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Computation of the p^6 order low-energy constants with tensor sources

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We present the results of computing the p^4 and p^6 order low-energy constants of the chiral Lagrangian with tensor sources for both two and three flavors pseudoscalar mesons. This is a generalization of our previous work on calculating the p^4 and p^6 order coefficients of the chiral Lagrangian without tensor sources in terms of the quark self-energy $\Sigma(p^2)$. We find that some p^6 order operators with tenor sources used in the literature are related to each other with the help of some epsilon relations. There leaves 100 independent terms for n-flavor, 94 terms for 3-flavor, and 67 terms for 2-flavor cases. We also find that the odd-intrinsic-parity chiral Lagrangian with tensor sources can not exist.

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I. INTRODUCTION

In the low-energy region, conventional perturbation theory is ineffective for the strong interaction. If we focus on the pseudoscalar mesons (π, K, η), chiral perturbation theory provides us an effective way to deal with the system. It can be applied not only to the strong interaction, but also to the weak and electromagnetic interactions. It was first introduced by Weinberg [1]. The idea was to expand the meson part Lagrangian in terms of powers of external momenta. Then, Gasser and Leutwyler [2, 3] extended it to the p^4 order, and built up the path integral formalism which enables us to compute the various Green's functions of the light-quark scalar, pseudoscalar, vector and axial vector currents in terms of the chiral Lagrangian. The formulation was generalized to the p^6 order later. The form of the normal (or even parity) part of the p^6 order chiral Lagrangian had been gotten in [4–6], and soon the anomalous (or odd parity) part's form [7, 8]. A latest and general review can be found in [9]. Unfortunately, the antisymmetric tensor currents were missed in the series works started from Gasser and Leutwyler. Although this may be partly due to the fact that tensor currents do not appear in the Standard Model (SM) Lagrangian, as discussed in Ref.[10], researches of hadron matrix elements and the study of interactions beyond SM may need the tensor currents. Further, antisymmetric tensor currents not only generate the conventional 1^{--} vector mesons, but also the more exotic 1^{+-} mesons. Therefore, the study of the antisymmetric currents can involve both of them and their interactions. More importantly, for the structure of the general currents $\overline{\psi}\Gamma\psi$, the 4×4 matrices Γ generally have 16 degrees of freedoms and ones usually choose 16 γ-matrices of $1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}$ to represent these freedoms. This implies that Γ can be expanded in terms of the 16 γ-matrices, and one is used to name the currents according to their γ-matrices structures. Due to this incompleteness of the γ-matrices structures, just scalar, pseudo scalar, vector and axial vector currents can not give the most general bilinear light-quark currents. Merely adding the tensor currents, we can get the set of the currents completely. Then the results of the Green's functions among currents are general. Five years ago, the form of chiral Lagrangian involving tensor currents had been discussed firstly in Ref.[10]. The results are the normal parts with tensor sources can start at the p^4 order, and both the p^4 and p^6 order chiral Lagrangian with tensor sources were obtained. But the odd-intrinsic-parity parts with tensor sources only starts at the p^8 order. Based on these results, more progresses are coming [11–13].

Within the chiral perturbation theory, when the orders of the momentum expansion increase, the number of the independent terms rise rapidly. For example, in the three flavors case, the p^4 order Lagrangian has 10 terms plus 2 contact terms, but in the p^6 order, which is 90 plus 3 contact terms. These independent terms cause large number of unknown low-energy constants (LECs), a summary of the numerical results of the LECs can be found in [14], which makes the discussions of the chiral Lagrangian high-momentum-order effects more and more difficult and complex, contrasting to adding tensor sources mentioned above. Originally, the LECs are fixed via the experiment data. Now, because of the more and more LECs in the higher order and the lack of enough experiments data, we can no longer solely rely on the insufficient experiment data to determine these LECs. Calculations of LECs from various models

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or underlying QCD then develop and become popular. In fact, not only the experiment data, but the theoretical calculations are also needed. Via them, we can check the correctness of the models or the theory appearing in the computation. As one of the members in the community of calculating LECs, we have calculated the p^4 order LECs in [15, 16], and then the p^6 order in [17, 18], including the normal and the anomalous parts, two and three flavors cases. In this paper, we will extend our work to the tensor sources and calculate all the LECs up to the p^6 order.

This paper is organized as follows: In Section II, we review our previous calculations on p^2 to p^6 order LECs and adding the tensor sources. In Section III, we collect the differences of conventions between our paper and Ref.[10] and discuss the possible dependent operators. In Section IV, we give our p^4 order results with tensor sources, and Section V presents our p^6 order results. Section VI is a summary.

II. THE CALCULATIONS ON p^2 TO p^6 ORDER LECS WITH TENSOR SOURCES

We have calculated the p^2 to p^6 order normal part's LECs without tensor sources in Ref.[17]. Using the same method, we can also deal with the tensor-source part. For convenience, we give a short introduction here, but adding tensor sources.

The difference from Ref.[17] is the external tensor sources $\bar{t}^{\mu\nu}$. Adding them in the original external sources v^μ, a^μ, s, p denoted as scalar, pseudoscalar, vector and axial vector sources, we get the complete sources set as follows

$$J = \not{v} + \not{a}\gamma_5 - s + ip\gamma_5 + \sigma_{\mu\nu}\bar{t}^{\mu\nu}. \tag{1}$$

From the original QCD, the lagrangian can be written as the QCD Lagrangian, $\mathcal{L}^0_{\mathrm{QCD}}$, plus the external sources part, and the generating functional reads

$$\begin{aligned} Z[J] &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\mathcal{D}A_\mu \exp\left\{i\int d^4x[\mathcal{L}^0_{\mathrm{QCD}} + \bar{\psi}J\psi]\right\} \\ &= \int \mathcal{D}U \exp\left\{i\int d^4x\mathcal{L}_{\mathrm{ChPT}}(U,J)\right\} = \int \mathcal{D}U e^{iS_{\mathrm{eff}}}, \end{aligned} \tag{2}$$

where ψ, Ψ and A_μ are light-quark, heavy-quark and gluon fields, U is the pseudoscalar meson field, $\mathcal{L}_{\mathrm{ChPT}}$ is the chiral Lagrangian, and S_{eff} is the effective action. Because this form of chiral Lagrangian is explicit U dependence in the high momentum orders, and is hard to investigate [4, 5] due to its complex U and J structures, we are used to make the chiral rotation to simplify the Lagrangian as [15, 17, 19]

$$J_\Omega = [\Omega P_R + \Omega^\dagger P_L][J + i\not{\partial}][\Omega P_R + \Omega^\dagger P_L] = \not{v}_\Omega + \not{a}_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 + \sigma_{\mu\nu}\bar{t}^{\mu\nu}_\Omega, \tag{3}$$

$$U = \Omega^2, \qquad P_L = \frac{1-\gamma_5}{2}, \qquad P_R = \frac{1+\gamma_5}{2}. \tag{4}$$

To separate the tensor sources to the even and odd parities, $t^{\mu\nu}_\pm$, we need the following tensor chiral projectors as Ref.[10]

$$P_R^{\mu\nu\lambda\rho} = \frac{1}{4}(g^{\mu\lambda}g^{\nu\rho} - g^{\nu\lambda}g^{\mu\rho} + i\epsilon^{\mu\nu\lambda\rho}), \tag{5}$$

$$P_L^{\mu\nu\lambda\rho} = (P_R^{\mu\nu\lambda\rho})^\dagger = \frac{1}{4}(g^{\mu\lambda}g^{\nu\rho} - g^{\nu\lambda}g^{\mu\rho} - i\epsilon^{\mu\nu\lambda\rho}), \tag{6}$$

$$t^{\mu\nu} = \frac{1}{2}(t^{\mu\nu}_+ + t^{\mu\nu}_-), \qquad t^{\dagger\mu\nu} = \frac{1}{2}(t^{\mu\nu}_+ - t^{\mu\nu}_-), \tag{7}$$

$$\bar{t}^{\mu\nu} = P_L^{\mu\nu\lambda\rho}t_{\lambda\rho} + P_R^{\mu\nu\lambda\rho}t^\dagger{}_{\lambda\rho}, \tag{8}$$

$$\sigma_{\mu\nu}\bar{t}^{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}t^{\mu\nu}_+ - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu\lambda\rho}t_{-,\lambda\rho} = \frac{1}{2}\sigma_{\mu\nu}(t^{\mu\nu}_+ - t^{\mu\nu}_-\gamma_5) = \sigma_{\mu\nu}\bar{t}'^{\mu\nu}, \tag{9}$$

$$\bar{t}'^{\mu\nu} \equiv \frac{1}{2}(t^{\mu\nu}_+ - t^{\mu\nu}_-\gamma_5). \tag{10}$$

To obtain the Eq.(9), we have used the γ matrix identity,

$$\sigma^{\mu\nu}\gamma_5 = \frac{i}{2}\epsilon^{\mu\nu\lambda\rho}\sigma_{\lambda\rho}, \tag{11}$$

and we introduce $\bar{t}'^{\mu\nu}$ for the future calculation convenience. After this operation, our symbols have the simple relations as [5, 10] (see Appendix A). Using the same method in [17, 19], we can obtain the effective action $S_{\rm eff}$ introduced in Eq.(2) from the first principle of QCD,

$$\begin{aligned} S_{\rm eff} \;=\; & -iN_c{\rm Tr}\ln[i\partial\!\!\!/+J_\Omega-\Pi_{\Omega c}]+iN_c{\rm Tr}\ln[i\partial\!\!\!/+J_\Omega]-iN_c{\rm Tr}\ln[i\partial\!\!\!/+J]+N_c{\rm Tr}[\Phi_{\Omega c}\Pi^T_{\Omega c}] \\ & +N_c\sum_{n=2}^{\infty}\int d^4x_1\cdots d^4x'_n\frac{(-i)^n(N_cg_s^2)^{n-1}}{n!}\bar{G}^{\sigma_1\cdots\sigma_n}_{\rho_1\cdots\rho_n}(x_1,x'_1,\cdots,x_n,x'_n)\Phi^{\sigma_1\rho_1}_{\Omega c}(x_1,x'_1)\cdots\Phi^{\sigma_n\rho_n}_{\Omega c}(x_n,x'_n)+O(\frac{1}{N_c}). \end{aligned} \tag{12}$$

Eq.(12) is the same as Eq.(1) in [17], but J_Ω, the external source J including currents and densities after Goldstone fields dependent chiral rotation Ω, includes the tensor sources. $\Phi_{\Omega c}$ and $\Pi_{\Omega c}$ are two-point rotated quark Green's function and the interaction part of two-point rotated quark vertices in the presence of the external sources, respectively; $\Pi_{\Omega c}$ is defined by

$$\Phi^{\sigma\rho}_{\Omega c}(x,y)\equiv\frac{1}{N_c}\langle\overline{\psi}^\sigma_\Omega(x)\psi^\rho_\Omega(y)\rangle=-i[(i\partial\!\!\!/+J_\Omega-\Pi_{\Omega c})^{-1}]^{\rho\sigma}(y,x),\qquad \psi_\Omega(x)\equiv[\Omega(x)P_L+\Omega^\dagger(x)P_R]\psi(x), \tag{13}$$

with subscript $_c$ denoting the classical field. $\bar{G}^{\sigma_1\cdots\sigma_n}_{\rho_1\cdots\rho_n}(x_1,x'_1,\cdots,x_n,x'_n)$ is the effective gluon n-point Green's function including gluon and heavy quark contributions and g_s is the strong coupling constant of QCD. It can be shown that the last two terms in the r.h.s. of Eq.(12) are independent of pseudoscalar meson field U or Ω and therefore are just irrelevant constants in the effective action. While the second and third terms are anomalous part contributions, since they represent the variations of the path integral measure of the light quark field ψ. The remaining first term is called the normal part contributions which relies on $\Pi_{\Omega c}$. The $\Phi_{\Omega c}$ and $\Pi_{\Omega c}$ are related by the first equation of (13) and determined by

$$\begin{aligned} & [\Phi_{\Omega c}+\tilde{\Xi}]^{\sigma\rho}+\sum_{n=1}^{\infty}\int d^4x_1d^4x'_1\cdots d^4x_nd^4x'_n\frac{(-i)^{n+1}(N_cg_s^2)^n}{n!}\overline{G}^{\sigma\sigma_1\cdots\sigma_n}_{\rho\rho_1\cdots\rho_n}(x,y,x_1,x'_1,\cdots,x_n,x'_n) \\ & \times\Phi^{\sigma_1\rho_1}_{\Omega c}(x_1,x'_1)\cdots\Phi^{\sigma_n\rho_n}_{\Omega c}(x_n,x'_n)=O(\frac{1}{N_c}), \end{aligned} \tag{14}$$

where $\tilde{\Xi}$ is a Lagrangian multiplier which insures the constraint ${\rm tr}_l[\gamma_5\Phi^T_{\Omega c}(x,x)]=0$. Eq.(14) is the Schwinger-Dyson equation (SDE) in the presence of the rotated external source. In Ref.[15], we have assumed the ansatz solution of (14) approximately by

$$\Pi^{\sigma\rho}_{\Omega c}(x,y)=[\Sigma(\overline{\nabla}^2_x)]^{\sigma\rho}\delta^4(x-y)\qquad\qquad \overline{\nabla}^\mu_x=\partial^\mu_x-iv^\mu_\Omega(x)\;, \tag{15}$$

where Σ is the quark self-energy which satisfies SDE (14) with vanishing rotated external source. Under the ladder approximation, this SDE in Euclidean space-time is reduced to the standard form of

$$\Sigma(p^2)-3C_2(R)\int\frac{d^4q}{4\pi^3}\frac{\alpha_s[(p-q)^2]}{(p-q)^2}\frac{\Sigma(q^2)}{q^2+\Sigma^2(q^2)}=0\;, \tag{16}$$

where $C_2(R)$ is the second order Casimir operator of the quark representation R. In our cases, quark is belonging to the $SU(N_c)$ fundamental representation, therefore $C_2(R)=(N_c^2-1)/2N_c$, and in the large N_c limit, we will neglect the second term of it. $\alpha_s(p^2)$ is the running coupling constant of QCD which depends on N_c and the number of quark flavors. With these approximations, the action (12) of the chiral Lagrangian becomes

$$S_{\rm eff}\approx-iN_c{\rm Tr}\ln[i\partial\!\!\!/+J_\Omega-\Sigma(\overline{\nabla}^2)]+iN_c{\rm Tr}\ln[i\partial\!\!\!/+J_\Omega]-iN_c{\rm Tr}\ln[i\partial\!\!\!/+J]+O(\frac{1}{N_c})\;. \tag{17}$$

We have proved that the normal parts of the second and the third terms are cancel each other [17]. Then in large N_c limit, for normal part we only need to calculate the first term.

$$S_{\rm eff}\approx-iN_c{\rm Tr}\ln[i\partial\!\!\!/+J_\Omega-\Sigma(\overline{\nabla}^2)] \tag{18}$$

Because in Minkovski space, it is not convenient to perform the compuation, we take the Wick rotation to change Eq.(18) to Euclidean Space, with the metric tensor $g^{\mu\nu}={\rm diag}(1,1,1,1)$.

$$\begin{aligned} & x^0|_M\to-ix^4|_E,\qquad x^i|_M\to x^i|_M, \\ & \gamma^0|_M\to\gamma^4|_E,\qquad \gamma^i|_M\to i\gamma^i|_E,\qquad \gamma_5|_M\to\gamma_5|_E, \end{aligned}$$

$$
\begin{aligned}
&s_\Omega|_M \to -s_\Omega|_E, \qquad p_\Omega|_M \to -p_\Omega|_E,\\
&\bar{t}_{\Omega,00}|_M \to -\bar{t}_{\Omega,44}|_E, \qquad \bar{t}_{\Omega,ij}|_M \to \bar{t}_{\Omega,ij}|_E, \qquad \bar{t}_{\Omega,0i}|_M \to i\bar{t}_{\Omega,4i}|_E, \qquad \bar{t}_{\Omega,i0}|_M \to i\bar{t}_{\Omega,i4}|_E. \qquad (19)
\end{aligned}
$$

$v_\Omega^\mu, a_\Omega^\mu$ transform as x^μ, and $\bar{t}_{\Omega,\mu\nu}$ are considering as (axial) vector-(axial) vector combined. With the help of Schwinger proper time method [20], the real part of the $\mathrm{Trln}[\cdots]$ in Euclidean space-time can be written as

$$
\begin{aligned}
&\mathrm{ReTr}\ln[\partial\!\!\!/ - i v\!\!\!/_\Omega - i a\!\!\!/_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 - \sigma_{\mu\nu}\bar{t}_\Omega^{\prime\dagger\mu\nu} + \Sigma(-\bar{\nabla}^2)]\\
\equiv\;& \mathrm{ReTr}\ln[D - \sigma_{\mu\nu}\bar{t}_\Omega^{\prime\mu\nu} + \Sigma(-\bar{\nabla}^2)]\\
=\;& \frac{1}{2}\mathrm{Tr}\ln\Big[[D^\dagger - \sigma_{\mu\nu}\bar{t}_\Omega^{\prime\dagger\mu\nu} + \Sigma(-\bar{\nabla}^2)][D - \sigma_{\mu\nu}\bar{t}_\Omega^{\prime\mu\nu} + \Sigma(-\bar{\nabla}^2)]\Big]\\
=\;& \frac{1}{2}\mathrm{Tr}\ln[O+N]\\
=\;& -\frac{1}{2}\lim_{\Lambda\to\infty}\int_{\frac{1}{\Lambda^2}}^\infty \frac{d\tau}{\tau}\mathrm{Tr}e^{-\tau(O+N)}(\text{remove const term})\\
=\;& -\frac{1}{2}\lim_{\Lambda\to\infty}\int_{\frac{1}{\Lambda^2}}^\infty \frac{d\tau}{\tau}\int d^4x\;\mathrm{tr}_f\langle x|e^{-\tau(O+N)}|x\rangle \qquad (20)\\
&D \equiv \partial\!\!\!/ - i v\!\!\!/_\Omega - i a\!\!\!/_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5. \qquad (21)
\end{aligned}
$$

Where a cutoff Λ is introduced into the theory to regularize the possible ultraviolet divergences. O is the old operator without tensor sources in [17], and N is the new operator with tensor sources

$$
\begin{aligned}
O \;=\;& [D^\dagger + \Sigma(-\bar{\nabla}^2)][D + \Sigma(-\bar{\nabla}^2)], \qquad (22)\\
N \;=\;& \overline{\nabla}_x^\lambda \bar{t}_\Omega^{\prime\mu\nu}\gamma^\lambda\sigma^{\mu\nu} - \bar{t}_\Omega^{\prime\dagger\mu\nu}\overline{\nabla}_x^\lambda\sigma^{\mu\nu}\gamma^\lambda + i a\!\!\!/_\Omega\bar{t}_\Omega^{\prime\mu\nu}\sigma^{\mu\nu}\gamma_5 + i\bar{t}_\Omega^{\prime\dagger\mu\nu}\sigma^{\mu\nu} a\!\!\!/_\Omega\gamma_5 + s_\Omega\bar{t}_\Omega^{\prime\mu\nu}\sigma^{\mu\nu} + \bar{t}_\Omega^{\prime\dagger\mu\nu}s_\Omega\sigma^{\mu\nu}\\
&+ip_\Omega\bar{t}_\Omega^{\prime\mu\nu}\sigma^{\mu\nu}\gamma_5 - i\bar{t}_\Omega^{\prime\dagger\mu\nu}p_\Omega\sigma^{\mu\nu}\gamma_5 - \Sigma(-\overline{\nabla}_x^2)\bar{t}_\Omega^{\prime\mu\nu}\sigma^{\mu\nu} - \bar{t}_\Omega^{\prime\dagger\mu\nu}\Sigma(-\overline{\nabla}_x^2)\sigma^{\mu\nu} + \bar{t}_\Omega^{\prime\dagger\mu\nu}\bar{t}_\Omega^{\prime\lambda\rho}\sigma_{\mu\nu}\sigma_{\lambda\rho}. \qquad (23)
\end{aligned}
$$

If we calculate Eq.(20) directly, it is not explicitly chiral covariant for each terms in the calculation. In order to recover the chiral covariant form to get the LECs, we must collect the relevant terms together by hand, which consumes too much time. Fortunately, we had found a method keeping the chiral covariance at each step of the low-energy expansion computation[21], and used it to obtain the normal part LECs without tensor sources succesfully[17]. We introduce it here briefly. Use the relations

$$
\begin{aligned}
k^\mu + i\bar{\nabla}_x^\mu \;=\;& e^{i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}}\Big(k^\mu + \tilde{F}^\mu(\bar{\nabla},\frac{\partial}{\partial k})\Big)e^{-i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}}\;, \qquad (24)\\
\tilde{F}^\mu \;=\;& \frac{1}{2}[\overline{\nabla}_x^\nu,\overline{\nabla}_x^\mu]\partial_k^\nu - \frac{i}{3}[\overline{\nabla}_x^\lambda,[\overline{\nabla}_x^\nu,\overline{\nabla}_x^\mu]]\partial_k^\lambda\partial_k^\nu - \frac{1}{8}[\overline{\nabla}_x^\rho,[\overline{\nabla}_x^\lambda,[\overline{\nabla}_x^\nu,\overline{\nabla}_x^\mu]]]\partial_k^\rho\partial_k^\lambda\partial_k^\nu\\
&+\frac{i}{30}[\overline{\nabla}_x^\sigma,[\overline{\nabla}_x^\rho,[\overline{\nabla}_x^\lambda,[\overline{\nabla}_x^\nu,\overline{\nabla}_x^\mu]]]]\partial_k^\sigma\partial_k^\rho\partial_k^\lambda\partial_k^\nu\\
&+\frac{1}{144}[\overline{\nabla}_x^\delta,[\overline{\nabla}_x^\sigma,[\overline{\nabla}_x^\rho,[\overline{\nabla}_x^\lambda,[\overline{\nabla}_x^\nu,\overline{\nabla}_x^\mu]]]]]\partial_k^\delta\partial_k^\sigma\partial_k^\rho\partial_k^\lambda\partial_k^\nu + O(p^7). \qquad (25)
\end{aligned}
$$

Substitute (24) into the integrand in (20), we change it to

$$
\begin{aligned}
&\mathrm{tr}_f\langle x|e^{-\tau(O(i\overline{\nabla}_x)+N(i\overline{\nabla}_x))}|x\rangle\\
&=\mathrm{tr}_f\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4k'}{(2\pi)^4}e^{ik'\cdot x}\langle k'|e^{-\tau(O(i\overline{\nabla}_x)+N(i\overline{\nabla}_x))}|k\rangle e^{-ik\cdot x}\\
&=\mathrm{tr}_f\int\frac{d^4k}{(2\pi)^4}e^{-\tau(O(k+i\overline{\nabla}_x)+N(k+i\overline{\nabla}_x))}\\
&=\mathrm{tr}_f\int\frac{d^4k}{(2\pi)^4}e^{-\tau(e^{i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}}\tilde{O}(k+i\overline{\nabla\!\!\!/})e^{-i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}}+e^{i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}}\tilde{N}(k+i\overline{\nabla\!\!\!/})e^{-i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}})}\\
&=\mathrm{tr}_f\int\frac{d^4k}{(2\pi)^4}e^{i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}}e^{-\tau(\tilde{O}(k+i\overline{\nabla\!\!\!/})+\tilde{N}(k+i\overline{\nabla\!\!\!/}))}e^{-i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}}\\
&=\mathrm{tr}_f\int\frac{d^4k}{(2\pi)^4}e^{-\tau(\tilde{O}(k+i\overline{\nabla\!\!\!/})+\tilde{N}(k+i\overline{\nabla\!\!\!/}))}. \qquad (26)
\end{aligned}
$$

Where $\tilde{O}$ is the original exponent without tensor sources, which can be found in Eq.(14) and (15) in Ref.[17], and $\tilde{N}$ is the new operator with tensor sources

$$\begin{aligned}\tilde{N} &= -i(k^\lambda+\tilde{F}^\lambda)\tilde{\bar{t}}^{\mu\nu}_\Omega\gamma^\lambda\sigma^{\mu\nu}+i\tilde{\bar{t}}^{\dagger\mu\nu}_\Omega(k^\lambda+\tilde{F}^\lambda)\sigma^{\mu\nu}\gamma^\lambda+i\tilde{a}^\lambda_\Omega\tilde{\bar{t}}^{\mu\nu}_\Omega\gamma^\lambda\sigma^{\mu\nu}\gamma_5+i\tilde{\bar{t}}^{\dagger\mu\nu}_\Omega\tilde{a}^\lambda_\Omega\sigma^{\mu\nu}\gamma^\lambda\gamma_5+\tilde{s}_\Omega\tilde{\bar{t}}^{\mu\nu}_\Omega\sigma^{\mu\nu}+\tilde{\bar{t}}^{\dagger\mu\nu}_\Omega\tilde{s}_\Omega\sigma^{\mu\nu}\\&\quad+i\tilde{p}_\Omega\tilde{\bar{t}}^{\mu\nu}_\Omega\sigma^{\mu\nu}\gamma_5-i\tilde{\bar{t}}^{\dagger\mu\nu}_\Omega\tilde{p}_\Omega\sigma^{\mu\nu}\gamma_5-\Sigma([k^\mu+\tilde{F}^\mu]^2)\tilde{\bar{t}}^{\mu\nu}_\Omega\sigma^{\mu\nu}-\tilde{\bar{t}}^{\dagger\mu\nu}_\Omega\Sigma([k^\mu+\tilde{F}^\mu]^2)\sigma^{\mu\nu}+\tilde{\bar{t}}^{\dagger\mu\nu}_\Omega\tilde{\bar{t}}^{\lambda\rho}_\Omega\sigma_{\mu\nu}\sigma_{\lambda\rho}, \qquad (27)\\ \tilde{\mathcal{O}} &= \mathcal{O}-i[\overline{\nabla}^\nu_x,\mathcal{O}]\partial^\nu_k-\frac{1}{2}[\overline{\nabla}^\lambda_x,[\overline{\nabla}^\nu_x,\mathcal{O}]]\partial^\nu_k\partial^\lambda_k+\frac{i}{6}[\overline{\nabla}^\rho_x,[\overline{\nabla}^\lambda_x,[\overline{\nabla}^\nu_x,\mathcal{O}]]]\partial^\nu_k\partial^\lambda_k\partial^\rho_k\\&\quad+\frac{1}{24}[\overline{\nabla}^\sigma_x,[\overline{\nabla}^\rho_x,[\overline{\nabla}^\lambda_x,[\overline{\nabla}^\nu_x,\mathcal{O}]]]]\partial^\sigma_k\partial^\nu_k\partial^\lambda_k\partial^\rho_k-\frac{i}{120}[\overline{\nabla}^\delta_x,[\overline{\nabla}^\sigma_x,[\overline{\nabla}^\rho_x,[\overline{\nabla}^\lambda_x,[\overline{\nabla}^\nu_x,\mathcal{O}]]]]]\partial^\delta_k\partial^\sigma_k\partial^\nu_k\partial^\lambda_k\partial^\rho_k+O(p^7). \qquad (28)\end{aligned}$$

Where $\tilde{\mathcal{O}}\equiv(\tilde{a}^\mu,\tilde{s},\tilde{p},\tilde{\bar{t}}^{\alpha\beta})^T$ and $\mathcal{O}\equiv(a^\mu_\Omega,s_\Omega,p_\Omega,\bar{t}'^{\alpha\beta}_\Omega)^T$. With Eq.(20) and (26), we get

$$\mathrm{ReTr}\ln[\partial\!\!\!/-i\not{v}_\Omega-i\not{a}_\Omega\gamma_5-s_\Omega+ip_\Omega\gamma_5-\sigma_{\mu\nu}\bar{t}^{\mu\nu}_\Omega+\Sigma(-\bar{\nabla}^2)]=-\frac{1}{2}\lim_{\Lambda\to\infty}\int_{\frac{1}{\Lambda^2}}^\infty\frac{d\tau}{\tau}\int d^4x\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\ e^{B+B_t}\cdot 1\,. \qquad (29)$$

B can be found in Eq.(17) of Ref.[17], and $B_t=-\tau\tilde{N}$. Expending Eq.(29) in terms of momentum powers, theoretically, we can get all order of the chiral Lagrangian. Before giving our result, we need to discuss the difference between our paper and Ref.[10].

III. CONVENTION DIFFERENCES AND INDEPEDENT OPERATORS

To match our original results and for convenience of our computation, we make the following changes in this paper.

- To match Ref.[5] and our original results in Ref.[17], we define

$$\chi_{\pm,\mu}=\nabla_\mu\chi_\pm-\frac{i}{2}\{\chi_\mp,u_\mu\}, \qquad (30)$$

comparing with

$$\chi_{\pm,\mu}=\nabla_\mu\chi_\pm, \qquad (31)$$

in Ref.[10]. Where χ_+,χ_- and u_μ are analogous to s_Ω,p_Ω and a_Ω in our symbol (see details in Appendix A).

- To match the coefficients' dimensions in a given order, i.e., all the coefficients in a given order have the same dimensions, we change $t^{\mu\nu}_\pm$ in Table 2 in [10] to $B_0t^{\mu\nu}_\pm$, analog of $\tau^{\mu\nu}$ defined in [10]. But [10] times b_0, a parameter is the analog of B_0 for tensor fields. Because b_0 is hard to be calculated, we use B_0 instead of b_0. Now, all the coefficients in the p^4 order are dimensionless, and in the p^6 order, their dimensions are GeV^{-2}.

- Ref.[10] does not consider the epsilon relations

$$\begin{aligned}\epsilon^{\lambda\rho\delta\mu}\epsilon_{\lambda\rho\delta}{}^\nu &= -6g^{\mu\nu}, \qquad (32)\\ \epsilon^{\sigma\delta\mu\nu}\epsilon_{\sigma\delta}{}^{\lambda\rho} &= -2g^{\mu\lambda}g^{\nu\rho}+2g^{\mu\rho}g^{\nu\lambda}, \qquad (33)\\ \epsilon^{\alpha\mu\nu\lambda}\epsilon_\alpha{}^{\rho\sigma\delta} &= -g^{\mu\rho}g^{\nu\sigma}g^{\lambda\delta}+g^{\mu\rho}g^{\nu\delta}g^{\lambda\sigma}-g^{\mu\sigma}g^{\nu\delta}g^{\lambda\rho}\\&\quad+g^{\mu\sigma}g^{\nu\rho}g^{\lambda\delta}-g^{\mu\delta}g^{\nu\rho}g^{\lambda\sigma}+g^{\mu\delta}g^{\nu\sigma}g^{\lambda\rho}. \qquad (34)\end{aligned}$$

Combining with Eq.(5.3) in [10],

$$\epsilon_{\mu\nu\alpha\beta}t^{\alpha\beta}_\pm=2it_{\mp,\mu\nu}, \qquad (35)$$

one can reduce $t_-t_-\to t_+t_+$ and $t_-t_+\to t_+t_-$, i.e., changing even t_- to corresponding t_+, and exchanging the order of t_+ and t_-. For example

$$t^{\mu\nu}_-t_{-,\mu\lambda}=\frac{1}{2}g^\nu{}_\lambda t^{\mu\rho}_+t_{+,\mu\rho}-t^\mu_{+\lambda}t_{+,\mu}{}^\nu. \qquad (36)$$

So even t_- terms and some $\langle \cdots t_- \cdots t_+ \cdots \rangle$ [1] terms are not independent ones. We substitute the epsilon relations in $Y_i, i = 23-30, 53, 56, 81, 83, 89, 91, 93, 104, 105, 109-111$, find that the most of these terms can lead new relations, except Y_{105}. From Y_{105}, we get

$$Y_{32} = \frac{1}{n_f} Y_{35} + \frac{8}{3} Y_{119}. \tag{37}$$

We list all the new relations in Appendix B. All the terms in the l.h.s of (B1) are considered being dependent, and being reduced. We find that totally, there are 20 additional dependent operators in n flavors case, 19 in 3 flavors and 11 in 2 flavors cases, leave 100 independent operators for n flavors, 94 for 3 flavors, and 67 for 2 flavors. In Section IV of Ref.[17], we have found a result that without quark self-energy, all the coefficients, except contact terms, must vanish. Now in the present work, if we similarly ignore the quark self-energy, without relation (32)-(35), we can not obtain these zero results. Instead, with relation (32)-(35), we do reproduce the vanishing result. This shows the importance of relation (32)-(35) in the computation.

- Also, with (32)-(35), adding (38)

$$\epsilon^{\mu\nu\lambda\rho}\epsilon^{\mu'\nu'\lambda'\rho'} = -\det(g^{\alpha\alpha'}), \qquad \alpha = \mu, \nu, \lambda, \rho \qquad \alpha' = \mu', \nu', \lambda', \rho', \tag{38}$$

one can remove all the epsilon in chiral Lagrangian as following. Firstly, even epsilon can be changed to $g_{\mu\nu}$, and odd epsilon can be reduce to one. Secondly, in one epsilon terms, one can change t_+ to t_- or t_- to t_+ with the help of (35), leave only two epsilon. Finally, using (32)-(34) and (38), all the epsilon can be removed. In other words, there does not exist the odd-intrinsic-parity part with tensor sources.

IV. THE p^4 ORDER CHIRAL LAGRANGIAN WITH TENSOR SOURCES

Using the same method as Ref.[17], we can expand the exponent in Eq.(29) to the order p^4. Ref.[10] had given us the p^4 order Lagrangian of form

$$\mathcal{L}_{4,t} = \Lambda_1 \langle t_+^{\mu\nu} f_{+\mu\nu} \rangle - i\Lambda_2 \langle t_+^{\mu\nu} u_\mu u_\nu \rangle + \Lambda_3 \langle t_+^{\mu\nu} t^+_{\mu\nu} \rangle + \Lambda_4 \langle t_+^{\mu\nu} \rangle^2 \equiv \sum_{n=1}^{4} \Lambda_n X_n. \tag{39}$$

Considering that our computation is done under large N_c limit, if we only expand Eq.(29), but do not consider the equations of motion,

$$\nabla_\mu u^\mu = \frac{i}{2}\left(\chi_- - \frac{\langle \chi_- \rangle}{N_f}\right), \tag{40}$$

terms in the chiral Lagrangian with two and more traces vanish. To avoid unnecessary complicities, in this paper we only write down those terms with one trace in the calculation

$$\mathcal{L}_{4,t} = \lambda_1 \mathrm{tr}_f[t_{+,\Omega,\mu\nu} t^{\mu\nu}_{+,\Omega}] + i\lambda_2 \mathrm{tr}_f[a_{\Omega,\mu} a_{\Omega,\nu} t^{\mu\nu}_{+,\Omega}] + \lambda_3 \mathrm{tr}_f[V_{\Omega,\mu\nu} t^{\mu\nu}_{+,\Omega}] + O\left(\frac{1}{N}\right) \equiv \sum_{n=1}^{3} \lambda_n x_n + O\left(\frac{1}{N}\right). \tag{41}$$

Expanding Eq.(29), We get the analytic results as

$$\lambda_1 = N_C \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{d\tau}{\tau} \int \frac{d^4k}{(2\pi)^4} e^{-\tau(k^2+\Sigma_k^2)} (-2\tau^2 \Sigma_k^2), \tag{42}$$

$$\lambda_2 = N_C \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{d\tau}{\tau} \int \frac{d^4k}{(2\pi)^4} e^{-\tau(k^2+\Sigma_k^2)} (-12\tau^2 \Sigma_k + 4\tau^3 k^2 \Sigma_k + 8\tau^3 \Sigma_k^3), \tag{43}$$

$$\lambda_3 = N_C \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{d\tau}{\tau} \int \frac{d^4k}{(2\pi)^4} e^{-\tau(k^2+\Sigma_k^2)} (2\tau^2 \Sigma_k - 2\tau^2 k^2 \Sigma_k'), \tag{44}$$

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[1] To avoid the confusion of our symbol with that used in Ref.[10], for more convenience both in the calculation and in the result, we use symbols both $\mathrm{tr}_f[\cdots]$ in the calculation and $\langle \cdots \rangle$ in the result to represent the operation of taking the flavor trace.

and the relation betweens λ_n and Λ_n are

$$\begin{aligned}
\Lambda_1 &= \frac{1}{2B_0}\lambda_3, && (45)\\
\Lambda_2 &= -\frac{1}{4B_0}\lambda_2 - \frac{1}{2B_0}\lambda_3, && (46)\\
\Lambda_3 &= \frac{1}{B_0^2}\lambda_1, && (47)\\
\Lambda_4 &= 0. && (48)
\end{aligned}$$

In Table I, we list our p^4 order LECs with tensor sources for cutoff $\Lambda = 1000^{+100}_{-100}$MeV. The 10% variation of the cutoff is considered in our calculation to examine the effects of cutoff dependence and the result change can be treat as the error of our calculations. The results are taken the values at $\Lambda = 1$GeV. The superscript is the difference caused at $\Lambda = 1.1$GeV and the subscript is the difference caused at $\Lambda = 0.9$GeV, i.e.[2],

$$\Lambda_{n,\Lambda=1\mathrm{GeV}}\Big|^{\Lambda_{n,\Lambda=1.1\mathrm{GeV}}-\Lambda_{n,\Lambda=1\mathrm{GeV}}}_{\Lambda_{n,\Lambda=0.9\mathrm{GeV}}-\Lambda_{n,\Lambda=1\mathrm{GeV}}}. \tag{49}$$

TABLE I: The obtained values of the p^4 order coefficients. The definition has some difference from [10]. The details can be found in Section III.

n	$10^3\Lambda_n(n_f=3)$	$10^3\Lambda_n(n_f=2)$
1	$12.89^{-0.76}_{+0.89}$	$13.03^{-0.75}_{+0.88}$
2	$-11.59^{+1.06}_{-1.49}$	$-11.75^{+1.05}_{-1.47}$
3	$-4.05^{+0.64}_{-0.97}$	$-4.18^{+0.65}_{-0.97}$

(50)

To compare with our original results, the parameters we use to get Table I are the same as Ref.[17]. We choose the running coupling constant from Ref.[22] to solve Eq.(16), and get the quark self-energy the same as FIG. 2 in Ref.[15], but adding two-flavor case. Except the quark self-energy, we need another input parameter F_0, the p^2 order coefficient in chiral Lagrangian. We choose $F_0 = 87$MeV to get F_π about 93MeV [17].

V. THE p^6 ORDER CHIRAL LAGRANGIAN WITH TENSOR SOURCES

Continuing our process in Section IV, we can obtain the p^6 order results directly. Before listing our results, we first introduce the existing results. Ref.[10] had given us the p^6 order Lagrangian as follows,

$$S_{\mathrm{eff}}\big|_{p^6,\text{tensor sources}} = \int d^4x \begin{cases} \sum\limits_{n=1}^{117} K_n^T Y_n + 3 \text{ contact terms} & n \text{ flavors} \\ \sum\limits_{n=1}^{110} C_n^T O_n + 3 \text{ contact terms} & \text{three flavors} \\ \sum\limits_{n=1}^{75} c_n^T P_n + 3 \text{ contact terms} & \text{two flavors} \end{cases}. \tag{51}$$

Here, we use the symbol Y_n, O_n, P_n to denote n, three and two flavors' independent monomials, which can be found in Table 2 in Ref.[10], and K_n^T, C_n^T, c_n^T for their coefficients. As the reasons in Section III and Appendix B, some of them are not independent. But we use the same numbers. If one monomial is not independent, we just neglect it.

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[2] Notice that Λ with subscript n, Λ_n, means the p^4 order coefficients in Eq.(39), but Λ without subscript is the cutoff in our calculation introduced in Eq.(20)

In our calculation, expanding Eq.(20) as the p^4 order, we only get one trace terms without the equation of motion

$$S_{\mathrm{eff}}|_{p^6,\mathrm{tensorsources}} = \int d^4x \bigg[\sum_{n=1}^{77} \mathcal{Z}_n^T \mathrm{tr}_f[\bar{O}_n] + O\bigg(\frac{1}{N_c}\bigg)\bigg]. \tag{52}$$

$\bar{O}_n$ are the p^6 order operators we could get in our calculation, and $\mathcal{Z}_n^T$ are the corresponding coefficients. For those operators with more than one derivatives, for example $\bar{O}_{66} = d_\mu V^{\mu}_{\Omega\;\nu} d_\lambda t^{\nu\lambda}_{+,\Omega}$, the derivatives are arranged in such a way that each of $V_{\Omega,\mu}{}^{\nu}$ and $t_{+,\Omega,\nu\lambda}$ has a derivative and we do not put two derivatives in one operator. We list all operators in Table II. With the help of computer, we can get the coefficients $\mathcal{Z}_n^T$, listing in Appendix C. Making the

TABLE II: The p^6 order operators $\bar{O}_n$

n	$\bar{O}_n$	n	$\bar{O}_n$	n	$\bar{O}_n$
1	$i\{a_{\Omega,\mu}a^{\mu}_{\Omega}, a_{\Omega,\nu}a_{\Omega,\lambda}\}t^{\nu\lambda}_{+,\Omega}$	27	$ia_{\Omega,\mu}[d_\nu a_{\Omega,\lambda}, d^\mu t^{\nu\lambda}_{+,\Omega}]$	53	$iV_{\Omega,\mu\nu}\{d_\lambda a^{\mu}_{\Omega}, t^{\nu\lambda}_{\Omega,-}\}$
2	$ia_{\Omega,\mu}a_{\Omega,\nu}(a^{\mu}_{\Omega}a_{\Omega,\lambda}t^{\nu\lambda}_{+,\Omega} + a_{\Omega,\lambda}a^{\nu}_{\Omega}t^{\mu\lambda}_{+,\Omega})$	28	$ia_{\Omega,\mu[}d_\nu a_{\Omega,\lambda}, d^\nu t^{\mu\lambda}_{\Omega,+}]$	54	$iV_{\Omega,\mu\nu}\{d_\lambda a^{\lambda}_{\Omega}, t^{\mu\nu}_{-,\Omega}\}$
3	$ia_{\Omega,\mu}a_{\Omega,\nu}a^{\nu}_{\Omega}a_{\Omega,\lambda}t^{\mu\lambda}_{+,\Omega}$	29	$ia_{\Omega,\mu}[d_\nu a_{\Omega,\lambda}, d^\lambda t^{\mu\nu}_{+,\Omega}]$	55	$s_\Omega[d_\mu a_{\Omega,\nu}, t^{\mu\nu}_{-,\Omega}]$
4	$ia_{\Omega,\mu}a_{\Omega,\nu}a_{\Omega,\lambda}a^{\mu}_{\Omega}t^{\nu\lambda}_{+,\Omega}$	30	$ia_{\Omega,\mu}\{d^\nu t_{+,\Omega,\nu\lambda}, t^{\mu\lambda}_{-,\Omega}\}$	56	$ip_\Omega[t^{\mu\nu}_{+,\Omega}, d_\mu a_{\Omega,\nu}]$
5	$a_{\Omega,\mu}a^{\mu}_{\Omega}[d_\nu a_{\Omega,\lambda}, t^{\nu\lambda}_{-,\Omega}]$	31	$ia_{\Omega,\mu}\{d_\nu t^{\mu}_{+,\Omega_{\lambda}}, t^{\nu\lambda}_{-,\Omega}\}$	57	$iV_{\Omega,\mu\nu}V^{\mu}_{\Omega\;\lambda}t^{\nu\lambda}_{+,\Omega}$
6	$a_{\Omega,\mu}a_{\Omega,\nu}(d^\mu a_{\Omega,\lambda}t^{\nu\lambda}_{-,\Omega} - t^{\mu}_{-,\Omega_{\lambda}}d^\nu a^{\lambda}_{\Omega})$	32	$ia_{\Omega,\mu}\{d^\mu t_{+,\Omega,\nu\lambda}, t^{\nu\lambda}_{-,\Omega}\}$	58	$iV_{\Omega,\mu\nu}t^{\mu}_{+,\Omega_{\lambda}}t^{\nu\lambda}_{+,\Omega}$
7	$a_{\Omega,\mu}a_{\Omega,\nu}(d^\nu a_{\Omega,\lambda}t^{\mu\lambda}_{-,\Omega} - t^{\nu}_{-,\Omega_{\lambda}}d^\mu a^{\lambda}_{\Omega})$	33	$ia_{\Omega,\mu}a_{\Omega,\nu}\{t^{\mu\nu}_{+,\Omega}, s_\Omega\}$	59	$iV_{\Omega,\mu\nu}t^{\mu}_{-,\Omega_{\lambda}}t^{\nu\lambda}_{-,\Omega}$
8	$a_{\Omega,\mu}a_{\Omega,\nu}(d_\lambda a^{\mu}_{\Omega}t^{\nu\lambda}_{-,\Omega} - t^{\mu}_{-,\Omega_{\lambda}}d^\lambda a^{\nu}_{\Omega})$	34	$ia_{\Omega,\mu}s_\Omega a_{\Omega,\nu}t^{\mu\nu}_{+,\Omega}$	60	$iV_{\Omega,\mu\nu}\{p_\Omega, t^{\mu\nu}_{-,\Omega}\}$
9	$a_{\Omega,\mu}a_{\Omega,\nu}(d_\lambda a^{\nu}_{\Omega}t^{\mu\lambda}_{-,\Omega} - t^{\nu}_{-\Omega_{\lambda}}d^\lambda a^{\mu}_{\Omega})$	35	$id_\mu a^{\mu}_{\Omega}[d_\mu a_{\Omega,\lambda}, t^{\nu\lambda}_{+,\Omega}]$	61	$V_{\Omega,\mu\nu}\{t^{\mu\nu}_{+,\Omega}, s_\Omega\}$
10	$a_{\Omega,\mu}a_{\Omega,\nu}\{d_\lambda a^{\lambda}_{\Omega}, t^{\mu\nu}_{-,\Omega}\}$	36	$id_\mu a_{\Omega,\nu}d^\mu a_{\Omega,\lambda}t^{\nu\lambda}_{+,\Omega}$	62	$it_{\Omega,\mu\nu}t^{\mu}_{+,\Omega_{\lambda}}t^{\nu\lambda}_{+,\Omega}$
11	$a_{\Omega,\mu}(d^\mu a_{\Omega,\nu}a_{\Omega,\lambda}t^{\nu\lambda}_{-,\Omega} + d_\nu a_{\Omega,\lambda}a^{\nu}_{\Omega}t^{\mu\lambda}_{-,\Omega})$	37	$id_\mu a_{\Omega,\nu}[d^\nu a_{\Omega,\lambda}, t^{\mu\lambda}_{+,\Omega}]$	63	$it_{\Omega,\mu\nu}t^{\mu}_{-,\Omega_{\lambda}}t^{\nu\lambda}_{-,\Omega}$
12	$a_{\Omega,\mu}(d_\nu a^{\mu}_{\Omega}a_{\Omega,\lambda}t^{\nu\lambda}_{-,\Omega} + d_\nu a_{\Omega,\lambda}a^{\lambda}_{\Omega}t^{\mu\nu}_{-,\Omega})$	38	$id_\mu a_{\Omega,\nu}d_\lambda a^{\nu}_{\Omega}t^{\mu\lambda}_{+,\Omega}$	64	$s_\Omega t_{+,\Omega,\mu\nu}t^{\mu\nu}_{+,\Omega}$
13	$a_{\Omega,\mu}d_\nu a^{\nu}_{\Omega}a_{\Omega,\lambda}t^{\mu\lambda}_{-,\Omega}$	39	$V_{\Omega,\mu\nu}(a^{\mu}_{\Omega}a_{\Omega,\lambda}t^{\nu\lambda}_{+,\Omega} - t^{\mu}_{+,\Omega_{\lambda}}a^{\lambda}_{\Omega}a^{\nu}_{\Omega})$	65	$s_\Omega t_{-,\Omega,\mu\nu}t^{\mu\nu}_{-,\Omega}$
14	$a_{\Omega,\mu}a^{\mu}_{\Omega}t_{+,\Omega,\nu\lambda}t^{\nu\lambda}_{+,\Omega}$	40	$V_{\Omega,\mu\nu}(a_{\Omega,\lambda}a^{\mu}_{\Omega}t^{\nu\lambda}_{+,\Omega} - t^{\mu}_{+,\Omega_{\lambda}}a^{\nu}_{\Omega}a^{\lambda}_{\Omega})$	66	$d_\mu V^{\mu}_{\Omega\;\nu}d_\lambda t^{\nu\lambda}_{+,\Omega}$
15	$a_{\Omega,\mu}a_{\Omega,\nu}t^{\mu}_{+,\Omega_{\lambda}}t^{\nu\lambda}_{+,\Omega}$	41	$V_{\Omega,\mu\nu}\{a_{\Omega,\lambda}a^{\lambda}_{\Omega}, t^{\mu\nu}_{+,\Omega}\}$	67	$d_\mu V_{\Omega,\nu\lambda}d^\mu t^{\nu\lambda}_{+,\Omega}$
16	$a_{\Omega,\mu}a_{\Omega,\nu}t^{\nu}_{+,\Omega_{\lambda}}t^{\mu\lambda}_{+,\Omega}$	42	$V_{\Omega,\mu\nu}(a^{\mu}_{\Omega}t^{\nu}_{+,\Omega_{\lambda}}a^{\lambda}_{\Omega} - a_{\Omega,\lambda}t^{\mu\lambda}_{+,\Omega}a^{\nu}_{\Omega})$	68	$d_\mu V_{\Omega,\nu\lambda}d^\nu t^{\nu\mu\lambda}_{+,\Omega}$
17	$a_{\Omega,\mu}a^{\mu}_{\Omega}t_{-,\Omega,\nu\lambda}t^{\nu\lambda}_{-,\Omega}$	43	$V_{\Omega,\mu\nu}a_{\Omega,\lambda}t^{\mu\nu}_{+,\Omega}a^{\lambda}_{\Omega}$	69	$d_\mu t^{\mu}_{+,\Omega_{\nu}}d_\lambda t^{\nu\lambda}_{+,\Omega}$
18	$a_{\Omega,\mu}a_{\Omega,\nu}t^{\mu}_{-,\Omega_{\lambda}}t^{\nu\lambda}_{-,\Omega}$	44	$ia_{\Omega,\mu}[t^{\mu}_{+,\Omega_{\nu}}, d^\nu p_\Omega]$	70	$d_\mu t_{+,\Omega,\nu\lambda}d^\mu t^{\nu\lambda}_{+,\Omega}$
19	$a_{\Omega,\mu}a_{\Omega,\nu}t^{\nu}_{-,\Omega_{\lambda}}t^{\mu\lambda}_{-,\Omega}$	45	$a_{\Omega,\mu}[t^{\mu}_{-,\Omega_{\nu}}, d^\nu s_\Omega]$	71	$d_\mu t_{+,\Omega,\nu\lambda}d^\nu t^{\mu\lambda}_{+,\Omega}$
20	$a_{\Omega,\mu}(t^{\mu}_{+,\Omega_{\nu}}a_{\Omega,\lambda}t^{\nu\lambda}_{+,\Omega} - t_{\Omega,\nu\lambda}a^{\nu}_{\Omega}t^{\mu\lambda}_{+,\Omega})$	46	$id_\mu a^{\mu}_{\Omega}\{t_{-,\Omega,\nu\lambda}, t^{\nu\lambda}_{+,\Omega}\}$	72	$d_\mu t^{\mu}_{-,\Omega_{\nu}}d_\lambda t^{\nu\lambda}_{-,\Omega}$
21	$a_{\Omega,\mu}t_{+,\Omega,\nu\lambda}a^{\mu}_{\Omega}t^{\nu\lambda}_{+,\Omega}$	47	$id_\mu a_{\Omega,\nu}\{t^{\mu}_{-,\Omega_{\lambda}}, t^{\nu\lambda}_{+,\Omega}\}$	73	$d_\mu t_{-,\Omega,\nu\lambda}d^\mu t^{\nu\lambda}_{-,\Omega}$
22	$a_{\Omega,\mu}(t^{\mu}_{-,\Omega_{\nu}}a_{\Omega,\lambda}t^{\nu\lambda}_{-,\Omega} - t_{\Omega,\nu\lambda}a^{\nu}_{\Omega}t^{\mu\lambda}_{-,\Omega})$	48	$id_\mu a_{\Omega,\nu}\{t^{\nu}_{-,\Omega_{\lambda}}, t^{\mu\lambda}_{+,\Omega}\}$	74	$d_\mu t_{-,\Omega,\nu\lambda}d^\nu t^{\mu\lambda}_{-,\Omega}$
23	$a_{\Omega,\mu}t_{-,\Omega,\nu\lambda}a^{\mu}_{\Omega}t^{\nu\lambda}_{-,\Omega}$	49	$iV_{\Omega,\mu\nu}\{a^{\mu}_{\Omega}, d_\lambda t^{\nu\lambda}_{-,\Omega}\}$	75	$a_{\Omega,\mu}a_{\Omega,\nu}\{t^{\mu\nu}_{-,\Omega}, p_\Omega\}$
24	$ia_{\Omega,\mu}[d^\mu a_{\Omega,\nu}d_\lambda, t^{\nu\lambda}_{+,\Omega}]$	50	$iV_{\Omega,\mu\nu}\{a_{\Omega,\lambda}, d^\mu t^{\nu\lambda}_{-,\Omega}\}$	76	$a_{\Omega,\mu}p_\Omega a_{\Omega,\nu}t^{\mu\nu}_{-,\Omega}$
25	$ia_{\Omega,\mu}[d_\nu a^{\mu}_{\Omega}, d_\lambda t^{\nu\lambda}_{+,\Omega}]$	51	$iV_{\Omega,\mu\nu}\{a_{\Omega,\lambda}, d^\lambda t^{\mu\nu}_{-,\Omega}\}$	77	$ip_\Omega\{t_{+,\Omega\mu\nu}, t^{\mu\nu}_{-,\Omega}\}$
26	$ia_{\Omega,\mu}[d_\nu a^{\nu}_{\Omega}, d_\lambda t^{\mu\lambda}_{+,\Omega}]$	52	$iV_{\Omega,\mu\nu}\{d^\mu a_{\Omega,\lambda}, t^{\nu\lambda}_{-,\Omega}\}$		

use of Table IV, the relation of our coefficients $\mathcal{Z}_n^T$ and K_n^T can be obtained directly, listed in Appendix D. Combining Appendix B, C, D and using the parameters in Section IV, we obtain both two and three flavors coefficients c_n^T and C_n^T, and list them in Table III. As the p^4 order, we write down the values at $\Lambda = 1\mathrm{GeV}$, and use the superscript and subscript denote the difference caused at $\Lambda = 1.1\mathrm{GeV}$ and $\Lambda = 0.9\mathrm{GeV}$.

$$C^T_{n,\Lambda=1\mathrm{GeV}}\Big|^{C^T_{n,\Lambda=1.1\mathrm{GeV}} - C^T_{n,\Lambda=1\mathrm{GeV}}}_{C^T_{n,\Lambda=0.9\mathrm{GeV}} - C^T_{n,\Lambda=1\mathrm{GeV}}} \qquad c^T_{n,\Lambda=1\mathrm{GeV}}\Big|^{c^T_{n,\Lambda=1.1\mathrm{GeV}} - c^T_{n,\Lambda=1\mathrm{GeV}}}_{c^T_{n,\Lambda=0.9\mathrm{GeV}} - c^T_{n,\Lambda=1\mathrm{GeV}}}. \tag{53}$$

Because of Appendix B, some terms are not independent, we denote their coefficients by a symbol "$-$". In Ref.[10], coefficients are multiplied by suitable power of b_0 to make them with the same dimensions. Since we can not find a simple way to obtain b_0. Instead of b_0, we use B_0 to match the dimensions.

TABLE III: The obtained values of the p^6 order LECs. C_n^T denote the three flavors coefficients, and c_n^T denote two flavors. n is the number of independent monomial in [10] with some difference. Some monomials are not independent, we denote their coefficients by a symbol "$-$" The details can be found in Section III.

i	$10^3\mathrm{GeV}^2 C_i^T$	j	$10^3\mathrm{GeV}^2 c_j^T$	i	$10^3\mathrm{GeV}^2 C_i^T$	j	$10^3\mathrm{GeV}^2 c_j^T$	i	$10^3\mathrm{GeV}^2 C_i^T$	j	$10^3\mathrm{GeV}^2 c_j^T$
1	$-0.68^{+0.13}_{-0.24}$			38	$9.04^{-0.54}_{+0.61}$	22	$6.31^{-0.34}_{+0.36}$	75	$0.00^{+0.00}_{+0.00}$	50	$0.00^{+0.00}_{+0.00}$
2	$-12.72^{+0.88}_{-1.05}$	1	$-16.03^{+1.22}_{-1.60}$	39	$-3.23^{+0.50}_{-0.75}$	23	$-8.98^{+0.89}_{-1.21}$	76	$-$		
3	$5.58^{-0.25}_{+0.18}$	2	$0.72^{+0.39}_{-0.89}$	40	$-3.64^{+0.25}_{-0.31}$			77	$-4.04^{+0.29}_{-0.36}$	51	$-3.97^{+0.28}_{-0.35}$
4	$2.89^{-0.34}_{+0.54}$			41	$0.00^{+0.00}_{+0.00}$			78	$-9.24^{+0.61}_{-0.70}$	52	$-9.69^{+0.63}_{-0.73}$
5	$0.00^{+0.00}_{+0.00}$			42	$4.98^{-0.35}_{+0.42}$	24	$5.00^{-0.35}_{+0.41}$	79	$14.54^{-0.88}_{+0.94}$	53	$14.80^{-0.87}_{+0.94}$
6	$0.00^{+0.00}_{+0.00}$			43	$-5.77^{+0.39}_{-0.46}$	25	$-5.82^{+0.39}_{-0.46}$	80	$0.00^{+0.00}_{+0.00}$		
7	$-0.83^{+0.09}_{-0.08}$	3	$-0.85^{+0.09}_{-0.08}$	44	$-11.08^{+0.79}_{-0.96}$	26	$-11.18^{+0.78}_{-0.95}$	81	$-22.86^{+3.54}_{-5.12}$	54	$-23.23^{+3.52}_{-5.09}$
8	$-0.24^{-0.00}_{+0.06}$	4	$-0.22^{-0.00}_{+0.06}$	45	$0.00^{+0.00}_{+0.00}$	27	$0.00^{+0.00}_{+0.00}$	82	$-$	55	$-$
9	$3.56^{-0.68}_{+1.19}$	5	$3.87^{-0.72}_{+1.24}$	46	$-0.01^{-0.16}_{+0.36}$	28	$-0.25^{-0.12}_{+0.31}$	83	$-20.23^{+3.14}_{-4.57}$	56	$-21.13^{+3.21}_{-4.66}$
10	$-14.06^{+2.27}_{-3.45}$	6	$-14.67^{+2.32}_{-3.51}$	47	$-40.61^{+6.31}_{-9.19}$	29	$-41.21^{+6.26}_{-9.10}$	84	$-$	57	$-$
11	$3.28^{-0.45}_{+0.55}$	7	$3.35^{-0.44}_{+0.55}$	48	$-$	30	$-$	85	$0.00^{+0.00}_{+0.00}$	58	$0.00^{+0.00}_{+0.00}$
12	$0.00^{+0.00}_{+0.00}$			49	$0.00^{+0.00}_{+0.00}$	31	$0.00^{+0.00}_{+0.00}$	86	$-$		
13	$0.00^{+0.00}_{+0.00}$			50	$0.00^{+0.00}_{+0.00}$	32	$0.00^{+0.00}_{+0.00}$	87	$38.66^{-2.27}_{+2.39}$	59	$38.48^{-2.20}_{+2.32}$
14	$0.00^{+0.00}_{+0.00}$			51	$-$	33	$-$	88	$1.37^{-0.18}_{+0.29}$	60	$1.51^{-0.19}_{+0.30}$
15	$0.00^{+0.00}_{+0.00}$			52	$3.11^{-0.13}_{+0.07}$	34	$3.06^{-0.12}_{+0.06}$	89	$-2.52^{+0.13}_{-0.13}$	61	$-2.48^{+0.13}_{-0.13}$
16	$0.00^{+0.00}_{+0.00}$			53	$-8.36^{+0.58}_{-0.69}$	35	$-8.35^{+0.56}_{-0.68}$	90	$-8.69^{+0.53}_{-0.59}$	62	$-8.43^{+0.50}_{-0.55}$
17	$0.00^{+0.00}_{+0.00}$	8	$0.00^{+0.00}_{+0.00}$	54	$5.68^{-0.47}_{+0.63}$	36	$5.96^{-0.48}_{+0.65}$	91	$4.99^{-0.31}_{+0.34}$	63	$5.14^{-0.32}_{+0.35}$
18	$0.00^{+0.00}_{+0.00}$	9	$0.00^{+0.00}_{+0.00}$	55	$2.42^{+0.03}_{-0.23}$	37	$2.18^{+0.06}_{-0.25}$	92	$14.91^{-0.93}_{+1.02}$	64	$15.12^{-0.92}_{+1.02}$
19	$-$	10	$-$	56	$4.72^{-0.47}_{+0.71}$	38	$4.92^{-0.48}_{+0.71}$	93	$-16.10^{+1.11}_{-1.30}$	65	$-16.09^{+1.09}_{-1.28}$
20	$-$	11	$-$	57	$0.00^{+0.00}_{+0.00}$			94	$0.00^{+0.00}_{+0.00}$		
21	$-$	12	$-$	58	$0.00^{+0.00}_{+0.00}$			95	$0.00^{+0.00}_{+0.00}$		
22	$-$			59	$0.00^{+0.00}_{+0.00}$			96	$0.00^{+0.00}_{+0.00}$		
23	$-$			60	$0.00^{+0.00}_{+0.00}$			97	$-$	66	$-$
24	$-$			61	$0.00^{+0.00}_{+0.00}$			98	$19.83^{-3.08}_{+4.48}$	67	$20.50^{-3.11}_{+4.52}$
25	$-$	13	$-$	62	$1.50^{-0.07}_{+0.06}$	39	$1.50^{-0.07}_{+0.05}$	99	$0.00^{+0.00}_{+0.00}$	68	$0.00^{+0.00}_{+0.00}$
26	$-4.13^{+0.72}_{-0.97}$	14	$-4.17^{+0.71}_{-0.94}$	63	$17.67^{-1.21}_{+1.44}$	40	$17.89^{-1.20}_{+1.43}$	100	$0.00^{+0.00}_{+0.00}$	69	$0.00^{+0.00}_{+0.00}$
27	$-$	15	$-$	64	$-6.33^{+0.46}_{-0.58}$	41	$-6.46^{+0.46}_{-0.58}$	101	$0.00^{+0.00}_{+0.00}$	70	$0.00^{+0.00}_{+0.00}$
28	$0.00^{+0.00}_{+0.00}$	16	$0.00^{+0.00}_{+0.00}$	65	$-6.74^{+0.48}_{-0.59}$	42	$-6.82^{+0.48}_{-0.59}$	102	$-$		
29	$0.00^{+0.00}_{+0.00}$	17	$0.00^{+0.00}_{+0.00}$	66	$0.00^{+0.00}_{+0.00}$	43	$0.00^{+0.00}_{+0.00}$	103	$-$		
30	$3.04^{-0.63}_{+0.99}$	18	$4.72^{-0.96}_{+1.49}$	67	$-3.83^{+0.59}_{-0.87}$	44	$-3.94^{+0.60}_{-0.87}$	104	$-$		
31	$0.00^{+0.00}_{+0.00}$	19	$0.00^{+0.00}_{+0.00}$	68	$-0.31^{+0.13}_{-0.22}$	45	$-1.28^{+0.19}_{-0.28}$	105	$5.18^{-0.80}_{+1.17}$	71	$5.31^{-0.81}_{+1.17}$
32	$0.00^{+0.00}_{+0.00}$			69	$7.16^{-0.32}_{+0.27}$	46	$7.14^{-0.31}_{+0.26}$	106	$3.50^{+0.79}_{-1.48}$	72	$3.08^{+0.83}_{-1.52}$
33	$0.00^{+0.00}_{+0.00}$			70	$0.00^{+0.00}_{+0.00}$	47	$0.00^{+0.00}_{+0.00}$	107	$-5.81^{+0.29}_{-0.24}$	73	$-5.64^{+0.26}_{-0.22}$
34	$0.73^{+0.58}_{-1.05}$	20	$0.49^{+0.60}_{-1.08}$	71	$0.00^{+0.00}_{+0.00}$	48	$0.92^{-0.06}_{+0.06}$	108	$6.07^{-0.41}_{+0.48}$	74	$6.05^{-0.40}_{+0.47}$
35	$8.89^{-0.02}_{-0.34}$	21	$8.62^{+0.01}_{-0.38}$	72	$0.00^{+0.00}_{+0.00}$			109	$6.57^{-0.44}_{+0.51}$	75	$6.63^{-0.44}_{+0.50}$
36	$0.00^{+0.00}_{+0.00}$			73	$-0.62^{+0.04}_{-0.04}$			110	$0.00^{+0.00}_{+0.00}$		
37	$0.00^{+0.00}_{+0.00}$			74	$-$	49	$-$				

Ref.[10] told us that operators contributing to the odd-intrinsic-parity part with tensor fields start from the p^8 order, and we show in Section III that the the odd-intrinsic-parity parts with tensor fields can not exist. So we have gotten all the LECs to the p^6 order, with scalar, pseudoscalar, vector, axial vector and tensor sources, including the normal and anomalous parts, two and three flavors cases. We found that in our method, all the contact terms' coefficients are divergent, except H_1 in the p^4 order normal part.

The calculation process is too complicated, to avoid some possible mistakes, the expansion in Eq.(29) and most of other calculations are done by computer. To check the correctness of our results, we examine them in some alternative ways. Firstly, because these results contain the original results in [17], if we switch off the tensor sources, as a check, we must recover the original results. Secondly, some terms in Table II have two parts, we calculate them separately. C, P and hermitian invariance constrain the two parts' coefficients equal(or with a minus sign). Our analytical results for the separate part must give the same coefficients. Thirdly, [17] told us that if we switch off the quark self-energy, all the LECs, except the contact terms', must be zeros. This is a strong restriction to our results. We found that this restriction can be realized only when we using the new relations given in Appendix A. With all above examinations, we are confident of the reliability of our numerical results for LECs.

VI. SUMMARY

To summarize our result: we extend our previous computation for LECs in Ref.[15, 17] to the case with tensor sources, and obtain all the p^4 and p^6 order LECs involving the tensor sources in the normal part of the chiral Lagrangian. We find that the operators given in the Ref.[10] are not all independent ones due to some relations among epsilon. Adding these relations, we can reduce 20 operators for n-flavor, 19 for 3-flavor case, and 11 for 2-flavor case, left 100 independent operators for n-flavor, 94 for 3-flavor, and 67 for 2-flavor case. Our LECs are presented with numerical values, both for two and three flavors cases. We also find that the the odd-intrinsic-parity parts chiral Lagrangian with tensor sources can not exist. So, until the p^6 order, we have already given all the LECs values. Although, when obtaining these values, we have made many approximations. As the first step of estimation for the value of these LECs, these results not only offer us the sign and order of magnitude , but also the quantitative information of LECs. Via the improvement in the computation procedure, we expect more precise results can be obtained in the future. Another coming research is applying the present chiral Lagrangian with tensor sources, adding the known LECs to various low energy (π, K, η) processes. We hope more physical results can be obtained.

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Appendix A: relations among our symbols and those used in Ref.[10]

To help in understanding the mutual relation among the definition of symbols in our formulation and those in Ref.[10], we give a comparison in Table IV.

TABLE IV: Comparisons between the symbols introduce in Ref.[10] (first and third columns) and corresponding ones defined in the present paper (second and fourth columns).

Ref. [10]	Present paper	Ref. [10]	Present paper
∇^μ	d^μ	χ_-^μ	$4iB_0 d^\mu p_\Omega - 4iB_0 s_\Omega a_\Omega^\mu - 4iB_0 a_\Omega^\mu s_\Omega$
u	Ω	$f_+^{\mu\nu}$	$2V_\Omega^{\mu\nu} - 2i(a_\Omega^\mu a_\Omega^\nu - a_\Omega^\nu a_\Omega^\mu)$
u^μ	$2a_\Omega^\mu$	$\nabla^\lambda f_+^{\mu\nu}$	$2d^\lambda V_\Omega^{\mu\nu} - 2id^\lambda(a_\Omega^\mu a_\Omega^\nu - a_\Omega^\nu a_\Omega^\mu)$
χ	χ	$f_-^{\mu\nu}$	$-2(d^\mu a_\Omega^\nu - d^\nu a_\Omega^\mu)$
χ_+	$4B_0 s_\Omega$	$\nabla^\lambda f_-^{\mu\nu}$	$-2(d^\lambda d^\mu a_\Omega^\nu - d^\lambda d^\nu a_\Omega^\mu)$
χ_+^μ	$4B_0 d^\mu s_\Omega + 4B_0 p_\Omega a_\Omega^\mu + 4B_0 a_\Omega^\mu p_\Omega$	$h^{\mu\nu}$	$2(d^\mu a_\Omega^\nu + d^\nu a_\Omega^\mu)$
χ_-	$4iB_0 p_\Omega$	Γ^μ	$-iv_\Omega^\mu$
$t_+^{\mu\nu}$	$t_{+,\Omega}^{\mu\nu}$	$t_-^{\mu\nu}$	$t_{-,\Omega}^{\mu\nu}$

Appendix B: new relations

In this appendix, we list the new relations, when using the epsilon relations in Section III. The l.h.s. of (B1) are considered being dependent, and being reduced.

$$
\begin{aligned}
Y_{23} &= \frac{1}{2}Y_9 - Y_{12} \\
Y_{24} &= \frac{1}{2}Y_9 - Y_{11} \\
Y_{25} &= Y_{10} - Y_{13} \\
Y_{26} &= \frac{1}{2}Y_{14} - Y_{15} \\
Y_{27} &= \frac{1}{2}Y_{16} - Y_{18} \\
Y_{28} &= \frac{1}{2}Y_{16} - Y_{17} \\
Y_{29} &= Y_{19} - Y_{20} \\
Y_{30} &= Y_{21} - Y_{22} \\
Y_{32} &= \frac{1}{N_f}Y_{35} + \frac{8}{3}Y_{119} \\
Y_{53} &= -\frac{1}{2}Y_{11} + \frac{1}{2}Y_{12} + \frac{1}{2}Y_{51} - Y_{52} + Y_{90} \\
Y_{56} &= \frac{1}{2}Y_{54} - Y_{55} \\
Y_{81} &= \frac{4}{3}Y_{119} \\
Y_{83} &= \frac{1}{2}Y_{36} - \frac{1}{2n_f}Y_{38} - Y_{82} \\
Y_{89} &= Y_{88} \\
Y_{91} &= Y_{90} \\
Y_{93} &= Y_{92} \\
Y_{104} &= \frac{4}{3}Y_{119} \\
Y_{109} &= \frac{1}{2}Y_{36} - \frac{1}{2N_f}Y_{38} - Y_{106} \\
Y_{110} &= -\frac{1}{2}Y_{82} + \frac{1}{2}Y_{92} + \frac{1}{2}Y_{106} + Y_{107} \\
Y_{111} &= -\frac{1}{4}Y_{36} + \frac{1}{4N_f}Y_{38} + \frac{1}{2}Y_{82} + \frac{1}{2}Y_{92} + \frac{1}{2}Y_{106} + Y_{108}
\end{aligned} \tag{B1}
$$

Appendix C: $\mathcal{Z}_n$ coefficients

$$
\begin{aligned}
\mathcal{Z}_1^T &= \int dK \Big[-10\tau^3\Sigma_k + \frac{10}{3}\tau^4 k^2\Sigma_k + \frac{40}{3}\tau^4\Sigma_k^3 - \frac{2}{9}\tau^5 k^4\Sigma_k - 2\tau^5 k^2\Sigma_k^3 - \frac{8}{3}\tau^5\Sigma_k^5 \Big], \\
\mathcal{Z}_2^T &= \int dK \Big[+10\tau^3\Sigma_k - \frac{8}{3}\tau^4 k^2\Sigma_k - \frac{40}{3}\tau^4\Sigma_k^3 + 2\tau^5 k^2\Sigma_k^3 + \frac{8}{3}\tau^5\Sigma_k^5 \Big], \\
\mathcal{Z}_3^T &= \int dK \Big[-10\tau^3\Sigma_k + 4\tau^4 k^2\Sigma_k + \frac{40}{3}\tau^4\Sigma_k^3 - \frac{2}{9}\tau^5 k^4\Sigma_k - 2\tau^5 k^2\Sigma_k^3 - \frac{8}{3}\tau^5\Sigma_k^5 \Big], \\
\mathcal{Z}_4^T &= \int dK \Big[-10\tau^3\Sigma_k + 2\tau^4 k^2\Sigma_k + \frac{40}{3}\tau^4\Sigma_k^3 + \frac{2}{9}\tau^5 k^4\Sigma_k - 2\tau^5 k^2\Sigma_k^3 - \frac{8}{3}\tau^5\Sigma_k^5 \Big],
\end{aligned}
$$

$$\mathcal{Z}_5^T = \int dK\Big[+4\tau^3\Sigma_k - \frac{2}{3}\tau^4k^2\Sigma_k - \frac{4}{3}\tau^4\Sigma_k^3 - \frac{1}{9}\tau^4k^4\Sigma_k'\Big],$$

$$\mathcal{Z}_6^T = \int dK\Big[-\frac{22}{3}\tau^3\Sigma_k + \frac{1}{3}\tau^3k^2\Sigma_k' + 2\tau^4k^2\Sigma_k + \frac{8}{3}\tau^4\Sigma_k^3\Big],$$

$$\mathcal{Z}_7^T = \int dK\Big[+6\tau^3\Sigma_k + \tau^3k^2\Sigma_k' - \frac{4}{3}\tau^4k^2\Sigma_k - \frac{8}{3}\tau^4\Sigma_k^3 - \frac{2}{9}\tau^4k^4\Sigma_k'\Big],$$

$$\mathcal{Z}_8^T = \int dK\Big[+4\tau^3\Sigma_k - \frac{4}{3}\tau^4k^2\Sigma_k - \frac{4}{3}\tau^4\Sigma_k^3 + \frac{1}{9}\tau^4k^4\Sigma_k'\Big],$$

$$\mathcal{Z}_9^T = \int dK\Big[-4\tau^3\Sigma_k + \frac{4}{3}\tau^4k^2\Sigma_k + \frac{4}{3}\tau^4\Sigma_k^3 - \frac{1}{9}\tau^4k^4\Sigma_k'\Big],$$

$$\mathcal{Z}_{10}^T = \int dK\Big[-6\tau^3\Sigma_k + \tau^3k^2\Sigma_k' + \frac{4}{3}\tau^4k^2\Sigma_k + \frac{8}{3}\tau^4\Sigma_k^3 - \frac{2}{9}\tau^4k^4\Sigma_k'\Big],$$

$$\mathcal{Z}_{11}^T = \int dK\Big[-\frac{20}{3}\tau^3\Sigma_k + \frac{2}{3}\tau^3k^2\Sigma_k' + \frac{4}{3}\tau^4k^2\Sigma_k + \frac{8}{3}\tau^4\Sigma_k^3 - \frac{2}{9}\tau^4k^4\Sigma_k'\Big],$$

$$\mathcal{Z}_{12}^T = \int dK\Big[+\frac{2}{3}\tau^4k^2\Sigma_k - \frac{2}{9}\tau^4k^4\Sigma_k'\Big],$$

$$\mathcal{Z}_{13}^T = \int dK\Big[+4\tau^3\Sigma_k + 2\tau^3k^2\Sigma_k' - \frac{2}{3}\tau^4k^2\Sigma_k - \frac{8}{3}\tau^4\Sigma_k^3 - \frac{4}{9}\tau^4k^4\Sigma_k'\Big],$$

$$\mathcal{Z}_{14}^T = \int dK\Big[+2\tau^2 - \frac{3}{2}\tau^3k^2 - 8\tau^3\Sigma_k^2 + \frac{2}{9}\tau^4k^4 + 2\tau^4k^2\Sigma_k^2 + \frac{8}{3}\tau^4\Sigma_k^4\Big],$$

$$\mathcal{Z}_{15}^T = \int dK\Big[-4\tau^2 + \tau^3k^2 + 16\tau^3\Sigma_k^2 + \frac{2}{9}\tau^4k^4 - 4\tau^4k^2\Sigma_k^2 - \frac{16}{3}\tau^4\Sigma_k^4\Big],$$

$$\mathcal{Z}_{16}^T = \int dK\Big[+4\tau^2 - 3\tau^3k^2 - 16\tau^3\Sigma_k^2 + \frac{2}{9}\tau^4k^4 + 4\tau^4k^2\Sigma_k^2 + \frac{16}{3}\tau^4\Sigma_k^4\Big],$$

$$\mathcal{Z}_{17}^T = \int dK\Big[-2\tau^2 + \frac{3}{2}\tau^3k^2 + 2\tau^3\Sigma_k^2 - \frac{2}{9}\tau^4k^4 - \frac{4}{3}\tau^4k^2\Sigma_k^2\Big],$$

$$\mathcal{Z}_{18}^T = \int dK\Big[+4\tau^2 - \tau^3k^2 - 4\tau^3\Sigma_k^2 - \frac{2}{9}\tau^4k^4 + \frac{4}{3}\tau^4k^2\Sigma_k^2\Big],$$

$$\mathcal{Z}_{19}^T = \int dK\Big[-4\tau^2 + 3\tau^3k^2 + 4\tau^3\Sigma_k^2 - \frac{2}{9}\tau^4k^4 - \frac{4}{3}\tau^4k^2\Sigma_k^2\Big],$$

$$\mathcal{Z}_{20}^T = \int dK\Big[+2\tau^2 - \tau^3k^2 - 8\tau^3\Sigma_k^2 + \frac{1}{9}\tau^4k^4 + 2\tau^4k^2\Sigma_k^2 + \frac{8}{3}\tau^4\Sigma_k^4\Big],$$

$$\mathcal{Z}_{21}^T = \int dK\Big[+\tau^2 - 4\tau^3\Sigma_k^2 - \frac{1}{9}\tau^4k^4 + \tau^4k^2\Sigma_k^2 + \frac{4}{3}\tau^4\Sigma_k^4\Big],$$

$$\mathcal{Z}_{22}^T = \int dK\Big[-2\tau^2 + \tau^3k^2 - \frac{1}{9}\tau^4k^4 + \frac{2}{3}\tau^4k^2\Sigma_k^2\Big],$$

$$\mathcal{Z}_{23}^T = \int dK\Big[-\tau^2 + \frac{1}{9}\tau^4k^4 + \frac{2}{3}\tau^4k^2\Sigma_k^2\Big],$$

$$\mathcal{Z}_{24}^T = \int dK\Big[+\frac{4}{3}\tau^3\Sigma_k - \tau^3k^2\Sigma_k' - \frac{2}{3}\tau^4k^2\Sigma_k + \frac{1}{9}\tau^4k^4\Sigma_k' + \frac{2}{3}\tau^4k^4\Sigma_k\Sigma_k'^2 + \frac{1}{9}\tau^5k^4\Sigma_k - \frac{4}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2\Big],$$

$$\mathcal{Z}_{25}^T = \int dK\Big[-\frac{2}{3}\tau^3\Sigma_k + \frac{2}{3}\tau^3k^2\Sigma_k'\Big],$$

$$\mathcal{Z}_{26}^T = \int dK\Big[-\frac{2}{3}\tau^3\Sigma_k + \frac{2}{3}\tau^4k^2\Sigma_k - \frac{4}{9}\tau^4k^4\Sigma_k\Sigma_k'^2 - \frac{1}{9}\tau^5k^4\Sigma_k + \frac{4}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2\Big],$$

$$\mathcal{Z}_{27}^T = \int dK\Big[+\frac{1}{3}\tau^3k^2\Sigma_k' + \frac{2}{3}\tau^4k^2\Sigma_k - \frac{1}{9}\tau^4k^4\Sigma_k' - \frac{2}{3}\tau^4k^4\Sigma_k\Sigma_k'^2 - \frac{1}{9}\tau^5k^4\Sigma_k + \frac{4}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2\Big],$$

$$\mathcal{Z}_{28}^T = \int dK\Big[-6\tau^3\Sigma_k + 10\tau^3k^2\Sigma_k\Sigma_k'^2 + \frac{8}{3}\tau^4k^2\Sigma_k + \frac{8}{3}\tau^4\Sigma_k^3 - \frac{1}{9}\tau^4k^4\Sigma_k' - \frac{10}{9}\tau^4k^4\Sigma_k\Sigma_k'^2 - \frac{40}{3}\tau^4k^2\Sigma_k^3\Sigma_k'^2$$

$$-\frac{2}{9}\tau^5k^4\Sigma_k - \frac{2}{3}\tau^5k^2\Sigma_k^3 + \frac{8}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2 + \frac{8}{3}\tau^5k^2\Sigma_k^5\Sigma_k'^2\Bigg],$$

$$\mathcal{Z}_{29}^T = \int dK\Bigg[+\frac{2}{3}\tau^3\Sigma_k + \frac{2}{3}\tau^4k^2\Sigma_k - \frac{4}{9}\tau^4k^4\Sigma_k\Sigma_k'^2 - \frac{1}{9}\tau^5k^4\Sigma_k + \frac{4}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2\Bigg],$$

$$\mathcal{Z}_{30}^T = \int dK\Bigg[+4\tau^2 - \frac{4}{3}\tau^3k^2 - 4\tau^3\Sigma_k^2\Bigg],$$

$$\mathcal{Z}_{31}^T = \int dK\Bigg[+\frac{8}{3}\tau^2 - \frac{2}{3}\tau^3k^2 - \frac{4}{3}\tau^3\Sigma_k^2\Bigg],$$

$$\mathcal{Z}_{32}^T = \int dK\Bigg[-3\tau^2 + \frac{7}{6}\tau^3k^2 + 2\tau^3\Sigma_k^2\Bigg],$$

$$\mathcal{Z}_{33}^T = \int dK\Bigg[-4\tau^2 + \tau^3k^2 + 16\tau^3\Sigma_k^2 - \frac{8}{3}\tau^4k^2\Sigma_k^2 - \frac{16}{3}\tau^4\Sigma_k^4\Bigg],$$

$$\mathcal{Z}_{34}^T = \int dK\Bigg[-4\tau^2 + 2\tau^3k^2 + 16\tau^3\Sigma_k^2 - \frac{8}{3}\tau^4k^2\Sigma_k^2 - \frac{16}{3}\tau^4\Sigma_k^4\Bigg],$$

$$\mathcal{Z}_{35}^T = \int dK\Bigg[-\frac{4}{3}\tau^3\Sigma_k + \tau^3k^2\Sigma_k' + \frac{2}{3}\tau^4k^2\Sigma_k - \frac{1}{9}\tau^4k^4\Sigma_k' - \frac{2}{3}\tau^4k^4\Sigma_k\Sigma_k'^2 - \frac{1}{9}\tau^5k^4\Sigma_k + \frac{4}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2\Bigg],$$

$$\mathcal{Z}_{36}^T = \int dK\Bigg[-6\tau^3\Sigma_k + 10\tau^3k^2\Sigma_k\Sigma_k'^2 + \frac{8}{3}\tau^4k^2\Sigma_k + \frac{8}{3}\tau^4\Sigma_k^3 - \frac{2}{9}\tau^4k^4\Sigma_k' - \frac{4}{3}\tau^4k^4\Sigma_k\Sigma_k'^2 - \frac{40}{3}\tau^4k^2\Sigma_k^3\Sigma_k'^2$$
$$-\frac{2}{9}\tau^5k^4\Sigma_k - \frac{2}{3}\tau^5k^2\Sigma_k^3 + \frac{8}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2 + \frac{8}{3}\tau^5k^2\Sigma_k^5\Sigma_k'^2\Bigg],$$

$$\mathcal{Z}_{37}^T = \int dK\Bigg[+\frac{1}{3}\tau^3k^2\Sigma_k' + \frac{2}{3}\tau^4k^2\Sigma_k - \frac{1}{9}\tau^4k^4\Sigma_k' - \frac{2}{3}\tau^4k^4\Sigma_k\Sigma_k'^2 - \frac{1}{9}\tau^5k^4\Sigma_k + \frac{4}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2\Bigg],$$

$$\mathcal{Z}_{38}^T = \int dK\Bigg[+\frac{2}{3}\tau^3\Sigma_k - \frac{2}{3}\tau^3k^2\Sigma_k'\Bigg],$$

$$\mathcal{Z}_{39}^T = \int dK\Bigg[-\frac{20}{3}\tau^3\Sigma_k + \frac{4}{3}\tau^3k^2\Sigma_k' + 2\tau^4k^2\Sigma_k + \frac{8}{3}\tau^4\Sigma_k^3 - \frac{2}{9}\tau^4k^4\Sigma_k'\Bigg],$$

$$\mathcal{Z}_{40}^T = \int dK\Bigg[+6\tau^3\Sigma_k + \frac{2}{3}\tau^3k^2\Sigma_k' - 2\tau^4k^2\Sigma_k - \frac{8}{3}\tau^4\Sigma_k^3 - \frac{2}{9}\tau^4k^4\Sigma_k'\Bigg],$$

$$\mathcal{Z}_{41}^T = \int dK\Bigg[+\frac{10}{3}\tau^3\Sigma_k - \frac{1}{3}\tau^3k^2\Sigma_k' - \tau^4k^2\Sigma_k - \frac{4}{3}\tau^4\Sigma_k^3 + \frac{2}{9}\tau^4k^4\Sigma_k'\Bigg],$$

$$\mathcal{Z}_{42}^T = \int dK\Bigg[+6\tau^3\Sigma_k + \frac{2}{3}\tau^3k^2\Sigma_k' - 2\tau^4k^2\Sigma_k - \frac{8}{3}\tau^4\Sigma_k^3 + \frac{2}{9}\tau^4k^4\Sigma_k'\Bigg],$$

$$\mathcal{Z}_{43}^T = \int dK\Bigg[+\frac{10}{3}\tau^3\Sigma_k + \frac{2}{3}\tau^3k^2\Sigma_k' - \tau^4k^2\Sigma_k - \frac{4}{3}\tau^4\Sigma_k^3 - \frac{2}{9}\tau^4k^4\Sigma_k'\Bigg],$$

$$\mathcal{Z}_{44}^T = \int dK\Bigg[-4\tau^2 + 2\tau^3k^2 + 4\tau^3\Sigma_k^2\Bigg],$$

$$\mathcal{Z}_{45}^T = \int dK\Bigg[+\frac{8}{3}\tau^2 - \frac{4}{3}\tau^3k^2 - \frac{4}{3}\tau^3\Sigma_k^2\Bigg],$$

$$\mathcal{Z}_{46}^T = \int dK\Bigg[-2\tau^2 + \frac{5}{6}\tau^3k^2 + 2\tau^3\Sigma_k^2\Bigg],$$

$$\mathcal{Z}_{47}^T = \int dK\Bigg[-\frac{8}{3}\tau^2 + \frac{5}{3}\tau^3k^2 + \frac{4}{3}\tau^3\Sigma_k^2\Bigg],$$

$$\mathcal{Z}_{48}^T = \int dK\Bigg[+6\tau^2 - \frac{8}{3}\tau^3k^2 - 4\tau^3\Sigma_k^2\Bigg],$$

$$\mathcal{Z}_{49}^T = \int dK\Bigg[+\frac{2}{3}\tau^3\Sigma_k - \frac{1}{9}\tau^4k^4\Sigma_k' - \frac{2}{9}\tau^4k^4\Sigma_k\Sigma_k'^2\Bigg],$$

$$\mathcal{Z}_{50}^T = \int dK\Bigg[+\frac{2}{3}\tau^2\Sigma_k' + \frac{4}{3}\tau^3\Sigma_k - \frac{4}{3}\tau^3\Sigma_k^2\Sigma_k' + 2\tau^3k^2\Sigma_k\Sigma_k'^2 - \frac{1}{9}\tau^4k^4\Sigma_k' - \frac{2}{9}\tau^4k^4\Sigma_k\Sigma_k'^2\Bigg],$$

$$\mathcal{Z}_{51}^T = \int dK\Big[+\tau^3\Sigma_k - \frac{7}{6}\tau^3k^2\Sigma_k' + \frac{1}{9}\tau^4k^4\Sigma_k' + \frac{2}{9}\tau^4k^4\Sigma_k\Sigma_k'^2\Big],$$

$$\mathcal{Z}_{52}^T = \int dK\Big[+\frac{2}{3}\tau^2\Sigma_k' + \frac{2}{3}\tau^3\Sigma_k + \frac{2}{3}\tau^3k^2\Sigma_k' - \frac{4}{3}\tau^3\Sigma_k^2\Sigma_k' - 2\tau^3k^2\Sigma_k\Sigma_k'^2 - \frac{1}{3}\tau^4k^2\Sigma_k + \frac{1}{9}\tau^4k^4\Sigma_k' + \frac{2}{9}\tau^4k^4\Sigma_k\Sigma_k'^2 + \frac{4}{3}\tau^4k^2\Sigma_k^3\Sigma_k'^2\Big],$$

$$\mathcal{Z}_{53}^T = \int dK\Big[+\frac{4}{3}\tau^3\Sigma_k - \frac{5}{3}\tau^3k^2\Sigma_k' + \frac{1}{9}\tau^4k^4\Sigma_k' + \frac{2}{9}\tau^4k^4\Sigma_k\Sigma_k'^2\Big],$$

$$\mathcal{Z}_{54}^T = \int dK\Big[+\frac{1}{3}\tau^3k^2\Sigma_k' - \frac{1}{9}\tau^4k^4\Sigma_k' - \frac{2}{9}\tau^4k^4\Sigma_k\Sigma_k'^2\Big],$$

$$\mathcal{Z}_{55}^T = \int dK\Big[+\frac{8}{3}\tau^2 - \frac{1}{3}\tau^3k^2 - \frac{4}{3}\tau^3\Sigma_k^2\Big],$$

$$\mathcal{Z}_{56}^T = \int dK\Big[+2\tau^2\Big],$$

$$\mathcal{Z}_{57}^T = \int dK\Big[-2\tau^3\Sigma_k + 2\tau^3k^2\Sigma_k' - 2\tau^3k^2\Sigma_k\Sigma_k'^2\Big],$$

$$\mathcal{Z}_{58}^T = \int dK\Big[-\frac{16}{3}\tau^2 + 3\tau^3k^2 + 4\tau^3\Sigma_k^2\Big],$$

$$\mathcal{Z}_{59}^T = \int dK\Big[+\frac{4}{3}\tau^2 - \tau^3k^2\Big],$$

$$\mathcal{Z}_{60}^T = \int dK\Big[-\tau^2\Big],$$

$$\mathcal{Z}_{61}^T = \int dK\Big[+3\tau^2 - \tau^3k^2 - 2\tau^3\Sigma_k^2\Big],$$

$$\mathcal{Z}_{62}^T = \int dK\Big[+4\tau^2\Sigma_k - 2\tau^3k^2\Sigma_k - \frac{8}{3}\tau^3\Sigma_k^3\Big],$$

$$\mathcal{Z}_{63}^T = \int dK\Big[-4\tau^2\Sigma_k + 2\tau^3k^2\Sigma_k\Big],$$

$$\mathcal{Z}_{64}^T = \int dK\Big[-6\tau^2\Sigma_k + 2\tau^3k^2\Sigma_k + 4\tau^3\Sigma_k^3\Big],$$

$$\mathcal{Z}_{65}^T = \int dK\Big[+2\tau^2\Sigma_k - 2\tau^3k^2\Sigma_k\Big],$$

$$\mathcal{Z}_{66}^T = \int dK\Big[+\frac{2}{3}\tau^2\Sigma_k' + \frac{1}{3}\tau^3k^2\Sigma_k' + \frac{2}{9}\tau^3k^4\Sigma_k'^3 - \frac{1}{9}\tau^4k^4\Sigma_k' - \frac{2}{9}\tau^4k^4\Sigma_k\Sigma_k'^2\Big],$$

$$\mathcal{Z}_{67}^T = \int dK\Big[+\frac{2}{3}\tau^3\Sigma_k - \frac{11}{18}\tau^3k^2\Sigma_k' - \frac{11}{9}\tau^3k^2\Sigma_k\Sigma_k'^2 - \frac{2}{9}\tau^3k^4\Sigma_k'^3 - \frac{1}{6}\tau^4k^2\Sigma_k + \frac{1}{9}\tau^4k^4\Sigma_k' + \frac{2}{9}\tau^4k^4\Sigma_k\Sigma_k'^2 + \frac{2}{3}\tau^4k^2\Sigma_k^3\Sigma_k'^2\Big],$$

$$\mathcal{Z}_{68}^T = \int dK\Big[-\frac{1}{9}\tau^3k^2\Sigma_k' + \frac{4}{9}\tau^3k^2\Sigma_k\Sigma_k'^2 - \frac{2}{9}\tau^3k^4\Sigma_k'^3 + \frac{1}{9}\tau^4k^4\Sigma_k' + \frac{2}{9}\tau^4k^4\Sigma_k\Sigma_k'^2\Big],$$

$$\mathcal{Z}_{69}^T = \int dK\Big[-\frac{1}{3}\tau^2 + \frac{1}{3}\tau^3k^2 - \frac{1}{18}\tau^4k^4 + \frac{2}{9}\tau^4k^4\Sigma_k^2\Sigma_k'^2\Big],$$

$$\mathcal{Z}_{70}^T = \int dK\Big[+\frac{5}{6}\tau^2 - \frac{1}{2}\tau^2k^2\Sigma_k'^2 - \frac{1}{2}\tau^3k^2 - \frac{2}{3}\tau^3\Sigma_k^2 + 2\tau^3k^2\Sigma_k^2\Sigma_k'^2 + \frac{1}{18}\tau^4k^4 + \frac{1}{6}\tau^4k^2\Sigma_k^2 - \frac{2}{9}\tau^4k^4\Sigma_k^2\Sigma_k'^2 - \frac{2}{3}\tau^4k^2\Sigma_k^4\Sigma_k'^2\Big],$$

$$\mathcal{Z}_{71}^T = \int dK\Big[-\frac{1}{3}\tau^2 - \frac{1}{3}\tau^3k^2 + \frac{1}{18}\tau^4k^4 - \frac{2}{9}\tau^4k^4\Sigma_k^2\Sigma_k'^2\Big],$$

$$\begin{aligned}
\mathcal{Z}^T_{72} &= \int dK\Big[+\frac{1}{3}\tau^2-\frac{1}{3}\tau^3k^2+\frac{1}{18}\tau^4k^4-\frac{2}{9}\tau^4k^4\Sigma_k^2\Sigma_k'^2\Big],\\
\mathcal{Z}^T_{73} &= \int dK\Big[-\frac{1}{2}\tau^2-\frac{1}{2}\tau^2k^2\Sigma_k'^2+\frac{1}{3}\tau^3k^2-\frac{1}{18}\tau^4k^4+\frac{2}{9}\tau^4k^4\Sigma_k^2\Sigma_k'^2\Big],\\
\mathcal{Z}^T_{74} &= \int dK\Big[+\frac{1}{3}\tau^2+\frac{1}{3}\tau^3k^2-\frac{1}{18}\tau^4k^4+\frac{2}{9}\tau^4k^4\Sigma_k^2\Sigma_k'^2\Big],\\
\mathcal{Z}^T_{75} &= \int dK\Big[-4\tau^2+\tau^3k^2+4\tau^3\Sigma_k^2\Big],\\
\mathcal{Z}^T_{76} &= \int dK\Big[-4\tau^2+2\tau^3k^2\Big],\\
\mathcal{Z}^T_{77} &= \int dK\Big[+2\tau^2\Sigma_k\Big],
\end{aligned} \tag{C1}$$

$$\int dK \equiv N_c\int\frac{d^4k}{(2\pi)^4}e^{-\tau(k^2+\Sigma_k^2)}\int_{\frac{1}{\Lambda^2}}^{\infty}\frac{d\tau}{\tau}. \tag{C2}$$

Appendix D: $\mathcal{Z}^T_n$ and K^T_n's relations

This appendix list the relations between our coefficients, $\mathcal{Z}^T_n$, and the coefficients in Ref.[10], K^T_n. Some coefficients vanish, because of the new relations in Appendix B.

$$\begin{aligned}
K^T_1 &= +\frac{1}{16B_0}\mathcal{Z}^T_1-\frac{1}{16B_0}\mathcal{Z}^T_{28}+\frac{1}{16B_0}\mathcal{Z}^T_{29}+\frac{1}{8B_0}\mathcal{Z}^T_{41}+\frac{1}{16B_0}\mathcal{Z}^T_{42}+\frac{1}{8B_0}\mathcal{Z}^T_{67}+\frac{1}{16B_0}\mathcal{Z}^T_{68}\\
K^T_2 &= +\frac{1}{16B_0}\mathcal{Z}^T_4+\frac{1}{8B_0}\mathcal{Z}^T_{40}+\frac{1}{8B_0}\mathcal{Z}^T_{43}+\frac{1}{16B_0}\mathcal{Z}^T_{57}\\
K^T_3 &= +\frac{1}{16B_0}\mathcal{Z}^T_3-\frac{1}{8B_0}\mathcal{Z}^T_{28}+\frac{1}{8B_0}\mathcal{Z}^T_{29}-\frac{1}{8B_0}\mathcal{Z}^T_{39}+\frac{1}{16B_0}\mathcal{Z}^T_{57}+\frac{1}{4B_0}\mathcal{Z}^T_{67}+\frac{1}{8B_0}\mathcal{Z}^T_{68}\\
K^T_4 &= +\frac{1}{16B_0}\mathcal{Z}^T_2+\frac{1}{8B_0}\mathcal{Z}^T_{28}-\frac{1}{8B_0}\mathcal{Z}^T_{29}+\frac{1}{16B_0}\mathcal{Z}^T_{39}-\frac{1}{16B_0}\mathcal{Z}^T_{40}-\frac{1}{16B_0}\mathcal{Z}^T_{42}-\frac{1}{16B_0}\mathcal{Z}^T_{57}-\frac{1}{4B_0}\mathcal{Z}^T_{67}-\frac{1}{8B_0}\mathcal{Z}^T_{68}\\
K^T_5 &= 0\\
K^T_6 &= 0\\
K^T_7 &= 0\\
K^T_8 &= 0\\
K^T_9 &= +\frac{1}{4B_0^2}\mathcal{Z}^T_{14}+\frac{1}{4B_0^2}\mathcal{Z}^T_{17}+\frac{1}{8B_0^2}\mathcal{Z}^T_{18}+\frac{1}{8B_0^2}\mathcal{Z}^T_{19}-\frac{1}{8B_0^2}\mathcal{Z}^T_{31}\\
K^T_{10} &= +\frac{1}{4B_0^2}\mathcal{Z}^T_{21}-\frac{1}{4B_0^2}\mathcal{Z}^T_{22}+\frac{1}{4B_0^2}\mathcal{Z}^T_{23}-\frac{1}{8B_0^2}\mathcal{Z}^T_{31}\\
K^T_{11} &= +\frac{1}{4B_0^2}\mathcal{Z}^T_{16}-\frac{1}{4B_0^2}\mathcal{Z}^T_{18}+\frac{1}{2B_0^2}\mathcal{Z}^T_{31}+\frac{1}{4B_0^2}\mathcal{Z}^T_{58}+\frac{1}{4B_0^2}\mathcal{Z}^T_{59}+\frac{1}{2B_0^2}\mathcal{Z}^T_{71}+\frac{1}{2B_0^2}\mathcal{Z}^T_{72}\\
K^T_{12} &= +\frac{1}{4B_0^2}\mathcal{Z}^T_{15}-\frac{1}{4B_0^2}\mathcal{Z}^T_{19}-\frac{1}{4B_0^2}\mathcal{Z}^T_{58}-\frac{1}{4B_0^2}\mathcal{Z}^T_{59}-\frac{1}{2B_0^2}\mathcal{Z}^T_{71}-\frac{1}{2B_0^2}\mathcal{Z}^T_{72}\\
K^T_{13} &= -\frac{1}{4B_0^2}\mathcal{Z}^T_{20}+\frac{1}{4B_0^2}\mathcal{Z}^T_{22}+\frac{1}{4B_0^2}\mathcal{Z}^T_{31}\\
K^T_{14} &= 0\\
K^T_{15} &= 0\\
K^T_{16} &= 0\\
K^T_{17} &= 0\\
K^T_{18} &= 0\\
K^T_{19} &= 0
\end{aligned}$$

$$
\begin{aligned}
K^T_{20} &= 0\\
K^T_{21} &= 0\\
K^T_{22} &= 0\\
K^T_{31} &= +\frac{1}{4B_0^3}\mathcal{Z}^T_{64}+\frac{1}{4B_0^3}\mathcal{Z}^T_{65}\\
K^T_{33} &= 0\\
K^T_{34} &= 0\\
K^T_{35} &= +\frac{1}{2N_fB_0^3}\mathcal{Z}^T_{77}\\
K^T_{36} &= 0\\
K^T_{37} &= 0\\
K^T_{38} &= 0\\
K^T_{39} &= +\frac{1}{16B_0}\mathcal{Z}^T_{26}-\frac{1}{16B_0}\mathcal{Z}^T_{28}+\frac{1}{16B_0}\mathcal{Z}^T_{29}+\frac{1}{16B_0^2}\mathcal{Z}^T_{33}-\frac{1}{16B_0^2}\mathcal{Z}^T_{44}+\frac{1}{8B_0^2}\mathcal{Z}^T_{61}-\frac{1}{16B_0}\mathcal{Z}^T_{66}+\frac{1}{8B_0}\mathcal{Z}^T_{67}+\frac{1}{16B_0}\mathcal{Z}^T_{68}\\
K^T_{40} &= +\frac{1}{8B_0}\mathcal{Z}^T_{26}-\frac{1}{8B_0}\mathcal{Z}^T_{28}+\frac{1}{8B_0}\mathcal{Z}^T_{29}+\frac{1}{16B_0^2}\mathcal{Z}^T_{34}-\frac{1}{8B_0^2}\mathcal{Z}^T_{44}-\frac{1}{8B_0}\mathcal{Z}^T_{66}+\frac{1}{4B_0}\mathcal{Z}^T_{67}+\frac{1}{8B_0}\mathcal{Z}^T_{68}\\
K^T_{41} &= 0\\
K^T_{42} &= 0\\
K^T_{43} &= +\frac{1}{16B_0}\mathcal{Z}^T_{10}-\frac{1}{16B_0^2}\mathcal{Z}^T_{45}+\frac{1}{16B_0}\mathcal{Z}^T_{50}+\frac{1}{8B_0}\mathcal{Z}^T_{51}-\frac{1}{8B_0}\mathcal{Z}^T_{54}+\frac{1}{8B_0^2}\mathcal{Z}^T_{60}-\frac{1}{16B_0^2}\mathcal{Z}^T_{75}\\
K^T_{44} &= +\frac{1}{16B_0}\mathcal{Z}^T_{13}-\frac{1}{8B_0}\mathcal{Z}^T_{45}-\frac{1}{8B_0}\mathcal{Z}^T_{50}-\frac{1}{16B_0^2}\mathcal{Z}^T_{76}\\
K^T_{45} &= -\frac{1}{8N_fB_0}\mathcal{Z}^T_{10}-\frac{1}{16N_fB_0}\mathcal{Z}^T_{13}-\frac{1}{4N_fB_0}\mathcal{Z}^T_{51}+\frac{1}{4N_fB_0}\mathcal{Z}^T_{54}\\
K^T_{46} &= 0\\
K^T_{47} &= -\frac{1}{16B_0}\mathcal{Z}^T_{7}-\frac{1}{16B_0}\mathcal{Z}^T_{9}+\frac{1}{16B_0}\mathcal{Z}^T_{49}+\frac{1}{8B_0}\mathcal{Z}^T_{50}-\frac{1}{16B_0}\mathcal{Z}^T_{52}-\frac{1}{16B_0}\mathcal{Z}^T_{53}\\
K^T_{48} &= -\frac{1}{16B_0}\mathcal{Z}^T_{6}-\frac{1}{16B_0}\mathcal{Z}^T_{8}-\frac{1}{16B_0}\mathcal{Z}^T_{50}-\frac{1}{8B_0}\mathcal{Z}^T_{51}+\frac{1}{16B_0}\mathcal{Z}^T_{52}+\frac{1}{16B_0}\mathcal{Z}^T_{53}\\
K^T_{49} &= -\frac{1}{16B_0}\mathcal{Z}^T_{11}-\frac{1}{16B_0}\mathcal{Z}^T_{12}+\frac{1}{16B_0}\mathcal{Z}^T_{49}-\frac{1}{16B_0}\mathcal{Z}^T_{50}-\frac{1}{8B_0}\mathcal{Z}^T_{51}\\
K^T_{50} &= 0\\
K^T_{51} &= -\frac{1}{4B_0^2}\mathcal{Z}^T_{31}+\frac{1}{B_0^2}\mathcal{Z}^T_{70}-\frac{1}{2B_0^2}\mathcal{Z}^T_{72}+\frac{1}{B_0^2}\mathcal{Z}^T_{73}+\frac{1}{2B_0^2}\mathcal{Z}^T_{74}\\
K^T_{52} &= +\frac{1}{B_0^2}\mathcal{Z}^T_{31}-\frac{1}{B_0^2}\mathcal{Z}^T_{69}+\frac{1}{B_0^2}\mathcal{Z}^T_{71}+\frac{1}{B_0^2}\mathcal{Z}^T_{72}-\frac{1}{B_0^2}\mathcal{Z}^T_{74}\\
K^T_{54} &= 0\\
K^T_{55} &= 0\\
K^T_{57} &= +\frac{1}{16B_0}\mathcal{Z}^T_{27}+\frac{1}{8B_0}\mathcal{Z}^T_{41}+\frac{1}{8B_0}\mathcal{Z}^T_{67}+\frac{1}{16B_0}\mathcal{Z}^T_{68}\\
K^T_{58} &= -\frac{1}{8B_0}\mathcal{Z}^T_{27}+\frac{1}{8B_0}\mathcal{Z}^T_{43}-\frac{1}{4B_0}\mathcal{Z}^T_{67}-\frac{1}{8B_0}\mathcal{Z}^T_{68}\\
K^T_{59} &= -\frac{1}{8B_0}\mathcal{Z}^T_{27}-\frac{1}{8B_0}\mathcal{Z}^T_{40}-\frac{1}{8B_0}\mathcal{Z}^T_{57}-\frac{1}{2B_0}\mathcal{Z}^T_{67}-\frac{1}{4B_0}\mathcal{Z}^T_{68}\\
K^T_{60} &= -\frac{1}{8B_0}\mathcal{Z}^T_{28}+\frac{1}{8B_0}\mathcal{Z}^T_{29}-\frac{1}{8B_0}\mathcal{Z}^T_{39}+\frac{1}{8B_0}\mathcal{Z}^T_{57}+\frac{1}{2B_0}\mathcal{Z}^T_{67}+\frac{1}{4B_0}\mathcal{Z}^T_{68}\\
K^T_{61} &= +\frac{1}{8B_0}\mathcal{Z}^T_{27}+\frac{1}{8B_0}\mathcal{Z}^T_{28}-\frac{1}{8B_0}\mathcal{Z}^T_{29}-\frac{1}{8B_0}\mathcal{Z}^T_{42}\\
K^T_{62} &= 0
\end{aligned}
$$

$$
\begin{aligned}
K_{63}^T &= 0\\
K_{64}^T &= 0\\
K_{65}^T &= 0\\
K_{66}^T &= 0\\
K_{67}^T &= 0\\
K_{68}^T &= 0\\
K_{69}^T &= +\frac{1}{16B_0}\mathcal{Z}_5^T - \frac{1}{16B_0}\mathcal{Z}_{49}^T\\
K_{70}^T &= +\frac{1}{16B_0}\mathcal{Z}_6^T - \frac{1}{16B_0}\mathcal{Z}_8^T + \frac{1}{16B_0}\mathcal{Z}_{50}^T + \frac{1}{8B_0}\mathcal{Z}_{51}^T - \frac{1}{16B_0}\mathcal{Z}_{52}^T + \frac{1}{16B_0}\mathcal{Z}_{53}^T\\
K_{71}^T &= +\frac{1}{16B_0}\mathcal{Z}_7^T - \frac{1}{16B_0}\mathcal{Z}_9^T + \frac{1}{16B_0}\mathcal{Z}_{49}^T + \frac{1}{16B_0}\mathcal{Z}_{52}^T - \frac{1}{16B_0}\mathcal{Z}_{53}^T\\
K_{72}^T &= -\frac{1}{16B_0}\mathcal{Z}_{11}^T + \frac{1}{16B_0}\mathcal{Z}_{12}^T - \frac{1}{16B_0}\mathcal{Z}_{49}^T - \frac{1}{16B_0}\mathcal{Z}_{50}^T - \frac{1}{8B_0}\mathcal{Z}_{51}^T\\
K_{73}^T &= 0\\
K_{74}^T &= -\frac{1}{16B_0^2}\mathcal{Z}_{55}^T + \frac{1}{8B_0^2}\mathcal{Z}_{61}^T\\
K_{75}^T &= -\frac{1}{8B_0}\mathcal{Z}_{54}^T + \frac{1}{16B_0^2}\mathcal{Z}_{55}^T + \frac{1}{8B_0^2}\mathcal{Z}_{60}^T\\
K_{76}^T &= +\frac{1}{16B_0}\mathcal{Z}_{26}^T + \frac{1}{16B_0}\mathcal{Z}_{27}^T - \frac{1}{16B_0}\mathcal{Z}_{35}^T + \frac{1}{16B_0^2}\mathcal{Z}_{55}^T - \frac{1}{16B_0^2}\mathcal{Z}_{56}^T - \frac{1}{16B_0}\mathcal{Z}_{66}^T + \frac{1}{8B_0}\mathcal{Z}_{67}^T + \frac{1}{16B_0}\mathcal{Z}_{68}^T\\
K_{77}^T &= 0\\
K_{78}^T &= 0\\
K_{79}^T &= 0\\
K_{80}^T &= +\frac{1}{4N_fB_0}\mathcal{Z}_{54}^T\\
K_{82}^T &= 0\\
K_{84}^T &= -\frac{1}{4B_0}\mathcal{Z}_{27}^T - \frac{1}{8B_0}\mathcal{Z}_{28}^T + \frac{1}{8B_0}\mathcal{Z}_{29}^T + \frac{1}{16B_0}\mathcal{Z}_{36}^T - \frac{1}{8B_0}\mathcal{Z}_{37}^T + \frac{1}{16B_0}\mathcal{Z}_{38}^T - \frac{1}{4B_0}\mathcal{Z}_{67}^T - \frac{1}{8B_0}\mathcal{Z}_{68}^T\\
K_{85}^T &= +\frac{1}{4B_0}\mathcal{Z}_{57}^T + \frac{1}{B_0}\mathcal{Z}_{67}^T + \frac{1}{2B_0}\mathcal{Z}_{68}^T\\
K_{86}^T &= -\frac{1}{8B_0}\mathcal{Z}_{52}^T + \frac{1}{8B_0}\mathcal{Z}_{53}^T\\
K_{87}^T &= 0\\
K_{88}^T &= +\frac{1}{B_0^2}\mathcal{Z}_{62}^T + \frac{1}{B_0^2}\mathcal{Z}_{63}^T\\
K_{90}^T &= -\frac{1}{2B_0^2}\mathcal{Z}_{31}^T - \frac{1}{2B_0^2}\mathcal{Z}_{58}^T - \frac{1}{2B_0^2}\mathcal{Z}_{59}^T - \frac{1}{B_0^2}\mathcal{Z}_{71}^T - \frac{1}{B_0^2}\mathcal{Z}_{72}^T\\
K_{92}^T &= 0\\
K_{94}^T &= -\frac{1}{2B_0}\mathcal{Z}_{66}^T + \frac{1}{B_0}\mathcal{Z}_{67}^T + \frac{1}{2B_0}\mathcal{Z}_{68}^T\\
K_{95}^T &= +\frac{1}{4B_0}\mathcal{Z}_{28}^T\\
K_{96}^T &= +\frac{1}{8B_0}\mathcal{Z}_{24}^T + \frac{1}{8B_0}\mathcal{Z}_{25}^T + \frac{1}{8B_0}\mathcal{Z}_{66}^T\\
K_{97}^T &= +\frac{1}{8B_0}\mathcal{Z}_{24}^T - \frac{1}{8B_0}\mathcal{Z}_{25}^T - \frac{1}{4B_0}\mathcal{Z}_{27}^T + \frac{1}{8B_0}\mathcal{Z}_{66}^T - \frac{1}{2B_0}\mathcal{Z}_{67}^T - \frac{1}{4B_0}\mathcal{Z}_{68}^T\\
K_{98}^T &= +\frac{1}{4B_0}\mathcal{Z}_{49}^T
\end{aligned}
$$

$$
\begin{aligned}
K_{99}^T &= +\frac{1}{4B_0}\mathcal{Z}_{51}^T \\
K_{100}^T &= -\frac{1}{4B_0}\mathcal{Z}_{50}^T \\
K_{101}^T &= 0 \\
K_{102}^T &= 0 \\
K_{103}^T &= 0 \\
K_{105}^T &= -\frac{1}{2B_0^2}\mathcal{Z}_{30}^T + \frac{1}{2B_0^2}\mathcal{Z}_{31}^T \\
K_{106}^T &= 0 \\
K_{107}^T &= 0 \\
K_{108}^T &= 0 \\
K_{112}^T &= -\frac{1}{8B_0^2}\mathcal{Z}_{45}^T \\
K_{113}^T &= +\frac{1}{8B_0}\mathcal{Z}_{26}^T - \frac{1}{8B_0}\mathcal{Z}_{28}^T + \frac{1}{8B_0}\mathcal{Z}_{29}^T - \frac{1}{8B_0^2}\mathcal{Z}_{44}^T - \frac{1}{8B_0}\mathcal{Z}_{66}^T + \frac{1}{4B_0}\mathcal{Z}_{67}^T + \frac{1}{8B_0}\mathcal{Z}_{68}^T \\
K_{114}^T &= +\frac{1}{8B_0}\mathcal{Z}_{28}^T - \frac{1}{8B_0}\mathcal{Z}_{29}^T + \frac{1}{16B_0}\mathcal{Z}_{36}^T + \frac{1}{8B_0}\mathcal{Z}_{37}^T + \frac{1}{16B_0}\mathcal{Z}_{38}^T - \frac{1}{4B_0}\mathcal{Z}_{67}^T - \frac{1}{8B_0}\mathcal{Z}_{68}^T \\
K_{115}^T &= +\frac{1}{8B_0}\mathcal{Z}_{27}^T + \frac{1}{16B_0}\mathcal{Z}_{36}^T - \frac{1}{16B_0}\mathcal{Z}_{38}^T + \frac{1}{4B_0}\mathcal{Z}_{67}^T + \frac{1}{8B_0}\mathcal{Z}_{68}^T \\
K_{116}^T &= +\frac{1}{8B_0}\mathcal{Z}_{52}^T + \frac{1}{8B_0}\mathcal{Z}_{53}^T \\
K_{117}^T &= 0 \\
K_{118}^T &= -\frac{2}{B_0^2}\mathcal{Z}_{31}^T \\
K_{119}^T &= +\frac{4}{3B_0^2}\mathcal{Z}_{31}^T + \frac{2}{3B_0^2}\mathcal{Z}_{32}^T - \frac{4}{3B_0^2}\mathcal{Z}_{46}^T - \frac{4}{3B_0^2}\mathcal{Z}_{47}^T + \frac{2}{3B_0^2}\mathcal{Z}_{48}^T + \frac{4}{3B_0^3}\mathcal{Z}_{77}^T \\
K_{120}^T &= +\frac{1}{4B_0^2}\mathcal{Z}_{55}^T
\end{aligned}
\tag{D1}
$$

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