Exact Non-singular Solution of Einstein Field Equations for Spherically Symmetric Quantized Spacetimes

Babur M. Mirza
Department of Mathematics,
Quaid-i-Azam University, Islamabad 45320. PK.

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Abstract

According to classical general relativity, a spherically symmetric static spacetime involves an essential singularity at $r = 0$ and a Schwarzschild singularity at $r = 2GM/c^2$. We show that when quantum field effects in terms of quantum stresses are included, Einstein field equations yield a non-trivial spacetime without any singularity.

1 Introduction

The existence of spacetime singularities in classical general relativity has been established under some very general conditions, both for spherically symmetric and axially symmetric vacuum solutions to Einstein field equation [1] and coupled Einstein-Maxwell equations [2]. In fact the Schwarzschild spacetime is the first case of spherically symmetric solutions of vacuum field equations corresponding uniquely to a static gravitational field (Birkhoff theorem [3]). Whereas it is possible to find solution to the Einstein field equations with matter-energy sources violating the no-hair theorem and its generalizations [4], physical implications of these solutions have remained special. Existence of singularities in classical field theory however entails many theoretical anomalies including the infinite mass (energy) problem. On the other hand material properties of a system cannot be separated from quantum aspects of the phenomena. Quantum effects such as wave-particle duality and quantum interference leaves any classical field theory incomplete when attempting to describe matter (energy) and its interactions. The question thus naturally arises whether in a spacetime description of gravity involving quantum effects spacetime singularities persist.

Here we show that within the framework of the general theory of relativity this is not necessarily the case. We show that even in the so-called vacuum field (exterior spacetime) case, Einstein field equations possess solutions that are singularity free. The derivation followed here rests on two assumptions. First, the quantum density function is the primary physical representation of the material properties of a system. Secondly, the quantum field can interact with the gravitational field. Thus in the exterior region spacetime can be modified by a non-zero quantum field density.

We use throughout the Greek indices to designate the spacetime coordinates t, r, θ, φ.

2 Einstein Field Equations

We consider the spacetime metric

$$ds^2 = e^{2\nu(r)}dt^2 - e^{2\lambda(r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2, \tag{1}$$

around a spherically symmetric static mass distribution with matter-energy stresses described by the covariant stress energy-momentum tensor

$$T_{\mu\nu} = \rho_m m v_\mu v_\nu + \rho m u_\mu u_\nu + P_{\mu\nu}, \tag{2}$$

with m being the elementary particle (field) mass, $\rho = \rho(\mathbf{r}, t)$ the quantum density function related to the wave function ψ as $\rho = |\psi|^2$, and $u_\mu \equiv -(D/\rho)\partial_\mu \rho$, $P_{\mu\nu} \equiv -(\hbar^2/2m)\partial_\mu\partial_\nu\rho$ where $D = \hbar/2m$. The first term in the stress tensor (2) corresponds to the Newtonian mass density ρ_m, and the last two terms represent quantum stresses in the system. The quantum stresses depend on the quantum density function and may not be zero for the exterior region. A reduction of (2) to the flat 3-space indicates that the $P_{\mu\nu}$ can be interpreted as due to shape stresses in the field, where as $\rho m u_\mu u_\nu$ corresponds to the stationaty flow stresses.

Putting into Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}, \tag{3}$$

the corresponding set of coupled non-identical equations for the stationary fields are

$$e^{-2\lambda}(1 - 2r\lambda') - 1 = 0, \tag{4}$$

$$e^{-2\lambda}(1 - 2r\lambda') - 1 = -\frac{8\pi G}{c^4} r^2 T_{rr}, \tag{5}$$

$$\nu'' + \nu'^2 - \lambda'\nu' + \frac{\nu' - \lambda'}{r} = 0, \tag{6}$$

where the prime indicates differentiation with respect to r.

3 Spacetime Geometry

From equation (5) we obtain

$$e^{-2\lambda} = 1 - \frac{2GM(r)}{c^2 r}, \tag{7}$$

where

$$M(r) \equiv \frac{4\pi}{c^2} \int_0^r r^2 T_{rr} dr, \tag{8}$$

is the total mass enclosed within a sphere of radius r. Also, we have $\lambda' = -\nu'$.

Let R denote the stellar radius, then the total mass can be written as

$$M_{total} = M(R) + \widetilde{M}, \tag{9}$$

where

$$M(R) \equiv \int_0^R \frac{4\pi r^2}{c^2} T_{rr} dr, \tag{10}$$

is the mass contained inside the star. In the exterior region there are no material stresses, therefore $T_{rr} = 0$ in equation (10). However, corresponding to the quantum field stresses in the region, we have for the quantum mass term

$$\widetilde{M} \equiv \int_R^\infty \frac{4\pi r^2}{c^2} \left(P_{rr} + \rho m {u_r}^2\right) dr. \tag{11}$$

To explicitly calculate the quantum mass, we make use of the energy conservation equation including the quantum potential $-\hbar^2 \left(\partial_{rr}(r\sqrt{\rho})\right) / \left(2mr\sqrt{\rho}\right)$ which gives,

$$\frac{\rho'}{r\rho} - \frac{\rho'^2}{4\rho^2} + \frac{\rho''}{2\rho} = -\frac{2mE}{\hbar^2}, \tag{12}$$

where ρ is the quantum density function for the mass distribution with total mass M and total energy E.

For the non-trivial case when ρ is not a constant, equation (12) gives for the quantum density function

$$\rho(r) = \frac{iA}{r^2} e^{-2r/\lambda\!\!\!-}, \tag{13}$$

where $\lambda\!\!\!-^{-1} = \sqrt{-2mE/\hbar^2}$ and $A = e^{2R/\lambda\!\!\!-}/2\pi\lambda\!\!\!-$ is the normalization constant for the exterior region (R, ∞). With $E > 0$, we therefore denote $\lambda\!\!\!- = i\alpha$ where $\alpha = \sqrt{\hbar^2/2mE}$, so that

$$\rho(r) = \frac{\cos[2(r-R)/\alpha]}{2\pi\alpha r^2}, \tag{14}$$

under the assumption that the quantum density function ρ is real.

Also, using equation (12), equation (11) can be written as

$$\widetilde{M} = -\frac{\hbar^2}{m} \int_R^\infty \frac{4\pi r^2}{c^2} \left(E\rho + \frac{1}{r}\frac{\partial \rho}{\partial r}\right) dr. \tag{15}$$

We notice that in the second term of equation (15) we have by integration by parts

$$\int_R^\infty (r\frac{\partial \rho}{\partial r})dr = r\rho \mid_R^\infty -1, \tag{16}$$

using the normalization of quantum density function over (R, ∞). The rhs of equation (16) is a constant, and in equation (15) can be absorbed by re-scaling $\widetilde{M}$. The remaining term thus gives for any point r in space

$$\widetilde{M} = -\frac{\hbar^2 M}{m} \sin \frac{2r}{\alpha}. \tag{17}$$

We denote in equation (17) $-\hbar^2/m$ by C, and re-scale $\widetilde{M}$ by C, and also r by $\pi\alpha$. Here the sign of C depends on the elementary mass m. We can now denote the total mass as determined in the exterior region by $\widetilde{M}$ and thus have for the spacetime metric

$$ds^2 = \Big(1 - \frac{r_S}{r} \sin 2\pi r\Big) dt^2 - \Big(1 - \frac{r_S}{r} \sin 2\pi r\Big)^{-1} dr^2 - d\Omega^2, \tag{18}$$

where $r_S \equiv 2GM/c^2$ and $d\Omega^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$. Since, except at $r = 0$, there is no point where $r_S \sin 2\pi r = r$ holds, the Schwarzschild singularity is removed. Also, as $r \to 0$, $\sin 2\pi r/r$ approaches 2π, and the coordinate singularity does not occur also.

4 Conclusions

The modified mass $\widetilde{M}$ depends on the signature of the elementary mass m, thus the case of similar mass sign implies a repulsive gravitational potential. The relative sign different between M and and m is indicative of whether the corresponding energy densities in the field are alike or opposite. The presence of mass m however does not violate the equivalence principle of general relativity, for the field potential has no explicit dependence on the field mass, in fact one can scale the physical units so that $-\hbar^2/m$ is unity. However, it indicates that the quantum effects enter the otherwise classical field potential solely by the quantum interference term $\sin 2\pi r$.

References

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