From Fermat principle to the Lobačevskii-Fok space in particle physics

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Abstract

The Poincaré model of the Lobačevskii geometry is derived from the Fermat principle. The Lobačevskii geometry is interpreted as the Lobačevskii-Fok velocity space geometry of moving particles. The relation of this geometry to the decay of the neutral pi-meson is considered. The generalization of the Lobačevskii geometry is performed and the new angle of parallelism is derived (Pardy, 2013). The light confined circularly in the optical medium is defined as the optical black hole. The existence of the centrifugal force acting on the photon is discussed.

Key words: The Fermat principle, light ray trajectories, optics, the Poincaré model of the Lobačevskii geometry, Lobačevskii-Fok geometry, generalized Lobačevskii geometry, optical black hole.

1 Introduction

The Fermat optical theorem states that the trajectory of light from point A to B in the optical medium is the trajectory performed during the minimal time. At the same time the trajectory of the optical ray from point A to point B with reflection on the mirror in the reflection point C is also performed during the minimal time. This principle can be generalized for an arbitrary number of reflection points.

The trajectory of light passing from point $A(x_1, y_1)$ to point $B(x_2, y_2)$ can be determined by the variational principle (Lavrentyev et al., 1950). It is the mathematical formulation of the Fermat principle and it states that the minimal time T of light passing from point $A(x_1, y_1)$ to point $B(x_2, y_2)$ is the result of the minimization of the functional

$$T(y, y') = \int_{x_1}^{x_2} \frac{ds}{v(y)} = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2}}{v(y)} dx, \tag{1}$$

where $v(y)$ is the velocity of light.

The functional $T(y, y')$ is the solution of the Euler-Lagrange equations with $y' = dy/dx$:

$$\frac{\partial T}{\partial y} - \frac{d}{dx}\left(\frac{\partial T}{\partial y'}\right) = 0. \tag{2}$$

If $v = Ay$, the solutions of eq. (2) are the circles forming the Poincaré model of the Lobačevskii geometry:

$$(x - C)^2 + y^2 = r^2. \tag{3}$$

Let us remark that the above method can be applied for determination of trajectory of light in the stratified medium or in medium with reflections on the boundary.

2 The Poincaré optical model of the Lobačevskii geometry

The Lobačevskii geometry is the integral part of the general geometry called non-euclidean geometry, or, hyperbolic geometry. The name non-Euclidean was used by Gauss to describe a system of geometry which differs from Euclid's in its properties of parallelism. Such a system was developed independently by Gauss in Germany, Bolyai in Hungary and Lobačevskii in Russia, many years ago. Another system, differing more radically from Euclid's, was suggested later by Riemann in Germany and Schlafli in Switzerland. The subject was unified by Klein, who gave the names parabolic, hyperbolic, and elliptic to the respective systems of Euclid, Gauss-Bolyai-Lobačevskii, and Riemann-Shlafli (Coxeter, 1998).

The substantial mathematical object in the Lobačevskii geometry is the angle of parallelism defined by Lobačevskii as follows. Given a point P and a line q. The Intersection of the perpendicular through P let be Q and $PQ = x$. The intersection of line p passing through P, with q, let be R and $QR = k$. Then, the angle RPQ for perpendicular distance x

$$\Pi(x) = 2 \tan^{-1} e^{-x/k}. \tag{4}$$

is known as the Lobačevskii formula for the angle of parallelism (Coxeter, 1998; Lobačevskii, 1914).

The Poincaré model of the Lobačevskii geometry is the physical model of the optical trajectories in a medium with the velocity of light $v = Ay$.

According to Hilbert (Hilbert, 1903; McCleary, 1994), it is not possible to realize the Lobačevskii geometry globally on surface with the constant negative curvature. The Beltrami realization of the Lobačevskii geometry is only partial one. The famous Russian mathematician Ostrogradskii never ackowledged the Lobačevskii geometry.

Folowing Bukreev (1951), we can investigate the Lobačevskii geometry and the Poincaré model of it using the pseudosphere (2D manifold) with metric

$$ds^2 = du^2 + e^{\frac{2u}{r}} dv^2. \quad (5)$$

by relations

$$v = x, \quad re^{-\frac{u}{r}} = y, \quad (6)$$

which mean that we get the line element as

$$ds^2 = \frac{r^2}{y^2}(dx^2 + dy^2), \quad (7)$$

which was used during the application of the Fermat principle of the minimal time.

The transformation (6) is the conformal mapping of the pseudo-spherical abstract surface (2-dimensional continuous differentiable manifold) into the upper Poincaré half plane in the Cartesian coordinates x, y. As an analogue to this situation it is possible to consider the conformal transformation of the 4-dimensional Einstein-Riemann gravitational manifold to the 4-dimensional Cartesian coordinates x, y, z, t of space-time. Let us still remark that there are many inversion transformation from the Cartesian Poincaré metric to the 2-dimensional manifold ds^2, to form the integral part of the optical models of the Lobačevskii geometry.

The trajectory of light in the Poincaré model is a trajectory passing from $A(x_1, y_1)$ to $B(x_2, y_2)$ and determined by the minimal time from $A(x_1, y_1)$ to $B(x_2, y_2)$. It is the result of the minimum of the functional (1).

The Poincaré circles (pseudo-straight lines) in his model are analogue of the straight lines in the Euclidean geometry.

The theorems following from the metric (7) (Bukreev, 1951) are valid in the Poincaré model of the Lobačevskii geometry:

Theorem 1: Only one half-circle passes through two points A, B in the Poincaré plane.

Theorem 2: The curvilinear segment AB in the Poincaré plane is of the shortest length.

Theorem 3: The parallels are two half-circles with the intersections on the x-axis.

Theorem 4: If point $A \notin q$ then there are $q_1 \parallel q$, $q_2 \parallel q$ passing through A, with $q_1 \neq q_2$.

Theorem 5: If point $A \notin q$, $q_1 \parallel q, q_2 \parallel q$, then q_1, q_2 divide the Poincaré plane in four different sectors I, II, III, IV.

Let us remark that the optical distance between point A and B is not equivalent to the mechanical distance realized by the nonelastic flexible fibre as the shortest distance between point A and B. The Poincaré model of geometry where the light velocity is $v = Ay$ is the interaction model of light with the optical medium.

It is elementary to see that if we define the Poincaré problem on a sphere, then we get so called spherical Poincaré model of the Lobačevskii geometry.

3 The Lobačevskii angle of parallelism from trigonometry

It is well known that Beltrami showed that the Lobačevskii trigonometry is the spherical trigonometry with the imaginary radius of the sphere. Or, $r \to ir$. Then instead of the trigonometrical cosine and sine relation on sphere,

$$\cos\frac{a}{r} = \cos\frac{b}{r}\cos\frac{c}{r} + \sin\frac{b}{r}\sin\frac{c}{r}\cos A, \tag{8}$$

$$\frac{\sin A}{\sin a/r} = \frac{\sin B}{\sin b/r} = \frac{\sin C}{\sin c/r}, \tag{9}$$

$$\cos A = -\cos B\cos C + \sin B\sin C\cos(a/r) \tag{10}$$

where r is the radius of sphere and a, b, c are lengths of sides of the triangle on the sphere and A, B, C are corresponding angles, the following relations of the Lobačevskii imaginary pangeometry follows from the Beltrami operation:

$$\cosh\frac{a}{r} = \cosh\frac{b}{r}\cosh\frac{c}{r} - \sinh\frac{b}{r}\sinh\frac{c}{r}\cos A \tag{11}$$

and

$$\frac{\sin A}{\sinh\frac{a}{r}} = \frac{\sin B}{\sinh\frac{b}{r}} = \frac{\sin C}{\sinh\frac{c}{r}}, \tag{12}$$

$$\cos A = -\cos B\cos C + \sin B\sin C\cosh\frac{a}{r}. \tag{13}$$

Now, we are prepared to derive the Lobačevskii function $\Pi(a)$, where a is BC in the Lobačevskii triangle ABC and the angle B is $\angle B = \pi/2$ and $\angle C = \Pi(a)$.

We have from (13)

$$1 = \sin\Pi(a)\cosh\frac{a}{r}. \tag{14}$$

On the other hand,

$$\cos\Pi(a) = \sqrt{1-\sin^2\Pi(a)} =$$

$$\sqrt{1-\frac{1}{\cosh^2\frac{a}{r}}} = \frac{\sqrt{\cosh^2\frac{a}{r}-1}}{\cosh\frac{a}{r}} = \frac{\sinh\frac{a}{r}}{\cosh\frac{a}{r}} = \tanh\frac{a}{r}. \tag{15}$$

Then, with $\tan\Pi = \sin\Pi/\cos\Pi$, we have

$$\tan^2\frac{\Pi(a)}{2} = \frac{1-\cos\Pi(a)}{1+\cos\Pi(a)} =$$

$$\frac{1-\tan\frac{a}{r}}{1+\tan\frac{a}{r}} = \frac{\cosh\frac{a}{r}-\sinh\frac{a}{r}}{\cosh\frac{a}{r}+\sinh\frac{a}{r}} = \frac{e^{-\frac{a}{r}}}{e^{\frac{a}{r}}} = e^{-\frac{2a}{r}}. \tag{16}$$

Or,

$$\tan\frac{\Pi(a)}{2} = e^{-\frac{a}{r}}. \tag{17}$$

The last formula is the famous one for the Lobačevskii angle $\Pi(a)$.

Let us remark that the angle of parallelism is immediately related to the decay of the neutral pi-meson to two gamma-photons, detected by the coincidence experimental method (Steinberger, et al., 1950). Or,

$$\pi^0 \quad \rightarrow \quad \gamma + \gamma. \tag{18}$$

The angle between velocities of the gamma photons in the rest system of neutral meson is evidently π. However according to the special theory of relativity the angle is transformed in the laboratory system according to the Lorentz transformation and it is smaller than π. It is equivalent to the statement that the Lobačevskii angle Π is smaller than $\pi/2$, or, $\Pi < \pi/2$. Such experiment can be considered as the confirmation of the Lobačevskii geometry in the elementary particle physics. Similarly the decay of the neutral η-meson $\eta^0 \rightarrow \gamma + \gamma$, axion $A^0 \rightarrow \gamma + \gamma$, or, the Higgs boson decay $H^0 \rightarrow \gamma + \gamma$, are the confirmation of the Lobačevskii geometry in the elementary particle physics and at present time can be tested in CERN.

The statement which is valid for the decay channel of the π^0-meson is valid by analogy also for all decay channels described in the Review of the Particle physics (Amsler et al., 2008).

4 The generalized Lobačevskii geometry

Theorem: The generalized Lobačevskii formulas for triangles in generalized Lobačevskii geometry follow from the spherical formulas (8), (9), (10) by transformation $r \rightarrow r + i\varrho$:

$$\cos\varphi_a \cosh\chi_a + i \sin\varphi_a \sinh\chi_a =$$

$$[\cos\varphi_b \cosh\chi_b + i \sin\varphi_b \sinh\chi_b][\cos\varphi_c \cosh\chi_c + i \sin\varphi_c \sinh\chi_c] \quad +$$

$$[\sin\varphi_b \cosh\chi_b + i \cos\varphi_b \sinh\chi_b][\sin\varphi_c \cosh\chi_c + i \cos\varphi_c \sinh\chi_c] \cos A, \tag{19}$$

$$\frac{\sin A}{\sin\varphi_a \cosh\chi_a + i \cos\varphi_a \sinh\chi_a} =$$

$$\frac{\sin B}{\sin\varphi_b \cosh\chi_b + i \cos\varphi_b \sinh\chi_b} =$$

$$\frac{\sin C}{\sin\varphi_c \cosh\chi_c + i \cos\varphi_c \sinh\chi_c}, \tag{20}$$

$$\cos A = -\cos B \cos C \quad +$$

$$\sin B \sin C[\cos\varphi_a \cosh\chi_a + i \sin\varphi_a \sinh\chi_a], \tag{21}$$

where

$$\varphi_a; \varphi_b; \varphi_c; = \frac{ar}{r^2 + \varrho^2}; \quad \frac{br}{r^2 + \varrho^2}; \quad \frac{cr}{r^2 + \varrho^2} \tag{22}$$

and

$$\chi_a; \chi_b; \chi_c; = \frac{a\varrho}{r^2 + \varrho^2}; \quad \frac{b\varrho}{r^2 + \varrho^2}; \quad \frac{c\varrho}{r^2 + \varrho^2}. \tag{23}$$

and ϱ is the new parameter of the new triangle on the 2D manifold and A, B, C are corresponding triangle angles.

It follows fro eq. (21) that when A, B, C are real quantities, then it is necessary to be $\sin\varphi_a = 0$, or, $\varphi_a = l\pi, l = 1, 2, 3....$ Similarly, $\sin\varphi_b = 0$, or, $\varphi_b = m\pi, m = 1, 2, 3...$, $\sin\varphi_c = 0$, or, $\varphi_c = n\pi, n = 1, 2, 3....$ It means that from the generalized Beltrami operation $r \to r + i\varrho$, the quantization of the generalized Lobačevskii geometry of the 2D-manifold follows.

Now, we can derive the generalized Lobačevskii function $\Pi(a)$, where a is BC in the generalized Lobačevskii triangle ABC, $\angle B = \pi/2$ and $\angle C = \Pi(a)$.

We have from eq. (21) for the generalized rectangular triangle:

$$1 = \sin\Pi(a)\cosh\chi_a, \tag{24}$$

where

$$\chi_a = \frac{a\varrho}{r^2 + \varrho^2} = \frac{\varrho}{r}\varphi_a = \frac{\varrho}{r}l\pi; \quad l = 1, 2, 3... \tag{25}$$

We have from eq. (24):

$$\tan\frac{\Pi(a)}{2} = e^{-\chi_a} \tag{26}$$

It is evident that in the limiting case $\varrho \to 0$, we get the Euclidean angle $\Pi(a) = \pi/2$. While the original Lobačevskii angle $\Pi(a)$ was confirmed in decay of the neutral pi-meson, the generalized Lobačevskii angle $\Pi(a)$ is expected to be confirmed in the high energy physics by experiments in CERN and it is not excluded that the new geometry will be revealed in the Little Bang (Dusling et al., 2011) if performed by LHC in CERN, or, in the vicinity of the galactical nucleus.

5 The Lobačevskii-Fok velocity space

We know from the special theory of relativity that the relative velocity of two particles with velocities $\mathbf{v}_1, \mathbf{v}_2$ is given by the formula (Landau et al., 1988) (c =1, for the velocity of light):

$$\mathbf{v}' = \frac{\sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2}}{1 - \mathbf{v}_1 \cdot \mathbf{v}_2}. \tag{27}$$

The last formula can be easily transformed for $\mathbf{v}_1 = \mathbf{v}$ and $\mathbf{v}_2 = \mathbf{v} + d\mathbf{v}$ to get new differential form which can be considered as the length element in the velocity space, where v_1, v_2, v_3 are so called the Beltrami coordinates (Fok, 1955; Kagan, 1947; ibid., 1948). Or,

$$dl_v^2 = \frac{(d\mathbf{v})^2 - (\mathbf{v} \times d\mathbf{v})^2}{(1 - v^2)^2} =$$

$$\frac{dv^2}{(1 - v^2)} + \frac{v^2}{(1 - v^2)}(d\theta^2 + \sin^2\theta d\varphi^2), \tag{28}$$

where θ and φ are polar and the azimutal angles of the velocity $\mathbf{v}$ in the spherical coordinate system.

Using the substitution $v = \tanh\chi$, we get the line element in the velocity space as

$$dl_v^2 = d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\varphi^2). \tag{29}$$

The last line element is from the geometrical point of view the element of the Lobačevskii space with the constant negative Gauss curvature.

The Lobačevskii space follows from the spherical element (Landau et al. 1988)

$$dl^2 = \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{30}$$

if we replace in the spherical metric the variable r by $r = a\sinh\chi$ and $a \to ia$. Then

$$dl_v^2 = a^2(d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\varphi^2). \tag{31}$$

The area of the sphere is $A = 4\pi a^2\sinh^2 r$ and volume V goes to infinity for r goes to infinity. So the Lobačevskii abstract space is identical with the Friedmann solution of the Einstein equations with the negative curvature.

Let us remark that if we perform transformation $r \to ir + \rho$ in formula (30) where we write $r^2 = r.r = r.r^*$, where $r^* = \rho - ir$, in the form

$$dl^2 = \frac{dr^2}{1 - \frac{r^2+\rho^2}{a^2}} + (r^2 + \rho^2)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{32}$$

which is the elementary generalization of the Lobaěvskii geometry line element.

Now, if we perform the substitution

$$r^2 + \rho^2 = a^2\sin^2\chi, \tag{33}$$

then the element (32) can be transformed into the form:

$$dl^2 = \frac{a^4\sin^2\chi d\chi^2}{a^2\sinh^2\chi - \rho^2} + a^2\sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2) \tag{34}$$

It may be easy to see that the last formula is adequate to the metric of the exotic cosmology with the new geometrical term ρ which should not be identified with the Einstein cosmological constant Λ.

6 The geodesic line in the Lobačevskii-Fok space of velocities

If we put $d\mathbf{v} = \dot{\mathbf{v}}dt$, for the line element in the Lobačevskii space we get from eq. (28) relation

$$\left(\frac{dl_v}{dt}\right)^2 = \frac{\dot{\mathbf{v}}^2}{1 - v^2} + \frac{(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{(1 - v^2)^2} = 2F, \tag{35}$$

where the symbol $2F$ is introduced by definition.

Then we write (Fok, 1955):

$$dl_v = \int_{t_1}^{t_2} \sqrt{2F}dt, \tag{36}$$

where $L = \sqrt{2F}$ is the Lagrange function, or

$$L = \sqrt{\frac{\dot{\mathbf{v}}^2}{1-v^2} + \frac{(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{(1-v^2)^2}}. \tag{37}$$

The geodetic line from time t_1 to time t_2 is the solution of the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}_k}\right) - \frac{\partial L}{\partial v_k} = 0; \quad k = 1,2,3, \tag{38}$$

or,

$$\frac{d}{dt}\left(\frac{1}{\sqrt{2F}}\frac{\partial F}{\partial \dot{v}_k}\right) - \frac{1}{\sqrt{2F}}\frac{\partial F}{\partial v_k} = 0; \quad k = 1,2,3. \tag{39}$$

We use here the parameter t which is an arbitrary parameter. The interpretation of it as time is of course possible. We choose it in such a way that

$$\frac{dF}{dt} = 0; \quad F = const. \tag{40}$$

Then equations (39) will be equivalent to

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{v}_k}\right) - \frac{\partial F}{\partial v_k} = 0; \quad k = 1,2,3. \tag{41}$$

As the function F is not the function of parameter t, then we can write:

$$\sum_k \dot{v}_k \frac{\partial F}{\partial \dot{v}_k} - F = const. \tag{42}$$

Eq. (42) is an analogue of the definition of energy in classical mechanics in generalized coordinates if we replace F by the Lagrange function of the massive point.

It follows from eq. (39):

$$\frac{\partial F}{\partial \dot{v}_k} = \frac{\dot{v}_k}{1-v^2} + \frac{v_k(\mathbf{v}\cdot\dot{\mathbf{v}})}{(1-v^2)^2}; \quad k = 1,2,3 \tag{43}$$

$$\frac{\partial F}{\partial v_k} = v_k\left(\frac{\dot{\mathbf{v}}^2}{(1-v^2)^2} + 2\frac{(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{(1-v^2)^3}\right) + \frac{\dot{v}_k(\mathbf{v}\cdot\dot{\mathbf{v}})}{(1-v^2)^2}; \quad k = 1,2,3. \tag{44}$$

Let us introduce the vector $\mathbf{w}$ by the definition

$$w_k = \frac{\dot{v}_k}{(1-v^2)}. \tag{45}$$

Then eqs. (43), (44) can be evidently written in the form

$$\frac{\partial F}{\partial \dot{v}_k} = w_k + \frac{v_k(\mathbf{v}\cdot\mathbf{w})}{(1-v^2)}; \quad k = 1,2,3 \tag{46}$$

$$\frac{\partial F}{\partial v_k} = v_k\left(w^2 + \frac{2(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{(1-v^2)}\right) + v_k(\mathbf{v}\cdot\mathbf{w}); \quad k = 1,2,3. \tag{47}$$

After t-derivation of eq. (47) and expressing $\dot{\mathbf{v}}$ as a function of $\mathbf{w}$, we get

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{v}_k} = \dot{w}_k + \frac{v_k(\mathbf{v}\cdot\dot{\mathbf{w}})}{(1-v^2)} \quad +$$

$$v_k\left(w^2 + \frac{2(\mathbf{v}\cdot\mathbf{w})^2}{(1-v^2)}\right) + v_k(\mathbf{v}\cdot\mathbf{w}); \quad k = 1,2,3. \tag{48}$$

After insertion of $\partial F/\partial v_k$ (47) and equation (48) into Lagrange equation (41), we get

$$\dot{w}_k + \frac{v_k(\mathbf{v}\cdot\dot{\mathbf{w}})}{(1-v^2)} = 0. \tag{49}$$

After multiplication of the last equation by v_k we get

$$\mathbf{v}\cdot\dot{\mathbf{w}} = 0, \tag{50}$$

from which equation follows $\dot{\mathbf{w}} = 0$, or, $\mathbf{w} = const$. However, $\mathbf{w}$ is collinear with $\dot{\mathbf{v}}$, then $\mathbf{w}\cdot\dot{\mathbf{v}} = 0$, Or,

$$\mathbf{w}\cdot\mathbf{v} = const. \tag{51}$$

It gives still two linearly independent integrals of the Lagrange equations.

7 The length of the straight segment in the Lobačevskii-Fok space

We consider the length AB as the shortest line segment from A to B in the Lobačevskii space. Let us introduce two vectors $\mathbf{v} = \mathbf{v}_1$ and $\mathbf{v} = \mathbf{v}_2$. Then the parametric form of the segment is

$$\mathbf{v} = \mathbf{v}_1 + \mu(\mathbf{v}_2 - \mathbf{v}_1); \quad 0 < \mu < 1. \tag{52}$$

After insertion of (52) into F in (35), we get

$$2F = \frac{(\mathbf{v}_2 - \mathbf{v}_1)^2 - (\mathbf{v}_1\times\mathbf{v}_2)^2}{(1-v^2)^2}\dot{\mu}^2. \tag{53}$$

Putting

$$a = \sqrt{(\mathbf{v}_2 - \mathbf{v}_1)^2 - (\mathbf{v}_1\times\mathbf{v}_2)^2}, \tag{54}$$

we get from eq. (36) the time integral from t_1 to t_2

$$dl_v = \int_0^1 \frac{a d\mu}{1-v^2} \tag{55}$$

Using substitution

$$\mu = \frac{(1-v_1^2)\xi}{1 - \mathbf{v}_1\cdot\mathbf{v}_2 + (\mathbf{v}_1\cdot\mathbf{v}_2 - v_1^2)\xi}, \tag{56}$$

we get

$$l_v = \int_0^1 \frac{abd\xi}{b^2 - a^2\xi^2} = \frac{1}{2}\ln\frac{b+a}{b-a}, \tag{57}$$

where $b = 1 - \mathbf{v}_1 \cdot \mathbf{v}_2$ and

$$\frac{a}{b} = \tanh l_v; \quad \left(\frac{a}{b}\right)^2 = \tanh^2 l_v. \tag{58}$$

Or,

$$\frac{(\mathbf{v}_2 - \mathbf{v}_1)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2}{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)} = \tanh l_v. \tag{59}$$

The left side of (59) is the relative velocity, So

$$|v'| = \tanh l_v. \tag{60}$$

Putting

$$v_1 = \tanh l_{1v}, \quad v_2 = \tanh l_{2v} \tag{61}$$

we get for $\mathbf{v}_1 \parallel \mathbf{v}_2$

$$V' = \tanh(l_{2v} - l_{1v}) = \frac{\tanh l_{2v} - \tanh l_{1v}}{1 - \tanh l_{2v} \cdot \tanh l_{1v}}, \tag{62}$$

Or,

$$V' = \frac{v_2 - v_1}{1 - v_1.v_2}, \tag{63}$$

which is the famous Einstein formula for the addition of velocities.

8 The Lobačevskii-Fok triangle in particle physics

Let us investigate the angle between vectors of the velocities velocities of the two bodies. Let the vectors are taken with regard to the point which is in the state of rest. The vectors are $\mathbf{v}_1$ and $\mathbf{v}_2$. Then the obligate formula for cosine of the angle of the two vectors is:

$$\cos(\mathbf{v}_1, \mathbf{v}_2) = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1| \cdot |\mathbf{v}_2|}. \tag{64}$$

However if the reference point of vectors is moving with the velocity $\mathbf{u}$, then the angle between vectors are given by the relativistic formula:

$$\cos\alpha = \cos(\mathbf{v}'_1, \mathbf{v}'_2) = \frac{\mathbf{v}'_1 \cdot \mathbf{v}'_2}{|\mathbf{v}'_1| \cdot |\mathbf{v}'_2|}, \tag{65}$$

where the prime symbols are vectors after the Lorentz transformations of the velocities. The Lorentz transformation of the velocities is as follows:

$$\mathbf{v}' = \frac{\mathbf{v} - \mathbf{u} + (a_{00} - 1)\frac{\mathbf{u}}{u^2}\left((\mathbf{u} \cdot \mathbf{v} - u^2)\right)}{a_{00}\,(1 + \mathbf{u} \cdot \mathbf{v})}. \tag{66}$$

If we express $\mathbf{v}'_1$ and $\mathbf{v}'_2$ and by $\mathbf{v}_1$ and by $\mathbf{v}_2$, we then get cosine of the angle α as it follows:

$$\cos\alpha = \frac{(\mathbf{v}_1 - \mathbf{u}) \cdot (\mathbf{v}_2 - \mathbf{u}) - (\mathbf{v}_1 \times \mathbf{u}) \cdot (\mathbf{v}_2 \times \mathbf{u})}{\sqrt{(\mathbf{v}_1 - \mathbf{u})^2 - (\mathbf{v}_1 \times \mathbf{u})^2} \cdot \sqrt{(\mathbf{v}_2 - \mathbf{u})^2 - (\mathbf{v}_2 \times \mathbf{u})^2}}. \tag{67}$$

This is the expression for the cosine of the angle of the triangle in the space of Lobačevskii. In other words, this is cosine of the angle in the vertex $\mathbf{u}$ in the triangle with the vertexes at points $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2$ where the relative velocities $\mathbf{v}_1 - \mathbf{u}$ and $\mathbf{v}_2 - \mathbf{u}$ are sides of the triangle and form the angle α.

It can be explained by the different way. The length element in the Lobačevskii space corresponding to $d\mathbf{v}$ is

$$dl_v^2 = \frac{(d\mathbf{v})^2 - (\mathbf{v} \times d\mathbf{v})^2}{(1-v^2)^2}, \tag{68}$$

And the length element in the Lobačevskii space corresponding to $\delta\mathbf{v}$ is

$$\delta l_v^2 = \frac{(\delta\mathbf{v})^2 - (\mathbf{v} \times \delta\mathbf{v})^2}{(1-v^2)^2}. \tag{69}$$

Then we can define the relation for the cosine between $d\mathbf{v}$ and $\delta\mathbf{v}$ by the relation:

$$dl_v \delta l_v \cos\alpha = \frac{d\mathbf{v}\delta\mathbf{v} - (\mathbf{v} \times d\mathbf{v})(\mathbf{v} \times \delta\mathbf{v})}{(1-v^2)^2}. \tag{70}$$

The angle between the relative velocities can be considered as the angle in the of the Lobačevskii triangle. If we have three bodies moving with velocities $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, then the corresponding triangle will have the vertexes in points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and the relative velocities are the sides of the triangle. This construction is the analogue of the non-relativistic case, but we have here the Lobačevskii triangle on the Lobačevskii 2D manifold. The generalization of the Lobačevskii triangle to the Euler-Lobačevskii tetrahedron is evident.

Lobačevskii, in his pangeometry, presents the idea (many years before Einstein) that his geometry is probably realized in the near vicinity of atoms and molecules and also in the cosmical space (Norden, 1956). Now, we see that his geometry is realized in particle physics of LHC in CERN.

9 Discussion

The article is a preamble of the unification of the Lobačevskii-Fok geometry with the physics of elementary particles. The starting point was the Fermat principle formulated by means of variational calculus. The Poincaré model of the Lobačevskii geometry was derived as the optical model of the interaction of light with optical medium in space. Beltrami showed that the Lobačevskii geometry follows from spherical geometry by the elementary (Beltrami) operation $r \to ir$. The operation is not involved in the famous Euler monograph on spherical geometry (Euler, 1896). The operation was in our article generalized to the operation $r \to ir + \varrho$, $r.r \to r.r^*$, in section 4 and by, $r \to r + i\rho$, in section 5. Symbols ϱ and ρ are introduced as the new geometrical constants, which should not to be identified with the Einstein cosmological constant Λ.

Fok formulated the Lobačevskii geometry physically as the geometry of the relativistic velocity space. From this approach follows the adequate description of the decay of the neutral pi-meson into two gamma photons.

The Fermat principle enables to get the circular optical trajectories, or in other words the confinement of light by optical medium - so called optical black hole. It may be easy to prove it.

Let be the index of refraction $n(r)$ in the Euclidean plane with polar coordinates r, φ. The explicit form of the Fermat principle

$$\delta \int n(r)ds = 0 \tag{71}$$

is (Marklund et al., 2002)

$$\delta \int n(r)\sqrt{1 + r^2 \left(\frac{d\varphi}{dr}\right)^2} dr = 0. \tag{72}$$

The last equation is equivalent to the Euler-Lagrange variational equation for the functional $F(\varphi, \varphi')$

$$F_\varphi - \frac{d}{dr} F_{\varphi'} = 0. \tag{73}$$

Or,

$$\frac{d}{dr}\left[n(r) \frac{r^2 d\varphi/dr}{\sqrt{1 + r^2 \left(\frac{d\varphi}{dr}\right)^2}} \right] = 0. \tag{74}$$

It is evident that the elimination of $d\varphi/dr$ is as follows:

$$\frac{d\varphi}{dr} = \pm \frac{C}{\sqrt{r^4 n^2(r) - C^2 r^2}}. \tag{75}$$

The circular trajectory is defined by equation $dr/d\varphi = 0$ from which follows the index of refraction for the so called optical black hole

$$n(r) = \frac{const}{r}. \tag{76}$$

There is the Bose-Einstein condensate where the optical light pulses travel with extremely small group velocity about 17 meters per second (Hau et al. 1999). This is the possible way for testing the optical black hole.

Without doubt, the monochromatic optical beam is composed from photons of energy $E = \hbar\omega$. While the rest mass of photon is zero, the relativistic mass follows from the Einstein relation $E = mc^2$. After identifying the relativity energy and quantum energy of photon we have

$$m = \frac{\hbar\omega}{c^2}. \tag{77}$$

The centrifugal force acting on photon moving with velocity v in optical medium along the circle with radius r is for the photon mass as follows:

$$F_{centrifugal} = \frac{\hbar\omega}{c^2} \frac{v^2}{r}. \tag{78}$$

The centrifugal force and the Kapitza effect (thermal fluctuations of the index of refraction) (Landau, et al., 1982) are the origin of the instability of the photon trajectory in the optical medium. So, the experimental investigation of the confinement of photon in the optical medium is meaningful at temperature $T \approx 0$. There is no doubt that the investigation of the photon trajectories is the crucial problem of the optical physics and it is interesting for all optical laboratories over the world.

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