_j(\omega)=S_j \tag{52}$$

with $c_{i\omega}:=\sum_{j=1}^m s_{ij}x_j\left(\omega\right)$.

To derive a Pareto optimal allocation that reflects the underlying game of strategy structure, it is first, important to notice that, given the entangled ket of equation (32), one may express a financial valuation from the density operator for the lottery such that, for the game's quantum Nash equilibrium, we have:

$$Tr\left(\hat{\rho}_{Lottery}\hat{x}_j\right) = \sum_{\omega=1}^{N} x_j(\omega)\frac{\left|\psi\left(f_\omega,\omega\right)\right|^2}{D} \tag{53}$$

this is the expected payoff for the j-th security, then, following **section 2.**'s framework, one may calculate a present value as one would for any other player's position, and write:

$$PV_j = D \times Tr\left(\hat{\rho}_{Economy}\hat{x}_j\right) = D\sum_{\omega=1}^{N} x_j(\omega)\frac{\left|\psi\left(f_\omega,\omega\right)\right|^2}{D} \tag{54}$$

it is then (financially) natural for the securities market agents to value the security in terms of this game payoff, such that the condition holds:

$$S_j = PV_j \tag{55}$$

which is financially sound since each state-contingent payoff is weighed by the quantum game price of that state, the state being, in this case, the result of the lottery which is, in turn, contingent upon the game of strategy's path.

Now, if the number of securities is equal to the number of players in the underlying game of strategy ($m = n$), and if each security is a financially transactionable exposure to the corresponding player's position in the game of strategy (*securitization of game positions*), in this case, one may consider that the security payoff is proportional to the player's payoff, the constant of proportionality θ giving the exposure (for instance, the proportion of equity is an example of an exposure to a company's value, the company being the player, the equity holders being the financial agents and the shares being the securities), formally, we may write $\hat{x}_i := \theta\hat{\pi}_i$.

Thus, by playing for the quantum Nash equilibrium the players are simultaneously maximizing the value of the corresponding security asset, and therefore, the portfolio value for any financial agent is maximized by the Nash optimizing adaptive behavior of the players involved in the underlying game of strategy.

Denoting $\left|\psi\left(f_\omega,\omega\right)\right|^2$ by ϕ_ω and letting, as before, $bel_i\left(f_\omega,\omega\right) = \phi_\omega$, by replacing (54) in (52), we obtain the optimum:

$$\sum_{\omega=1}^{N}\phi_\omega\frac{\frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i\omega}}}{\sum_{\omega=1}^{N}\phi_\omega\frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i0}}}x_j(\omega) = D\sum_{\omega=1}^{N}\frac{1}{D}\phi_\omega x_j(\omega) \tag{56}$$

which can be rewritten as:

$$\sum_{\omega=1}^{N}\phi_\omega x_j(\omega)\left(\frac{\frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i\omega}}}{D\sum_{\omega=1}^{N}\phi_\omega\frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i0}}} - \frac{1}{D}\right) = 0 \tag{57}$$

One solution, in particular, for (56) results from taking the argument within the brakets to be zero, $\frac{\frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i\omega}}}{D\sum_{\omega=1}^{m}\phi_\omega\frac{\partial u_i(c_{i0},c_{i\omega})}{\partial c_{i0}}} - \frac{1}{D} = 0$, which leads to the inter-temporal Pareto optimal consumption plan condition:

$$\frac{\partial u_i(c_{i0}, c_{i\omega})}{\partial c_{i\omega}} = D\sum_{\omega=1}^{m}\frac{1}{D}\phi_\omega\frac{\partial u_i(c_{i0}, c_{i\omega})}{\partial c_{i0}} \tag{58}$$

Therefore, even in incomplete securities' markets, as long as there is a security exposure for each game player's position that is proportional to the payoffs on that position, and assuming that the quantum Nash equilibrium is played, one obtains a consumption Pareto optimal plan for securities prices reflecting the quantum Nash equilibrium of the game.

The optimal consumption plan depends upon two behaviors: a quantum Nash optimizing behavior on the part of the players and an equilibrium valuation behavior on the part of the securities' market agents, such that the securities' market agents reflect, in each securities' price, the financial present value of the exposure at the corresponding quantum Nash equilibrium. Therefore, the adaptiveness of the financial market system is inextricably interweaved with the adaptiveness of the business game playing system.

4 Conclusion

Financial decisions, regarding the inter-temporal allocation of wealth and of financial resources, does not take place within a single pure lottery game structure, rather, while financial agents manage asset portfolios, trying to optimize their intertemporal securities' management, the value drivers for these assets come from a complex adaptive dynamics in which companies play games of strategy with their stakeholders, adapting to opportunities and responding to threats, expanding their strengths and addressing their weaknesses, managing their business risk. Business strategic decisions affect directly the probabilities associated with different financial scenarios and, therefore, directly affect asset allocation and valuation issues.

This state of affairs (business and financial) was the main point of concern for the present article, dealing with the effects upon the traditional financial optimization problems, when one abandons the assumption of a pure lottery game towards a lottery whose results, and therefore probability profiles, are entangled with a game of strategy where players enact a quantum Nash equilibrium.

In order to operationalize this problem it became necessary to address how one might obtain an Arrow-Debreu price structure from the game of strategy itself and, at the same time, entangle that structure with a lottery game, such that all of the financial lottery state-contingent prices coincide with the quantum game equilibrium solution Arrow-Debreu prices.

Such a financial approach to game theory, along with the quantum game equilibrium theorem, derived in **section 2.**, thus, allowed for the quantum formalism to be applied towards a greater effectiveness in the integration between business game theory and financial decision theory, showing how Pareto optimal allocations in a securities market may reflect the game of strategy's quantum Nash equilibrium playing, a result that is independent from any completeness assumption that would restrict the number of linearly independent securities to equal the number of lottery states.

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