JACOB'S LADDERS AND THE $\tilde{Z}^2$-TRANSFORMATION OF THE ORTHOGONAL SYSTEM OF TRIGONOMETRIC FUNCTIONS

JAN MOSER

Abstract. It is shown in this paper that there is a continuum set of orthogonal systems relative to the weight function $\tilde{Z}^2(t)$. The corresponding integrals cannot be obtained in known theories of Balasubramanian, Heath-Brown and Ivic.

1. The first result

1.1. In this paper we obtain some new properties of the signal

$$Z(t)=e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right) \tag{1.1}$$

that is generated by the Riemann zeta-function, where

$$\vartheta(t)=-\frac{t}{2}\ln\pi+\mathrm{Im}\ln\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)=\frac{t}{2}\ln\frac{t}{2\pi}-\frac{t}{2}-\frac{\pi}{8}+\mathcal{O}\left(\frac{1}{t}\right). \tag{1.2}$$

Let us remind that

$$\tilde{Z}^2(t)=\frac{\mathrm{d}\varphi_1(t)}{\mathrm{d}t},\ \varphi_1(t)=\frac{1}{2}\varphi(t) \tag{1.3}$$

where

$$\tilde{Z}^2(t)=\frac{Z^2(t)}{2\Phi'_\varphi[\varphi(t)]}=\frac{Z^2(t)}{\left\{1+\mathcal{O}\left(\frac{\ln\ln t}{\ln t}\right)\right\}\ln t} \tag{1.4}$$

(see [12], (5.1)-(5.3)) and $\varphi_1(T),\ T\geq T_0[\varphi_1]$ is the Jacob's ladder.

1.2. It is known that the system of trigonometric functions

$$\left\{1,\cos\left(\frac{\pi}{l}t\right),\sin\left(\frac{\pi}{l}t\right),\dots,\cos\left(\frac{\pi}{l}nt\right),\sin\left(\frac{\pi}{l}nt\right),\dots\right\} \tag{1.5}$$

is the orthogonal system on the segment $[0,2l]$. In this direction the following theorem holds true.

Theorem 1. Let $\mathcal{J}(2l)=\varphi_1\{\mathring{\mathcal{J}}(2l)\}$, where

$$\begin{aligned}&\mathcal{J}(2l)=\mathcal{J}(2l,K)=[2lK,2l(K+1)],\\&\mathring{\mathcal{J}}(2l)=\mathring{\mathcal{J}}(2l,K)=\left[\overset{\circ}{\widehat{2lK}},\overset{\circ}{\widehat{2l(K+1)}}\right],\ 2lK\geq T_0[\varphi_1],\\&2l\in\left(0,\frac{T}{\ln T}\right];\ K\in\mathbb{N}.\end{aligned}$$

Key words and phrases. Riemann zeta-function.

Then the system of functions

$$\left\{1, \cos\left(\frac{\pi}{l}\varphi_1(t)\right), \sin\left(\frac{\pi}{l}\varphi_1(t)\right), \dots, \cos\left(\frac{\pi}{l}n\varphi_1(t)\right), \sin\left(\frac{\pi}{l}n\varphi_1(t)\right), \dots\right\} \tag{1.6}$$

is the orthogonal system on $\mathring{\mathcal{J}}(2l)$ with respect to the weight function $\tilde{Z}^2(t)$, i.e. the following new system of integrals

$$\begin{aligned}
&\int_{\mathring{\mathcal{J}}(2l)} \cos\left(\frac{\pi}{l}m\varphi_1(t)\right)\cos\left(\frac{\pi}{l}n\varphi_1(t)\right)\tilde{Z}^2(t)\mathrm{d}t = \left\{\begin{array}{ccc} 0 & , & m\neq n, \\ l & , & m=n, \end{array}\right. \\
&\int_{\mathring{\mathcal{J}}(2l)} \sin\left(\frac{\pi}{l}m\varphi_1(t)\right)\sin\left(\frac{\pi}{l}n\varphi_1(t)\right)\tilde{Z}^2(t)\mathrm{d}t = \left\{\begin{array}{ccc} 0 & , & m\neq n, \\ l & , & m=n, \end{array}\right. \\
&\int_{\mathring{\mathcal{J}}(2l)} \sin\left(\frac{\pi}{l}m\varphi_1(t)\right)\cos\left(\frac{\pi}{l}n\varphi_1(t)\right)\tilde{Z}^2(t)\mathrm{d}t = 0, \\
&\int_{\mathring{\mathcal{J}}(2l)} \cos\left(\frac{\pi}{l}n\varphi_1(t)\right)\tilde{Z}^2(t)\mathrm{d}t = 0, \\
&\int_{\mathring{\mathcal{J}}(2l)} \sin\left(\frac{\pi}{l}n\varphi_1(t)\right)\tilde{Z}^2(t)\mathrm{d}t = 0
\end{aligned} \tag{1.7}$$

for all $m,n\in\mathbb{N}$ is obtained, where

$$t-\varphi_1(t)\sim(1-c)\pi(t), \tag{A}$$

$$2l(K+1)<\overset{\circ}{\widehat{2lK}}, \tag{B}$$

$$\rho\{\mathcal{J}(2l);\mathring{\mathcal{J}}(2l)\}\sim(1-c)\pi(t)\to\infty, \tag{C}$$

as $K\to\infty$, and ρ denotes the distance of the corresponding segments, c is the Euler constant and $\pi(t)$ is the prime-counting function.

Remark 1. Theorem 1 gives the contact point between the functions $\zeta\left(\frac{1}{2}+it\right)$, $\pi(t)$, $\varphi_1(t)$ and the orthogonal system of trigonometric functions.

Remark 2. It is clear that the formulae (1.7) - for the modulated function $\tilde{Z}^2(t)$ - cannot be obtained in the known theories of Balasubramanian, Heath-Brown and Ivic (comp. [1]).

This paper is a continuation of the series [2]-[15].

2. New method of the quantization of the Hardy-Littlewood integral (a special case)

2.1. We obtain from the first two formulae in (1.7)

$$\begin{aligned}
&\int_{\mathring{\mathcal{J}}(2l)} \cos^2\left(\frac{\pi}{l}m\varphi_1(t)\right)\tilde{Z}^2(t)\mathrm{d}t = \frac{1}{2}|\mathcal{J}(2l)|, \\
&\int_{\mathring{\mathcal{J}}(2l)} \sin^2\left(\frac{\pi}{l}m\varphi_1(t)\right)\tilde{Z}^2(t)\mathrm{d}t = \frac{1}{2}|\mathcal{J}(2l)|
\end{aligned} \tag{2.1}$$

for all $m\in\mathbb{N}$. Next, from (2.1) we obtain

Corollary 1.

$$\int_{\mathring{\mathcal{J}}(2l)} \tilde{Z}^2(t)\mathrm{d}t = |\mathcal{J}(2l)|;\ |\mathcal{J}(2l)|=2l. \tag{2.2}$$

2.2. Let us consider now the problem concerning the solid of revolution corresponding to the graph of the function (comp. [5])

$$\tilde{Z}(t),\ t\in[\overset{\circ}{\widehat{2lK}},+\infty),\ 2lK>T_0[\varphi_1].$$

Problem. To divide this solid of revolution on parts of equal volumes.

From (2.2) we obtain the resolution of this problem.

Corollary 2. Since

$$\text{(a)}\qquad [\overset{\circ}{\widehat{2lK}},+\infty)=\bigcup_{r=1}^{\infty}\mathring{\mathcal{J}}(2l,r),\ \mathring{\mathcal{J}}(2l,r)=[\overset{\circ}{\widehat{2l(K+r-1)}},\overset{\circ}{\widehat{2l(K+r)}}],$$

$$\text{(b)}\qquad \pi\int_{\mathring{\mathcal{J}}(2l,r)}\tilde{Z}^2(t)\mathrm{d}t=2\pi l,\ r=1,2,3,\dots\ ,$$

it follows that the sequence of points

$$\{\overset{\circ}{\widehat{2l(K+r-1)}}\}_{r=2}^{+\infty}$$

is the resolution to the Problem for arbitrary fixed $2l\in(0,T/\ln T]$.

3. Generalization of the formula (2.2)

3.1. The following theorem holds true.

Theorem 2. Let

$$\mathcal{J}(T,U)=[T,T+U],\ J(T,U)=\varphi_1\{\mathring{\mathcal{J}}(T,U)\};\ \mathring{\mathcal{J}}(T,U)=[\mathring{T},\overset{\circ}{\widehat{T+U}}].$$

Then

$$\int_{\mathring{\mathcal{J}}(T,U)}\tilde{Z}^2(t)\mathrm{d}t=|\mathcal{J}(T,U)|=U, \tag{3.1}$$

for every $T\geq T_0[\varphi_1],\ U\in(0,T/\ln T]$.

Remark 3. From (3.1) the general method for quantization of the Hardy-Littlewood integral follows (comp. Corollary 2: $2lK\to\forall\ T\geq T_0[\varphi_1],\ \mathcal{J}(2l)\to\mathcal{J}(T,U)$).

Next, we obtain, using the mean-value theorem in (3.1)

Corollary 3.

$$\tilde{Z}^2(\xi)=\frac{|\mathcal{J}(T,U)|}{|\mathring{\mathcal{J}}(T,U)|},\ \xi\in\xi(\mathring{T},\overset{\circ}{\widehat{T+U}}),\ \tilde{Z}(\xi)\neq 0, \tag{3.2}$$

i.e.

$$\tilde{Z}^2(\xi):1=|\mathcal{J}(T,U)|:|\mathring{\mathcal{J}}(T,U)|.$$

3.2. Let $\{[T',T'+1]\}$ stands for the continuum set of segments $[T',T'+1]\subset [T,T+T/\ln T]$. Since

$$\frac{1}{|\overset{\circ}{\mathcal{J}}(T',1)|}=\tilde{Z}^2(\xi),\ \xi=\xi(T')\in(\overset{\circ}{T'},\overset{\circ}{\widehat{T'+1}})$$

then by the Riemann-Siegel formula

$$Z(t)=2\sum_{n\leq\sqrt{\frac{t}{2\pi}}}\frac{1}{\sqrt{n}}\cos\{\vartheta(t)-t\ln n\}+\mathcal{O}(t^{-1/4})$$

we obtain (see (1.4))

Corollary 4.

$$\frac{1}{\sqrt{|\overset{\circ}{\mathcal{J}}(T',1)|}}\sim\frac{2}{\sqrt{\ln\xi}}\left|\sum_{n\leq\sqrt{\frac{\xi}{2\pi}}}\frac{1}{\sqrt{n}}\cos\{\vartheta(\xi)-\xi\ln n\}+\mathcal{O}(\xi^{-1/4})\right| \tag{3.3}$$

where $\xi=\xi(T')$.

Remark 4. The formula (3.3) describes the complicated oscillations of the value $|\overset{\circ}{\mathcal{J}}(T',1)|$ generated by the nonlinear transformation $\mathcal{J}(T',1)=\varphi_1\{\overset{\circ}{\mathcal{J}}(T',1)\}$.

4. Proof of Theorems 1 and 2

4.1. Let us remind that the following lemma is true (see [12], (5.1)-(5.3))

Lemma. For every integrable function (in the Lebesgue sense) $f(x),\ x\in[\varphi_1(T),\varphi_1(T+U)]$ the following is true

$$\int_T^{T+U}f[\varphi_1(t)]\tilde{Z}^2(t)\mathrm{d}t=\int_{\varphi_1(T)}^{\varphi_1(T+U)}f(x)\mathrm{d}x,\ U\in(0,T/\ln T], \tag{4.1}$$

where $t-\varphi_1(t)\sim(1-c)\pi(t)$.

Remark 5. The formula (4.1) is true also in the case when the integral on the right-hand side of eq. (4.1) is convergent but not absolutely (in the Riemann sense).

4.2. If $\varphi_1\{[\overset{\circ}{T},\overset{\circ}{\widehat{T+U}}]\}=[T,T+U]$ then we obtain from (4.1) the following formula

$$\int_{\overset{\circ}{T}}^{\overset{\circ}{\widehat{T+U}}}f[\varphi_1(t)]\tilde{Z}^2(t)\mathrm{d}t=\int_T^{T+U}f(x)\mathrm{d}x,\ U\in(0,T/\ln T]. \tag{4.2}$$

Next, in the case $[T,T+U]=[2lK,2lK+2l]=\mathcal{J}(2l)$, we have

$$\int_{\mathcal{J}(2l)}F(t)\mathrm{d}t=\int_0^{2l}F(t)\mathrm{d}t \tag{4.3}$$

for every (integrable) $2l$-periodic function $F(t)$. Then from the known formulae

$$\begin{aligned}
&\int_{\mathcal{J}(2l)}\cos\left(\frac{\pi}{l}m\varphi_1(t)\right)\cos\left(\frac{\pi}{l}n\varphi_1(t)\right)\mathrm{d}t=\left\{\begin{array}{ccc}0 & , & m\not=n, \\ l & , & m=n,\end{array}\right. \\
&\int_{\mathcal{J}(2l)}\sin\left(\frac{\pi}{l}m\varphi_1(t)\right)\sin\left(\frac{\pi}{l}n\varphi_1(t)\right)\mathrm{d}t=\left\{\begin{array}{ccc}0 & , & m\not=n, \\ l & , & m=n,\end{array}\right. \\
&\int_{\mathcal{J}(2l)}\sin\left(\frac{\pi}{l}m\varphi_1(t)\right)\cos\left(\frac{\pi}{l}n\varphi_1(t)\right)\mathrm{d}t=0, \\
&\int_{\mathcal{J}(2l)}\cos\left(\frac{\pi}{l}n\varphi_1(t)\right)\mathrm{d}t=0,\quad \int_{\mathcal{J}(2l)}\sin\left(\frac{\pi}{l}n\varphi_1(t)\right)\mathrm{d}t=0,\quad m,n\in\mathbb{N},
\end{aligned}$$

by the $\tilde{Z}^2$-transformation (see (4.2), (4.3); $[\mathring{T},\overset{\circ}{\widehat{T+U}}]=\mathring{\mathcal{J}}(2l)$) the formulae (1.7) follow. The properties (B), (C) in Theorem 1 are identical with [13], (A1), (B1).

4.3. The formula (3.1) follows from (4.2) in the case $f(x)\equiv 1$.

5. Another type of the orthogonal systems

It follows from (4.2) that the continuum set $\mathcal{S}(T,2l)$ of the systems

$$\begin{aligned}
&\left\{|\tilde{Z}(t)|,|\tilde{Z}(t)|\cos\left(\frac{\pi}{l}(\varphi_1(t)-T)\right),|\tilde{Z}(t)|\sin\left(\frac{\pi}{l}(\varphi_1(t)-T)\right),\dots,\right. \\
&\left.|\tilde{Z}(t)|\cos\left(\frac{\pi}{l}n(\varphi_1(t)-T)\right),|\tilde{Z}(t)|\sin\left(\frac{\pi}{l}n(\varphi_1(t)-T)\right),\dots\right\}, \\
&t\in[\mathring{T},\overset{\circ}{\widehat{T+2l}}]
\end{aligned}$$

for all

$$T\geq T_0[\varphi_1],\ 2l\in(0,T/\ln T]$$

is the set of orthogonal systems on $[\mathring{T},\overset{\circ}{\widehat{T+2l}}]$.

Remark 6. Let us call the elements of the system $\mathcal{S}(T,2l)$ for fixed $2l\in(0,T/\ln T]$ and for all $T\geq T_0[\varphi_1]$ as *the clones* of the known orthogonal trigonometric system

$$\left\{1,\cos\left(\frac{\pi}{l}t\right),\sin\left(\frac{\pi}{l}t\right),\dots,\cos\left(\frac{\pi}{l}nt\right),\sin\left(\frac{\pi}{l}nt\right),\dots\right\},\ t\in[0,2l].$$

I would like to thank Michal Demetrian for helping me with the electronic version of this work.

References

[1] A. Ivic, 'The Riemann zeta-function', A Willey-Interscience Pub., New York, 1985.

[2] J. Moser, 'Jacob's ladders and the almost exact asymptotic representation of the Hardy-Littlewood integral', (2008), arXiv:0901.3973.

[3] J. Moser, 'Jacob's ladders and the tangent law for short parts of the Hardy-Littlewood integral', (2009), arXiv:0906.0659.

[4] J. Moser, 'Jacob's ladders and the multiplicative asymptotic formula for short and microscopic parts of the Hardy-Littlewood integral', (2009), arXiv:0907.0301.

[5] J. Moser, 'Jacob's ladders and the quantization of the Hardy-Littlewood integral', (2009), arXiv:0909.3928.

[6] J. Moser, 'Jacob's ladders and the first asymptotic formula for the expression of the sixth order $|\zeta(1/2+i\varphi(t)/2)|^4|\zeta(1/2+it)|^2$', (2009), arXiv:0911.1246.

[7] J. Moser, 'Jacob's ladders and the first asymptotic formula for the expression of the fifth order $Z[\varphi(t)/2+\rho_1]Z[\varphi(t)/2+\rho_2]Z[\varphi(t)/2+\rho_3]\hat{Z}^2(t)$ for the collection of disconnected sets', (2009), arXiv:0912.0130.

[8] J. Moser, 'Jacob's ladders, the iterations of Jacob's ladder $\varphi_1^k(t)$ and asymptotic formulae for the integrals of the products $Z^2[\varphi_1^n(t)]Z^2[\varphi^{n-1}(t)]\cdots Z^2[\varphi_1^0(t)]$ for arbitrary fixed $n\in\mathbb{N}$' (2010), arXiv:1001.1632.

[9] J. Moser, 'Jacob's ladders and the asymptotic formula for the integral of the eight order expression $|\zeta(1/2+i\varphi_2(t))|^4|\zeta(1/2+it)|^4$', (2010), arXiv:1001.2114.

[10] J. Moser, 'Jacob's ladders and the asymptotically approximate solutions of a nonlinear diophantine equation', (2010), arXiv: 1001.3019.

[11] J. Moser, 'Jacob's ladders and the asymptotic formula for short and microscopic parts of the Hardy-Littlewood integral of the function $|\zeta(1/2+it)|^4$', (2010), arXiv:1001.4007.

[12] J. Moser, 'Jacob's ladders and the nonlocal interaction of the function $|\zeta(1/2+it)|$ with $\arg\zeta(1/2+it)$ on the distance $\sim(1-c)\pi(t)$', (2010), arXiv: 1004.0169.

[13] J. Moser, 'Jacob's ladders and the $\tilde{Z}^2$ - transformation of polynomials in $\ln\varphi_1(t)$', (2010), arXiv: 1005.2052.

[14] J. Moser, 'Jacob's ladders and the oscillations of the function $|\zeta\left(\frac{1}{2}+it\right)|^2$ around the main part of its mean-value; law of the almost exact equality of the corresponding areas', (2010), arXiv: 1006.4316

[15] J. Moser, 'Jacob's ladders and the nonlocal interaction of the function $Z(t)$ with the function $\tilde{Z}^2(t)$ on the distance $\sim(1-c)\pi(t)$ for a collection of disconneted sets', (2010), arXiv: 1006.5158

Department of Mathematical Analysis and Numerical Mathematics, Comenius University, Mlynska Dolina M105, 842 48 Bratislava, SLOVAKIA

E-mail address: jan.mozer@fmph.uniba.sk