The Equivalence Principle Applicability Boundaries,Measurability and UVD in QFT

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This paper presents, within the scope of the earlier introduced measurability concept, a study of the ultra-violet behavior quantum field theories. It is demonstrated that in the case of the natural assumptions there are no ultra-violet divergences in these theories. In so doing, the methods of a lattice quantum-field theory are used. Applicability of the obtained results to different energy scales is discussed.

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I. INTRODUCTION

In this paper the author continues a study of QFT and gravity within the scope of the **measurability** concept introduced in his previous works [1]–[7]. In [8] it has been demonstrated that, in the context of the equivalence-principle applicability boundaries (EPAB), a perturbation theory for QFT in terms of **measurable** quantities as applied to the scalar model φ^4 is free from ultra-violet divergences (UVD).
The principal objective of the present manuscript is to extend the methods and results from [8] to the theories of a more general type, in particular to the Yang-Mills fields interacting with fermions.
The structure of this paper is as follows.
Section 2 presents in short the preliminary information necessary for understanding of the essence of this work. In Section 3 the correlation between the **measurability** and the lattice approach in QFT is revealed. Section 4 is central in this presentation. In its first part the principal result associated with the model φ^4 from [8] is elucidated. In the second part of this section it is shown that the methods and results presented in the first part may be generalized to the quantum theory of gauge fields interacting with fermion fields of the matter. Moreover, the general mechanism enabling one to demonstrate, within the scope of EPAB, the absence of UVD in a perturbation theory of any Lattice Quantum Field Theory (LQFT),and of its continuous limit.

II. IMPORTANT PRELIMINARY INFORMATION BRIEFLY

A. Measurability Concept in Quantum Theory and Gravity

In this Section we briefly consider some of the results from [1]–[7] which are essential for subsequent studies. Without detriment to further consideration, in the initial definitions we lift some unnecessary restrictions and make important specifications.
Presently, many researchers are of the opinion that at very high energies (Plank's or trans-Planck's) the ultraviolet cutoff exists that is determined by some maximal momentum.
Therefore, it is further assumed that there is a maximal bound for the measurement momenta $p = p_{max}$ represented as follows:

$$p_{max} \doteq p_\ell = \hbar/\ell, \tag{1}$$

where ℓ is some small length and $\tau = \ell/c$ is the corresponding time. Let us call ℓ the *primary* length and τ the *primary* time.
Without loss of generality, we can consider ℓ and τ at Plank's level, i.e. $\ell \propto l_p, \tau = \kappa t_p$, where the numerical constant κ is on the order of 1. Consequently, we have $E_\ell \propto E_p$ with the corresponding proportionality factor, where $E_\ell \doteq p_\ell c$.

Explanation. In the theory under study it is not assumed from the start that there exists some minimal length l_{min} and that ℓ is such. In fact, the minimal length is defined with the use of Heisenberg's Uncertainty Principle (HUP) $\Delta x \cdot \Delta p \geq \frac{1}{2}\hbar$ or of its generalization to high (Planck) energies – Generalized Uncertainty Principle (GUP) [9]–[17], for example, of the form [9]

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_p^2 \frac{\Delta p}{\hbar}, \tag{2}$$

where α' is a constant on the order of 1. Evidently this formula (2) initially leads to the minimal length $\widetilde{\ell}$ on the order of the Planck length $\widetilde{\ell} \doteq 2\sqrt{\alpha'} l_p$. Besides, other forms of GUP [17] also lead to the minimal length.

Thus, we should note that in all the works l_{min} is actually (but not explicitly) introduced on the basis of some measuring procedure (different forms of the Generalized Uncertainty Principle (GUP)). In any form GUP in turn is a high-energy generalization of HUP. But in the original proof of HUP a planar geometry of the initial space-time was actively used [18]. Extension of this principle to other pairs of conjugate variables is also valid only for quantum mechanics in the planar geometry space [19]. As HUP is a local principle, at low energies in the curved space-time, by virtue of Einstein's Equivalence Principle, we can consider that in a fairly small neighborhood of any point the geometry is planar an hence HUP is valid too. But all the results obtained point to the fact that l_{min} should be at a level of l_p,i.e. $l_{min} \propto l_p$, or even should be smaller. As noted in the Section 2 of [7], at the Planck scales Einstein's Equivalence Principle is obviously inapplicable, and there is no way to use the measuring procedure ignoring the space geometry at these scales. Meantime, none of the GUP forms [17] makes an effort to include it and is hardly completely correct. Moreover, there are some serious arguments against GUP as demonstrated in Section IX of the review paper [17]. The foregoing considerations support argumentation against the introduction of l_{min} from the start.

Because of this, in the present work the validity of this principle is not implied from the start too. GUP is given merely as an example. As p_{max} (1) is taken at Planck's level, it is clear that HUP is inapplicable. Taking this into consideration, the existence of a certain minimal length $\widetilde{\ell}$ is not mandatory. So, we start from the *primary* length ℓ and the *primary* time τ. The whole formalism, developed in [1]–[6] on condition that ℓ is the minimal length, is valid for the case when ℓ is the *primary* length but now we can lift the formal requirement for involvement of l_{min} in the theory from the start.

There is one more barrier for the use of l_{min} in the theory as indicated in [16] and other works (for example, [17]). In the above-mentioned papers, it has been noted that there is a nonzero minimal uncertainty in position, i.e. l_{min} implies that there is no physical state which is a position eigenstate since an eigenstate would, of course, have zero uncertainty in position. So, in this case in a quantum theory we have the momentum representation rather than the position representation, and the quantum theory becomes very depleted.

The question arises whether the introduction of p_{max} is naturally associated with the involvement of a minimal length. But this is the case only when at the energies E_{max} corresponding to p_{max} we have the substantiated measuring procedure. Unfortunately, this is not the case.

Note that in the canonical QFT in continuous space-time (i.e. without l_{min})[20]–[23] measurements of the contributions in the loop amplitudes involve the standard cut-off procedure for some large (maximal) momentum $p_{cut} \doteq p_{max}$. Then it is demonstrated that the theory at low energies $p \ll p_{cut}$ is in fact independent of the selection of $p_{cut} \doteq p_{max}$. Of course, the theory still remains to be continuous [20]–[23]. In this case we make another step forward, relating the corresponding length $\ell = \hbar/p_{max}$ to p_{max} and constructing on its basis a low-energy theory very close to the initial continuous theory. Now we have the naturally derived parameter ℓ for the construction of a high-energy deformation of this theory at the energies $E \approx E_{max}$ within the scope of determining the physical theory deformation [24]. So, we start from the *primary* length ℓ and the *primary* time τ. The whole formalism, developed in [1]–[6] on condition that ℓ is the minimal length, is valid for the case when ℓ is the *primary* length but now we can lift the formal requirement for involvement of l_{min} in the theory from the start.

In what follows we mainly make references to [6] and [7]. In particular, the basic definitions **Primary Measurability**,**Generalized Measurability**, **Primarily Measurable Quantities(PMQ)**, **Primarily Measurable Momenta(PMM)**, **Generalized Measurable Quantities(GMQ)** and the like are given in Section II of [6].

Besides, in Section III from [6] it has been demonstrated how, at low energies $E \ll E_p$, the arbitrary metric $g_{\mu\nu}(x)$ may be derived in terms of **measurable** quantities.

It should be noted that in virtue of assumption in [7] *observables* in **measurable** theory are **Primarily Measurable Quantities**.

B. Relativistic Invariance, Equivalence Principle Applicability Boundary, and QFT in Flat Space

The canonical quantum field theory (QFT) [20]– [23] is a local relativistically-invariant theory considered in continuous space-time with a plane geometry, i.e with the local Minkowskian metric $\eta_{\mu\nu}(\overline{x})$. And this assumption is valid for all the energy range. Still, it is quite clear that the quantum processes associated with QFT (particle collisions, decay,...) can introduce perturbations into the space-time geometry, varying its curvature. But as QFT is a local theory, a strong Equivalence Principle (EP) [25] enables one, in a **sufficiently small** region $\mathcal{V}_r$ of the fixed point, to consider space-time as a flat space in this case too. Consequently, we naturally think about the applicability boundary of this principle. In Section 2 of [7] this problem has been thoroughly studied.

In essence, **sufficiently small** $\mathcal{V}_r$ means that the region $\mathcal{V}^{'}$, for which $\overline{x} \in \mathcal{V}^{'}_{r'} \subset \mathcal{V}_r$ with $r' < r$ (here r, r' are characteristic spatial sizes of $\mathcal{V}_r$ and $\mathcal{V}'_r$, respectively), satisfies the condition $g_{\mu\nu}(\overline{x}) \equiv \eta_{\mu\nu}(\overline{x})$, where $\eta_{\mu\nu}(\overline{x})$ is Minkowskian metric. In this way we can construct the sequence

$$... \subset \mathcal{V}^{''}_{r''} \subset \mathcal{V}^{'}_{r'} \subset \mathcal{V}_r, \\ ... < r^{''} < r' < r \qquad (3)$$

The problem arises, is there any lower limit for the sequence in formula (3)?

The answer is positive. Currently, there is no doubt that at very high energies (on the order of Planck energies $E \approx E_p$), i.e. on Planck scales, $l \approx l_p$ quantum fluctuations of any metric $g_{\mu\nu}(\overline{x})$ are so high that in this case the geometry determined by $g_{\mu\nu}(\overline{x})$ is replaced by the "geometry" following from **quantum foam** that is defined by great quantum fluctuations of $g_{\mu\nu}(\overline{x})$,i.e. by the characteristic spatial sizes of the quantum-gravitational region (for example, [26]–[31]). The above-mentioned geometry is drastically differing from the locally smooth geometry of continuous space-time and EP in it is no longer valid [32]–[39]. Actually,

the **quantum foam** is not geometry in a common sense as locally it is determined by a set of different metrics, each of which is taken into consideration with its statistical weight [29].
From this it follows that the region $\mathcal{V}_{\overline{r},\overline{t}}$ with the characteristic spatial size $\overline{r} \approx l_p$ (and hence with the temporal size $\overline{t} \approx t_p$) is the lower (approximate) limit for the sequence in (3).
In this way EP has the applicability boundary that, at least, lies in the region of Planck energies and hence the relativistic invariance must be violated at the same energy scales because its applicability necessitates space-time with the locally flat geometry, just supported by EP.
It should be noted that initially strong EP has been formulated for the macroscopic case (i.e. for the space-time domains of great size) that is beyond quantum consideration. On extension of this principle to microscopic domains, the problem of its applicability boundaries is absolutely natural.
It is difficult to find the exact lower limit for the sequence in formula (3) – it seems to be dependent on the processes under study. Section 2 of [7] presents the arguments that it should be associated with the energy scales $E \ll E_p$.
Therefore, it is assumed that the Equivalence Principle is valid for the locally smooth space-time and this suggests that all the energies E of the particles in the most general form meet the necessary condition

$$E \ll E_p. \tag{4}$$

As validity of RI requires the applicability of EP, we can consider the condition (4) a necessary condition for the validity of RI. Then, if not stipulated otherwise, we can assume that the condition (4) is valid.
The canonical quantum field theory (QFT) [20]– [23] is a local theory considered in continuous space-time with a local plane geometry, i.e with the Minkowskian metric $\eta_{\mu\nu}(\overline{x})$. In addition, it is assumed that all objects in QFT are point-like. However, as noted above, this assumption will be true to a certain limit: the assumptions that (a) even local space-time geometry is plane and (b) all objects in QFT are point-like have natural applicability boundaries directly specifying the EP applicability boundary.
Within the scope of the canonical QFT, the process of passage to more higher energies without a change in the local curvature has no limits [20]–[23], just this fact is the reason for ultraviolet divergences in QFT.
However, on passage to the Planck energies $E \approx E_p$ (Planck scales $l \approx l_p$), the space in the Planck neighborhood $\mathcal{V}_{\overline{r},\overline{t}}$ of the point $\overline{x}$ one cannot consider flat even locally and in this case (as noted above) EP is not valid.
Then we introduce the following assumption:

Assumption 2.1
In the canonical QFT in calculations of the quantities it is wrong to sum (or same consider within a single sum) the contributions corresponding to space-time manifolds with locally nonzero or zero curvatures since these contributions are associated with different processes: (1) with the existence of a gravitational field that, in principle, can hardly be excluded; (2) in the absence of a gravitational field.
From the start, we can isolate the case when EP is valid and hence RI takes place (at sufficiently low energies, specifically satisfying the condition (4)) from the cases when EP becomes invalid (for example, Planck energies $E \approx E_p$).

Remark 2.2
According to **Assumption 2.1**, we should consider two limiting cases:
(a)low energies $E \ll E_p$ and
(b)very high (essentially maximal) energies $E \approx E_p$.
Then it should be noted that, as all the experimentally involved energies E are low, they satisfy condition a). Specifically, for LHC maximal energies are $\sim 10TeV = 10^4 GeV$, that is by 15 orders of magnitude lower than the Planck energy $\sim 10^{19} GeV$.
Moreover, the characteristic energy scales of all fundamental interactions also satisfy condition a). Indeed, in the case of strong interactions this scale is $\Lambda_{QCD} \sim 200MeV$; for electroweak interactions this scale is determined by the vacuum average of a Higgs boson and equals $v \approx 246GeV$; finally, the scale of the (Grand Unification Theory (GUT)) M_{GUT} lies in the range of $\sim 10^{14}GeV - -10^{16}GeV$. It is obvious that all the above figures satisfy condition a).
Thus, only the expected characteristic energy scale of quantum gravity satisfies condition b).

From **Remark 2.2** it directly follows that even very high energies arising on unification of all the interaction types $M_{GUT} \approx 10^{14}GeV- \sim 10^{16}GeV$,(except of gravitational),satisfy the condition (4).
At the same time, it is clear that the RI validity requirement in canonical QFT [20]– [23],due to the action of Lorentz boost (or same hyperbolic rotations) (formula (3) in [40]), results in however high momenta and energies. But it has been demonstrated that unlimited growth of the momenta and energies is impossible because in this case we fall within the energy region, where the conventional quantum field theory is invalid. This section supports the validity of the fact in the general case of the canonical QFT in continuous space-time as well.
Note that at the present time there are experimental indications that RI is violated in QFT on passage to higher energies (for example, [41]). Besides, one should note important recent works associated with EP applicability boundaries and violation in nuclei and atoms at low energies (for example [42]). We can mention other works indicating the applicability boundaries of EP for specific processes, especially associated with the context of this paper (for example,

[43],[44]). Proceeding from the above, the requirement for RI and EP is possible only within the scope of the condition (4).
Due to the condition $E \ll E_p$ and to the results of Section 2 in [7], all conclusions made in this section are valid both for the canonical Quantum theory in continuous space-time, [20]– [23], and for its **measurable** analog in Section 2 of [7],(Subsection 2.1 in the present paper).

Remark 2.3
Why in canonical QFT it is so important never forget about the fact that space-time has a flat geometry, or the same possesses the Minkowskian metric $\eta_{\mu\nu}(\overline{x})$? Simply, in the contrary case we should refuse from some fruitful methods and from the results obtained by these methods in canonical QFT, in particular from Wick rotation [23]. In fact, in this case the time variable is replaced by $t \mapsto it \doteq t_E$, and the Minkowskian metric $\eta_{\mu\nu}(\overline{x})$ is replaced by the four-dimensional Euclidean metric

$$ds^2 = dt_E^2 + dx^2 + dy^2 + dz^2. \qquad (5)$$

Clearly, such replacement is possible only in the case when from the start space-time (locally) has a flat geometry, i. e. possesses the Minkowskian metric $\eta_{\mu\nu}(\overline{x})$. This is another argument supporting the key role of the EP applicability boundary. Otherwise, when we go beyond this boundary, Wick rotation becomes invalid. Naturally, some other methods of canonical QFT will lose their force too.
2.2.1 *In this paper we consider two limiting energy scales:$E \ll E_p$ and $E \approx E_p$. Of course, the whole energy range $0 < E \leq E_p$ is not reduced to these scales. But, assuming that the onset of the Universe had started from the energies close to the Planck energies E_p and its expansion was very fast, the above boundary is reasonable. An additional argument in support is the fact that, as noted in* **Remark 2.2**, *the energy ranges for all the fundamental models combining various interactions are associated with these scales.*
2.2.2 It is clear that the equivalence-principle applicability boundaries (EPAB) in each specific case are dependent on the particular processes studied in particle physics. *In what follows we consider only the energy range $E \ll E_p$ assuming that the common EPAB lies within $0 < E \leq\approx 10^{-2}E_p$.*

III. QFT IN MEASURABLE FORM AS LATTICE FIELD THEORY

From Section 4 in [7] it follows directly that the **measurable** approach generates a Lattice Quantum Field Theory (LQFT). Hereinafter we use the symbols, terms, and results from the (LQFT) [45],[46].
Then it is assumed that the theory under study is considered in a sufficiently large hypercubic box with the edge length L and space-time size L^4, where $L = N_L\ell, N_L \gg 1$. In general, not necessarily N_L is an integer number.
For convenience, let us introduce the following:

$$\Omega \doteq L^4. \qquad (6)$$

Assuming that t varies over the interval $0 \leq t \leq T, T \neq L$, (6) will take the form (formula (2.3) in [46])

$$\Omega \doteq VT = L^3T. \qquad (7)$$

In what follows, when not stated otherwise, we assume $T = L$ and hence formula (7) takes the form (6).
Without loss of generality, we can assume that all integer N_{x_μ} are equal to each other and are equal to some integer $N \gg 1$ maximally high in the absolute value. Then, according to the present consideration, in the **measurable** form there arises a lattice model of the position representation with $a = \ell/N$, where a is the lattice distance or same lattice spacing (section 2.5 from [45]).
In line with the general approach, in LQFT we have [45]

$$L = aM = \frac{\ell}{N}M, i.e. \frac{L}{M} = \frac{\ell}{N}, \qquad (8)$$

where $M \gg 1$ is an integer number. It is obvious that $M/N = N_L$, i.e. $M \gg N_L$.
As L is great, also without loss of generality, it is assumed that the periodic boundary conditions (formula (2.58) in [45])) are valid

$$\phi(x + L) = \phi(x). \qquad (9)$$

Then all formulae of LQFT in the position representation (Sections 2.5,2.6 in [45]) are valid for the **measurable** form of a continuous theory. And formula (2.54) from [45]

$$\sum_x f(x) \rightarrow \int_0^L d^4x f(x), M \rightarrow \infty, a = \frac{L}{M}, \quad L \quad fixed \qquad (10)$$

may be rewritten for such consideration with substitution of $f(x) \rightarrow \mathcal{L}$ under the integration sign for $f(x) \rightarrow \mathcal{L}_{meas,\{N\}}$ within the summation, and $a \rightarrow \ell/N$, where $\mathcal{L}$ and $\mathcal{L}_{meas,\{N\}}$ are correspondent formulae in continuous and measurable cases.
Since ℓ is also a fixed quantity, it is clear that the conditions $M \rightarrow \infty$ and $N \rightarrow \infty$ in the case under study are equivalent, representing the thermodynamic limit that gives a continuous pattern. Note that in this case we can use the results from Sections 2.5 and 2.6 of [45],assigning a_t as the temporal lattice distance $a_t \doteq \tau/N_t$, where τ/N_t is taken from formula (4) in [6].
Thus, in the coordinate representation the studied lattice of **measurable** quantities may be regarded as a canonical space-time lattice of LQFT, with the spacing $a = \ell/N$ and temporal distance $a_t = \tau/N_t$.

In this case all the basic operators in Sections 2.5 and 2.6 of [45] have their analogs in the present work. Specifically, finite-differences operators $\partial_\mu\varphi_x, \partial'_\mu\varphi_x$ from formulae (2.55),(2.56) in [45] and formulae of Section 2.3 in [47] in the present paper correspond to the operators $\frac{\mathbf{\Delta}}{\mathbf{\Delta_N}}$ from formula (9) in [6] for positive and negative values of $\mathbf{N}$. The transfer-operator $\hat{T}$ may be constructed for the lattice of interest, with the spacing $a = \ell/N$ and temporal distance $a_t = \tau/N_t$, in accordance with formulae (2.71),(2.74) of [45], so all the formulae from Section 2.6 in [45] are valid for this case. We assume that $a_t = a$.

For the lattice values of momenta, in the momentum representation, according to formula (2.81) in [45], we have

$$p_\mu(latt) = n_\mu \frac{2\pi}{L}, \tag{11}$$

where n_μ are integers.

Consequently, the lattice edge in the momentum representation $\Delta p_\mu(latt)$ adopts the value

$$\Delta p_\mu(latt) = \frac{2\pi}{L} \propto \frac{1}{N_L}, \tag{12}$$

where it is assumed that $\hbar = 1$.

At the same time, the integer numbers n_μ are varying in magnitude over the interval $[0, N_L N]$,, where $N_L N = L/a$ (formula (2.82) in [45]). As a result, in the case of interest a maximum value of the momentum along any axis will be given by

$$p_{latt,max} = \frac{\pi}{a} = \frac{\pi}{\ell/N} = \frac{\pi N}{\ell} \doteq \Lambda. \tag{13}$$

Formula (13) gives an explicit expression for a maximal lattice momentum $p_{latt,max} = \Lambda$. To be more exact, the momenta are restricted to the so-called first Brillouin zone (**BZ**) $\mathcal{B}$ (formula (1.218) from [46])

$$\mathcal{B} \doteq \{p|\frac{-\pi}{a} < p_\mu \leq \frac{\pi}{a}\}. \tag{14}$$

It is clear that $p_{latt,max} = \Lambda \gg p_\ell$. As follows from formula (13), $\Lambda \propto N p_\ell, N \gg 1$,i.e. the boundary of **BZ** Λ passes far beyond the region of the physical energy values.

But due to the condition $E \ll E_p$, we consider only a low-energy part of the lattice, the momenta of which are given as $p \approx \frac{\hbar}{N^*\ell}$ with $|N^*| \gg 1$. Because of this, in the case under study only particular momenta may be maximal (so-called "maximally reachable" momentum) $p_{max,reach}$ and $p_{max,reach} \ll p_{latt,max}$.

In this way **BZ** in formula (14) is narrowed significantly

$$-p_{max,reach} \leq p_\mu \leq p_{max,reach}, \tag{15}$$

where $p_{max,reach} \ll p_\ell$.

As $a = \ell/N$,where $N \gg 1$, when the mass m is fixed, am is close to zero and hence the correlation length ξ (formula (1.224) in [46])

$$\xi \equiv \frac{1}{am} = \frac{N}{\ell m} \tag{16}$$

is finite but very great. Passage to a continuum limit $\xi \to \infty$ means going to $N \to \infty$. In this case, within the constant factor m^{-1}, we have

$$\xi = \frac{N}{\ell} \propto N p_\ell \approx N p_{pl} \propto p_{latt,max} = \Lambda. \tag{17}$$

From formulae (13),(15) it follows directly that

$$p_{max,reach} = \frac{p_l}{\tilde{N}} = \frac{\Lambda}{N\tilde{N}}, N \gg 1, \tilde{N} \gg 1. \tag{18}$$

Then, proceeding from the formulae above, in the case of interest (**BZ**) $\mathcal{B}$ (14) is narrowed to $\mathcal{B}_\mathcal{N}$

$$\mathcal{B}_\mathcal{N} \doteq \{p|\frac{-\pi}{N\tilde{N}a} < p_\mu \leq \frac{\pi}{N\tilde{N}a}\}, N \gg 1, \tilde{N} \gg 1. \tag{19}$$

Lattice summation in the general case is given by formula (2.7) from [46]

$$\int_{p\in\mathcal{B}} \doteq \int_{\mathcal{B}} \equiv \frac{1}{a^4\Omega} \sum_{p\in\mathcal{B}}. \tag{20}$$

In the case under study the lattice summation takes the form

$$\int_{p\in\mathcal{B}_\mathcal{N}} \doteq \int_{\mathcal{B}_\mathcal{N}} \equiv \frac{1}{a^4\Omega} \sum_{p\in\mathcal{B}_\mathcal{N}}. \tag{21}$$

Respectively, on passage to the thermodynamic limit $L \to \infty, T \to \infty$, in the general case we arrive at formula (2.8) in [47]

$$\int_{p\in\mathcal{B}} = \frac{1}{(2\pi)^4} \int_{\frac{-\pi}{a}}^{\frac{\pi}{a}} d^4p. \tag{22}$$

In the case of interest (22) is transformed to

$$\int_{p\in\mathcal{B}_\mathcal{N}} = \frac{1}{(2\pi)^4} \int_{\frac{-\pi}{N\tilde{N}a}}^{\frac{\pi}{N\tilde{N}a}} d^4p. \tag{23}$$

Remark 3.1

As a rule, in the literature devoted to LQFT it is assumed that the lattice edge a is equal to 1. Then the formula for the first Brillouin zone $\mathcal{B}$ (14) is of the form

$$\mathcal{B} \doteq \{p| -\pi < p_\mu \leq \pi\}. \tag{24}$$

Whereas for the "short-cut" Brillouin zone $\mathcal{B}_\mathcal{N}$ (19) we have

$$\mathcal{B}_\mathcal{N} \doteq \{p|\frac{-\pi}{N\tilde{N}} < p_\mu \leq \frac{\pi}{N\tilde{N}}\}, N \gg 1, \tilde{N} \gg 1, \tag{25}$$

with the corresponding changes in all other formulae.

IV. PERTURBATION THEORY IN CONTINUOUS AND MEASURABLE CASES

A. Simple Scalar Model φ^4

The canonical Lagrangian for model φ^4 in continuous space-time has the form [20]

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu \varphi)^2 - \frac{m_0^2}{2}\,\varphi^2 - \frac{\mathrm{g}_0}{4!}\varphi^4 \tag{26}$$

where $\mathcal{L}_0 \doteq \frac{1}{2}((\partial_\mu\varphi)^2 - m_0^2\varphi^2)$ is free fields Lagrangian and $\mathcal{L}_I \doteq -\frac{\mathrm{g}}{4!}\varphi^4$ is interaction Lagrangian and g is a dimensionless constant (in four dimensions).
Using in measurable form operator $\frac{\mathbf{\Delta}}{\mathbf{\Delta} \mathbf{N}_{\mathbf{x}_\mu}}$ from formula (9) in [6], we can easily obtain, instead of $\mathcal{L}$ its **measurable** form

$$\mathcal{L}_{meas,\{N\}} = \frac{1}{2}\,(\frac{\mathbf{\Delta}}{\mathbf{\Delta} \mathbf{N}_{\mathbf{x}_\mu}}\varphi_{meas})^2 - \frac{1}{2}m_0^2\varphi_{meas}^2 - \\ -\frac{\mathrm{g}_0}{4!}\varphi_{meas}^4 \tag{27}$$

and instead $\mathcal{L}_0$ with the corresponding *Klein–Gordon equation* or *KGE*

$$(\Box + m_0^2)\,\phi = 0 \tag{28}$$

their **measurable** forms

$$\mathcal{L}_{meas,\{N\},0} = \frac{1}{2}\,(\frac{\mathbf{\Delta}}{\mathbf{\Delta} \mathbf{N}_{\mathbf{x}_\mu}}\phi_{meas})^2 - \frac{m_0^2}{2}\,\phi_{meas}^2 \tag{29}$$

and

$$(\Box_{\mathbf{N}_{\mathbf{x}_\mu}} + m_0^2)\,\phi_{meas} = 0\ . \tag{30}$$

Within the scope of a perturbation theory, let us consider examples of Feynman diagrams, which give UVD for the φ^4-model in canonical QFT in continuous space-time [20] – [23], to find what are the correspondences with a **measurable** picture.
Now we consider one-loop corrections for the two- and four-vertex functions:

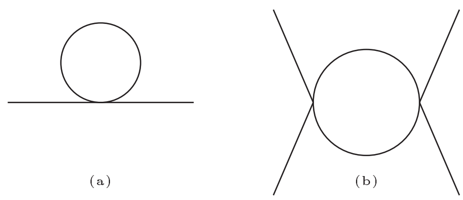

FIG. 1: Diagrams (a) and (b)

.

Then the quantity $G(0)$, quadratically divergent over the momentum k (associated with the diagram (a) in Fig.1, formula (9.1) in [20])

$$G(0) = \mathrm{g}_0 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\tilde{G}(k) = \mathrm{g}_0 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{k^2 - m_0^2} \tag{31}$$

corresponds in a **measurable** picture to the integral, finite over k with $|N^*| = \infty$

$$G(0, N_*) \doteq \mathrm{g}_0 \int\limits_{-p_{N_*}}^{p_{N_*}} \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{k^2 - m_0^2}. \tag{32}$$

Similarly, another divergent diagram–graph of the order $O(\mathrm{g}^2)$, whose contribution is represented by the logarithmically divergent integral (formula (9.2) in [20])

$$\mathrm{g}_0^2 \int \frac{\mathrm{d}^4k}{(2\pi)^8}\,\frac{1}{(k^2 - m_0^2)((p_1 + p_2 - k)^2 - m_0^2)} \tag{33}$$

in a **measurable** consideration will be associated with the finite quantity

$$\mathrm{g}_0^2 \int\limits_{-p_{N'_*}}^{p_{N'_*}} \frac{\mathrm{d}^4k}{(2\pi)^8}\,\frac{1}{(k^2 - m_0^2)((p_1 + p_2 - k)^2 - m_0^2)}. \tag{34}$$

(Here in the **measurable** case the right-hand sides of formulae (32),(34) should have the corresponding sums instead of the integrals but,in virtue the final part of Section 3 the sums may be replaced by the corresponding integrals).
It should be noted that we can pass to Euclidean space-time by means of Wick rotation (**Remark 2.4**) for better convergence of the integrals. Then, with the help of an analytical extension, we can return to Minkowskian space-time. This is a standard method both for QFT and LQFT [45],[46].The continuum action of the theory (26) in Euclidean space-time is of the form (formula (2.17) from [47])

$$S = \int d^4x(\frac{1}{2}\,(\partial_\mu\varphi)^2 + \frac{m_0^2}{2}\,\varphi^2 + \frac{\mathrm{g}_0}{4!}\varphi^4), \tag{35}$$

and the corresponding lattice action has the following form:

$$S_{meas,\{N\}} = a^4 \sum_x (\frac{1}{2}\,(\frac{\mathbf{\Delta}}{\mathbf{\Delta N}}\varphi_{meas})^2 + \frac{1}{2}m_0^2\varphi_{meas}^2 + \\ + \frac{\mathrm{g}_0}{4!}\varphi_{meas}^4). \tag{36}$$

For the lattice values of momenta, in the momentum representation take place the formulae (11),(12). However, here the difficulty arises – the corresponding lattice in the momentum representation on L^4 is uniform with the lattice spacing in formula (12).
In the considered case the lattice of **measurable** momenta is nonuniform with the lattice spacing

$$\triangle p_\mu(meas) = \frac{1}{(N^* - \kappa)(N^* - \kappa \mp 1)\ell}, \tag{37}$$

where κ is an integer number, $|\kappa| \ll |N^*| \gg 1$.
As shown in [7], in order to use the results from [45], it is required that the condition

$$\triangle p_\mu(latt) \approx \triangle p_\mu(meas) \tag{38}$$

be fulfilled.
As follows from formula (37) and [7] this is the case when

$$N_L \approx (N^*)^2. \tag{39}$$

This condition is quite natural considering that L may be chosen no matter how large but finite.
Now in the same way we consider the momentum representation and Fourier transformation of the above mentioned lattice (formula (1.171) in [46])

$$G(x-y;a) = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \tilde{G}(p;a) = \\ = \int_{p\in\mathcal{B}} \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \tilde{G}(p;a). \tag{40}$$

Then we can use the results of [46] to find, how well a continuous propagator of the momentum representation is approximated by the "lattice" propagator in this representation. As it has been noted, all calculations in [46] are first performed in Euclidean space-time and followed by the analytical extension to Minkowskian space.
In virtue, using formula (1.173) from [46], we have

$$\tilde{G}(p;a) = \{\sum_{\mu=1}^{4} a^{-2} 4\sin^2\frac{ap_\mu}{2} + m^2\}^{-1}. \tag{41}$$

But it has been shown that in the case under study the momenta p are taken only from the subset $\mathcal{B}_\mathcal{N}$. Consequently, $p_\mu \propto 1/N_\mu, |N_\mu| \gg 1$. As $a = \ell/N, N \gg 1$, the argument of the function $\sin^2$ is $\propto 1/(NN_\mu)$, i.e. it is very close to zero. Further we use a simple property – $\sin x \approx x$ for x close to 0. Immediately, within a high accuracy, by formula (41) we can obtain

$$\tilde{G}(p;a) = \{\sum_{\mu=1}^{4} a^{-2} 4\frac{a^2p_\mu^2}{4} + m^2\}^{-1} = \\ = \{\sum_{\mu=1}^{4} p_\mu^2 + m^2\}^{-1} = (p^2+m^2)^{-1} \tag{42}$$

in a good agreement with the corresponding formula in a continuous picture, i.e. for $a \to 0$ ([46], formula (1.178)). Thus, these computations once again demonstrate that in a **measurable** form at low energies $E \ll E_p$ the theory studied is to a high accuracy coincident with the corresponding theory in the continuous case.

Perturbation theory and Feynman rules for present lattice are analogous to a continuous theory but they have the interaction term

$$S^{(1)} = a^4 \sum_x \frac{g_0}{4!}\varphi^4_{meas}. \tag{43}$$

As distinct from a continuous consideration, by the lattice approach all Feynman graphs satisfy the following properties in momentum space ([46],p.64) in the general case:

each line is associated with the propagator $\tilde{\Delta}(q) \equiv (m_0^2 + \hat{q}^2)^{-1}$;

each vertex is an end point of four lines and is associated with the factor $-g_0$;

at in inner vertices momentum conservation holds modulo 2π;

loop momenta should to be integrated over the first Brillouin zone $\mathcal{B}$ with the integration measure $\int_{p\in\mathcal{B}}$;

there is in overall factor $(2\kappa)^{-n/2}$ resulting from our normalization of the lattice scalar fields

formally UVD appear only in the continuum limit, i.e. when $a \to 0$.

Note that in the second point $-g_0$ should be replaced by $-g_{0,\mathcal{B}_\mathcal{N}}$, and it seems that the fourth item should be replaced by

loop momenta should be integrated over the short-cut Brillouin zone,$\mathcal{B}_\mathcal{N}$ with the integration measure $\int_{p\in\mathcal{B}_\mathcal{N}}$.

As, for $N \gg 1$, the lattice edge $a = \ell/N$ is very small and hence the correlation length ξ (formula (16)) is very great but not infinite, the indicated lattice in the space-time and momentum representation is actually not distinct from a continuous consideration for the momenta satisfying **BZ** $\mathcal{B}$ (14).

Thus, as directly follows from formula (19), we should include the contributions made *only* by very small momenta p in $\mathcal{B}$,i.e. for $p \in \mathcal{B}_\mathcal{N}$. Taking this into account, further we use the known formulae of LQFT for small momenta (Section 2 in [46])).
We assume that the field $\varphi(x)$ in a *symmetric phase*

$$\langle\varphi(x)\rangle = 0, \tag{44}$$

i.e. Z_2-symmetry of $\varphi(x) \mapsto -\varphi(x)$ is the case, whereas Green's functions with an odd number of arguments vanish.
As it has been correctly noted in Section 2 of [21]:
"...Renormalization has its own intrinsic physical basis and is not brought about solely by the necessity to expurgate infinities. Even in a totally finite theory we would still have to renormalize physical quantities".
This is associated with the fact that the theoretical initial (*bare*) quantities (mass m_0,charge q_0 and so on) can differ drastically from the real (physical) quantities (m_R,q_R and so on).But because in this case in the **measurable** picture at energies $E \ll E_p$ a low-energy part of the

lattice is involved, very close to continuous space-time, there is a possibility to derive QFT without infinities, when renormalization of the theory is understood as a passage from some finite quantities to the other.
Next, we present briefly the results from [8].
In the general case a one-loop correction to the two-vertex function (diagram (a)) takes the form ([47],p.53):

$$\Gamma^{(2)}(p,-p) = -(\hat{p}^2 + m_0^2) - \frac{\mathrm{g}_0}{2} J_1(m_0) \equiv \equiv -(\hat{p}^2 + m_R^2), \qquad (45)$$

where, as a rule, the term $\mathcal{O}(\mathrm{g}_0^2)$ in the right-hand side is omitted and the designations from Section 2 in [46] are used: $\tilde{\Delta}(q) \equiv (m_0^2 + \hat{q}^2)^{-1}, J_n(m_0) \equiv \int\limits_{\mathcal{B}(q)} \tilde{\Delta}(q)^n$. Here where m_R is the renormalized mass in the general case and $\mathcal{B}(\tilde{q})$ is (**BZ**) for the variable $\tilde{q}$.
But, proceeding from the earlier results, in considered case it follows that $\Gamma^{(2)}(p,-p)$ should be replaced by

$$\Gamma^{(2)}(p,-p,\mathcal{B_N}) = -(\hat{p}^2 + m_{0,\mathcal{B_N}}^2) - \frac{\mathrm{g}_{0,\mathcal{B_N}}}{2} J_1(m_0,\mathcal{B_N}) \equiv -(\hat{p}^2 + m_{R,\mathcal{B_N}}^2), \qquad (46)$$

where $p \in \mathcal{B_N}$,

$$J_n(m_0,\mathcal{B_N}) \equiv \int\limits_{\mathcal{B_N}(q)} \tilde{\Delta}(q)^n, \qquad (47)$$

and $m_{0,\mathcal{B_N}}, \mathrm{g}_{0,\mathcal{B_N}}$ – corresponding *bare* mass and coupling constant within $\mathcal{B_N}$. Here, $\mathcal{B_N}(\tilde{q})$ is the narrowed (**BZ**) $\mathcal{B_N}$ for the variable $\tilde{q}$, and in the right side (46) there is no term $\mathcal{O}(\mathrm{g}_{0,\mathcal{B_N}}^2)$ and $m_{R,\mathcal{B_N}}$ are the experimental values of mass obtained for the energies on the order of $\mathcal{B_N}$. Naturally, we can suppose that the renormalized (i.e. experimental) values of mass m_R and coupling constant g_R at energies $E \ll E_p$ *should not* depend on the whole domain of $\mathcal{B}$, the limiting values of which are much greater than E_p. Besides, in any region satisfying the condition $E \ll E_p$ they are independent of this domain and hence we have $m_{R,\mathcal{B_N}} = m_R, \mathrm{g}_{\mathrm{R},\mathcal{B_N}} = \mathrm{g}_R$.
In virtue of the condition $m_{R,\mathcal{B_N}} = m_R$ and considering the terms $\mathcal{O}(\mathrm{g}_0^2), \mathcal{O}(\mathrm{g}_{0,\mathcal{B_N}}^2)$, we can rewrite formula (45) as (formula (2.93) in [46])

$$m_R^2 = m_0^2 + \frac{\mathrm{g}_0}{2} J_1(m_0) + \mathcal{O}(\mathrm{g}_0^2), \qquad (48)$$

and formula (46) as

$$m_{R,\mathcal{B_N}}^2 = m_R^2 = m_{0,\mathcal{B_N}}^2 + \frac{\mathrm{g}_{0,\mathcal{B_N}}}{2} J_1(m_{0,\mathcal{B_N}},\mathcal{B_N}) + +\mathcal{O}(\mathrm{g}_{0,\mathcal{B_N}}^2). \qquad (49)$$

Similar calculations may be performed for the coupling constant too. Specifically, let $\Gamma_R^{(4)}(p_1,p_2,p_3,p_4)$ be the renormalized four-point function. Then, for the renormalized coupling constant g_R, we have ([46],formula (2.96))

$$\mathrm{g}_R = -\Gamma_R^{(4)}(0,0,0,0) = \mathrm{g}_0 - \frac{3}{2}\mathrm{g}_0^2 J_2(m_0) + \mathcal{O}(\mathrm{g}_0^3), \qquad (50)$$

And, since $g_{R,\mathcal{B_N}} = g_R$, we have

$$\mathrm{g}_{\mathrm{R},\mathcal{B_N}} = \mathrm{g}_R = -\Gamma_{R,\mathcal{B_N}}^{(4)}(0,0,0,0) = = \mathrm{g}_{0,\mathcal{B_N}} - \frac{3}{2}\mathrm{g}_{0,\mathcal{B_N}}^2 J_2(m_{0,\mathcal{B_N}},\mathcal{B_N}) + \mathcal{O}(\mathrm{g}_{0,\mathcal{B_N}}^3). \qquad (51)$$

As follows from the four last equations, since left sides of each pair of these equations are equal, whereas the integrals $J_1(m_0)$ and $J_1(m_0,\mathcal{B_N})$ and hence $J_2(m_0)$ and $J_2(m_0,\mathcal{B_N})$ are greatly differing (because in the second case the integration domain is drastically narrowed), the quantities $m_0, m_{0,\mathcal{B_N}}$ and $\mathrm{g}_0, \mathrm{g}_{0,\mathcal{B_N}}$ should also differ from each other. And this really is the case.
According to formulae (2.110),(2.111) from [46] in the general case, for *bare* quantities in the one-loop order we have

$$m_0^2 = m_R^2 + \frac{\mathrm{g}_R}{2} J_1(m_R) + \mathcal{O}(\mathrm{g}_R^2)$$
$$\mathrm{g}_0 = \mathrm{g}_R + \frac{3}{2}\mathrm{g}_R^2 J_2(m_R) + \mathcal{O}(\mathrm{g}_R^3). \qquad (52)$$

Then, considering the equalities, we can rewrite $m_{R,\mathcal{B_N}} = m_R, \mathrm{g}_{R,\mathcal{B_N}} = \mathrm{g}_R$ (52) in the one-loop order in the **measurable** picture under study as follows:

$$m_{0,\mathcal{B_N}}^2 = m_R^2 + \frac{\mathrm{g}_R}{2} J_1(m_R,\mathcal{B_N}) + \mathcal{O}(\mathrm{g}_R^2)$$
$$\mathrm{g}_{0,\mathcal{B_N}} = \mathrm{g}_R + \frac{3}{2}\mathrm{g}_R^2 J_2(m_R,\mathcal{B_N}) + \mathcal{O}(\mathrm{g}_R^3). \qquad (53)$$

BZ $\mathcal{B_N}$ is a narrow low-energy (in fact central) part of the total **BZ** $\mathcal{B}$. From this it follows that the integrals $J_1(m_R,\mathcal{B_N}), J_2(m_R,\mathcal{B_N})$ are low-energy components of the integrals $J_1(m_R), J_2(m_R)$, respectively,and hence they are small.
As it has been noted in [47], by the lattice approach ultraviolet divergence (UVD) in QFT appear on passage to a theory in continuous space-time, i.e. for $a \to 0$. However, in this **measurable** picture we study the lattice per se rather than the continuum limit. As this takes place, UVD of a continuous theory in this case are associated with the quantities lying beyond the boundary of E_p and, in particular, beyond that of the narrowed **BZ**, i.e. $\mathcal{B_N}$. Because we are most interested in the experimental (renormalized) quantities of m_R, g_R which are coincident in the cases $\mathcal{B_N}$ and $\mathcal{B}$ and defined within the energy

range $E \ll E_p$, formula (53) demonstrates that *bare* quantities can be also defined at low energies $E \ll E_p$ and in terms of "narrow" **BZ** $\mathcal{B}_{\mathcal{N}}$. For the two-loop order the foregoing algorithm remains valid, excepting greater complexity of the formulae (for example formula (2.85) in [46]).
It is important that all formulae of a perturbation theory in the two-loop order in a **measurable** consideration can be derived in the same way as in the one-loop order by substitution of the short-cut Brillouin zone $\mathcal{B}_{\mathcal{N}}$ for the corresponding integrals around loop momenta over the first Brillouin zone $\mathcal{B}$.
It should be noted that the case of symmetry violation (44),i.e. $\langle\varphi(x)\rangle \neq 0$ (Section 2.2.3 in [46]) has no principal differences from our consideration. We can derive all the basic formulae in the **measurable** picture at low energies $E \ll E_p$ replacing the Brillouin zone $\mathcal{B}$ by the short-cut Brillouin zone $\mathcal{B}_{\mathcal{N}}$ in all the relevant formulae in Section 2.2.3 from [46].
Next we consider the limiting transition of this LQFT in the general case to a theory in continuous space-time, i.e. when $a \to 0$. As $a = \ell/N, N \gg 1$, we get $N \to \infty$, and from formula (14)it is inferred that full (**BZ**) $\mathcal{B} \to \infty$. It is obvious that the right and left sides of formulae (45),(52),..., where we have full (**BZ**) $\mathcal{B}$, tend to infinity. Precisely this is demonstration of UVD in canonical QFT in continuous space-time.
Since we are interested particularly in the short-cut Brillouin zone $\mathcal{B}_{\mathcal{N}}$ that is invariable, due to formulae (19) (or same (25)), the left and right sides of the corresponding formulae (46),((53)),... for $N \to \infty$ always are finite limited quantities and hence we have no UVD on passage to the continuum limit in the present consideration.
The principal distinction of the earlier results, e.g. [46],[47], from those obtained in this paper is the fact that in the previous works bare quantities m_0 and g_0 take infinite values on passage to the continuum limit, as is accepted by canonical QFT in continuous space-time (for example, Section 10.2 in [22]), whereas in this paper they are finite quantities obtained within the energy range $E \ll E_p$.

B. Gauge-invariante Lagrangians With the Fermions Fields

The above-mentioned results for the scalar model φ^4 are also valid for the theory of a more general type, in particular, for the Yang-Mills fields. In the lattice form, for scalar fields we can use the well-known and evaluated methods [48],for example, the Wick rotation from Minkowski space to imaginary times ((4.1) as in [48]):

$$\begin{aligned} x_0^E &= i x_0^M, \\ k_0^E &= -i k_0^M, \end{aligned} \tag{54}$$

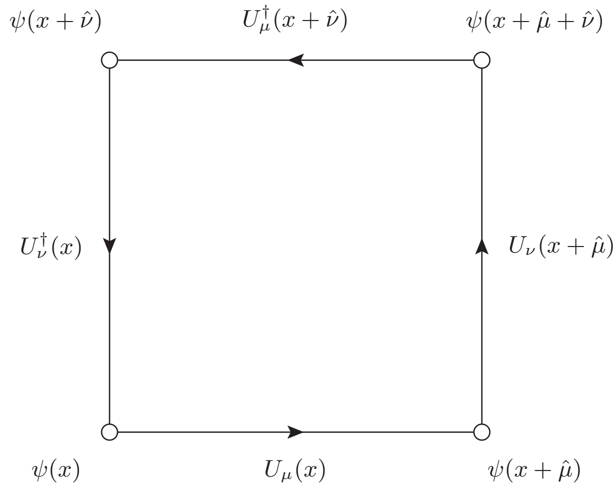

FIG. 2: The plaquette.

where the second line of this formula presents the Wick rotation in momentum space.
Then, for the statistical sum, the Wick rotation gives the Euclidean functional integral e^{-S_E} ((4.2) in [48])

$$e^{iS_M} \longrightarrow e^{-S_E} \tag{55}$$

the convergence of which is much better than the initial functional integral e^{iS_M}. Besides, in this case the corresponding Feynman integral of QFT, in fact, becomes the partition function of the corresponding statistical system.
In the case under study, the lattice gauge theories are most conveniently considered with the use of the approach proposed by K.Wilson [49], because it retains the gage invariance. And this is very important as, in the **measurable** form, the gauge invariance may be retained too, see Section 4.4 of [7].
Thus in the **measurable** picture we can use all the formula associated with the lattice gauge theory, specifically, the Wilson formalism [48].
We start with Euclidean action of Yang-Mills fields interaction with fermions in continuous space-time ((5.1) in [48])

$$\begin{aligned} S = \int d^4x \Big[&\overline{\psi}(x)\Big(\not{D} + m_f\Big)\psi(x) + \\ &+\frac{1}{2} Tr\Big[F_{\mu\nu}(x)F_{\mu\nu}(x)\Big]\Big] \end{aligned} \tag{56}$$

Then its discretization according to Wilson takes the form ((5.2) given in [48])

$$
\begin{aligned}
S_W &= a^4 \sum_x \Bigg[-\frac{1}{2a} \sum_\mu \Big[\overline{\psi}(x)(r-\gamma_\mu)U_\mu(x)\psi(x+a\hat{\mu}) \\
&\quad + \overline{\psi}(x+a\hat{\mu})(r+\gamma_\mu)U^\dagger_\mu(x)\psi(x)\Big] + \overline{\psi}(x)\left(m_0 + \frac{4r}{a}\right)\psi(x)\Bigg] \\
&\quad + \frac{1}{g_0^2}\, a^4 \sum_{x,\mu\nu} \Big[N_c - \mathrm{Re}Tr[U_\mu(x)U_\nu(x+a\hat{\mu})U^\dagger_\mu(x+a\hat{\nu})U^\dagger_\nu(x)]\Big] \qquad (57)
\end{aligned}
$$

where $x = an$, $0 < r \leq 1$ and, as usual, $\not{D} \doteq \gamma^\mu D_\mu$.
In what follows the notation is similar to that from Section 5 in [48] for the lattice spacing $a = \ell/N, |N| \gg 1$.

Then, according to (5.12) in [48] and by virtue of formulae (14)–(21) of this paper, in the general case for the Fourier transforms of the lattice we have

$$
\begin{aligned}
\psi(x) &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4}\, e^{ixp}\, \psi(p) = \int\limits_{\mathcal{B}} \frac{d^4p}{(2\pi)^4}\, e^{ixp}\, \psi(p), \\
\overline{\psi}(x) &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4}\, e^{-ixp}\, \overline{\psi}(p) = \int\limits_{\mathcal{B}} \frac{d^4p}{(2\pi)^4}\, e^{-ixp}\, \overline{\psi}, \\
A_\mu(x) &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} e^{i(x+a\hat{\mu}/2)k}\, A_\mu(k) = \int\limits_{\mathcal{B}} \frac{d^4k}{(2\pi)^4} e^{i(x+a\hat{\mu}/2)k}\, A_\mu. \qquad (58)
\end{aligned}
$$

But, considering the results of Section 4 and, particularly, formulae (19),(23),(25), the formula (58) may be rewritten as

$$
\begin{aligned}
\psi(x) &= \int\limits_{\mathcal{B}_{\mathcal{N}}} \frac{d^4p}{(2\pi)^4}\, e^{ixp}\, \psi(p), \\
\overline{\psi}(x) &= \int\limits_{\mathcal{B}_{\mathcal{N}}} \frac{d^4p}{(2\pi)^4}\, e^{-ixp}\, \overline{\psi}, \\
A_\mu(x) &= \int\limits_{\mathcal{B}_{\mathcal{N}}} \frac{d^4k}{(2\pi)^4} e^{i(x+a\hat{\mu}/2)k}\, A_\mu. \qquad (59)
\end{aligned}
$$

Consequently, the Kronecker delta in position space in the general case is as follows:

$$
\delta_{xy} = a^4 \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4} e^{i(x-y)p} = a^4 \int\limits_{\mathcal{B}} \frac{d^4p}{(2\pi)^4} e^{i(x-y)p}. \qquad (60)
$$

and in considered case

$$
\delta_{xy} = a^4 \int\limits_{\mathcal{B}_{\mathcal{N}}} \frac{d^4p}{(2\pi)^4} e^{i(x-y)p}. \qquad (61)
$$

Here, the lattice spacing a is not normalized to 1 purposely, i.e., $\mathcal{B}_{\mathcal{N}}$ is given by (19).

Note that the inverse Fourier transforms in the considered **measurable** case are of the same form as in the general case and are given by formula (5.13) in [48]

$$
\begin{aligned}
\psi(p) &= a^4 \sum_x e^{-ixp}\psi(x) \\
\overline{\psi}(p) &= a^4 \sum_x e^{ixp}\overline{\psi}(x) \\
A_\mu(k) &= a^4 \sum_x e^{-i(x+a\hat{\mu}/2)k} A_\mu(x) \qquad (62)
\end{aligned}
$$

and correspondingly

$$
\delta^{(4)}(p) = \frac{a^4}{(2\pi)^4} \sum_x e^{-ixp}. \qquad (63)
$$

However, in the general case $p \in \mathcal{B}, k \in \mathcal{B}$ and in the **measurable** consideration $p \in \mathcal{B}_{\mathcal{N}}, k \in \mathcal{B}_{\mathcal{N}}$.

Remark 4.1
It is convenient to use formula (25) for $\mathcal{B}_{\mathcal{N}}$ when the value of a is fixed. Since $\frac{\pm\pi}{N\tilde{N}a} = \frac{\pm\pi}{N\tilde{N}\ell/N} = \frac{\pm\pi}{\tilde{N}\ell}$, $\mathcal{B}_{\mathcal{N}}$ may be represented as a domain with the boundaries which are evidently independent of a

$$
\mathcal{B}_{\mathcal{N}} \doteq \{p | \frac{-\pi}{\tilde{N}\ell} < p_\mu \leq \frac{\pi}{\tilde{N}\ell}, \tilde{N} \gg 1\} \doteq \mathcal{B}_{\tilde{N}}. \qquad (64)
$$

In this way formula (64) indicates that a "width" (or same size) of $\mathcal{B}_{\tilde{N}}$ depends only on the number $\tilde{N}$, i.e., on EPAB. For gauge theories, in the general case we can use the same methods as in Section 4.1 with due regard for the results of Sections 2,3. Specifically, for the first-order Wilson action at gauge coupling in the general case of the quark-quark-gluon vertex in momentum space we have (formula (5.16) in [48]):

$$\begin{aligned}
S_{qqg} &= -\frac{ig_0}{2}a^4\sum_{x,\mu}\Big(\overline{\psi}(x)(r-\gamma_\mu)A_\mu(x)\psi(x+a\hat{\mu})-\overline{\psi}(x+a\hat{\mu})(r+\gamma_\mu)A_\mu(x)\psi(x)\Big)\\
&= -\frac{ig_0}{2}a^4\sum_{x,\mu}\int\limits_{\mathcal{B}}\frac{d^4p}{(2\pi)^4}\int\limits_{\mathcal{B}}\frac{d^4k}{(2\pi)^4}\int\limits_{\mathcal{B}}\frac{d^4p'}{(2\pi)^4}\,e^{ix(p+k-p')}e^{iak_\mu/2}\times \qquad (65)\\
&\times\Big(\overline{\psi}(p')(r-\gamma_\mu)A_\mu(k)\psi(p)e^{iap_\mu}-\overline{\psi}(p')e^{-iap'_\mu}(r+\gamma_\mu)A_\mu(k)\psi(p)\Big)=\\
&= \frac{ig_0}{2}\sum_{\mu}\int\limits_{\mathcal{B}}\frac{d^4p}{(2\pi)^4}\int\limits_{\mathcal{B}}\frac{d^4k}{(2\pi)^4}\int\limits_{\mathcal{B}}\frac{d^4p'}{(2\pi)^4}\,(2\pi)^4\delta^{(4)}(p+k-p')e^{iak_\mu/2}\\
&\times\Big(\overline{\psi}(p')\gamma_\mu A_\mu(k)\psi(p)(e^{iap_\mu}+e^{-iap'_\mu})+r\overline{\psi}(p')A_\mu(k)\psi(p)(-e^{iap_\mu}+e^{-iap'_\mu})\Big)\\
&= \frac{ig_0}{2}\sum_{\mu}\int\limits_{\mathcal{B}}\frac{d^4p}{(2\pi)^4}\int\limits_{\mathcal{B}}\frac{d^4k}{(2\pi)^4}\int\limits_{\mathcal{B}}\frac{d^4p'}{(2\pi)^4}\,(2\pi)^4\delta^{(4)}(p+k-p')e^{iak_\mu/2}\\
&\times\Bigg(\overline{\psi}(p')\gamma_\mu A_\mu(k)\psi(p)e^{iap_\mu/2}e^{-iap'_\mu/2}\cdot 2\cos\frac{a(p+p')_\mu}{2}\\
&\quad+r\overline{\psi}(p')A_\mu(k)\psi p)e^{iap_\mu/2}e^{-iap'_\mu/2}\cdot(-2i)\sin\frac{a(p+p')_\mu}{2}\Bigg).
\end{aligned}$$

Consequently, in the considered pattern, taking into account **Remark 4.1**, we have

$$\begin{aligned}
S_{qqg,N} &= \frac{ig_0}{2}\sum_{\mu}\int\limits_{\mathcal{B}_{\tilde{N}}}\frac{d^4p}{(2\pi)^4}\int\limits_{\mathcal{B}_{\tilde{N}}}\frac{d^4k}{(2\pi)^4}\int\limits_{\mathcal{B}_{\tilde{N}}}\frac{d^4p'}{(2\pi)^4}\,(2\pi)^4\delta^{(4)}(p+k-p')e^{iak_\mu/2}\\
&\times\Bigg(\overline{\psi}(p')\gamma_\mu A_\mu(k)\psi(p)e^{iap_\mu/2}e^{-iap'_\mu/2}\cdot 2\cos\frac{a(p+p')_\mu}{2}\\
&\quad+r\overline{\psi}(p')A_\mu(k)\psi p)e^{iap_\mu/2}e^{-iap'_\mu/2}\cdot(-2i)\sin\frac{a(p+p')_\mu}{2}\Bigg). \qquad (66)
\end{aligned}$$

On going to the continuous limit, in the general case, i.e., for formula (65), we have

$$\lim_{a\to 0}S_{qqg}=\int_{-\infty}^{\infty}\frac{d^4p}{(2\pi)^4}\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}\int_{-\infty}^{\infty}\frac{d^4p'}{(2\pi)^4}\,(2\pi)^4\delta^{(4)}(p+k-p')\,\cdot$$
$$\cdot ig_0\sum_{\mu}\overline{\psi}(p')\gamma_\mu A_\mu(k)\psi(p). \qquad (67)$$

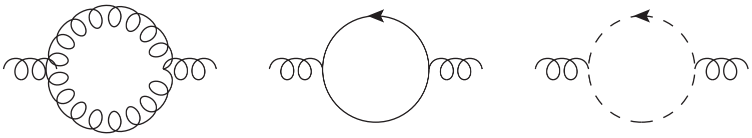

FIG. 3: Loop diagrams for the self-energy of the gluon on the lattice,which have a continuum analog.

In this case we get

$$\lim_{a\to 0} S_{qqg.N} = \int\limits_{\mathcal{B}_{\tilde{N}}} \frac{d^4p}{(2\pi)^4} \int\limits_{\mathcal{B}_{\tilde{N}}} \frac{d^4k}{(2\pi)^4} \int\limits_{\mathcal{B}_{\tilde{N}}} \frac{d^4p'}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p+k-p') \cdot$$

$$\cdot ig_0 \sum_{\mu} \overline{\psi}(p')\gamma_\mu A_\mu(k)\psi(p). \tag{68}$$

The quantities in formulae (66) and (68) are finite as summation in the corresponding formulae is performed for the same finite domain of momenta $\mathcal{B}_{\tilde{N}}$. The difference is in the fact that the first is dependent on the lattice spacing a, whereas the second is not. But, as $a = \ell/N, N \gg 1$ is very small, the value of $S_{qqg,N}$ is close to $\lim_{a\to 0} S_{qqg.N}$. From point 2.2.2 of Section 2 it follows that the number $\tilde{N} \gg 1$ determines the applicability boundaries of EPAB and we always have $\tilde{N} \geq 10^2$. In the general case $\tilde{N}$ is a variable and it should be dependent on the processes under study.

All the methods considered in Sections 3,4.1 and 4.2 are also good for loop Feynman diagrams, for example, for simple loop diagrams in Figure 3. In this case, similar to formulae of Sections 4.1,4.2, the Brillouin zone $\mathcal{B}$ is contracted to $\mathcal{B}_{\tilde{N}}$,and the renormalized (experimental) quantities $m_{f,R}$ and g_R (coupling constant g_0) are coincident with $m_{f,R,\mathcal{B}_\mathcal{N}}$ and with $g_{R,\mathcal{B}_\mathcal{N}}$. In line with Section 4.1, at $a \to 0$, the initial Brillouin zone $\mathcal{B} \to \infty$. However, $\mathcal{B}_{\tilde{N}}$ remains invariant, as follows form formula (64) indicating that $\mathcal{B}_{\tilde{N}}$ is independent of a.
Then, similar to formulae (52),(53), *bare* the quantities m_0 and g_0 may be expressed in terms of the finite quantities $m_{f,R,\mathcal{B}_\mathcal{N}}$ and $g_{R,\mathcal{B}_\mathcal{N}}$, and the integrals over the finite domain $\mathcal{B}_{\tilde{N}}$. This means that they remain finite on passage to the continuous limit $a \to 0$.
From this it follows directly that in this case, similar to the scalar model φ^4, all the loop contributions in a perturbation theory may be expressed in terms of the finite quantities and integrals from the bounded (narrow,central) Brillouin zone $m_{f,R,\mathcal{B}_\mathcal{N}}$ that remains invariant for $a \to 0$ and leads to the finite perturbation theory both in the lattice variant and on passage to the continuous limit.
It should be noted that in the lattice consideration the number of loop Feynman diagrams may be greater. Specifically, apart from the diagrams shown in Figure 3 which have their analog in the continuous case, we can have pure lattice diagrams, in particular, to ensure the theory gauge invariance (for example, lower row in Figure 6 in [48]). Of course, all the above calculations are valid for these diagrams too.

V. CONCLUSION

5.1 It is clear that for all other models of QFT in the general case the calculations and the results of Section 4 are valid. Specifically, in the lattice representation, for all models in each particular case we should determine the following:

5.1.1 EPAB for the specific model and the corresponding number $\tilde{N}$ from formula (64);

5.1.2 finite values of the corresponding lattice loop contributions for the "contracted" Brillouin zone $\mathcal{B}_{\tilde{N}}$;

5.1.3 finite values of the quantities *bare* for all the parameters of the model $m_0, g_0, ...$ in terms of the experimental values of these parameters $m_R, g_R, ...$ and of the above-mentioned finite loop contributions derived within the energy ranges from $\mathcal{B}_{\tilde{N}}$;

5.1.4 in this case passing to the continuous limit means replacing of the lattice loop contributions in point **5.1.2** by the corresponding integrals over $\mathcal{B}_{\tilde{N}}$.

So, due to the results from Section 4 and points **5.1.1–5.1.4**, both the lattice model and its continuous limit are free from the ultraviolet divergences.
It should be noted that, when in the **measurable** picture we take into account in point **5.1.2** the lattice loop contributions only for **observable** quantities, they will be smaller than in the canonical case of the lattice model in line with the results from Section III of [8].

5.2 The proposed approach is based on a precise evaluation of EPAB, i.e., of the number $\tilde{N}$, in every particular case. In the paradigm of the local plane geometry, the growing precision of evaluation of $\tilde{N}$ should be associated with more and more higher orders of the quantum corrections.
As noted in **Remark 2.2**,initially, it has been assumed that we should consider only two (in a sense extreme) energy regions: low energies $E \ll E_p$ and very high energies $E \approx E_p$. Still, it is not impossible that the problem solved may lead to some intermediate energies lying in the interval between the two indicated regions. It is clear that, from the beginning or by definition, these energies should have lower and upper bounds. As in this case EP would be violated, QFT in this energy range would be a theory with the particular locally-unavoidable nontrivial metric $\tilde{g}_{\mu\nu}(x)$ (unreduced to the Minkowskian metric $\eta_{\mu\nu}(x)$) that should be dependent on the processes under study. Then this theory would be the ultraviolet *finite* QFT in curved space-time [50].

5.3 The subject-matter of this work is associated with such a problem as experimental detection of a quantum foam. At the present time, the relevant results are still inconsistent [51] due to inadequate capacity of the data available and insufficient precision of the experiments. (The situation resembles that in chronology of the detection of gravitational waves relict including.)
Nevertheless, an active search in this direction is in progress [52], [53].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this work.

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