

Tidal interactions and
principle of corresponding states:
from micro to macro cosmos.
A century after van der Waals' Nobel Prize

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Abstract

The current attempt is aimed to honor the first centennial of Johannes Diderik van der Waals (VDW) awarding Nobel Prize in Physics. The VDW theory of ordinary fluids is reviewed in the first part of the paper, where special effort is devoted to the equation of state and the law of corresponding states. In addition, a few mathematical features involving properties of cubic equations are discussed, for appreciating the intrinsic beauty of the VDW theory. A theory of astrophysical fluids is shortly reviewed in the second part of the paper, grounding on the tensor virial theorem for two-component systems, and an equation of state is formulated with a convenient choice of reduced variables. Additional effort is devoted to particular choices of density profiles, namely a simple guidance case and two cases of astrophysical interest. The related macroisothermal curves are found to be qualitatively similar to VDW isothermal curves below the critical threshold and, for sufficiently steep density profiles, a critical macroisothermal curve exists, with a single horizontal inflexion point. Under the working hypothesis of a phase transition (assumed to be gas-stars) for astrophysical fluids, similar to the vapour-liquid

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phase transition in ordinary fluids, the location of gas clouds, stellar systems, galaxies, cluster of galaxies, on the plane scanned by reduced variables, is tentatively assigned. A brief discussion shows how van der Waals' two great discoveries, namely a gas equation of state where tidal interactions between molecules are taken into account, and the law of corresponding states, related to microcosmos, find a counterpart with regard to macrocosmos. In conclusion, after a century since the awarding of the Nobel Prize in Physics, van der Waals' ideas are still valid and helpful to day for a full understanding of the universe.

keywords - galaxies: evolution - dark matter: haloes.

1 Introduction

One century ago (1910), the Nobel Prize in Physics was awarded to Johannes Diderik van der Waals (hereafter quoted as VDW). In his doctoral thesis (1873) the ideal gas equation of state was generalized for embracing both the gaseous and the liquid state, where these two states of aggregation not only merge into each other in a continuous manner, but are in fact of the same nature. With respect to ideal gases, the volume of the molecules and the intermolecular tidal forces were taken into account.

The VDW equation was later reformulated in terms of reduced (dimensionless) variables (1880), which allows the description of all substances in terms of a single equation. In other words, the state of any substance, defined by the values of reduced volume, reduced pressure, and reduced temperature, is independent of the nature of the substance. This result is known as the law of corresponding states.

The VDW equation of state in dimensional and reduced form, served as a guide during experiments which ultimately led to hydrogen (1898) and helium (1908) liquefaction. The Cryogenic Laboratory at Leiden had developed under the influence of VDW's theories. For further details on VDW's biography refer to specialized textbooks (e.g., Nobel Lectures 1967).

The current paper has been written in honor of the first centennial of VDW awarding Nobel Prize in Physics. The ideal and VDW equation of state, both in dimensional and reduced form, are reviewed, and a number of features are analysed in detail, in Section 2. Counterparts to ideal and VDW equations of state for astrophysical fluids, or macrogases, are briefly summarized and compared with the classical formulation in Section 3. The discussion and the conclusion are drawn in Section 4.

2 Equation of state of ordinary fluids

Let ordinary fluids be conceived as fluids which can be investigated in laboratory. The simplest description is provided by the theory of ideal gas, where the following restrictive assumptions are made: (i) particles are identical spheres; (ii) the number of particles is extremely large; (iii) the motion of particles is random; (iv) collisions between particles or with the wall of the box are perfectly elastic; (v) interactions between particles or with the wall of the box are null.

The equation of state of ideal gases may be written under the form (e.g., Landau and Lifchitz, 1967, Chap. IV, §42, hereafter quoted as LL67):

$$pV = kNT \quad ; \quad (1)$$

where p is the pressure, V the volume, T the temperature, N the particle number, and k the Boltzmann constant.

In getting a better description of ordinary fluids, the above assumption (v) is relaxed and tidal interactions between particles are taken into consideration. The VDW generalization of the equation of state of ideal gases, Eq. (1), reads (van der Waals, 1873):

$$\left(p + A \frac{N^2}{V^2} \right) (V - NB) = kNT \quad ; \quad (2)$$

where A and B are constants which depend on the nature of the particles. More specifically, the presence of an attractive interaction between particles reduces both the force and the frequency of particle-wall collisions: the net effect is a reduction of the pressure, proportional to the square numerical density, expressed as $A(N/V)^2$. On the other hand, the whole volume of the box, V , is not accessible to particles, in that they are conceived as identical spheres: the free volume within the box is $V - NB$, where B is the volume of a single sphere. For further details refer to specific textbooks (e.g., LL67, Chap. VII, §74).

The isothermal ($T = \text{const}$) curves for ideal gases are hyperbolas with axes, $p = \mp V$, conformly to Eq. (1). In VDW theory of real gases, the isothermal curves exhibit two extremum points below a threshold, which reduce to a single horizontal inflexion point when a critical temperature is attained, as shown in Fig. 1. Well above the critical isothermal curve, $T \gg T_c$, the trends exhibited by ideal and VDW gases look very similar. Below the critical isothermal curve, $T < T_c$, the behaviour of VDW gases is different with respect to ideal gases and, in addition, the related isothermal curves provide a wrong description within a specific region where saturated vapour and liquid phases coexist. Further details are shown in Fig. 2. Above the critical

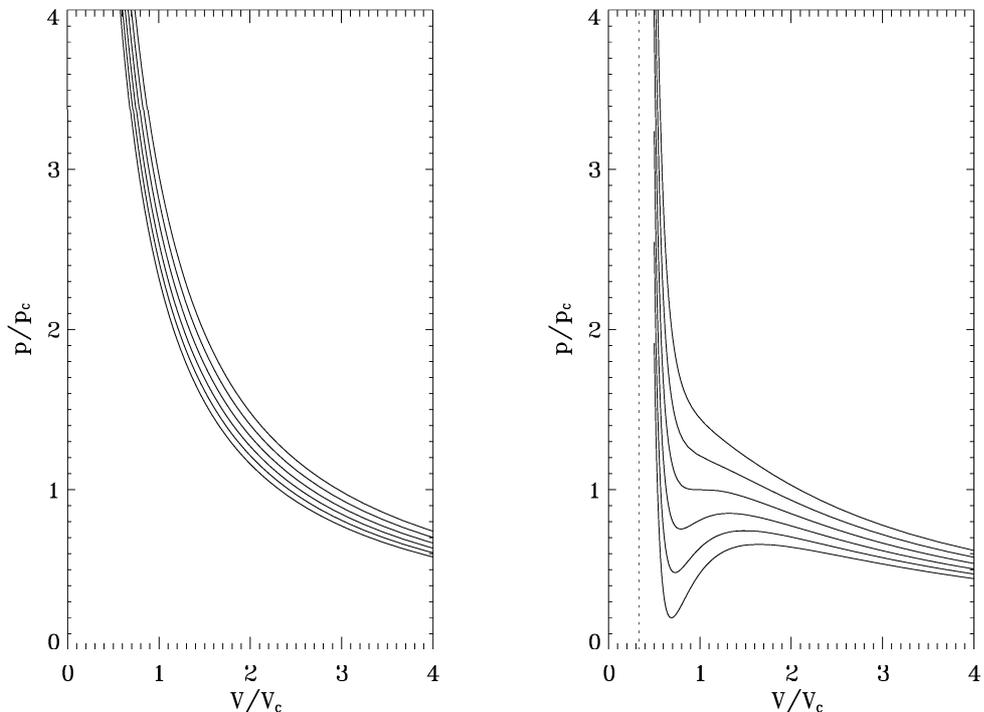


Figure 1: Isothermal curves related to ideal (left panel) and VDW (right panel) gases, respectively. Isothermal curves (from bottom to top) correspond to $T/T_c = 20/23, 20/22, 20/21, 20/20, 20/19, 20/18$. No extremum point exists above the critical isothermal curve, $T/T_c = 1$.

isothermal curve ($T = T_c$) the trend is similar with respect to ideal gases. Below the critical isothermal curve and on the right of the dashed curve, the supersaturated vapour still behaves as an ideal gas. Below the critical isothermal curve and on the left of the dashed curve, the liquid shows little change in volume as the pressure rises. Within the bell-shaped area bounded by the dashed curve, the liquid phase is in equilibrium with the saturated vapour phase. A diminished volume implies smaller saturated vapour fraction and larger liquid fraction at constant pressure, and vice versa. The VDW equation of state is no longer valid in this region. The dashed curve (including the central branch) is the locus of intersections between VDW and real isothermal curves, the latter being related to constant pressure where liquid and vapour phases coexist. The dotted curve is the locus of VDW isothermal extremum points.

A specific ($T/T_c = 20/23$) VDW and corresponding real isothermal curve, are represented in Fig. 3. The VDW isothermal curve and the real isothermal curve coincide within the range, $V \leq V_A$ and $V \geq V_E$. The VDW isothermal curve exhibits two extremum points: a minimum, **B**, and a maximum, **D**, while the real isothermal curve is flat, within the range, $V_A \leq V \leq V_E$. Configurations related to the VDW isothermal curve within the range, $V_A \leq V \leq V_B$ (due to tension forces acting on the particles yielding superheated liquid), and $V_D \leq V \leq V_E$ (due to the occurrence of undercooled vapour), may be obtained under special conditions, while configurations within the range, $V_B \leq V \leq V_D$, are always unstable. The volumes, V_A and V_E , correspond to the maximum value in presence of the sole liquid phase and the minimum value in presence of the sole vapour phase, respectively.

The surfaces, **ABC** and **CDE**, are equal, as first inferred by Maxwell (e.g., Rostagni, 1957, Chap. XII, §19). The VDW and real isothermal curves represented in Fig. 3 being related to the same temperature, T , the cycle, **ABCDECA**, is completely both isothermal and reversible, and the work, W , performed therein cannot be positive to avoid violation of the second law of the thermodynamics. The cycles, **ABCA** and **CDEC**, occurring in counter-clockwise and clockwise sense, respectively, are also completely both isothermal and reversible. Accordingly, $W_{ABCDECA} = W_{ABCA} - W_{CDEC} \leq 0$. A similar procedure, related to the reversed cycle, **ACEDCBA**, yields $W_{ACEDCBA} = W_{CEDC} - W_{CBAC} \leq 0$. Then $W_{ABCDECA} = W_{ACEDCBA} = 0$, which implies $W_{ABCA} = W_{CDEC} = W_{CEDC} = W_{CBAC}$ and, in turn, the equality between the related surfaces. For further details refer to specific textbooks (e.g., LL67, Chap. VIII, §85).

In order to simplify both notation and calculations, it is convenient to deal with (dimensionless) reduced variables (e.g., Rostagni, 1957, Chap. XII, §16; LL67, Chap. VIII, §85). To this aim, the first step is the knowledge of the parameters related to the critical point, V_c, p_c, T_c . Using the VDW equation of state, Eq. (2), the pressure and its first and second partial derivatives, with respect to the volume, read:

$$p = \frac{kNT}{V - NB} - A\frac{N^2}{V^2} ; \quad N = \text{const} ; \quad (3)$$

$$\left(\frac{\partial p}{\partial V}\right)_{V,T} = -\frac{kNT}{(V - NB)^2} + 2A\frac{N^2}{V^3} ; \quad (4)$$

$$\left(\frac{\partial^2 p}{\partial V^2}\right)_{V,T} = \frac{2kNT}{(V - NB)^3} - 6A\frac{N^2}{V^4} ; \quad (5)$$

where the domain is $V > NB$, $V = NB$ is a vertical asymptote, and $p = 0$ is a horizontal asymptote. The critical isothermal corresponds to the highest

temperature allowing a liquid phase, which occurs therein only at the critical point. The critical isothermal curve exhibits neither a minimum nor a maximum, which are replaced by a horizontal inflexion point coinciding with the critical point. Accordingly, $(\partial p/\partial V)_{V_c, T_c} = 0$, $(\partial^2 p/\partial V^2)_{V_c, T_c} = 0$, and $p_c = kNT_c/(V_c - NB) - AN^2/V_c^2$. The solution of the related system is:

$$V_c = 3NB \quad ; \quad (6)$$

$$T_c = \frac{8}{27} \frac{A}{B} \frac{1}{k} \quad ; \quad (7)$$

$$p_c = \frac{1}{27} \frac{A}{B^2} \quad ; \quad (8)$$

$$Z_c = \frac{p_c V_c}{kNT_c} = \frac{3}{8} \quad ; \quad (9)$$

where, in general, the compressibility factor, $Z = pV/(kNT)$, defines the degree of departure from the behaviour of ideal gases, for which $Z = 1$, according to Eq.(1). For further details refer to specific textbooks (e.g., Rostagni, 1957, Chap. XII, §20; LL67, Chap. VIII, §85).

With regard to the reduced variables:

$$\mathcal{V} = \frac{V}{V_c} \quad ; \quad \phi = \frac{p}{p_c} \quad ; \quad \mathcal{T} = \frac{T}{T_c} \quad ; \quad (10)$$

the ideal gas equation of state, Eq.(1), and the VDW equation of state, Eq.(2), reduce to:

$$\phi \mathcal{V} = \frac{8}{3} \mathcal{T} \quad ; \quad (11)$$

$$\left(\phi + \frac{3}{\mathcal{V}^2} \right) \left(\mathcal{V} - \frac{1}{3} \right) = \frac{8}{3} \mathcal{T} \quad ; \quad \mathcal{V} > \frac{1}{3} \quad ; \quad (12)$$

and Eqs.(3), (4), and (5), reduce to:

$$\phi = \frac{8\mathcal{T}}{3\mathcal{V} - 1} - \frac{3}{\mathcal{V}^2} \quad ; \quad (13)$$

$$\left(\frac{\partial \phi}{\partial \mathcal{V}} \right)_{\mathcal{V}, \mathcal{T}} = -\frac{24\mathcal{T}}{(3\mathcal{V} - 1)^2} + \frac{6}{\mathcal{V}^3} \quad ; \quad (14)$$

$$\left(\frac{\partial^2 \phi}{\partial \mathcal{V}^2} \right)_{\mathcal{V}, \mathcal{T}} = \frac{144\mathcal{T}}{(3\mathcal{V} - 1)^3} - \frac{18}{\mathcal{V}^4} \quad ; \quad (15)$$

where, for assigned \mathcal{T} , the domain of the function, $\phi(\mathcal{V})$, is $\mathcal{V} > 1/3$, $\mathcal{V} = 1/3$ is a vertical asymptote, and $\phi = 0$ is a horizontal asymptote. In the

special case of the critical point, $\mathcal{V} = 1$, $\mathcal{T} = 1$, $\mathcal{p} = 1$, the partial derivatives are null, as expected.

The extremum points, via Eq. (14), are defined by the relation:

$$f(\mathcal{V}) = \frac{(3\mathcal{V} - 1)^2}{4\mathcal{V}^3} = \mathcal{T} \quad ; \quad (16)$$

which is satisfied at the critical point, as expected. The function on the left-hand side of Eq. (16) has two extremum points: a minimum at $\mathcal{V} = 1/3$ (outside the physical domain) and a maximum at $\mathcal{V} = 1$, where $\mathcal{T} = 1$. Accordingly, Eq. (16) is never satisfied for $\mathcal{T} > 1$, which implies no extremum point for related isothermal curves, as expected. The contrary holds for $\mathcal{T} < 1$, where it can be seen that the third-degree equation associated to Eq. (14) has three real solutions, related to extremum points. One lies outside the physical domain, which implies $\mathcal{V} \leq 1/3$. The remaining two are obtained as the intersections between the curve, $f(\mathcal{V})$, expressed by Eq. (16), and the straight line, $y = \mathcal{T}$, keeping in mind that $f(1/3) = 0$, $f(1) = 1$, and $\lim_{\mathcal{V} \rightarrow +\infty} f(\mathcal{V}) = 0$.

The third-degree equation associated to Eq. (14), may be ordered as:

$$\mathcal{V}^3 - 9a\mathcal{V}^2 + 6a\mathcal{V} - a = 0 \quad ; \quad (17a)$$

$$a = \frac{1}{4\mathcal{T}} \quad ; \quad (17b)$$

with regard to the standard formulation (e.g., Spiegel, 1968, Chap. 9):

$$x^3 + a_1x^2 + a_2x + a_3 = 0 \quad ; \quad (18)$$

the discriminants of Eq. (17a) are:

$$Q = \frac{3a_2 - a_1^2}{9} = a(2 - 9a) \quad ; \quad (19)$$

$$R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54} = \frac{a(1 - 18a + 54a^2)}{2} \quad ; \quad (20)$$

$$D = Q^3 + R^2 = \frac{a^2(1 - 4a)}{4} \quad ; \quad (21)$$

where $D = 0$ in the special case of the critical isothermal curve ($\mathcal{T} = 1$, $a = 1/4$), $D < 0$ for $\mathcal{T} < 1$, and $D > 0$ for $\mathcal{T} > 1$. Accordingly, three coincident real solutions exist if $D = 0$, three (at least two) different real solutions if $D < 0$, one real (outside the physical domain) and two complex conjugate if $D > 0$.

The three real solutions ($D \leq 0$) may be expressed as (e.g., Spiegel, 1968, Chap. 9):

$$\mathcal{V}_1 = 2\sqrt{-Q} \cos\left(\pi + \frac{\theta}{3}\right) - \frac{1}{3}a_1 ; \quad (22a)$$

$$\mathcal{V}_2 = 2\sqrt{-Q} \cos\left(\pi + \frac{\theta}{3} + \frac{2\pi}{3}\right) - \frac{1}{3}a_1 ; \quad (22b)$$

$$\mathcal{V}_3 = 2\sqrt{-Q} \cos\left(\pi + \frac{\theta}{3} + \frac{4\pi}{3}\right) - \frac{1}{3}a_1 ; \quad (22c)$$

$$\theta = \arctan \frac{\sqrt{-D}}{R} ; \quad (22d)$$

where $a_1 = -9a$ and, in the special case of the critical isothermal curve, $a = 1/4$, $Q = -1/16$, $D = 0$, which implies $\mathcal{V}_0 = \min(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3)$, $\mathcal{V}_A = \mathcal{V}_B = \mathcal{V}_C = \mathcal{V}_D = \mathcal{V}_E = \max(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3)$. In the special case, $\mathcal{T} \rightarrow 0$, Eq. (17a) reduces to a second-degree equation whose solutions are $\mathcal{V}_{01} = \mathcal{V}_{02} = 1/3$, while the related function is otherwise divergent as $a \rightarrow +\infty$. In general, the extremum points of VDW isothermal curves ($\mathcal{T} \leq 1$) occur at $\mathcal{V} = \mathcal{V}_B$ (minimum) and $\mathcal{V} = \mathcal{V}_D$ (maximum), $\mathcal{V}_B \leq \mathcal{V}_D$. As $\mathcal{T} \rightarrow 0$, $\mathcal{V}_B \rightarrow 1/3$, $\mathcal{V}_D \rightarrow +\infty$, where, in all cases, $1/3 < \mathcal{V}_B \leq 1 \leq \mathcal{V}_D$.

The two areas defined by the intersection of a generic VDW isothermal curve ($\mathcal{T} \leq 1$) and related real isothermal curves (see Fig. 3), are expressed as:

$$W_1 = \int_{\mathcal{V}_A}^{\mathcal{V}_C} p_C dV - \int_{\mathcal{V}_A}^{\mathcal{V}_C} p dV = p_C V_C \left[\phi_C(\mathcal{V}_C - \mathcal{V}_A) - \int_{\mathcal{V}_A}^{\mathcal{V}_C} \phi d\mathcal{V} \right] ; \quad (23a)$$

$$W_2 = \int_{\mathcal{V}_C}^{\mathcal{V}_E} p dV - \int_{\mathcal{V}_C}^{\mathcal{V}_E} p_C dV = p_C V_C \left[\int_{\mathcal{V}_C}^{\mathcal{V}_E} \phi d\mathcal{V} - \phi_C(\mathcal{V}_E - \mathcal{V}_C) \right] ; \quad (23b)$$

and the substitution of Eq. (13) into (23) allows explicit expressions for the integrals. The result is:

$$\frac{W_1}{p_C V_C} = \phi_C(\mathcal{V}_C - \mathcal{V}_A) - \frac{8}{3} \mathcal{T} \ln \frac{3\mathcal{V}_C - 1}{3\mathcal{V}_A - 1} + \frac{3(\mathcal{V}_C - \mathcal{V}_A)}{\mathcal{V}_A \mathcal{V}_C} ; \quad (24a)$$

$$\frac{W_2}{p_C V_C} = \frac{8}{3} \mathcal{T} \ln \frac{3\mathcal{V}_E - 1}{3\mathcal{V}_C - 1} - \frac{3(\mathcal{V}_E - \mathcal{V}_C)}{\mathcal{V}_C \mathcal{V}_E} - \phi_C(\mathcal{V}_E - \mathcal{V}_C) ; \quad (24b)$$

and the condition, $W_1 = W_2$, after some algebra reads (Caimmi 2010, hereafter quoted as C10):

$$\phi_C = \frac{8}{3} \frac{\mathcal{T}}{\mathcal{V}_E - \mathcal{V}_A} \ln \frac{3\mathcal{V}_E - 1}{3\mathcal{V}_A - 1} - \frac{3}{\mathcal{V}_A \mathcal{V}_E} ; \quad (25)$$

where, for a selected isothermal curve, the unknowns are $\phi_C = \phi_A = \phi_E$, Ψ_A , and Ψ_E .

The reduced volumes, Ψ_A , Ψ_C , Ψ_E , see Fig. 3, may be considered as intersections between a VDW isothermal curve ($\mathcal{T} < 1$) and a horizontal straight line, $\phi = \phi_C$, in the $(\mathcal{O}\Psi\phi)$ plane. In other words, Ψ_A , Ψ_C , Ψ_E , are the real solutions of the third-degree equation:

$$\Psi^3 - \left(\frac{1}{3} + \frac{8}{3} \frac{\mathcal{T}}{\phi_C} \right) \Psi^2 + \frac{3}{\phi_C} \Psi - \frac{1}{\phi_C} = 0 ; \quad (26)$$

which has been deduced from Eq. (13), particularized to $\phi = \phi_C$. The related solutions may be calculated using Eqs. (22). The last unknown, ϕ_C , is determined from Eq. (25).

An inspection of Fig. 3 shows that the points, **A** and **E**, are located on the left of the minimum, **B**, and on the right of the maximum, **D**, respectively. Keeping in mind the above results, the following inequality holds: $\Psi_A \leq \Psi_B \leq 1 \leq \Psi_D \leq \Psi_E$, which implies further investigation on the special case, $\Psi_C = 1$. The particularization of the VDW equation of state, Eq. (13), to the point, **C** = **C**₁, assuming $\Psi_{C_1} = 1$, yields:

$$\mathcal{T} = \frac{\phi_{C_1} + 3}{4} ; \quad (27)$$

and Eq. (26) reduces to:

$$\Psi^3 - (1 + 2b)\Psi^2 + 3b\Psi - b = 0 ; \quad (28a)$$

$$b = \frac{1}{\phi_{C_1}} ; \quad (28b)$$

with regard to the generic third-degree equation, Eq. (18), the three solutions, x_1 , x_2 , x_3 , satisfy the relations (e.g., Spiegel, 1968, Chap. 9):

$$x_1 + x_2 + x_3 = -a_1 ; \quad (29a)$$

$$x_1x_2 + x_2x_3 + x_3x_1 = a_2 ; \quad (29b)$$

$$x_1x_2x_3 = -a_3 ; \quad (29c)$$

where, in the case under discussion:

$$a_1 = -1 - 2b ; \quad a_2 = 3b ; \quad a_3 = -b ; \quad (30a)$$

$$x_1 = \Psi_A ; \quad x_2 = \Psi_{C_1} = 1 ; \quad x_3 = \Psi_E ; \quad (30b)$$

and the substitution of Eqs. (30) into two among (29) yields:

$$\Psi_A = b - \sqrt{b^2 - b} ; \quad (31a)$$

$$\Psi_E = b + \sqrt{b^2 - b} ; \quad (31b)$$

and the combination of Eqs. (27), (28b), and (31) produces:

$$\mathcal{V}_A = \frac{1 - 2\sqrt{1 - \mathcal{T}}}{4\mathcal{T} - 3} ; \quad \mathcal{T} \leq 1 ; \quad (32a)$$

$$\mathcal{V}_E = \frac{1 + 2\sqrt{1 - \mathcal{T}}}{4\mathcal{T} - 3} ; \quad \mathcal{T} \leq 1 ; \quad (32b)$$

which, together with $\mathcal{V}_{C_1} = 1$, are the abscissae of the intersection points between a selected VDW isothermal curve in the $(\mathcal{O}\mathcal{V}\mathcal{p})$ plane and the straight line, $\mathcal{p} = \mathcal{p}_{C_1}$, in the special case under discussion.

The substitution of Eqs. (32) into (25), the last being related to the real isothermal curve, yields:

$$\frac{\mathcal{T}}{\sqrt{1 - \mathcal{T}}} \ln \frac{3 - 2\mathcal{T} + 3\sqrt{1 - \mathcal{T}}}{3 - 2\mathcal{T} - 3\sqrt{1 - \mathcal{T}}} = 6 ; \quad (33)$$

which holds only for the critical isothermal curve, $\mathcal{T} = 1$. Accordingly, the abscissa of the intersection point, C, between a selected VDW isothermal curve and related real isothermal curve, see Fig. 3, cannot occur at $\mathcal{V}_C = 1$ unless the critical isothermal curve is considered. Then the third-degree equation, Eq. (26), must be solved in the general case by use of Eqs. (22). The results are shown in Tab. 1, where the following parameters (in reduced variables) are listed for each VDW isothermal curve, see Fig. 3: the temperature, \mathcal{T} , the lower volume limit, \mathcal{V}_A , for which the liquid and vapour phase coexist; the extremum point (minimum) volume, \mathcal{V}_B ; the intermediate volume, \mathcal{V}_C , for which the pressure equals its counterpart related to the corresponding lower and upper volume limit, for which the liquid and vapour phase coexist; the extremum point (maximum) volume, \mathcal{V}_D ; the upper volume limit, \mathcal{V}_E , for which the liquid and vapour phase coexist; the extremum point (minimum) pressure, \mathcal{p}_B ; the pressure, $\mathcal{p}_A = \mathcal{p}_C = \mathcal{p}_E$, related to the horizontal real isothermal curve; the extremum point (maximum) pressure, \mathcal{p}_D . The locus of the intersections between VDW and real isothermal curves is represented in Fig. 2 as a trifold curve, where the left, the right, and the middle branch correspond to \mathcal{V}_A , \mathcal{V}_E , and \mathcal{V}_C , respectively. The common starting point coincides with the critical point. The locus of the VDW isothermal curve extremum points is represented in Fig. 2 as a dotted curve starting from the critical point, where the left and the right branch corresponds to minimum and maximum points, respectively.

A fluid state can be represented in reduced variables as $(\mathcal{V}, \mathcal{p}, \mathcal{T})$, where one variable may be expressed as a function of the remaining two, by use

Table 1: Values of parameters, \mathcal{T} , \mathcal{V}_A , \mathcal{V}_B , \mathcal{V}_C , \mathcal{V}_D , \mathcal{V}_E , \mathcal{P}_B , \mathcal{P}_C , \mathcal{P}_D , within the range, $0.85 \leq \mathcal{T} \leq 0.99$, using a step, $\Delta \mathcal{T} = 0.01$. Additional values are computed near the critical point, to increase the resolution. The true value of the reduced temperature on the last row is $\mathcal{T} = 0.9999$ or $10 \mathcal{T} = 9.999$. All values equal unity at the critical point. Index captions: A, C, E - intersections between VDW and real isothermal curves; B - extremum point of minimum; D - extremum point of maximum. Extremum points are related to VDW isothermal curves, while their real counterparts are flat in presence of both liquid and vapour phase. To save aesthetics, 01 on head columns stands for unity.

$10 \mathcal{T}$	$10 \mathcal{V}_A$	$10 \mathcal{V}_B$	$01 \mathcal{V}_C$	$01 \mathcal{V}_D$	$01 \mathcal{V}_E$	$10 \mathcal{P}_B$	$10 \mathcal{P}_C$	$10 \mathcal{P}_D$
8.50	5.5336	6.7168	1.1453	1.7209	3.1276	0.4963	5.0449	6.2055
8.60	5.6195	6.8003	1.1337	1.6821	2.9545	1.2750	5.3125	6.4005
8.70	5.7116	6.8883	1.1225	1.6436	2.7909	2.0346	5.5887	6.6011
8.80	5.8106	6.9814	1.1116	1.6052	2.6360	2.7752	5.8736	6.8076
8.90	5.9176	7.0804	1.1009	1.5669	2.4889	3.4965	6.1674	7.0205
9.00	6.0340	7.1860	1.0905	1.5285	2.3488	4.1984	6.4700	7.2401
9.10	6.1615	7.2994	1.0804	1.4900	2.2151	4.8807	6.7816	7.4669
9.20	6.3022	7.4221	1.0706	1.4511	2.0869	5.5430	7.1021	7.7014
9.30	6.4593	7.5561	1.0610	1.4117	1.9634	6.1849	7.4318	7.9443
9.40	6.6369	7.7040	1.0516	1.3715	1.8438	6.8058	7.7707	8.1963
9.50	6.8412	7.8697	1.0425	1.3300	1.7271	7.4049	8.1188	8.4584
9.60	7.0819	8.0593	1.0336	1.2867	1.6118	7.9811	8.4762	8.7319
9.70	7.3756	8.2830	1.0249	1.2404	1.4960	8.5328	8.8429	9.0185
9.80	7.7554	8.5611	1.0164	1.1892	1.3761	9.0576	9.2191	9.3209
9.90	8.3091	8.9461	1.0081	1.1278	1.2430	9.5510	9.6048	9.6437
9.95	8.7471	9.2353	1.0040	1.0876	1.1618	9.7830	9.8012	9.8157
9.98	9.1727	9.5049	1.0016	1.0540	1.0972	9.9158	9.9202	9.9240
9.99	9.4018	9.6456	1.0008	1.0377	1.0670	9.9585	9.9600	9.9614
9.99	9.8035	9.8856	1.0001	1.0117	1.0204	9.9960	9.9960	9.9960

of the reduced ideal gas equation of state, Eq. (11), or the reduced VDW equation of state, Eq. (12). The formulation in terms of reduced variables, Eqs. (10), makes the related equation of state universal i.e. it holds for any fluid. Similarly, the Lane-Emden equation expressed in polytropic (dimensionless) variables, describes the whole class of polytropic gas spheres with assigned polytropic index, in hydrostatic equilibrium (e.g., Chandrasekhar 1939, Chap. IV, §4).

The states of two fluids with equal $(\mathcal{V}, \mathcal{p}, \mathcal{T})$, are defined as corresponding states. The mere existence of an equation of state yields the following result.

Law of corresponding states. *Given two fluids, the equality between two among three reduced variables, \mathcal{V} , \mathcal{p} , \mathcal{T} , implies the equality between the remaining related reduced variables i.e. the two fluids are in corresponding states.*

The law was first formulated by van der Waals in 1880. For further details refer to specific textbooks (e.g., LL67, Chap. VIII, §85).

3 Equation of state of astrophysical fluids

Let macrogases be defined as two-component fluids which interact only gravitationally. For assigned density profiles, the virial theorem can be formulated for each subsystem, where the potential energy is the sum of the self potential energy of the component under consideration, and the tidal energy induced by the other one. The virial theorem for subsystem can be expressed as a macrogas equation of state in terms of dimensionless variables, X_V , X_p , X_T , related to axis ratio, mass ratio, virial (i.e. self + tidal) potential energy ratio, respectively. The result is (C10):

$$X_p X_V F_X(X_p, X_V) = X_T ; \quad (34a)$$

$$X_p = m^2 ; \quad X_V = \frac{1}{y} ; \quad X_T = \phi ; \quad (34b)$$

where the function, F_X , depends on the selected density profiles, m is the (outer to inner component) mass ratio, y is the (outer to inner component) axis ratio along a generic direction, ϕ is the (outer to inner component) virial energy ratio, and the density profiles are restricted to be homeoidally striated. The variables, X_V , X_p , X_T , play a similar role as the volume, the pressure, and the temperature, for ordinary fluids. Accordingly, X_V , X_p , X_T , may be defined as macrovolume, macropressure, and macrotemperature, respectively. For further details refer to the parent paper (C10).

Macroisothermal curves on the $(OX_V X_p)$ plane exhibit a similar trend with respect to VDW isothermal curves on the (OVp) plane, with two main differences. First, no critical point occurs for sufficiently mild density profiles, where all macroisothermal curves are characterized by two extremum points, one maximum and one minimum. Second, a critical macroisothermal curve appears for sufficiently steep density profiles, above (instead of below) which macroisothermal curves exhibit extremum points. For further details refer to the parent paper (C10) and an earlier attempt (Caimmi and Valentinuzzi 2008).

The last inconvenient may be avoided turning Eq. (34) into the following:

$$Y_p Y_V F_Y(Y_p, Y_V) = Y_T \quad ; \quad (35a)$$

$$Y_p = \frac{1}{X_p} \quad ; \quad Y_V = \frac{1}{X_V} \quad ; \quad Y_T = \frac{1}{X_T} \quad ; \quad (35b)$$

$$F_Y(Y_p, Y_V) = F_X(X_p, X_V) \quad ; \quad (35c)$$

as suggested in the parent paper (C10).

The existence of a phase transition moving along a selected macroisothermal curve, where the path is a horizontal line (“real” macroisothermal curve) instead of a curve including the extremum points (“actual” macroisothermal curve), must necessarily be assumed as a working hypothesis, due to the analogy between VDW isothermal curves and macroisothermal curves. Unlike the VDW equation of state, Eq. (3), the theoretical macrogas equation of state, Eq. (35a), is not analytically integrable, which implies the procedure used for determining a selected macroisothermal curve, must be numerically performed.

The main steps are (i) calculate the intersections, $Y_{V_A}, Y_{V_C}, Y_{V_E}, Y_{V_A} < Y_{V_C} < Y_{V_E}$, between the generic horizontal line in the $(OY_V Y_p)$ plane, $Y_p = \text{const}$, and the theoretical macrogas equation of state, within the range, $Y_{p_B} < Y_p < Y_{p_D}$, where B and D denote the extremum points of minimum and maximum, respectively; (ii) calculate the area of the regions, ABC and CDE; (iii) find the special value, $Y_p = Y_{p_C}$, which makes the two areas equal; (iv) trace the real macroisothermal curve as a horizontal line connecting the points, $(Y_{V_A}, Y_{p_A}), (Y_{V_C}, Y_{p_C}), (Y_{V_E}, Y_{p_E}), Y_{p_A} = Y_{p_C} = Y_{p_E} = Y_{p_c}$. For further details refer to an earlier attempt (C10).

The procedure related to point (ii) above is rather cumbersome and should be performed again with the new variables, Y_V, Y_p , and Y_T , with respect to an earlier attempt (C10). For this reason, the current paper shall be restricted to theoretical macroisothermal curves and related extremum points. In order to preserve the analogy with ideal and VDW gases, the tidal potential energy shall be excluded and included, respectively, in the formulation of the virial

theorem and related equation of state. The following cases shall be dealt with: UU macrogases, where no critical point occurs; HH macrogases, where the critical point occurs; HN/NH macrogases, where the critical point occurs.

In presence of the critical point, Eq. (35) may be translated into reduced variables, as:

$$\mathcal{Y}_p \mathcal{Y}_V F_Y(\mathcal{Y}_p, \mathcal{Y}_V) \frac{Y_{pc} Y_{Vc}}{Y_{Tc}} = \mathcal{Y}_T \quad ; \quad (36a)$$

$$\mathcal{Y}_p = \frac{Y_p}{Y_{pc}} \quad ; \quad \mathcal{Y}_V = \frac{Y_V}{Y_{Vc}} \quad ; \quad \mathcal{Y}_T = \frac{Y_T}{Y_{Tc}} \quad ; \quad (36b)$$

$$F_Y(\mathcal{Y}_p, \mathcal{Y}_V) = F_Y(\mathcal{Y}_p Y_{pc}, \mathcal{Y}_V Y_{Vc}) \quad ; \quad (36c)$$

where Y_{pc} , Y_{Vc} , Y_{Tc} , are the values of the variables related to the critical point. The counterpart of Eq. (36a) for ideal macrogases reads:

$$\mathcal{Y}_p \mathcal{Y}_V G_Y(\mathcal{Y}_p, \mathcal{Y}_V) \frac{Y_{pc} Y_{Vc}}{Y_{Tc}} = \mathcal{Y}_T \quad ; \quad (37)$$

where $G_Y(\mathcal{Y}_p, \mathcal{Y}_V)$ is the expression of $F_Y(\mathcal{Y}_p, \mathcal{Y}_V)$ where the interaction terms are omitted. For further details refer to an earlier attempt (C10). Accordingly, the equation of state for ideal macrogases where $G_Y(\mathcal{Y}_p, \mathcal{Y}_V) Y_{pc} / (Y_{Vc} Y_{Tc}) = 3/8$, coincides with its counterpart related to ideal gases, conformly to Eq. (11).

Macroisothermal curves related to IUU (tidal potential energy excluded) and AUU (tidal potential energy included) macrogases, are plotted in Fig. 4, left and right panel, respectively, for values of the macrotemperature, $Y_T = 20/23, 20/22, 20/21, 20/20, 20/19, 20/18$, from bottom to top. The coordinates, Y_V, Y_p, Y_T , may be conceived as normalized to their fictitious critical counterparts, $Y_{Vc} = 1, Y_{pc} = 1, Y_{Tc} = 1$ (C10). The comparison with ideal and VDW gases, plotted in Fig. 1, shows a similar trend, except the absence of a critical macroisothermal curve, above which the extremum points disappear.

Macroisothermal curves related to IHH (tidal potential energy excluded) and AHH (tidal potential energy included) macrogases, are plotted in Fig. 5, left and right panels, respectively, for infinitely extended subsystems and values of the reduced macrotemperature, $\mathcal{Y}_T = Y_T / Y_{Tc} = 20/23, 20/22, 20/21, 20/20, 20/19, 20/18$, from bottom to top. The general case of bounded subsystems makes only little changes. The comparison with ideal and VDW gases, plotted in Fig. 1, shows a similar trend where macroisothermal curves are more extended along the horizontal direction with respect to isothermal curves.

Macroisothermal curves related to IHN/NH (tidal potential energy excluded) and AHN/NH (tidal potential energy included) macrogases, are plotted in Fig. 6, left and right panels, respectively, for infinitely extended subsystems and values of the reduced macrotemperature, $Y_T = Y_T/Y_{T_c} = 20/23, 20/22, 20/21, 20/20, 20/19, 20/18$, from bottom to top. The general case of bounded subsystems makes only little changes for AHN/NH macrogases, while the scale change tends to disappear for IHN/NH macrogases. The comparison with ideal and VDW gases, plotted in Fig. 1, shows a similar trend where macroisothermal curves are more extended along the horizontal direction with respect to isothermal curves, and the occurrence of a scale difference for ideal macrogases. The last is due to a mass divergence for infinitely extended N density profiles, which makes tidal effects highly increase.

The comparison between the VDW critical isothermal curve and its counterparts related to HH and HN/NH macrogases is shown in Fig. 7. The broken curve is the same as in Fig. 2. Accordingly, the vapour and the liquid phase of ordinary fluids coexist within the bell-shaped region bounded by the broken curve. Both HH and HN/NH macroisothermal curves are more extended along the horizontal direction with respect to VDW isothermal curves, which implies a more flattened counterpart of the above mentioned bell-shaped region. The critical point belongs to all curves.

4 Discussion and conclusion

Tidal interactions between neighbouring bodies span across the whole admissible range of lengths in nature: from, say, atoms and molecules to galaxies and clusters of galaxies i.e. from micro to macrocosmos. Ordinary fluids are collisional, which makes the stress tensor be isotropic and the velocity distribution obey the Maxwell's law. Tidal interactions (electromagnetic in nature) therein act between colliding particles (e.g., LL67, Chap. VII, §74). Astrophysical fluids are collisionless, which makes the stress tensor be anisotropic and the velocity distribution no longer obey the Maxwell's law. Tidal interactions (gravitational in nature) therein act between a single particle and the system as a whole (e.g., C10).

In both cases, an equation of state can be formulated in reduced variables: the VDW equation for ordinary fluids and an equation which depends on the density profiles for astrophysical fluids. For sufficiently mild density profiles, macroisothermal curves are characterized by the occurrence of two extremum points, similarly to isothermal curves where a transition from liquid to gaseous phase takes place, or vice versa. For sufficiently steep density profiles, a critical macroisothermal curve exhibits a single horizontal inflex-

ion point, which defines the critical point. Macroisothermal curves below and above the critical one, show two or no extremum point, respectively, in complete analogy with VDW isothermal curves. In any case, the existence of an equation of state in reduced variables implies the validity of the law of corresponding states for macrogases with assigned density profiles.

For astrophysical fluids, the existence of a phase transition must necessarily be assumed as a working hypothesis by analogy with ordinary fluids. The phase transition has to be conceived between gas and stars, and the $(OY_V Y_p)$ plane may be divided into three parts, namely (i) a region bounded by the critical macroisothermal curve on the left of the critical point, and the locus of onset of phase transition on the right of the critical point, where only gas exists; (ii) a region bounded by the critical macroisothermal curve on the left of the critical point, the locus of onset of phase transition on the left of the critical point, and the vertical axis, where only stars exist; (iii) a region bounded by the locus of onset of phase transition, and the horizontal axis, where gas and stars coexist. The locus of onset of phase transition, not shown in Fig. 7 for reasons explained above, is similar to its counterpart related to ordinary fluids, represented by the bell-shaped curve in Fig. 7, but more extended along the horizontal direction.

In this view, elliptical and S0 galaxies lie on (ii) region unless hosting hot interstellar gas, and the same holds for globular clusters; spiral, irregular, and dwarf spheroidal galaxies lie on (iii) region, and the same holds for cluster of galaxies; gas clouds in absence of star formation lie on (i) region, and the same holds for hypothetical galaxies with no stars.

In conclusion, van der Waals' two great discoveries, more specifically a gas equation of state where tidal interactions between molecules are taken into account, and the law of corresponding states, related to microcosmos, find a counterpart with regard to macrocosmos. After a century since the awarding of the Nobel Prize in Physics, van der Waals' ideas are still valid and helpful to day for a full understanding of the universe.

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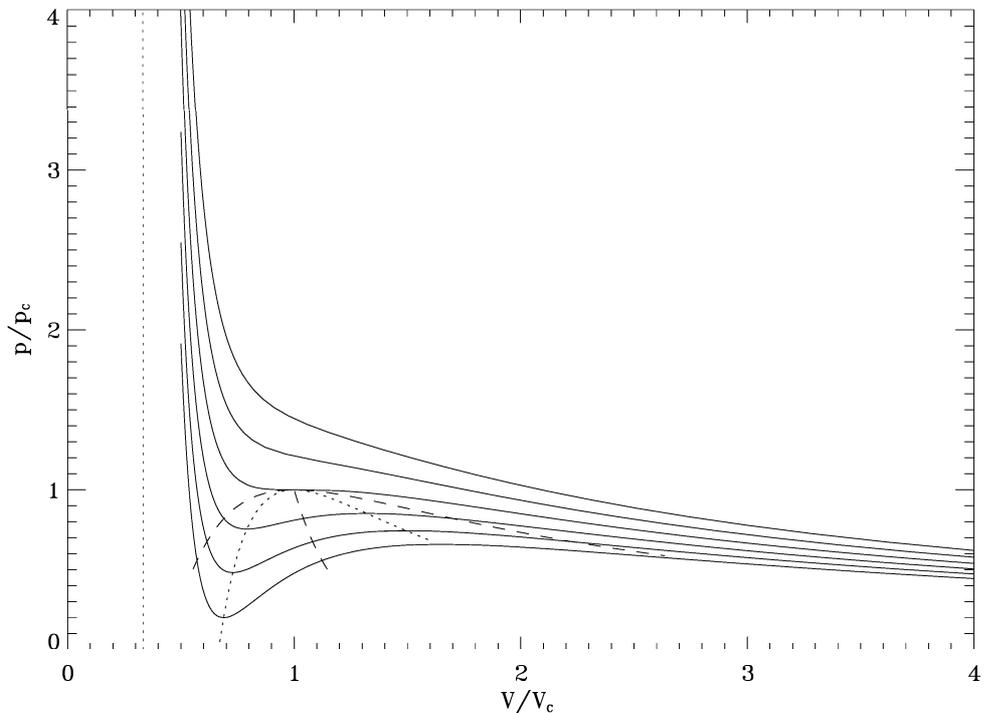


Figure 2: Same as in Fig. 1 (right panel), where the occurrence (within the bell-shaped area bounded by the dashed curve) of saturated vapour is considered. Above the critical isothermal curve ($T = T_c$) the trend is similar with respect to ideal gases. Below the critical isothermal curve and on the right of the dashed curve, the gas still behaves as an ideal gas. Below the critical isothermal curve and on the left of the dashed curve, the liquid shows little change in volume as the pressure rises. Within the bell-shaped area bounded by the dashed curve, the liquid phase is in equilibrium with the saturated vapour phase. A diminished volume implies smaller saturated vapour fraction and larger liquid fraction at constant pressure, and vice versa. The VDW equation of state is no longer valid in this region. The dashed curve (including the central branch) is the locus of intersection between VDW and real isothermal curves, the latter being related to constant pressure where liquid and vapour phases coexist. The dotted curve is the locus of VDW isothermal extremum points.

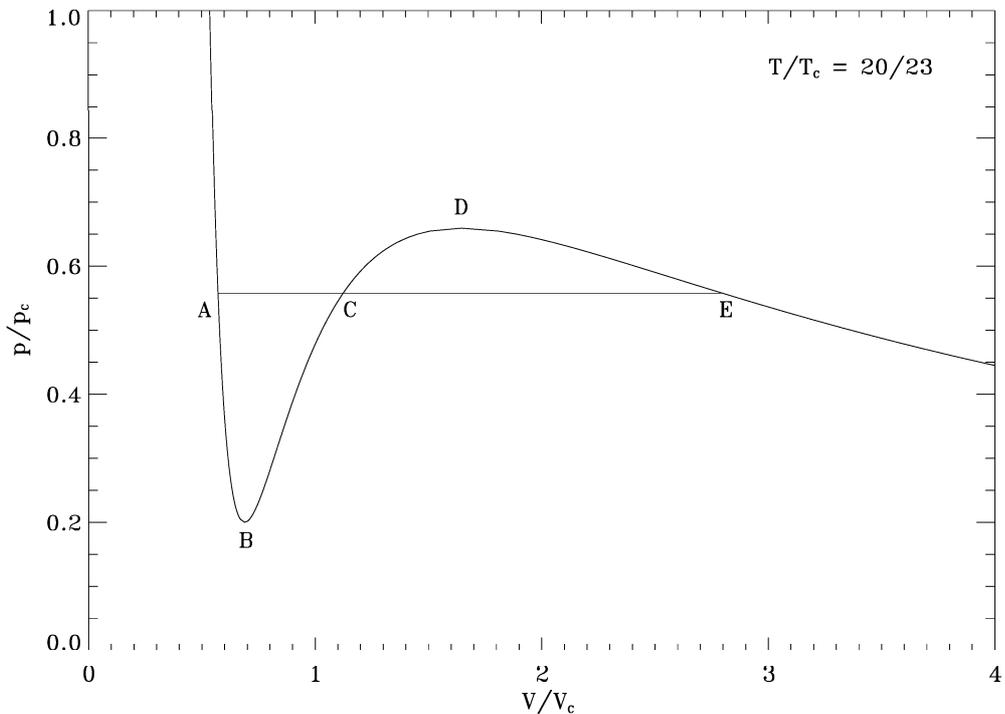


Figure 3: A specific ($T/T_c = 20/23$) VDW and corresponding real isothermal curve. The above mentioned curves coincide within the range, $V \leq V_A$ and $V \geq V_E$. The VDW isothermal curve exhibits two extremum points: a minimum, B , and a maximum, D , while the real isothermal curve is flat within the range, $V_A \leq V \leq V_E$. Configurations related to the VDW isothermal curve within the range, $V_A \leq V \leq V_B$ (due to tension forces acting on the particles yielding superheated liquid), and $V_D \leq V \leq V_E$ (due to the occurrence of undercooled vapour), may be obtained under special conditions, while configurations within the range, $V_B \leq V \leq V_D$, are always unstable. The volumes, V_A and V_E , correspond to the maximum value in presence of the sole liquid phase and the minimum value in presence of the sole vapour phase, respectively. The regions, ABC and CDE , have equal area. For further details refer to the text.

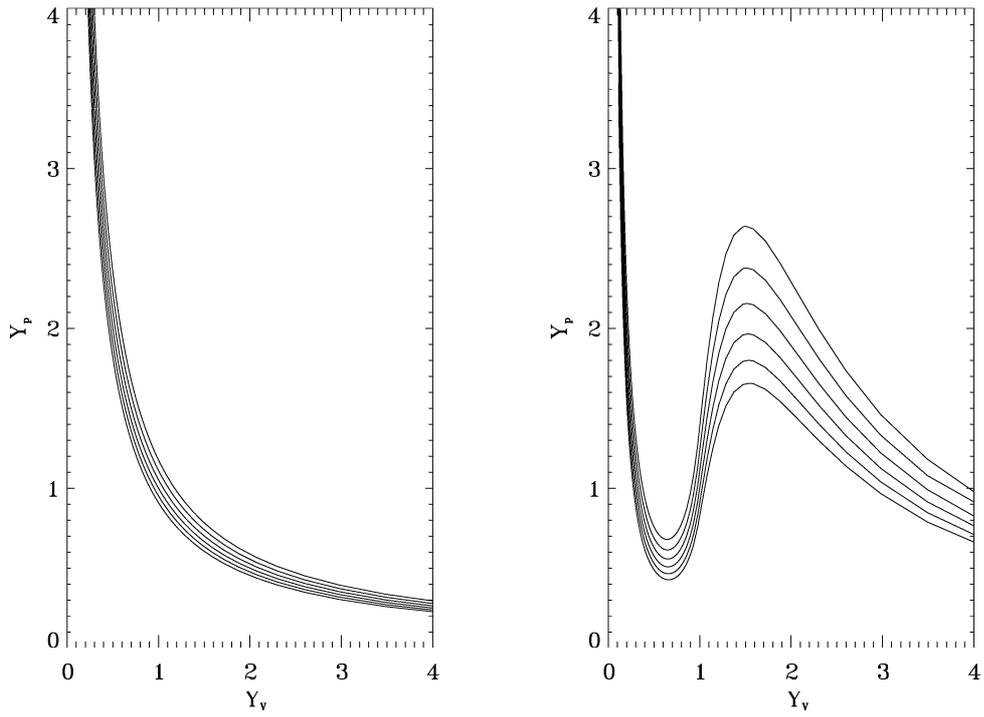


Figure 4: Macroisothermal curves related to IUU (left panel) and AUU (right panel) macrogases, respectively. Macroisothermal curves (from bottom to top) correspond to $Y_T = 20/23, 20/22, 20/21, 20/20, 20/19, 20/18$. No critical macroisothermal curve exists, above which the extremum points disappear. The coordinates, Y_V, Y_p, Y_T , may be conceived as normalized to their fictitious critical counterparts, $Y_{V_c} = 1, Y_{p_c} = 1, Y_{T_c} = 1$.

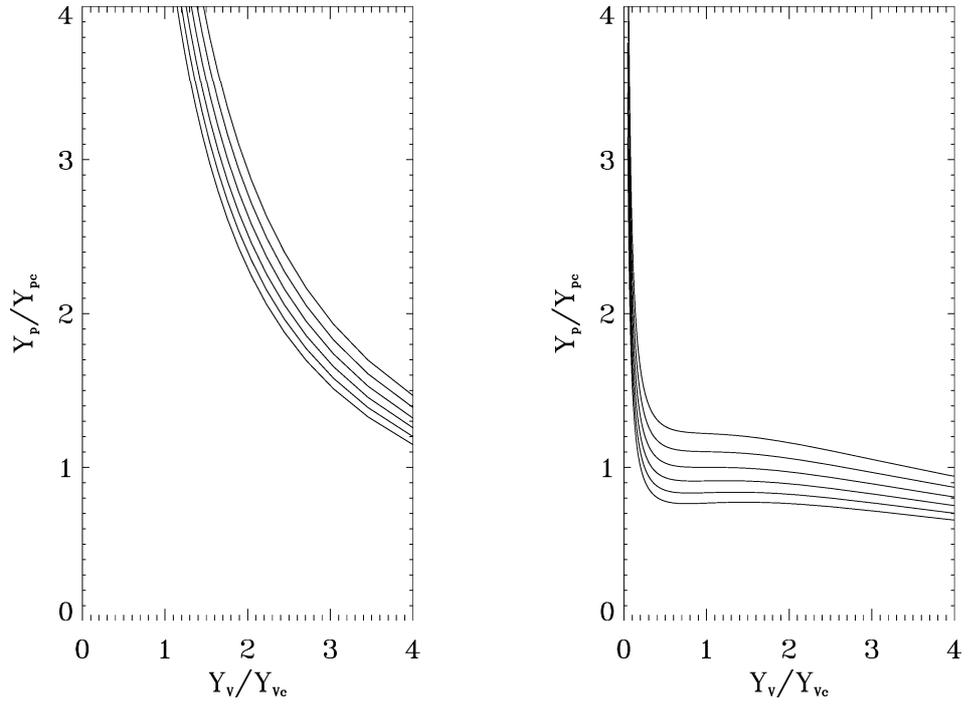


Figure 5: Macroisothermal curves ($Y_p = Y_p/Y_{pc}$ vs. $Y_v = Y_v/Y_{vc}$) related to IHH (left panels) and AHH (right panels) macrogases, respectively, for infinitely extended subsystems. Macroisothermal curves (from bottom to top) correspond to $Y_T = Y_T/Y_{Tc} = 20/23, 20/22, 20/21, 20/20, 20/19, 20/18$. The general case of bounded subsystems makes only little changes.

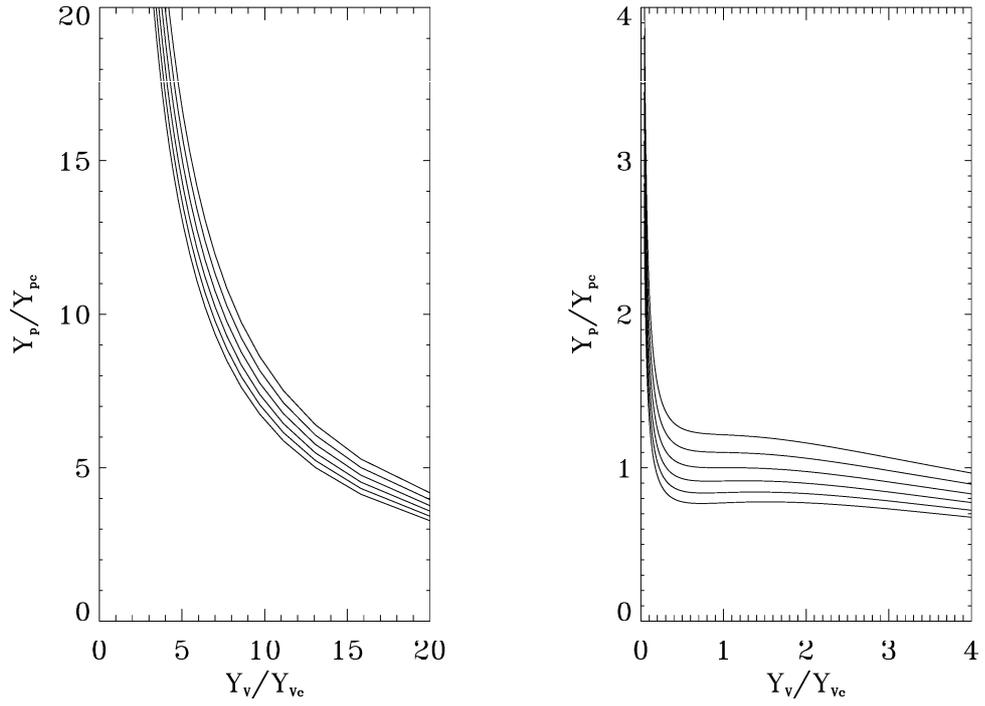


Figure 6: Macroisothermal curves ($Y_p = Y_p/Y_{pc}$ vs. $Y_v = Y_v/Y_{vc}$) related to IHN/NH (left panels, to be noted the scale difference) and AHN/NH (right panels) macrogases, respectively, for infinitely extended subsystems. Macroisothermal curves (from bottom to top) correspond to $Y_T = Y_T/Y_{Tc} = 23/20, 22/20, 21/20, 20/20, 19/20, 18/20$. The general case of bounded subsystems makes only little changes for AHN/NH macrogases, while the scale difference tends to disappear for IHN/NH macrogases.

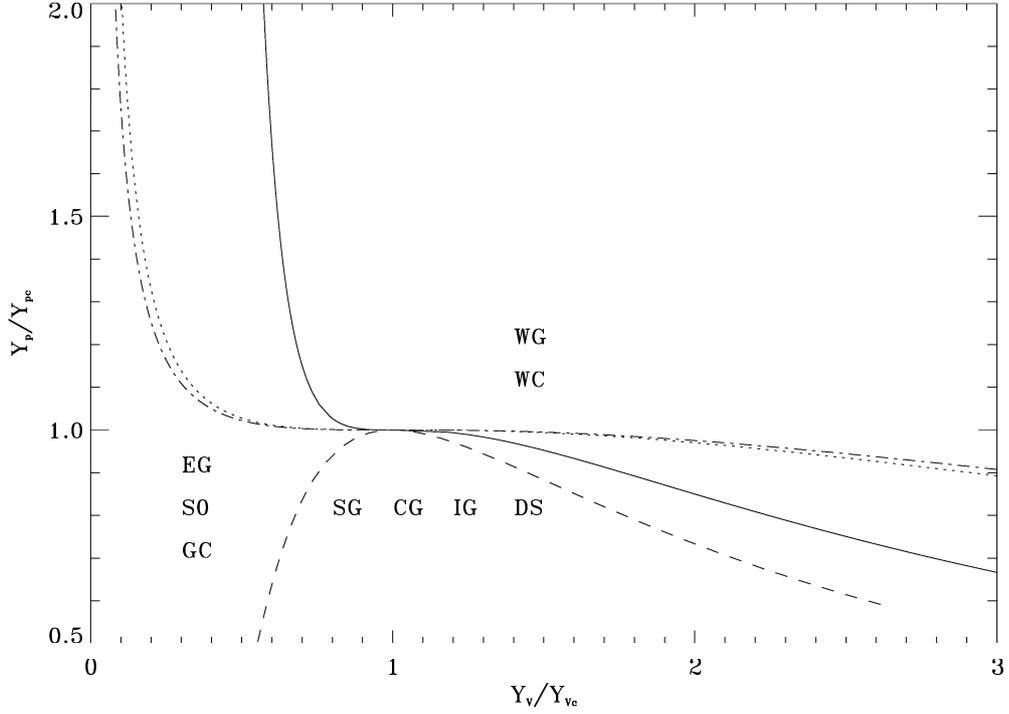


Figure 7: Comparison between VDW critical isothermal curve (full), HH critical macroisothermal curve (dotted) and HN/NH critical macroisothermal curve (dot-dashed). With regard to ordinary fluids, the vapour and the liquid phase coexist within the bell-shaped region bounded by the dashed curve and, in addition, $Y_v = V$, $Y_p = p$. More extended (along the horizontal direction) bell-shaped regions are expected for HH and HN/NH macroisothermal curves. The critical point belongs to all curves. Different letters denote the expected location of different astrophysical systems. Caption: EG - elliptical galaxies; S0 - lenticular galaxies; SG - spiral galaxies including barred; IG - irregular galaxies; DS - dwarf spheroidal galaxies; GC - globular clusters; CG - clusters of galaxies; WC - wholly gaseous clouds i.e. in absence of star formation; WG - (hypothetical) wholly gaseous galaxies i.e. in absence of star formation.