DERIVATIONS ON IDEALS IN COMMUTATIVE AW^*-ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a commutative AW^*-algebra, let $S(\mathcal{A})$ be the $*$-algebra of all measurable operators affiliated with $\mathcal{A}$, let $\mathcal{I}$ be an ideal in $\mathcal{A}$, let $s(\mathcal{I})$ be the support of the ideal $\mathcal{I}$ and let $\mathbb{Y}$ be a quasi-normed solid subspace in $S(\mathcal{A})$. We show that any derivation from $\mathcal{I}$ into $\mathbb{Y}$ is always trivial. At the same time, there exist non-zero derivations from $\mathcal{I}$ into $S(\mathcal{A})$, if and only if the Boolean algebra of all projections from $s(\mathcal{I})\mathcal{A}$ is not σ-distributive.

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1. Introduction

It is well known ([12], Lemma 4.1.3) that every derivation on a C^*-algebra is norm continuous and, when a C^*-algebra $\mathcal{A}$ is commutative, any derivation on $\mathcal{A}$ is trivial. In particular, for a commutative von Neumann algebra and a commutative AW^*-algebra $\mathcal{A}$ any derivation $\delta\colon \mathcal{A} \to \mathcal{A}$ is identically zero. Development of the theory of algebras $S(\mathcal{A})$ of measurable operators affiliated with a von Neumann algebra or a AW^*-algebra $\mathcal{A}$ ([4], [13]) allowed to construct and study the new significant examples of $*$-algebras of unbounded operators. One of interesting problems in this theory is the problem of description of derivations acting in $S(\mathcal{A})$. For a von Neumann algebra and an AW^*-algebra $\mathcal{A}$ it is known ([11], ([12], 4.1.6)) that any derivation $\delta\colon \mathcal{A} \to \mathcal{A}$ is inner, i.e. it has a form $\delta(x) = [a, x] = ax - xa$ for some $a \in \mathcal{A}$ and for all $x \in \mathcal{A}$. For the algebra $S(\mathcal{A})$ it is not the same. In case of a commutative von Neumann algebra $\mathcal{A}$ in [2] it was established that any derivation on $S(\mathcal{A})$ is inner, i.e. trivial, if and only if $\mathcal{A}$ is an atomic algebra. For a commutative AW^*-algebra $\mathcal{A}$ there are non-zero derivations on $S(\mathcal{A})$, if and only if the Boolean algebra of all projections from $\mathcal{A}$ is not σ-distributive [8]. In case of a von Neumann algebra $\mathcal{A}$ of type I, all derivations from $S(\mathcal{A})$ into $S(\mathcal{A})$ are described in [1]. The next step in the study of properties of derivations in operator algebras has become the research of derivations acting on an ideal in a von Neumann algebra $\mathcal{A}$ with values in a Banach solid space in $S(\mathcal{A})$ [3]. In particular, in [3] it is proven that any derivation from a commutative von Neumann algebra $\mathcal{A}$ with values in a Banach solid space $\mathbb{Y} \subset S(\mathcal{A})$ is always trivial.

In this paper we consider derivations acting on an ideal $\mathcal{I}$ in a commutative AW^*-algebra $\mathcal{A}$ (respectively, in the algebra $C(Q)$ of all continuous real-valued functions defined on the Stone space Q of a complete Boolean algebra $\mathcal{B}$) with values in a solid space $\mathbb{Y} \subset S(\mathcal{A})$ (respectively, in $E \subset C_\infty(Q)$). It is proven that for a quasi-normed solid space $\mathbb{Y}$ any derivation $\delta\colon \mathcal{I} \to \mathbb{Y}$ is

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always trivial. At the same time, there exist non-zero derivations from $\mathcal{I}$ into $S(\mathcal{A})$ (respectively, in $C_\infty(Q)$), if and only if the Boolean algebra of all projections from $\mathcal{A}$ (respectively, $\mathcal{B}$) is not σ-distributive (compare [8]). In particular, if $\mathcal{A}$ is a commutative von Neumann algebra, then any derivation $\delta\colon \mathcal{I} \to S(\mathcal{A})$ is trivial, if and only if the algebra $\mathcal{A}$ is atomic (compare [2]).

2. Preliminaries

Let $\mathcal{B}$ be an arbitrary complete Boolean algebra with zero $\mathbf{0}$ and unity $\mathbf{1}$. For an arbitrary non-zero $e \in \mathcal{B}$ set $\mathcal{B}_e = \{q \in \mathcal{B} : q \leqslant e\}$. With respect to the partial order induced from $\mathcal{B}$ the set $\mathcal{B}_e$ is a Boolean algebra with zero $\mathbf{0}$ and unity e.

A set $D \subset \mathcal{B}$ minorizes a subset $E \subset \mathcal{B}$ if for each non-zero $e \in E$ there exist $\mathbf{0} \neq q \in D$ such that $q \leqslant e$. We need the following important property of complete Boolean algebras.

Theorem 2.1 ([9], 1.1.6)**.** *Let $\mathcal{B}$ be a complete Boolean algebra, let $\mathbf{0} \neq e \in \mathcal{B}$ and let D be a minorant subset to $\mathcal{B}_e$. Then there exists a disjoint subset $D_1 \subset D$ such that*

(i). $\sup D_1 = e = \sup D$*;*

(ii). For each element $q \in D_1$ there is an element $p \in D$ satisfying $q \leqslant p$.

A non-zero element q from a Boolean algebra $\mathcal{B}$ is called an atom, if $\mathcal{B}_e = \{\mathbf{0}, q\}$. A Boolean algebra $\mathcal{B}$ is called atomic if, for every non-zero $e \in \mathcal{B}$ there exists an atom $q \in \mathcal{B}$ such that $q \leqslant e$. Each complete atomic Boolean algebra is isomorphic to the Boolean algebra 2^Δ of all subsets of the set Δ of all atoms in $\mathcal{B}$ ([14], ch. III, §2).

A Boolean algebra $\mathcal{B}$ is said to be continuous if $\mathcal{B}$ does not have atoms. If $\mathcal{B}$ is a complete Boolean algebra and Δ is non-empty set of all atoms from $\mathcal{B}$, then for $e = \sup \Delta$ we have that $\mathcal{B}_e$ is an atomic Boolean algebra and $\mathcal{B}_{\mathbf{1}-e}$ is a continuous Boolean algebra.

Denote by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{N}^\mathbb{N}$ the set of all mappings from $\mathbb{N}$ into $\mathbb{N}$. A σ-complete Boolean algebra $\mathcal{B}$ is called σ-distributive, if for any double sequence $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ in $\mathcal{B}$ the following condition holds

$$\bigvee_{n\in\mathbb{N}} \bigwedge_{m\in\mathbb{N}} e_{n,m} = \bigwedge_{\varphi\in\mathbb{N}^\mathbb{N}} \bigvee_{n\in\mathbb{N}} e_{n,\varphi(n)}.$$

Each complete atomic Boolean algebra is σ-distributive ([8], 3.1). In general, the converse does not hold. Moreover, there exist continuous σ-distributive complete Boolean algebras ([9], 5.1.7). At the same time, the following proposition holds.

Proposition 2.2. *If $\mathcal{B}$ is a continuous σ-complete Boolean algebra and there exists a finite strictly positive countably additive measure μ on $\mathcal{B}$, then $\mathcal{B}$ is not σ-distributive.*

Proof. Since the measure μ is strictly positive, i.e. $\mu(e) > 0$ for $e \neq \mathbf{0}$, the Boolean algebra $\mathcal{B}$ has a countable type ([14], ch. I, §6), i.e. any set of non-zero pairwise disjoint elements from $\mathcal{B}$ is at most countable. Hence, $\mathcal{B}$ is a complete Boolean algebra ([14], ch. III, §2). Since $\mathcal{B}$ is a continuous Boolean algebra, it follows that for any $n \in \mathbb{N}$ exists a finite set of pairwise disjoint elements $\{e_k^{(n)}\}_{k=1}^n \subset \mathcal{B}$ such that $\sup\limits_{1\leq k\leq n} e_k^{(n)} = \mathbf{1}$ and $\mu(e_k^{(n)}) = \frac{\mu(\mathbf{1})}{n}$ for all $k = 1, \ldots, n$ ([14], ch.III, §2). If the Boolean algebra $\mathcal{B}$ is σ-distributive, then, according to item 5.1.3 from [9], there exists a partition $\{q_i\}$ of unity refined from $\{e_k^{(n)}\}_{k=1}^n, n \in \mathbb{N}$. It means that for fixed i and n there exists $e_{k_{i,n}}^{(n)} \geqslant q_i, k_{i,n} \in \{1,\dots,n\}$. Since $\mu(e_{k_{i,n}}^{(n)}) = \frac{\mu(\mathbf{1})}{n} \to 0$ for $n \to \infty$, we have $q_i = 0$ for all i, that contradicts to the equality $\sup\limits_i q_i = \mathbf{1}$. Hence, the Boolean algebra $\mathcal{B}$ is not σ-distributive. $\square$

A complete Boolean algebra $\mathcal{B}$ is said to be multinormed, if the set of all finite completely additive measures separates the points of $\mathcal{B}$ ([9], 1.2.9). If a Boolean algebra $\mathcal{B}$ is multinormed, then there exists a partition $\{e_i\}_{i\in I}$ of unity $\mathbf{1}$ such that the Boolean algebra $\mathcal{B}_{e_i}$ has a finite strictly positive countably additive measure for all $i \in I$ ([9], 1.2.10). Therefore, from Proposition 2.2 we have the following

Corollary 2.3. *A multinormed Boolean algebra $\mathcal{B}$ is σ-distributive, if and only if $\mathcal{B}$ is an atomic Boolean algebra.*

Let Q be the Stone space of a complete Boolean algebra $\mathcal{B}$. Denote by $C_\infty(Q)$ the set of all continuous functions $x : Q \to [-\infty, +\infty]$ assuming the values $\pm\infty$ possibly on a nowhere-dense set. The space $C_\infty(Q)$ with naturally defined algebraic operations and partial order is an algebra over the field $\mathbb{R}$ of real numbers and is a universally complete vector lattice. The identically one function $\mathbf{1}_Q$ is the unity of the algebra $C_\infty(Q)$ and an order-unity of the vector lattice $C_\infty(Q)$ ([9], 1.4.2).

An element $x \in C_\infty(Q)$ is an idempotent, i.e. $x^2 = x$, if and only if

$$x(t) = \chi_{Q(e)}(t) = \begin{cases} 1, t \in Q(e) \\ 0, t \notin Q(e) \end{cases} =: e(t)$$

for some clopen set $Q(e) \subset Q$ corresponding to the element $e \in \mathcal{B}$, in addition, $e \leqslant q \Leftrightarrow e(t) \leqslant q(t)$ for all $t \in Q$, where $e, q \in \mathcal{B}$. Thus, the Boolean algebra $\mathcal{B}$ is identified with the Boolean algebra of all idempotents from $C_\infty(Q)$. In this identification the unity $\mathbf{1}$ of $\mathcal{B}$ coincides with the function $\mathbf{1}_Q$, and zero $\mathbf{0}$ of $\mathcal{B}$ coincides with identically zero function. Further we suppose that $\mathcal{B} \subset C_\infty(Q)$, and the algebra $C_\infty(Q)$ we denote by $L^0(\mathcal{B})$.

As in an arbitrary vector lattice, for every $x \in L^0(\mathcal{B})$ denote by $x_+ := x \vee 0$ (respectively, $x_- := -(x \wedge 0)$) the positive (negative) part of x and by $|x| := x_+ + x_-$ denote the modulus of x. The set of all positive elements of $L^0(\mathcal{B})$ is denoted by $L^0_+(\mathcal{B})$.

For every $x \in L^0(\mathcal{B})$ define the support of x by the equality $s(x) = \mathbf{1} - \sup\{e \in \mathcal{B} : ex = 0\}$. It is clear that $s(x) \in \mathcal{B}$ and $s(q) = q$ for all $q \in \mathcal{B}$. Note, that an idempotent $q \in \mathcal{B}$ is the support of $x \in L^0(\mathcal{B})$, if and only if $qx = x$ and from $e \in \mathcal{B}, ex = x$, it follows that $e \geqslant q$.

It is easy to see that supports have the following properties:

Proposition 2.4. *If $x, y \in L^0(\mathcal{B}), 0 \neq \lambda \in \mathbb{R}$, then*

(i). $s(\lambda x) = s(x)$;

(ii). $s(xy) = s(x)s(y)$;

(iii). $s(|x|) = s(x)$;

(iv). If $xy = 0$, then $s(x + y) = s(x) + s(y)$, in particular, $s(x) = s(x_+) + s(x_-)$.

For an arbitrary non-empty subset $E \subset L^0(\mathcal{B})$ define the support $s(E)$ of E by setting $s(E) = \sup\{s(x)\colon x \in E\}$.

A non-zero linear subspace X of $L^0(\mathcal{B})$ is called a solid space, if $x \in X, y \in L^0(\mathcal{B})$ and $|y| \leqslant |x|$ implies that $y \in X$. If X is a solid space in $L^0(\mathcal{B})$ and $s(X) = \mathbf{1}$, then X is said to be a fully solid space in $L^0(\mathcal{B})$.

Denote by $C(Q)$ the algebra of all continuous functions on Q with values in $\mathbb{R}$. It is clear that $C(Q)$ is a subalgebra and a fully solid space in $L^0(\mathcal{B})$, in addition, $C(Q)$ is a Banach algebra with the norm $\|x\|_\infty = \sup\limits_{t \in Q} |x(t)|, x \in C(Q)$. As usual, a subalgebra $\mathcal{A}$ of $C(Q)$ is called an ideal, if $xa \in \mathcal{A}$ for all $x \in C(Q), a \in \mathcal{A}$.

Proposition 2.5. *(i). A linear subspace X in $L^0(\mathcal{B})$ is solid, if and only if $C(Q)X = X$;*

(ii). If X is a solid space and $X \subset C(Q)$, then X is an ideal in $C(Q)$, in particular, X is a subalgebra of $C_\infty(Q)$. Conversely, if X is an ideal in $C(Q)$, then X is a solid space in $L^0(\mathcal{B})$.

Proof. (i) It is clear that $X = \mathbf{1}X \subset C(Q)X$. Let X be a solid space in $L^0(\mathcal{B})$ and let $y \in C(Q), x \in X$. Select $c > 0$ such that $|y| \leqslant c\mathbf{1}$. Then $|yx| \leqslant c|x|$, that implies $yx \in X$. Consequently, $C(Q)X \subset X$, and therefore $C(Q)X = X$.

Conversely, if $C(Q)X = X$, $x \in X, y \in L^0(\mathcal{B})$ and $|y| \leqslant |x|$, then $|y| = a|x|$, where $a \in C(Q)$ and $0 \leqslant a \leqslant \mathbf{1}$. Hence, $|y| \in X$, and therefore $y = (s(y_+) - s(y_-))|y| \in X$.

(ii) If X is a solid space and $X \subset C(Q)$, then $C(Q)X = X$ (see item (i)), and therefore X is an ideal in $C(Q)$. Conversely, if X is an ideal in $C(Q)$, then $C(Q)X = X$ and, by item (i), X is a solid space in $L^0(\mathcal{B})$. $\square$

Proposition 2.6. *Let X be a fully solid space in $L^0(\mathcal{B})$. Then*

(i). For every non-zero $p \in \mathcal{B}$ there exists $0 \neq e \in X \cap \mathcal{B}$ such that $e \leqslant p$.

(ii). There exists a partition $\{e_i\}_{i \in I}$ of unity contained in X.

Proof. (i) Since $s(X) = \sup\{s(x) : x \in X\} = \mathbf{1}$, for every given $0 \neq p \in \mathcal{B}$ there exists a non-zero $x_0 \in X$, such that $s(|x_0|)p \neq 0$. For every $\lambda > 0$ consider the spectral idempotent $e_\lambda(|x_0|) = \{t \in Q : |x_0(t)| \leqslant \lambda\}$ of $|x_0|$. Since $\lambda(\mathbf{1} - e_\lambda(|x_0|)) \leqslant (\mathbf{1} - e_\lambda(|x_0|))|x_0|$ and X is a solid space in $L^0(\mathcal{B})$, it follows that $(\mathbf{1} - e_\lambda(|x_0|)) \in X$. Using the convergence $(\mathbf{1} - e_\lambda(|x_0|)) \uparrow s(|x_0|)$ for $\lambda \to 0$, we have $(\mathbf{1} - e_\lambda(|x_0|))p \uparrow s(|x_0|)p \neq 0$. Hence, there exists $\lambda_0 > 0$ such that $e = (\mathbf{1} - e_{\lambda_0}(|x_0|))p \neq 0$, in addition, $e \in X$ and $e \leqslant p$.

Item (ii) follows from Theorem 2.1 and item (i). $\square$

Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is either the field $\mathbb{C}$ of complex numbers, or the field $\mathbb{R}$ of real numbers. A mapping $\|\cdot\| : X \to \mathbb{R}$ is a quasi-norm, if there exists $C \geqslant 1$ such that for all $x, y \in X, \alpha \in \mathbb{K}$ the following properties hold

1) $\|x\| \geqslant 0, \|x\| = 0 \Leftrightarrow x = 0$;
2) $\|\alpha x\| = |\alpha|\|x\|$;
3) $\|x + y\| \leqslant C(\|x\| + \|y\|)$.

A quasi-norm $\|\cdot\|_X$ on a solid space X is said to be monotone, if $x, y \in X, |y| \leqslant |x|$ implies that $\|y\|_X \leqslant \|x\|_X$. A solid space X in $L^0(\mathcal{B})$ is called a quasi-normed solid space, if X is equipped with a monotone quasi-norm.

If $(X, \|\cdot\|_X)$ is a quasi-normed solid space in $L^0(\mathcal{B})$, then from the inequality $|yx| \leqslant \|y\|_\infty |x|$ it follows that $\|yx\|_X \leqslant \big\|\|y\|_\infty |x|\big\|_X = \|y\|_\infty \|x\|_X$ for all $y \in C(Q), x \in X$.

For every non-zero $e \in \mathcal{B}$ consider the subalgebra $eL^0(\mathcal{B}) = \{ex : x \in L^0(\mathcal{B})\}$. If $Q(e)$ is a clopen set corresponding to the idempotent e in the Stone space Q of complete Boolean algebra $\mathcal{B}$, then $Q_e = Q(e) \cap Q$ is the Stone space of the Boolean algebra $\mathcal{B}_e$. In addition, the mapping $\Phi \colon eL^0(\mathcal{B}) \to L^0(\mathcal{B}_e)$ defined by the equality $\Phi(x) = x|_{Q_e}$, where $x|_{Q_e}$ is the restriction of a function x on the compact Q_e, is an algebraic and lattice isomorphism of $eL^0(\mathcal{B})$ onto $L^0(\mathcal{B}_e)$. Consequently, for every solid space X in $L^0(\mathcal{B})$ and every $0 \neq e \in \mathcal{B}$ the image $Y = \Phi(eX)$ is a solid space in $L^0(\mathcal{B}_e)$, in addition, if $\|\cdot\|_X$ is a monotone quasi-norm on X, then $\|y\|_Y = \|\Phi^{-1}(y)\|_X$ is a monotone quasi-norm on Y. Hence the following proposition holds.

Proposition 2.7. *If $(X, \|\cdot\|_X)$ is a quasi-normed solid space in $L^0(\mathcal{B}), 0 \neq e \in \mathcal{B}$, then $(\Phi(eX), \|\cdot\|_{\Phi(eX)})$ is a quasi-normed solid space in $L^0(\mathcal{B}_e)$, where $\|\Phi(ex)\|_{\Phi(eX)} = \|ex\|_X$ for all $x \in X$. Moreover, $\Phi(eX)$ is a fully solid space in $L^0(\mathcal{B}_e)$, if $e = s(X)$.*

Now, consider the complexification $L^0_{\mathbb{C}}(\mathcal{B}) = L^0(\mathcal{B}) \oplus iL^0(\mathcal{B})$ (with i standing for the imaginary unity) of a real vector lattice $L^0(\mathcal{B})$ (see [9], 1.3.11). As usual, for an element $z = x + iy, x, y \in L^0(\mathcal{B})$ the adjoint element of z is defined by the equality $\overline{z} = x - iy$, in addition, $\mathrm{Re} z = \frac{1}{2}(z + \overline{z}) = x$ is called the real part of z and $\mathrm{Im} z = \frac{1}{2i}(z - \overline{z}) = y$ is called the imaginary part of z. The modulus $|z|$ of every element $x \in L^0_{\mathbb{C}}(\mathcal{B})$ is defined by the equality $|z| := \sup\{\mathrm{Re}(e^{i\theta} z) : 0 \leqslant \theta < 2\pi\}$. An element z of $L^0_{\mathbb{C}}(\mathcal{B})$ may be interpreted as continuous function $z \colon Q \to \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, assuming the values ∞ possibly on a nowhere-dense set, where $\overline{\mathbb{C}}$ is the one-point compactification of $\mathbb{C}$. In addition, the algebraic operations in $L^0_{\mathbb{C}}(\mathcal{B})$ coincide with pointwise algebraic operations on the functions from $L^0_{\mathbb{C}}(\mathcal{B})$, defined up to non-where dense sets. In particular, $L^0_{\mathbb{C}}(\mathcal{B})$ is a commutative $*$-algebra and the modulus $|z|$ of an element $z \in L^0_{\mathbb{C}}(\mathcal{B})$ is defined by the equality $|z|(t) = (\overline{z}(t) z(t))^{\frac{1}{2}} = ((\mathrm{Re} z(t))^2 + (\mathrm{Im} z(t))^2)^{\frac{1}{2}}$ for all t from some open everywhere dense set from Q. The selfadjoint part $(L^0_{\mathbb{C}}(\mathcal{B}))_h = \{z \in L^0_{\mathbb{C}}(\mathcal{B}) : z = \overline{z}\}$ of a complex vector lattice $L^0_{\mathbb{C}}(\mathcal{B})$ coincides with $L^0(\mathcal{B})$. The algebra $\mathbb{C}(Q)$ of all continuous complex functions $z \colon Q \to \mathbb{C}$ coincides with the complexification $C(Q) \oplus iC(Q)$ and $\mathbb{C}(Q)$ is a commutative C^*-algebra with the norm $\|z\|_\infty = \sup\limits_{t \in Q} |z(t)|$.

A solid space $\mathbb{X}$ and a quasi-normed solid space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ in $L^0_{\mathbb{C}}(\mathcal{B})$ are defined as in $L^0(\mathcal{B})$. It is clear that for a solid space $\mathbb{X}$ in $L^0_{\mathbb{C}}(\mathcal{B})$ the set $X := \mathbb{X}_h = \mathbb{X} \cap L^0(\mathcal{B})$ is a solid space in $L^0(\mathcal{B})$, in addition, $\mathbb{X} = X \oplus iX$. If $\|\cdot\|_{\mathbb{X}}$ is a monotone quasi-norm on $\mathbb{X}$, then $(X, \|\cdot\|_{\mathbb{X}})$ is a quasi-normed solid space in $L^0(\mathcal{B})$.

Conversely, if X is a solid space in $L^0(\mathcal{B})$ and $\|\cdot\|_X$ is a monotone quasi-norm on X, then $\mathbb{X} = X \oplus iX$ is a solid space in $L^0_{\mathbb{C}}(\mathcal{B}), \mathbb{X}_h = X$ and the function $\|z\|_{\mathbb{X}} = \||z|\|_X, z \in \mathbb{X}$ is a monotone quasi-norm on $\mathbb{X}$.

3. Derivations on solid spaces in $L^0(\mathcal{B})$

A linear mapping δ from $L^0(\mathcal{B})$ (respectively, $L^0_{\mathbb{C}}(\mathcal{B})$) into $L^0(\mathcal{B})$ (respectively, $L^0_{\mathbb{C}}(\mathcal{B})$) is called a derivation if

$$\delta(xy) = \delta(x)y + x\delta(y) \tag{1}$$

for all $x, y \in L^0(\mathcal{B})$ (respectively, $x, y \in L^0_{\mathbb{C}}(\mathcal{B})$).

If a complete atomic Boolean algebra $\mathcal{B}$ has a countable type, then there exists a strictly positive countable additive measure on $\mathcal{B}$, and therefore any derivation from $L^0(\mathcal{B})$ (respectively, $L^0_{\mathbb{C}}(\mathcal{B})$) into $L^0(\mathcal{B})$ (respectively, $L^0_{\mathbb{C}}(\mathcal{B})$) is trivial ([2], Theorem 3.3). The following theorem gives a necessary and sufficient condition for existence of non-zero derivations.

Theorem 3.1 ([8],§3, Corollary 3.5)**.** *For a complete Boolean algebra $\mathcal{B}$ the following conditions are equivalent:*

(i). $\mathcal{B}$ is a σ-distributive Boolean algebra;

(ii). There are no non-zero derivations from $L^0_{\mathbb{C}}(\mathcal{B})$ into $L^0_{\mathbb{C}}(\mathcal{B})$.

In case, when the Boolean algebra $\mathcal{B}$ is multinormed, $\mathcal{B}$ is σ-distributive if and only if $\mathcal{B}$ is an atomic Boolean algebra $\mathcal{B}$ (see Corollary 2.3). Therefore for a multinormed Boolean algebra $\mathcal{B}$, there exist nonzero derivations on $L^0_{\mathbb{C}}(\mathcal{B})$, if and only if the Boolean algebra $\mathcal{B}$ is not atomic (this fact was also established independently of [8] in ([2], Theorem 3.4)).

By Theorem 3.1, in case when $\mathcal{B}$ is not a σ-distributive Boolean algebra there exists non-zero derivations from $L^0_{\mathbb{C}}(\mathcal{B})$ into $L^0_{\mathbb{C}}(\mathcal{B})$. Later, in Section 5, we show that for any complete Boolean algebra $\mathcal{B}$ every derivation from a solid space $X \subset C(Q)$ into a quasi-normed solid space $Y \subset L^0(\mathcal{B})$ (see the definition below) is always trivial.

Let X, Y be solid spaces in $L^0(\mathcal{B})$, $X \subset C(Q)$. By Proposition 2.5, $C(Q)Y = Y$, i.e. $xy \in Y$ for all $x \in X, y \in Y$.

A linear mapping $\delta\colon X \to Y$ is called a derivation if condition (1) hold for all $x, y \in X$. A (complex) derivation of solid spaces $\mathbb{X}$ and $\mathbb{Y}$ in $L^0_{\mathbb{C}}(\mathcal{B})$, where $\mathbb{X} \subset \mathbb{C}(Q)$, is defined in the same manner.

Proposition 3.2 (compare [2], §2, Proposition 2.3)**.** *If $\delta\colon X \to Y$ is a derivation from X into Y, then for all $e \in X \cap \mathcal{B}, x \in X$ the equalities $\delta(e) = 0$ and $\delta(ex) = e\delta(x)$ hold.*

Proof. If $e \in X \cap \mathcal{B}$, then $\delta(e) = \delta(e^2) = \delta(e)e + e\delta(e) = 2e\delta(e)$, and therefore $e\delta(e) = 2e\delta(e)$, i.e. $e\delta(e) = 0$, that implies the equality $\delta(e) = 0$. Further, for $x \in X$ we have $\delta(ex) = \delta(e)x + e\delta(x) = e\delta(x)$. $\square$

Let $\mathbb{X} = X \oplus iX, \mathbb{Y} = Y \oplus iY$ be solid spaces in $L^0_{\mathbb{C}}(\mathcal{B}), \mathbb{X} \subset \mathbb{C}(Q), X = \mathbb{X}_h, Y = \mathbb{Y}_h$ and let δ be a complex derivation from $\mathbb{X}$ into $\mathbb{Y}$. Consider the mappings $\delta_{\mathbb{Re}}\colon X \to Y$ and $\delta_{\mathbb{Im}}\colon X \to Y$, defined by the equalities:

$$\delta_{\mathbb{Re}}(x) = \frac{\delta(x) + \overline{\delta(x)}}{2}, \delta_{\mathbb{Im}}(x) = \frac{\delta(x) - \overline{\delta(x)}}{2i}, x \in X.$$

Then $\delta_{\mathbb{Re}}, \delta_{\mathbb{Im}}$ are (real) derivations from X into Y, in addition, $\delta = \delta_{\mathbb{Re}} + i\delta_{\mathbb{Im}}$.

Theorem 3.1 implies the following

Corollary 3.3. *For a complete Boolean algebra $\mathcal{B}$ the following conditions are equivalent:*

(i). $\mathcal{B}$ is a σ-distributive Boolean algebra;

(ii). There are no non-zero derivations from $L^0(\mathcal{B})$ into $L^0(\mathcal{B})$.

Proof. $(i) \Rightarrow (ii)$. If δ is an arbitrary (real) derivation from $L^0(\mathcal{B})$ into $L^0(\mathcal{B})$, then $\hat{\delta}(x+iy) = \delta(x) + i\delta(y), x, y \in L^0(\mathcal{B})$ is a (complex) derivation from $L^0_{\mathbb{C}}(\mathcal{B})$ into $L^0_{\mathbb{C}}(\mathcal{B})$. By Theorem 3.1, we have $\hat{\delta} = 0$, and therefore $\delta = 0$.

$(ii) \Rightarrow (i)$. Since any derivation $\delta\colon L^0_{\mathbb{C}}(\mathcal{B}) \to L^0_{\mathbb{C}}(\mathcal{B})$ has a form $\delta = \delta_{\mathbb{Re}} + i\delta_{\mathbb{Im}}$, where $\delta_{\mathbb{Re}}, \delta_{\mathbb{Im}}$ are (real) derivations from $L^0(\mathcal{B})$ into $L^0(\mathcal{B})$, it follows that there are no non-zero derivations from $L^0_{\mathbb{C}}(\mathcal{B})$ into $L^0_{\mathbb{C}}(\mathcal{B})$. By Theorem 3.1, the Boolean algebra $\mathcal{B}$ is σ-distributive. $\square$

By Corollary 2.3, for a multinormed Boolean algebras $\mathcal{B}$, there exist non-zero derivations on $L^0(\mathcal{B})$, if and only if the Boolean algebra $\mathcal{B}$ is not atomic.

We need the following useful property of derivations on $L^0(\mathcal{B})$.

Theorem 3.4. *If $\delta\colon L^0(\mathcal{B}) \to L^0(\mathcal{B})$ is a non-zero derivation, then there exist a sequence $\{a_n\}_{n=1}^\infty$ in $C(Q)$ and a non-zero idempotent $q \in \mathcal{B}$ such that $|a_n| \leqslant \mathbf{1}$ and $|\delta(a_n)| \geqslant nq$ for all $n \in \mathbb{N}$.*

In the proof of Theorem 3.4 we use the notion of a cyclic set in $L^0(\mathcal{B})$ given below.

Let $\{e_i\}_{i\in I}$ be a partition of unity in $\mathcal{B}, x_i \in L^0(\mathcal{B}), i \in I$, where I is an arbitrary indexing set. Select a unique $x \in L^0(\mathcal{B})$ such that $e_i x = e_i x_i$ for all $i \in I$ (the uniqueness of x follows from the equality $\sup\limits_{i\in I} e_i = \mathbf{1}$ ([9], 7.3.1)). The element x is called a mixing of the family $\{x_i\}_{i\in I}$ by the partition of unity $\{e_i\}_{i\in I}$ and is denoted by $\operatorname*{mix}\limits_{i\in I}(e_i x_i)$. If $x_i \geqslant 0$ for all $i \in I$, then $\operatorname*{mix}\limits_{i\in I}(e_i x_i) = \sup\limits_{i\in I} e_i x_i$.

The set of all mixing of families of elements from $E \subset L^0(\mathcal{B})$ is called a cyclic envelope of a subset E in $L^0(\mathcal{B})$ and is denoted by $\operatorname{mix}(E)$. If $\operatorname{mix}(E) = E$, then E is said to be a cyclic subset in $L^0(\mathcal{B})$ ([10], 0.3.5).

Lemma 3.5. *If $E \subset L^0_+(\mathcal{B}), E = \operatorname{mix}(E)$ and E is an order unbounded set in $L^0(\mathcal{B})$, there exist $0 \neq q \in \mathcal{B}, \{x_n\}_{n=1}^\infty \subset E$ such that $qx_n \geqslant nq$ for all $n \in \mathbb{N}$.*

Proof of the lemma. Let us show that there exists a non-zero idempotent $q \in \mathcal{B}$ such that the set pE is an order unbounded in $L^0(\mathcal{B})$ for all $0 \neq p \in \mathcal{B}_q$. If this is not the case, for every $0 \neq q \in \mathcal{B}$ there exist $0 \neq p_q \in \mathcal{B}_q, a_p \in p_q L^0_+(\mathcal{B})$ such that $0 \leqslant p_q x \leqslant a_p$ for all $x \in E$. By Theorem 2.1, there exists a partition of unity $\{e_i\}_{i\in I}$ such that $e_i \leqslant p_{q_i}$ for some $0 \neq q_i \in \mathcal{B}$, $i \in I$. Set $a = \operatorname*{mix}\limits_{i\in I}(e_i a_{p_{q_i}})$. Then $a \in L^0_+(\mathcal{B})$ and for $x \in E$ the relationships hold $e_i x = e_i p_{q_i} x \leqslant e_i a_{p_{q_i}} = e_i a$ for all $i \in I$, that implies $0 \leqslant x \leqslant a$. It means that E is order bounded in $L^0(\mathcal{B})$, which is a contradiction. Consequently, there exists a non-zero idempotent $q \in \mathcal{B}$ such that pE is order unbounded in $L^0(\mathcal{B})$ for all $0 \neq p \in \mathcal{B}_q$.

Fix $n \in \mathbb{N}$ and for every $0 \neq p \in \mathcal{B}_q$ select an element $x_{n,p} \in pE$ which is not dominated by the element np. It means that there exists $0 \neq r_p \in \mathcal{B}_p$ such that $r_p x_{n,p} \geqslant nr_p$. Using Theorem

2.1 again, select a partition $\{z_j\}_{j\in J}$ of the element q such that $z_j \leqslant r_{p_j}$ for some $r_{p_j} \in \mathcal{B}_q$. Since E is a cyclic subset, then $x_n = \underset{j\in J}{\text{mix}}(z_j x_{n,p_j}) \in qE$, in addition,

$$z_j x_n = z_j x_{n,p_j} = z_j r_{p_j} x_{n,p_j} \geqslant n z_j r_{p_j} = n z_j$$

for all $j \in J$. Hence $qx_n = x_n \geqslant nq$ for all $n \in \mathbb{N}$. $\square$

Let us proceed to the proof of Theorem 3.4.

Assume that the set $E = \{|\delta(a)| : a \in L^0(\mathcal{B}), |a| \leqslant \mathbf{1}\}$ is order bounded in $L^0(\mathcal{B})$, i.e. there exists $x \in L^0_+(\mathcal{B})$ such that $|\delta(a)| \leqslant x$ for all $a \in L^0(\mathcal{B})$ with $|a| \leqslant \mathbf{1}$. Let us show that in this case $\delta = 0$.

Let $\{y_n\} \in C(Q)$ and $t_n = \|y_n - y\|_\infty \to 0$ for some $y \in C(Q)$. Then

$$|\delta(y_n) - \delta(y)| = |\delta(y_n - y)| = t_n \left|\delta\left(\frac{y_n - y}{t_n}\right)\right| \leqslant t_n x \tag{2}$$

for all $n \in \mathbb{N}$ with $t_n \neq 0$.

Since $\delta(e) = 0$ for all $e \in \mathcal{B}$ (see Proposition 3.2), it follows that $\delta(x) = 0$ for all step elements $x = \sum\limits_{i=1}^{n} \lambda_i e_i$, where $\lambda_i \in \mathbb{R}, e_i \in \mathcal{B}, i = \overline{1,n}, n \in \mathbb{N}$. For every $b \in C(Q)$ there exists a sequence of step elements $\{x_n\}_{n=1}^\infty$ such that $\|b - x_n\|_\infty \to 0$. Due to (2), we have $|\delta(b)| = |\delta(b) - \delta(x_n)| \leqslant \|b - x_n\|_\infty x$, that implies $\delta(b) = 0$.

Now, let b be an arbitrary element from $L^0(\mathcal{B})$. Select a partition of unity $\{e_n\}_{n=1}^\infty$ such that $e_n b \in C(Q)$ for all $n \in \mathbb{N}$. Since $e_n\delta(b) = \delta(e_n b) = 0$ for all $n \in \mathbb{N}$ (see Proposition 3.2), it follows $\delta(b) = 0$.

Thus, the set E is order unbounded in $L^0(\mathcal{B})$. Let us show that $E = \text{mix}(E)$. Let $\{e_i\}_{i\in I}$ be a partition of unity and let $\{x_i\}_{i\in I}$ be a family of elements from E. Since $x_i = |\delta(a_i)|, a_i \in C(Q), |a_i| \leqslant \mathbf{1}$, then $e_i x_i = |e_i \delta(a_i)| = |\delta(e_i a_i)|, i \in I$. Setting $a = \underset{i\in I}{\text{mix}}(e_i a_i)$ we have that $a \in C(Q), |a| \leqslant \mathbf{1}$ and

$$e_i|\delta(a)| = |e_i\delta(a)| = |\delta(e_i a)| = |\delta(e_i a_i)| = e_i x_i, i \in I,$$

i.e. $\underset{i\in I}{\text{mix}}(e_i x_i) = |\delta(a)| \in E$. Consequently, $E = \text{mix}(E)$. Therefore, by Lemma 3.5, there exist $0 \neq q \in \mathcal{B}, \{a_n\} \in C(Q)$ with $|a_n| \leqslant \mathbf{1}$, such that $|\delta(a_n)| \geqslant q|\delta(a_n)| \geqslant nq$ for all $n \in \mathbb{N}$. $\square$

4. Extension of derivations

In this section we give the construction of extension of any derivation $\delta\colon X \to L^0(\mathcal{B})$, acting on an ideal of the algebra $C(Q)$, up to a derivation $\hat{\delta}\colon L^0(\mathcal{B}) \to L^0(\mathcal{B})$ (compare [2], Theorem 3.1).

Theorem 4.1. *Let $\mathcal{B}$ be a complete Boolean algebra, let Q be the Stone space of $\mathcal{B}$, let X be an ideal in the algebra $C(Q)$ and let $\delta\colon X \to L^0(\mathcal{B})$ be a derivation. Then there exists a derivation $\hat{\delta}\colon L^0(\mathcal{B}) \to L^0(\mathcal{B})$ such that $\hat{\delta}(x) = \delta(x)$ for all $x \in X$. In addition, if $s(X) = \mathbf{1}$, then such derivation $\hat{\delta}$ is unique.*

Proof. Firstly, let us assume that $X = C(Q)$. For every $x \in L^0(\mathcal{B})$ there exists a partition of unity $\{e_n\}_{n\in\mathbb{N}}$ such that $e_n x \in C(Q)$ for all $n \in \mathbb{N}$. Set $\hat{\delta}(x) = \underset{n\in\mathbb{N}}{\mathrm{mix}}(e_n\delta(e_n x))$. Let us show that this definition does not depend on a choice of the partition of unity $\{e_n\}_{n\in\mathbb{N}}$. If $\{q_n\}_{n\in\mathbb{N}}$ is another partition of unity, for which $q_n x \in C(Q)$ for all $n \in \mathbb{N}$ and $y = \underset{n\in\mathbb{N}}{\mathrm{mix}}(q_n\delta(q_n x))$, then

$$e_m q_n y = e_m q_n \delta(q_n x) = q_n e_m \delta(e_m q_n x) = q_n e_m \delta(e_m x) = q_n e_m \hat{\delta}(x)$$

for all $n, m \in \mathbb{N}$. Since $\sup_{m\in\mathbb{N}} e_m = \sup_{n\in\mathbb{N}} q_n = \mathbf{1}$, then $y = \hat{\delta}(x)$. Thus, the mapping $\hat{\delta}\colon L^0(\mathcal{B}) \to L^0(\mathcal{B})$ is correctly defined.

If $x, y \in L^0(\mathcal{B})$ and $\{e_n\}_{n\in\mathbb{N}}, \{p_n\}_{n\in\mathbb{N}}$ are partitions of unity such that $e_n x, p_n y \in C(Q)$ for all $n \in \mathbb{N}$, then

$$\begin{aligned} e_n p_m \hat{\delta}(x+y) &= e_n p_m \delta(e_n p_m (x+y)) = e_n p_m \delta(e_n p_m x) + e_n p_m \delta(e_n p_m y) = \\ &= e_n p_m \big(\delta(e_n x) + \delta(p_m y)\big) = e_n p_m (\hat{\delta}(x) + \hat{\delta}(y)) \end{aligned}$$

for all $n, m \in \mathbb{N}$. Consequently, $\hat{\delta}(x+y) = \hat{\delta}(x) + \hat{\delta}(y)$. Similarly, it is established that $\hat{\delta}(\lambda x) = \lambda\hat{\delta}(x)$ for all $\lambda \in \mathbb{R}$. Thus, $\hat{\delta}$ is a linear mapping. Further

$$\begin{aligned} e_n p_m \hat{\delta}(xy) &= e_n p_m \delta((e_n x)(p_m y)) = e_n p_m \big(\delta(e_n x) p_m y + e_n x \delta(p_m y)\big) = \\ &= e_n p_m (\hat{\delta}(x) y + x\hat{\delta}(y)) \end{aligned}$$

for all $n, m \in \mathbb{N}$, that implies the equality $\hat{\delta}(xy) = \hat{\delta}(x)y + x\hat{\delta}(y)$.

Consequently, $\hat{\delta}\colon L^0(\mathcal{B}) \to L^0(\mathcal{B})$ is a derivation, in addition, for $x \in C(Q)$ and a partition of unity $\{e_n\}_{n\in\mathbb{N}}$ the equalities $e_n\delta(x) = e_n\delta(e_n x) = e_n\hat{\delta}(x), n \in \mathbb{N}$ hold, i.e. $\delta(x) = \hat{\delta}(x)$.

Assume that $\delta_1\colon L^0(\mathcal{B}) \to L^0(\mathcal{B})$ is another derivation, for which $\delta_1(x) = \delta(x)$ for all $x \in C(Q)$. Then, by Proposition 3.2, for $x \in L^0(\mathcal{B})$ and a partition of unity $\{e_n\}_{n\in\mathbb{N}}$ such that $e_n x \in C(Q), n \in \mathbb{N}$ we have $e_n\delta_1(x) = e_n\delta_1(e_n x) = e_n\hat{\delta}(e_n x) = e_n\hat{\delta}(x)$ for all $n \in \mathbb{N}$. Since $\sup_{n\in\mathbb{N}} e_n = \mathbf{1}$, it follows that $\delta_1(x) = \hat{\delta}(x)$ for all $x \in L^0(\mathcal{B})$, i.e. $\delta_1 = \hat{\delta}_1$.

Now, let X be an arbitrary ideal in $C(Q)$ such that $s(X) = \mathbf{1}$. Due to Proposition 2.5(ii), X is fully solid space in $L^0(\mathcal{B})$. By Proposition 2.6(ii) there exists a partition of unity $\{e_i\}_{i\in I}$ contained in X. For all $i \in I, \lambda \in \mathbb{R}$ we have that $\lambda e_i \in X$, that implies the inclusion $e_i C(Q) \subset X$ for all $i \in I$.

Define the mapping $\overline{\delta}\colon C(Q) \to L^0(\mathcal{B})$ by setting

$$\overline{\delta}(x) = \underset{i\in I}{\mathrm{mix}}(e_i\delta(e_i x)), x \in C(Q).$$

As above it is established that $\overline{\delta}$ is a derivation from $C(Q)$ into $L^0(\mathcal{B})$, in addition, $\overline{\delta}(x) = \delta(x)$ for all $x \in X$. Assume that $\delta_2\colon C(Q) \to L^0(\mathcal{B})$ is another derivation for which $\delta_2(x) = \delta(x), x \in X$. If $x \in C(Q)$, then $e_i x \in X$ for all $i \in I$ and

$$e_i\delta_2(x) = e_i\delta_2(e_i x) = e_i\delta(e_i x) = e_i\overline{\delta}(e_i x) = e_i\overline{\delta}(x),$$

i.e. $\delta_2 = \overline{\delta}$. Thus, $\overline{\delta}$ is a unique derivation from $C(Q)$ into $L^0(\mathcal{B})$ such that $\overline{\delta}(x) = \delta(x)$ for all $x \in X$. From the first part of the proof it follows that there exists a unique derivation $\hat{\delta}\colon L^0(\mathcal{B}) \to L^0(\mathcal{B})$ such that $\hat{\delta}(x) = \overline{\delta}(x) = \delta(x)$ for all $x \in X$.

Now, consider an arbitrary ideal X in $C(Q)$. Let $e = s(X) \neq \mathbf{1}$ and Φ is an algebraic and lattice isomorphism from $eL^0(\mathcal{B})$ onto $L^0(\mathcal{B}_e)$ (see Section 2). Let us show that $\delta(X) \subset eL^0(\mathcal{B})$. By setting $q = \mathbf{1} - e$, assume that $q\delta(x) \neq 0$ for some $x \in X$. Then $g = s(q\delta(x)) = qs(\delta(x)) \neq 0$. Since $\Phi(X)$ is a fully solid space in $L^0(\mathcal{B}_e)$, by Proposition 2.6(i), there exists a non-zero $e_0 \in X \cap \mathcal{B}_e$, such that $e_0 \leqslant g$. Hence,

$$0 \neq e_0 = e_0 qs(\delta(x)) = e_0(\mathbf{1} - e)s(\delta(x)) = e_0 s(\delta(x)) - e_0 es(\delta(x)) = 0,$$

which is a contradiction. Thus, $q\delta(x) = 0$ for all $x \in X$, i.e. $\delta(x) = e\delta(x) \in eL^0(\mathcal{B})$, and therefore $\delta(X) \subset eL^0(\mathcal{B})$.

Consequently, there is correctly defined a mapping $\delta_0 \colon \Phi(X) \to \Phi(eL^0(\mathcal{B})) = L^0(\mathcal{B}_e)$ by the equality $\delta_0(\Phi(x)) = \Phi(\delta(x))$. It is clear that δ_0 is a derivation from $\Phi(X)$ with values in $L^0(\mathcal{B}_e)$. Since $\Phi(X)$ is a fully solid space in $L^0(\mathcal{B}_e)$, then, from the proven above, it follows that there exists a derivation $\hat{\delta}_0 \colon L^0(\mathcal{B}_e) \to L^0(\mathcal{B}_e)$ such that $\hat{\delta}_0(\Phi(x)) = \delta_0(\Phi(x))$ for all $x \in X$. Consider the mapping $\hat{\delta}_1 \colon eL^0(\mathcal{B}) \to eL^0(\mathcal{B})$, defined by the equality $\hat{\delta}_1(ex) = \Phi^{-1}(\hat{\delta}_0(\Phi(ex))), x \in L^0(\mathcal{B})$. It is clear that $\hat{\delta}_1$ is a linear mapping, in addition, for all $x, y \in L^0(\mathcal{B})$ we have

$$\begin{aligned}\hat{\delta}_1(exey) &= \Phi^{-1}(\hat{\delta}_0(\Phi(ex)\Phi(ey))) = \Phi^{-1}\big(\hat{\delta}_0(\Phi(ex))\Phi(ey) + \Phi(ex)\hat{\delta}_0(\Phi(ey))\big) = \\ &= \Phi^{-1}(\hat{\delta}_0(\Phi(ex)))ey + ex\Phi^{-1}(\hat{\delta}_0(\Phi(ey))) = \hat{\delta}_1(ex)ey + ex\hat{\delta}_1(ey).\end{aligned}$$

Consequently, $\hat{\delta}_1$ is a derivation from $eL^0(\mathcal{B})$ into $eL^0(\mathcal{B})$, in addition,

$$\hat{\delta}_1(x) = \hat{\delta}_1(ex) = \Phi^{-1}(\hat{\delta}_0(\Phi(x))) = \Phi^{-1}(\delta_0(\Phi(x))) = \delta(x)$$

for all $x \in X$.

Extend a derivation $\hat{\delta}_1 \colon eL^0(\mathcal{B}) \to eL^0(\mathcal{B})$ up to a derivation $\hat{\delta} \colon L^0(\mathcal{B}) \to L^0(\mathcal{B})$ by setting $\hat{\delta}(x) = \hat{\delta}_1(ex)$. It is clear that $\hat{\delta}$ is a derivation from $L^0(\mathcal{B})$ into $L^0(\mathcal{B})$ and $\hat{\delta}(x) = \hat{\delta}_1(x) = \delta(x)$ for all $x \in X$. □

Let X be an arbitrary nonzero ideal in $C(Q)$ with the support $e = s(X)$. Let us show that, when the Boolean algebra $\mathcal{B}_e$ is not σ-distributive, there exists a non-zero derivation δ from X into $L^0(\mathcal{B})$. Indeed, in this case, by Corollary 3.3, there exists a non-zero derivation $\delta \colon L^0(\mathcal{B}_e) \to L^0(\mathcal{B}_e)$. Let Φ be an algebraic and lattice isomorphism from $eL^0(\mathcal{B})$ onto $L^0(\mathcal{B}_e)$. Consider the restriction δ_0 of the derivation δ on $\Phi(X)$. Since $\Phi(X)$ is a fully solid space in $L^0(\mathcal{B}_e)$ (see Proposition 2.7), it follows by Theorem 4.1 that there exists a unique derivation $\hat{\delta}_0 \colon L^0(\mathcal{B}_e) \to L^0(\mathcal{B}_e)$ such that $\hat{\delta}_0(\Phi(x)) = \delta_0(\Phi(x))$ for all $x \in X$. Since $\delta(\Phi(x)) = \delta_0(\Phi(x))$ for all $x \in X$, due to the uniqueness of the derivation $\hat{\delta}_0$, we have $\delta = \hat{\delta}_0$. If δ_0 is a trivial derivation, then, due to the construction of $\hat{\delta}_0$ (see the proof of Theorem 4.1), we have that $\hat{\delta}_0$ is also trivial, and therefore $\delta = 0$, which is a contradiction. Consequently, δ_0 is a non-zero derivation from $\Phi(X)$ with values in $L^0(\mathcal{B}_e)$. Construct the mapping $\overline{\delta} \colon X \to L^0(\mathcal{B})$ by setting $\overline{\delta}(x) = \Phi^{-1}(\delta_0(\Phi(x)))$. As in the proof of Theorem 4.1, it is established that the mapping $\overline{\delta}$ is a derivation. In addition, it is clear that $\overline{\delta}$ is a non-zero derivation from X with values in $L^0(\mathcal{B})$.

Similarly, it is established that, when $\mathcal{B}_e$ is a σ-distributive Boolean algebra, then any derivation δ from an ideal X with values in $L^0(\mathcal{B})$ is always trivial. Thus, the following version of Theorem 3.1 holds.

Theorem 4.2. *Let $\mathcal{B}$ be a complete Boolean algebra, let Q be the Stone space of $\mathcal{B}$, let X be an ideal in $C(Q)$ (respectively, $\mathbb{X}$ an ideal in $\mathbb{C}(Q)$). The following conditions are equivalent:*

(i). The Boolean algebra $\mathcal{B}_{s(X)}$ (respectively, $\mathcal{B}_{s(\mathbb{X}_h)}$) is σ-distributive;

(ii). There are no non-zero derivations from X (respectively, $\mathbb{X}$) with values in $L^0(\mathcal{B})$ (respectively, $L^0_{\mathbb{C}}(\mathcal{B})$).

5. Main result

In this section we prove that any derivation from an ideal $X \subset C(Q)$ into a quasi-normed solid space $Y \subset L^0(\mathcal{B})$ is always trivial.

Theorem 5.1. *Let $\mathcal{B}$ be an arbitrary complete Boolean algebra, let Q be the Stone space of $\mathcal{B}$, let X be an ideal in $C(Q)$ and let Y be a quasi-normed solid space in $L^0(\mathcal{B})$. Then any derivation $\delta\colon X \to Y$ is trivial.*

Proof. Firstly assume that $X = C(Q)$. Let $\delta\colon C(Q) \to Y$ be a non-zero derivation. By Theorem 4.1, the derivation δ extends up to a derivation $\hat{\delta}\colon L^0(\mathcal{B}) \to L^0(\mathcal{B})$. Since δ is a non-zero derivation, $\hat{\delta}$ is also a non-zero derivation from $L^0(\mathcal{B})$ into $L^0(\mathcal{B})$, in addition, $\hat{\delta}(C(Q)) = \delta(C(Q)) \subset Y$.

By Theorem 3.4, there exist a sequence $\{a_n\}_{n=1}^{\infty} \subset C(Q)$ and a non-zero idempotent $q \in \mathcal{B}$ such that $|a_n| \leqslant \mathbf{1}$ and

$$|\hat{\delta}(qa_n)| = |q\hat{\delta}(a_n)| = q|\hat{\delta}(a_n)| \geqslant nq \tag{3}$$

for all $n \in \mathbb{N}$.

Assume that q has a form $q = \sup\limits_{1\leq i\leq k} q_i$, where q_i are atoms in $\mathcal{B}, k \in \mathbb{N}$. Then $\mathcal{B}_q$ is an atomic Boolean algebra, and hence, the Boolean algebra $\mathcal{B}_q$ is σ-distributive. Consider an algebraic and lattice isomorphism $\Phi\colon qL^0(\mathcal{B}) \to L^0(\mathcal{B}_q)$ from Section 2. Define mapping $\delta_q\colon L^0(\mathcal{B}_q) \to L^0(\mathcal{B}_q)$ by setting

$$\delta_q(x) = \Phi(\hat{\delta}(q\Phi^{-1}(x))) = \Phi(q\hat{\delta}(\Phi^{-1}(x))), x \in L^0(\mathcal{B}_q).$$

It is clear that δ_q is a linear mapping, in addition,

$$\begin{aligned}\delta_q(xy) = \Phi(q\hat{\delta}(\Phi^{-1}(x)\Phi^{-1}(y))) = \Phi\big(q\big(\hat{\delta}(\Phi^{-1}(x))\Phi^{-1}(y) + \Phi^{-1}(x)\hat{\delta}(\Phi^{-1}(y))\big)\big) =\\ = \Phi(q\hat{\delta}(\Phi^{-1}(x)))y + x\Phi(q\hat{\delta}(\Phi^{-1}(y))) = \delta_q(x)y + x\delta_q(y).\end{aligned}$$

Consequently, δ_q is a derivation from $L^0(\mathcal{B}_q)$ into $L^0(\mathcal{B}_q)$ and, according to Corollary 3.3, $\delta_q \equiv 0$, that contradicts to (3).

It means that there exist a countably partition $\{e_n\}_{n\in\mathbb{N}}$ of q such that $e_n \neq 0$ for all $n \in \mathbb{N}$. Since Y is a solid space in $L^0(\mathcal{B})$, the inclusion $\hat{\delta}(qa_n) \in Y$ and the inequality $q \leqslant \frac{1}{n}|\hat{\delta}(qa_n)|$ (see (3)) imply that $q \in Y$. Since $0 \leqslant e_n \leqslant q$, it follows that $e_n \in Y$ for all $n \in \mathbb{N}$, in addition, $\|e_n\|_Y > 0$.

For every $n \in \mathbb{N}$ select an integer m_n such that $m_n\|e_n\|_Y > n$. After that select $x \in q(L^0_+(\mathcal{B}))$ such that $e_n x = e_n a_{m_n}$. Since $|e_n x| = e_n|a_{m_n}| \leqslant e_n$ for all $n \in \mathbb{N}$, $\sup\limits_{n\in\mathbb{N}} e_n = q$ and $qx = x$, it follows that $|x| \leqslant q$, i.e. $0 \leqslant x \in qC(Q)$.

We have that

$$|e_n\hat{\delta}(x)| = |\hat{\delta}(e_n x)| = |\hat{\delta}(e_n a_{m_n})| = e_n|\hat{\delta}(a_{m_n})| \overset{(3)}{\geqslant} m_n e_n.$$

Hence, due to the selection of m_n, we have

$$\|\hat{\delta}(x)\|_Y = \|e_n\|_\infty\|\hat{\delta}(x)\|_Y \geqslant \|e_n\hat{\delta}(x)\|_Y \geqslant \|m_n e_n\|_Y = m_n\|e_n\|_Y > n$$

for all $n \in \mathbb{N}$. Consequently, the derivation $\delta\colon C(Q) \to Y$ is trivial.

Now, let X be an arbitrary ideal in $C(Q)$. By proposition 2.5(ii), X is a solid space in $L^0(\mathcal{B})$. Firstly let us assume that X is a fully solid space in $L^0(\mathcal{B})$. By Proposition 2.6(ii) there exists a partition $\{e_i\}_{i\in I}$ of unity contained in X. For all $i \in I, \lambda \in \mathbb{R}$ we have $\lambda e_i \in X$, that implies $e_i C(Q) \subset X$ for all $i \in I$.

Let Q_{e_i} be the Stone space of the Boolean algebra $\mathcal{B}_{e_i}$ and let Φ_i be an algebraic and lattice isomorphism from $e_i L^0(\mathcal{B})$ into $L^0(\mathcal{B}_{e_i})$ for which $\Phi_i(x) = x|_{Q_{e_i}}$ (see Section 2). Since $\Phi_i^{-1}(C(Q_{e_i})) = e_i C(Q) \subset X$, there is defined a mapping $\delta_i\colon C(Q_{e_i}) \to L^0(\mathcal{B}_{e_i})$ by the equality $\delta_i(x) = \Phi_i(e_i\delta(\Phi_i^{-1}(x))), x \in C(Q_{e_i})$. As above, it is established that δ_i is a derivation from $C(Q_{e_i})$ into $L^0(\mathcal{B}_{e_i})$, in addition, $\delta_i(C(Q_{e_i})) \subset \Phi_i(e_i Y)$ for all $i \in I$. By Proposition 2.7, $(\Phi_i(e_i Y), \|\cdot\|_{\Phi_i(e_i Y)})$ is a quasi-normed solid space in $L^0(\mathcal{B}_{e_i})$. Therefore, from the proven above it follows that $\delta_i \equiv 0$ for all $i \in I$. Thus, for every $x \in X$ we have that

$$e_i\delta(x) = e_i\delta(e_i x) = e_i\delta(\Phi_i^{-1}(\Phi_i(e_i x))) = e_i\Phi^{-1}(\delta_i(\Phi_i(ex))) = 0, i \in I,$$

i.e. δ is trivial.

Now, let X be an arbitrary solid space in $L^0(\mathcal{B}), X \subset C(Q)$, $e = s(X) = \sup\{s(x) : x \in X\} \neq \mathbf{1}$ and let Φ be an algebraic and lattice isomorphism from $eL^0(\mathcal{B})$ onto $L^0(\mathcal{B}_e)$. As in the end of the proof of Theorem 4.1, it is established that $\delta(X) \subset eY$. By Proposition 2.7, $(\Phi(eY), \|\cdot\|_{\Phi(eY)})$ is a quasi-normed solid space in $L^0(\mathcal{B}_e)$, and $\Phi(X) = \Phi(eX)$ is a fully solid space in $L^0(\mathcal{B}_e)$, in addition, $\delta_0(x) = \Phi(\delta(\Phi^{-1}(x))), x \in \Phi(X)$ is a derivation from the fully solid space $\Phi(X)$ into the quasi-normed solid space $\Phi(eY)$. From the proven above, we have that $\delta_0 \equiv 0$, and therefore $\delta\colon X \to Y$ is also trivial. $\square$

Now, consider the complex case. Let $\mathbb{X}$ be an ideal in the algebra $\mathbb{C}(Q)$ and let $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be a quasi-normed solid space in $L^0_{\mathbb{C}}(\mathcal{B})$. Since any derivation $\delta\colon \mathbb{X} \to \mathbb{Y}$ has a form $\delta = \delta_{\mathbb{Re}} + i\delta_{\mathbb{Im}}$, where $\delta_{\mathbb{Re}}, \delta_{\mathbb{Im}}$ are derivations from the ideal space $X = \mathbb{X}_h \subset C(Q)$ into the quasi-normed solid space $Y = \mathbb{Y}_h$, Theorem 5.1 implies the following

Corollary 5.2. *Let $\mathcal{B}$ be an arbitrary complete Boolean algebra, let Q be the Stone space of $\mathcal{B}$, let $\mathbb{X}$ be an ideal in $\mathbb{C}(Q)$ and let $\mathbb{Y}$ be a quasi-normed solid space in $L^0_{\mathbb{C}}(\mathcal{B})$. Then any derivation $\delta\colon \mathbb{X} \to \mathbb{Y}$ is trivial.*

Recall that, by Theorem 4.2, for nonzero ideal X (respectively, $\mathbb{X}$) in $C(Q)$ (respectively, $\mathbb{C}(Q)$) in case, when the Boolean algebra $\mathcal{B}_{s(X)}$ (respectively, $\mathcal{B}_{s(\mathbb{X}_h)}$) is not σ-distributive, there exist non-zero derivations δ from X (respectively, from $\mathbb{X}$) with values in $L^0(\mathcal{B})$ (respectively, $L^0_C(\mathcal{B})$).

Now, give a version of Corollary 5.2 for commutative AW^*-algebras. An AW^*-algebra is a C^*-algebra which is simultaneously a Baer $*$-algebra ([9], 7.5.2). If $\mathcal{A}$ is a commutative AW^*-algebra, then the lattice $P(\mathcal{A}) = \{p \in \mathcal{A} : p = p^* = p^2\}$ of all projections from $\mathcal{A}$ is a complete Boolean algebra [5], in particular, if $\mathcal{A}$ is a commutative von Neumann algebra, then the Boolean algebra $P(\mathcal{A})$ is multinormed. In this case, AW^*-algebra $\mathcal{A}$ is $*$-isomorphic to the C^*-algebra $\mathbb{C}(Q(P(\mathcal{A})))$, where $Q(P(\mathcal{A}))$ is the Stone space of the Boolean algebra $P(\mathcal{A})$ ([9], 7.4.3, 7.5.2). Denote by $S(\mathcal{A})$ the $*$-algebra of all measurable operators affiliated with AW^*-algebra $\mathcal{A}$ (see e.g. [4], [5]). It is known [4], that for a commutative AW^*-algebra $\mathcal{A}$ the $*$-algebra $S(\mathcal{A})$ is $*$-isomorphic to the $*$-algebra $L^0_{\mathbb{C}}(P(\mathcal{A}))$. Therefore, Theorem 4.2, Corollaries 2.3 and 5.2 imply the following

Theorem 5.3. *Let $\mathcal{A}$ be a commutative AW^*-algebra (respectively, commutative von Neumann algebra), let $\mathcal{I}$ be an ideal in $\mathcal{A}$. Then*

(i). The Boolean algebra $P(s(\mathcal{I}_h)\mathcal{A})$ of all projections from $s(\mathcal{I}_h)\mathcal{A}$ is σ-distributive (respectively, atomic), if and only if any derivation from $\mathcal{I}$ into $S(\mathcal{A})$ is trivial;

(ii). If $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ is a quasi-normed solid space in $S(\mathcal{A})$, then any derivation from $\mathcal{I}$ into $\mathbb{Y}$ is trivial.

Give one more illustration of Theorem 5.1. Let (Ω, Σ, μ) be a measurable space with a complete σ-finite measure μ, let $L^0(\Omega)$ be the algebra of all measurable real-valued functions defined on (Ω, Σ, μ), and let $L^\infty(\Omega)$ be the subalgebra of all essentially bounded functions from $L^0(\Omega)$ (functions that are equal almost everywhere are identified). Denote by t_μ the topology of convergence locally in measure μ in $L^0(\Omega)$. Convergence $f_n \xrightarrow{t_\mu} f, f_n, f \in L^0(\Omega)$ means that $f_n\chi_A \to f\chi_A$ in measure for any $A \in \Sigma$ with $\mu(A) < \infty$. By Proposition 3.2, and because of density of the subalgebra of step functions in the algebra $L^0(\Omega)$ with respect to the topology t_μ, any t_μ-continuous derivation $\delta\colon L^0(\Omega) \to L^0(\Omega)$ is trivial.

Consider an arbitrary non-zero ideal X in the algebra $L^\infty(\Omega)$ and Banach solid space $(Y, \|\cdot\|_Y)$ of measurable functions on (Ω, Σ, μ) (see e.g. [6], ch.IV, §3). The examples of such solid spaces are L_p-spaces, $p \geqslant 1$, the Orlicz, Marcinkiewicz spaces, symmetric spaces of measurable functions on (Ω, Σ, μ) [7]. Theorems 4.2, 5.1 and Corollary 2.3 imply the following

Theorem 5.4. *(i). Any derivation from X into Y is trivial;*

(i). If (Ω, Σ, μ) does not have atoms, then there exists a non-zero derivation from X into $L^0(\Omega)$, which is not continuous with respect to the topology t_μ;

(iii). If (Ω, Σ, μ) is an atomic measure space, then any derivation from X into $L^0(\Omega)$ is trivial.

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