

The Petrovskii correctness and semigroups of operators

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Abstract

Let $P(\partial/\partial x)$ be an $m \times n$ matrix whose entries are PDO on \mathbb{R}^n with constant coefficients, and let $\mathcal{S}(\mathbb{R}^n)$ be the space of infinitely differentiable rapidly decreasing functions on \mathbb{R}^n . It is proved that $P(\partial/\partial x)|_{(\mathcal{S}(\mathbb{R}^n))^m}$ is the infinitesimal generator of a (C_0) -semigroup $(S_t)_{t \geq 0} \subset L((\mathcal{S}(\mathbb{R}^n))^m)$ if and only if $P(\partial/\partial x)$ satisfies the Petrovskii correctness condition. Moreover, if it is the case, then $(S_t)_{t \geq 0}$ is an exponential semigroup whose characteristic exponent is equal to the stability index of $P(\partial/\partial x)$. Similar statements are also proved for some other function spaces on \mathbb{R}^n , and for the space of tempered distributions.

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1. Introduction. The matricial differential operator $P(\partial/\partial x)$ and the corresponding Cauchy problem

Let $m, n, d \in \mathbb{N}$. Denote by M_m the ring of $m \times m$ matrices with complex entries. Suppose that for every multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ of length $|\alpha| = \alpha_1 + \dots + \alpha_n \leq d$ there is given a matrix $A_\alpha \in M_m$. Consider the polynomial $P(X) = P(X_1, \dots, X_n) \in M_m[X_1, \dots, X_n]$ of n variables X_1, \dots, X_n with coefficients in M_m defined by the formula

$$P(X) = \sum_{|\alpha| \leq d} A_\alpha X^\alpha, \quad X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}. \quad (1.1)$$

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Substituting

$$X_k = \frac{\partial}{\partial x_k}, \quad k = 1, \dots, n,$$

we obtain the matricial partial differential operator on \mathbb{R}^n with constant coefficients:

$$P\left(\frac{\partial}{\partial x}\right) = \sum_{|\alpha| \leq d} A_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha, \quad \left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}. \quad (1.2)$$

Substituting

$$X_k = i\xi_k \in \mathbb{R}, \quad \xi_k \in \mathbb{R}, \quad k = 1, \dots, n,$$

we obtain the symbol of $P(\partial/\partial x)$, i.e. the $m \times m$ matrix

$$\tilde{P} = \tilde{P}(\xi) = \sum_{|\alpha| \leq d} i^{|\alpha|} A_\alpha \xi^\alpha, \quad \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}, \quad (1.3)$$

whose entries are scalar complex polynomials on \mathbb{R}^n .

If E is a space of C^m -valued functions or distributions on \mathbb{R}^n , then one can consider the Cauchy problem for E -valued functions $u(\cdot)$ of a real variable:

$$\begin{aligned} \frac{du(t)}{dt} &= P\left(\frac{\partial}{\partial x}\right)u(t) \quad \text{for } t \geq 0 \text{ (or for } t \in \mathbb{R}), \\ u(0) &= u_0. \end{aligned} \quad (1.4)$$

where $u_0 \in E$ is given. Some sort of well posedness of such a Cauchy problem consists in the fact that the operator $P(\partial/\partial x)$ considered on the domain $\{u \in E : P(\partial/\partial x)u \in E\}$ is the infinitesimal generator of a one-parameter semigroup $(S_t)_{t \geq 0} \subset L(E)$ (or group $(G_t)_{t \in \mathbb{R}} \subset L(E)$) of class (C_0) .

The subsequent Section 2 is devoted to $E = (Z'_n)^m$. Then the space of the Fourier transforms $\mathcal{F}^{-1}E = (\mathcal{F}^{-1}Z'_n)^m = (\mathcal{D}'(\mathbb{R}^n))^m$ is invariant with respect to multiplication by arbitrary elements of $C^\infty(\mathbb{R}^n; M_m)$, and this implies that the operator $P(\partial/\partial x)|_{(Z'_n)^m}$ is the infinitesimal generator of a one-parameter group $(G_t)_{t \in \mathbb{R}} \subset L((Z'_n)^m)$ of class (C_0) . The main result of the present paper is formulated in Section 3 where several spaces E are considered with $\mathcal{F}^{-1}E$ not invariant with respect to multiplication by arbitrary elements of $C^\infty(\mathbb{R}^n; M_m)$. Then, in order to prove that a suitable restriction of $P(\partial/\partial x)$ generates a (C_0) -semigroup $(S_t)_{t \geq 0} \subset L(E)$, one must assume *something* about \tilde{P} , and, for each of the spaces E considered, this *something* appears to be the Petrovskiĭ correctness condition.

Recall that if E is an l.c.v.s. and $L(E)$ is the algebra of continuous linear operators on E , then a parametrized family $(S_t)_{t \geq 0} \subset L(E)$ is called a *one-parameter semigroup of class (C_0)* if it satisfies the following three conditions:

- (i) $S_{t_1+t_2} = S_{t_1}S_{t_2}$ for every $t_1, t_2 \in [0, \infty[$,
- (ii) $S_0 = \mathbb{1}$, the unity of $L(E)$,
- (iii) for every $u \in E$ the map $[0, \infty[\ni t \mapsto S_t u \in E$ (called the trajectory of u or/and of $(S_t)_{t \geq 0}$) is continuous.

The *infinitesimal generator* of the (C_0) -semigroup $(S_t)_{t \geq 0} \subset L(E)$ is the linear operator A from E into E with domain $D(A)$ such that

$$D(A) = \left\{ u \in E : \lim_{t \downarrow 0} \frac{1}{t}(S_t u - u) \text{ exists in the topology of } E \right\},$$

$$Au = \lim_{t \downarrow 0} \frac{1}{t}(S_t u - u) \quad \text{for } u \in D(A).$$

Notice that $D(A)$ is dense in E , and $D(A) = E$ if and only if all the trajectories of the (C_0) -semigroup belong to $C^\infty([0, \infty[; E)$. The name “generator” is justified by the fact that if a (C_0) -semigroup $(S_t)_{t \geq 0} \subset L(E)$ is locally equicontinuous, then it is uniquely determined by A . (See the proof of the uniqueness theorem in Section 2.) For one-parameter groups of linear operators things are similar.

2. The one-parameter group $(G_t)_{t \in \mathbb{R}} \subset L((Z'_n)^m)$ generated by $P(\partial/\partial x)$

In the present section the matrices $A_\alpha \in M_m$, $|\alpha| \leq p$, are arbitrary. This is important for the proof of necessity of the Petrovskiĭ correctness condition in Theorem 1(iii) of Section 3.

Since $\phi(t, \xi) := \exp(t\tilde{P}(\xi))$ satisfies the differential equation $\frac{d}{dt}\phi = \tilde{P}(\xi)\phi$, the theorem on differentiation of a solution of an ODE with respect to a parameter ([H], Sec. V.4, Corollary 4.1) implies that $\phi \in C^\infty(\mathbb{R}^{n+1}; M_m)$. This conclusion may also be (not very easily) obtained by term by term differentiation of the series $\exp(t\tilde{P}(\xi)) = \mathbb{1} + \sum_{k=1}^{\infty} \frac{t^k}{k!} \tilde{P}(\xi)^k$. Let $\mathcal{D}'(\mathbb{R}^n)$ be the space of distributions on \mathbb{R}^n endowed with the topology of uniform convergence on bounded subsets of $C_0^\infty(\mathbb{R}^n)$. For every $T \in \mathcal{D}'(\mathbb{R}^n)$ the mapping $C^\infty(\mathbb{R}^n) \ni \varphi \mapsto \varphi T \in \mathcal{D}'(\mathbb{R}^n)$ is continuous. Consequently, the formula

$$\tilde{G}_t f = e^{t\tilde{P}} f, \quad t \in \mathbb{R}, f \in (\mathcal{D}'(\mathbb{R}^n))^m,$$

determines a one-parameter group $(\tilde{G}_t)_{t \in \mathbb{R}} \subset L((\mathcal{D}'(\mathbb{R}^n))^m)$ of class (C_0) all of whose trajectories belong to $C^\infty(\mathbb{R}; (\mathcal{D}'(\mathbb{R}^n))^m)$. The infinitesimal generator of this one-parameter group is the multiplication operator $\tilde{P}|_{(\mathcal{D}'(\mathbb{R}^n))^m} \in L((\mathcal{D}'(\mathbb{R}^n))^m)$. It is easy to prove that the one-parameter group $(\tilde{G}_t)_{t \in \mathbb{R}} \subset L((\mathcal{D}'(\mathbb{R}^n))^m)$ is locally equicontinuous, i.e. for every compact $K \subset \mathbb{R}$ the family of operators $\{\tilde{G}_t : t \in K\} \subset L((\mathcal{D}'(\mathbb{R}^n))^m)$ is equicontinuous.

Let \mathcal{F} be the n -dimensional Fourier transformation defined by

$$(\mathcal{F}\varphi)(x) = \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \varphi(\xi) d\xi \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ and } x \in \mathbb{R}^n$$

where $\langle x, \xi \rangle = \sum_{k=1}^n x_k \xi_k$. Then \mathcal{F} is an automorphism of $\mathcal{S}(\mathbb{R}^n)$ with inverse $\mathcal{F}^{-1} = (2\pi)^{-n} \mathcal{F}^\vee$ where \vee denotes the reflection in 0. Define $Z_n := \mathcal{F}C_c^\infty(\mathbb{R}^n)$. Then $Z_n = \mathcal{F}C_c^\infty(\mathbb{R}^n) = \mathcal{F}^{-1}C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$. The space Z_n consists of those functions belonging to $\mathcal{S}(\mathbb{R}^n)$ which have holomorphic extension onto \mathbb{C}^n with growth properties characterized by the Paley–Wiener–Schwartz theorem (see [H2], Theorem 7.3.1). The topology in Z_n is that transported by \mathcal{F} from $C_c^\infty(\mathbb{R}^n)$. The restriction $\mathcal{F}|_{C_c^\infty(\mathbb{R}^n)}$ is a topological isomorphism of $C_c^\infty(\mathbb{R}^n)$ onto Z_n , and $\mathcal{F}|_{Z_n} = (2\pi)^n \mathcal{F}^{-1\vee}|_{Z_n}$ is a topological isomorphism of Z_n onto $C_c^\infty(\mathbb{R}^n)$. The space Z'_n is defined as the strong dual of Z_n , and the dual mapping of $\mathcal{F}|_{Z_n} : Z_n \rightarrow C_c^\infty(\mathbb{R}^n)$ is an isomorphism of $\mathcal{D}'(\mathbb{R}^n)$ onto Z'_n . This last isomorphism *extends* $\mathcal{F}|_{C_c^\infty(\mathbb{R}^n)}$ and for this reason it is still denoted by \mathcal{F} . It follows that

$$\langle \mathcal{F}T, u \rangle_{Z'_n \times Z_n} = \langle T, \mathcal{F}u \rangle_{\mathcal{D}'(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n)} \quad \text{for every } T \in \mathcal{D}'(\mathbb{R}^n) \text{ and } u \in Z_n,$$

or, what is the same,

$$\begin{aligned} \langle \mathcal{F}T, \mathcal{F}\varphi \rangle_{Z'_n \times Z_n} &= (2\pi)^n \langle T, \varphi^\vee \rangle_{\mathcal{D}'(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n)} \\ &\text{for every } T \in \mathcal{D}'(\mathbb{R}^n) \text{ and } \varphi \in C_c^\infty(\mathbb{R}^n). \end{aligned}$$

For every $k = 1, \dots, n$ the operator $\partial/\partial x_k : Z'_n \rightarrow Z'_n$ is defined as dual to the operator $-\partial/\partial x_k : Z_n \rightarrow Z_n$, so that $(\partial/\partial x_k)\mathcal{F}T = \mathcal{F}(i\xi_k \cdot T)$ for every $T \in \mathcal{D}'(\mathbb{R}^n)$.

The actions of \mathcal{F} on $(\mathcal{D}'(\mathbb{R}^n))^m$ and of \mathcal{F}^{-1} on $(Z'_n)^m$ are coordinatewise. From our assertions concerning the one-parameter group $(\tilde{G}_t)_{t \in \mathbb{R}}$ it follows that *the formula*

$$G_t = \mathcal{F}e^{t\tilde{P}}\mathcal{F}^{-1}, \quad t \in \mathbb{R}, \quad (2.1)$$

determines a locally equicontinuous one-parameter group $(G_t)_{t \in \mathbb{R}} \subset L((Z'_n)^m)$ of class (C_0) all of whose trajectories belong to $C^\infty(\mathbb{R}; (Z'_n)^m)$ and whose infinitesimal generator is $\mathcal{F}\tilde{P}\mathcal{F}^{-1}|_{(Z'_n)^m} = P(\partial/\partial x)|_{(Z'_n)^m}$.

If $u_0 \in (Z'_n)^m$ and $u(t) = G_t u_0$ for $t \in \mathbb{R}$, then $u(\cdot) \in C^\infty(\mathbb{R}; (Z'_n)^m)$ and $u(\cdot)$ is a solution of the Cauchy problem (1.4). The subsequent theorem shows that this Cauchy problem has no other $(Z'_n)^m$ -valued solutions.

Uniqueness Theorem. *Let $t_0 \in]0, \infty]$ and let I be equal to either $[0, t_0[$ or $] -t_0, 0]$. If the map $I \ni t \mapsto u(t) \in (Z'_n)^m$ belongs to $C^1(I; (Z'_n)^m)$ and*

$$\frac{du(t)}{dt} = P\left(\frac{\partial}{\partial x}\right)u(t) \quad \text{for every } t \in I,$$

then

$$u(t) = G_t u(0) \quad \text{for every } t \in I.$$

Proof. We will prove this theorem for $I = [0, t_0[$, the proof for $I =] -t_0, 0]$ being similar. Fix any $t \in]0, t_0[$ and let $\tau \in [0, t]$. Then

$$\lim_{h \rightarrow 0} G_{t-\tau} \frac{1}{h} (G_{-h} u(\tau) - u(\tau)) = -G_{t-\tau} P\left(\frac{\partial}{\partial x}\right)u(\tau)$$

with limit in the topology in $(Z'_n)^m$. Furthermore, by local equicontinuity of the one-parameter group $(G_t)_{t \in \mathbb{R}} \subset L((Z'_n)^m)$, the map $\mathbb{R} \times (Z'_n)^m \ni (t, u) \mapsto G_t u \in (Z'_n)^m$ is continuous, and so

$$\lim_{[-\tau, t-\tau] \ni h \rightarrow 0} G_{t-\tau-h} \frac{1}{h} (u(\tau+h) - u(\tau)) = G_{t-\tau} P\left(\frac{\partial}{\partial x}\right)u(\tau).$$

Consequently,

$$\lim_{[-\tau, t-\tau] \ni h \rightarrow 0} \frac{1}{h} [G_{t-\tau-h} u(\tau+h) - G_{t-\tau} u(\tau)] = 0.$$

This shows that for every $t \in]0, t_0]$ the function $[0, t] \ni \tau \mapsto G_{t-\tau} u(\tau) \in (Z'_n)^m$ has derivative vanishing everywhere on $[0, t]$ (the derivative at the ends of $[0, t]$ being one-sided). Consequently, $\frac{d}{d\tau} [G_{t-\tau} u(\tau)](\varphi) = 0$ for every $\tau \in [0, t]$ and $\varphi \in (Z_n)^m$, whence $[u(t) - G_t u(0)](\varphi) = [G_{t-\tau} u(\tau)](\varphi)|_{\tau=0}^{\tau=t} = 0$, and so $u(t) = G_t u(0)$.

Notice that the above argument resembles one used in the proof of E. R. van Kampen's uniqueness theorem for solutions of ordinary differential equations. See [K] and [H], Sec. III.7.

Remark. For every $t \in \mathbb{R}$ one has $G_t(Z_n)^m \subset (Z_n)^m$ and $G_t(\mathcal{FE}'(\mathbb{R}^n))^m \subset (\mathcal{FE}'(\mathbb{R}^n))^m$. The restricted operators G_t constitute one-parameter (C_0) -groups $(G_t|_{(Z_n)^m})_{t \in \mathbb{R}} \subset L((Z_n)^m)$ and $(G_t|_{(\mathcal{FE}'(\mathbb{R}^n))^m})_{t \in \mathbb{R}} \subset L((\mathcal{FE}'(\mathbb{R}^n))^m)$ having properties analogous to those of $(G_t)_{t \in \mathbb{R}} \subset L((Z'_n)^m)$. Similarly to Z_n , the space $\mathcal{FE}'(\mathbb{R}^n)$ has a direct analytical characterization: its elements are those functions which belong to $\mathcal{O}_M(\mathbb{R}^n)$ (the space of slowly increasing C^∞ -functions on \mathbb{R}^n) and have holomorphic extensions onto \mathbb{C}^n with growth properties characterized by the Paley–Wiener–Schwartz theorem ([S], Chapter VII, Theorem XVI; [H2], Theorem 7.3.1).

3. The main result

As in Section 1, take a polynomial $P(X) = P(X_1, \dots, X_n) \in M_m[X_1, \dots, X_n]$. Let $P(\partial/\partial x)$ be the corresponding matricial partial differential operator with constant coefficients, and let $\tilde{P} = \tilde{P}(\xi) = P(i\xi)$ be the symbol of $P(\partial/\partial x)$. Define the *stability index* ω_0 of $P(\partial/\partial x)$ by the formula

$$\omega_0 = \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi)), \xi \in \mathbb{R}^n\} \quad (3.1)$$

where $\sigma(\tilde{P}(\xi))$ denotes the spectrum of the matrix $\tilde{P}(\xi) \in M_m$.

Let E denote one of the following spaces of \mathbb{C}^m -valued functions or distributions on \mathbb{R}^n :

- (i) $E = (\mathcal{S}(\mathbb{R}^n))^m$ where $\mathcal{S}(\mathbb{R}^n)$ is the L. Schwartz space of rapidly decreasing infinitely differentiable functions on \mathbb{R}^n ,
- (ii) $E = (\mathcal{S}'(\mathbb{R}^n))^m$ where $\mathcal{S}'(\mathbb{R}^n)$ is the L. Schwartz space of tempered distributions on \mathbb{R}^n equipped with the topology of uniform convergence on bounded subsets of $\mathcal{S}(\mathbb{R}^n)$,
- (iii) $E = (C_b^\infty(\mathbb{R}^n))^m$ where $C_b^\infty(\mathbb{R}^n)$ is the space of those bounded infinitely differentiable functions on \mathbb{R}^n whose partial derivatives are all bounded on \mathbb{R}^n ,
- (iv) $E = (H^\infty(\mathbb{R}^n))^m$ where $H^\infty(\mathbb{R}^n)$ consists of those infinitely differentiable functions on \mathbb{R}^n which belong to $L^2(\mathbb{R}^n)$ together with all their partial derivatives,
- (v) $E = \{u \in (L^2(\mathbb{R}^n))^m : (P(\partial/\partial x))^k u \in (L^2(\mathbb{R}^n))^m \text{ for } k = 1, 2, \dots\}$ where partial derivatives are meant in the sense of distributions and the topology of E is determined by the system of seminorms $\|u\|_k = \|(P(\partial/\partial x))^k u\|_{(L^2(\mathbb{R}^n))^m}$, $k = 0, 1, \dots$

Theorem 1. Let $P(\partial/\partial x)$ be the matricial partial differential operator with constant coefficients corresponding to a polynomial $P(X) \in M_m[X_1, \dots, X_n]$ and let ω_0 be the stability index of $P(\partial/\partial x)$. Fix whichever of the five spaces E listed above. Then the following two conditions are equivalent:

- (a) $\omega_0 < \infty$,
- (b) the operator $P(\partial/\partial x)|_E$ is the infinitesimal generator of a (C_0) -semigroup $(S_t)_{t \geq 0} \subset L(E)$.

If $\omega_0 < \infty$, then the (C_0) -semigroup $(S_t)_{t \geq 0} \subset L(E)$ occurring in (b) is unique and $S_t = G_t|_E$ for every $t \geq 0$, where $(G_t)_{t \in \mathbb{R}} \subset L((Z'_n)^m)$ is the (C_0) -group from Section 2. Furthermore, if $\omega_0 < \infty$, then the semigroup $(S_t)_{t \geq 0}$ is of exponential type, i.e. for every real sufficiently large ω the semigroup $(e^{-\omega t} S_t)_{t \geq 0} \subset L(E)$ is equicontinuous, and

$$\omega_0 = \omega_E$$

where

$$\omega_E = \inf\{\omega \in \mathbb{R} : \text{the semigroup } (e^{-\omega t} S_t)_{t \geq 0} \subset L(E) \text{ is equicontinuous}\}.$$

Remarks. 1. In each of the cases (i)–(v) the space E is continuously imbedded in $(Z'_n)^m$ so that if $\omega_0 < \infty$, then the equality $S_t = G_t|_E$ and the uniqueness of the (C_0) -semigroup $(S_t)_{t \geq 0} \subset L(E)$ generated by $P(\partial/\partial x)|_E$ are consequences of (b) and of the uniqueness theorem from Section 2.

2. In case (iii) the equivalence (a) \Leftrightarrow (b) follows from results of I. G. Petrovskii [P] not involving one-parameter semigroups. Condition (a), now called the *Petrovskii correctness condition*, was used in [P] in a seemingly weaker form which was later proved to be equivalent to (a), according to a conjecture formulated in [P]. From Theorem 1(i) it follows that if (a) holds, then $S_t = \mathcal{F}e^{t\tilde{P}}\mathcal{F}^{-1} = T_t*$ for every $t \in [0, \infty[$ where $T_t \in (\mathcal{S}'(\mathbb{R}^n))^m$. Formulas (8.5)–(8.6) in Sec. 7.8 of A. Friedman's book [F] exhibit the structure of distributions T_t and yield some results of type (a) \Rightarrow (b) not related to exponential semigroups, generalizing the above-mentioned results of Petrovskii.

3. One may call ω_E the *characteristic exponent* or the *equicontinuity index* of the semigroup $(S_t)_{t \geq 0} \subset L(E)$. A semigroup $(S_t)_{t \geq 0} \subset L(E)$ with ω_E finite may be called an *exponential semigroup*. The exponential (C_0) -semigroups in an l.c.v.s. reduce (by multiplication by real exponential function of the parameter t) to equicontinuous (C_0) -semigroups for which a theory of Hille–Yosida type is presented in Chapter IX of the monograph of K. Yosida [Y].

In a Banach space all one-parameter (C_0) -semigroups are exponential, but a similar statement is not true for non-normed spaces. See for instance [Ki], p. 4.

4. The idea of using one-parameter semigroups in connection with Cauchy's problem (1.4) is taken from papers of G. Birkhoff, T. Mullikin and T. Ushijima [B-M], [B] and [U], and from Section I.8 of S. G. Krein's monograph [Kr]. Let us stress that in [B] the equality $\omega_0 = \omega_E$ is discussed.

5. A result similar to Theorem 1(i) concerning Cauchy's problem for the equation

$$P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) = 0$$

where $P(X_0, X_1, \dots, X_n) = \sum_{k=0}^m P_k(X_1, \dots, X_n)X_0^k$ is a scalar polynomial of $n + 1$ variables is stated in the book of J. Rauch [R] as Theorem 2 on p. 128. Since the polynomial $P_m(X_1, \dots, X_n)$ need not reduce to a constant, the theorem of Rauch does not follow from Theorem 1(i) (or vice versa).

6. The spaces E in cases (i)–(iv) are standard, not depending on $P(\partial/\partial x)$. The space E in case (v), depending on $P(\partial/\partial x)$, was introduced by T. Ushijima [U] who proved the equivalence (a) \Leftrightarrow (b) in this case. Let $X = (L^2(\mathbb{R}^n))^m$, and let A be the operator from X into X with domain $D(A)$ such that $D(A) = \{u \in X : P(\partial/\partial x)u \in X\}$ and $Au = P(\partial/\partial x)u$ for $u \in D(A)$. Then for the space E of Theorem 1(v) one has $E = D(A^\infty) := \bigcap_{k=1}^\infty D(A^k)$ and, in the terminology of [U], assertion (b) of Theorem 1(v) means that the operator A is $D(A^\infty)$ -well posed. The $D(A^\infty)$ -well posedness of an operator A from a Banach space into itself is one of the central notions of the ACP-theoretical paper [U]. In Sec. 1.3 of [Ki] the notion of $D(A^\infty)$ -well posedness is also elucidated by some facts not mentioned in [U].

7. Results similar to (a) \Rightarrow (b) \wedge ($\omega_E \leq \omega_0$) of Theorems 1(iii) and 1(iv) constitute a part of Theorem 4.1 of [H-H-N] obtained by means of one-parameter regularized semigroups of operators and Fourier multipliers.

4. Banach and Hilbert spaces adapted to $P(\partial/\partial x)$. Constructions of G. Birkhoff and S. D. Eidelman–S. G. Krein

In Theorem 1 all the spaces E are not normed. In order to compare Theorem 1 with earlier results related to some special Banach and Hilbert spaces E depending on $P(\partial/\partial x)$ let us recall two constructions:

- (vi) $E = \mathcal{B}_{\mathcal{N},p}$ where $p \in [1, \infty[$ and $\mathcal{B}_{\mathcal{N},p}$ is the Banach space introduced by G. Birkhoff and T. Mullikin in [B-M] and [B],

(vii) $E = \mathcal{L}_B$ where \mathcal{L}_B is the Hilbert space constructed by S. D. Eidelman and S. G. Krein.

$\mathcal{B}_{\mathcal{N},p}$ depends upon the K. Baker map \mathcal{N} whose existence was proved in [Ba], and which is a Borel measurable map $\mathcal{N} : \mathbb{R}^n \rightarrow M_m$ such that for every $\xi \in \mathbb{R}^n$ the matrix $\mathcal{N}(\xi)$ is invertible and $\mathcal{N}(\xi)\tilde{P}(\xi)\mathcal{N}(\xi)^{-1}$ is a Jordan matrix whose diagonal elements belong to $\sigma(\tilde{P}(\xi))$, directly over-diagonal elements are equal to zero or one, and all other elements are equal to zero. Given a Baker map \mathcal{N} one defines $\mathcal{B}_{\mathcal{N},p}$ as a linear subset of $(Z'_n)^m$ consisting of all those elements u of $(Z'_n)^m$ for which the distribution $\mathcal{F}^{-1}u \in (\mathcal{D}'(\mathbb{R}^n))^m$ is a \mathbb{C}^m -valued Lebesgue measurable function on \mathbb{R}^n such that

$$\|u\|_{\mathcal{N},p} = \left(\int_{\mathbb{R}^n} \|\mathcal{N}(\xi)(\mathcal{F}^{-1}u)(\xi)\|_{\mathbb{C}^m}^p d\xi \right)^{1/p} < \infty. \quad (4.1)$$

Equipped with the norm $\|\cdot\|_{\mathcal{N},p}$, $\mathcal{B}_{\mathcal{N},p}$ is a Banach space continuously imbedded in $(Z'_n)^m$, and $\mathcal{B}_{\mathcal{N},2}$ is a Hilbert space. If the Baker function $\mathcal{N} : \mathbb{R}^n \rightarrow M_m$ is bounded on \mathbb{R}^n and $p \in [2, \infty[$, then, by the Hausdorff–Young theorem, $\mathcal{B}_{\mathcal{N},p} \supset (L^q(\mathbb{R}^n))^m$, where $q = p/(p-1) \in]1, 2]$.

Construction of \mathcal{L}_B , described in Section I.8 of the monograph [Kr], requires the assumption that $\omega_0 < \infty$. If $\omega_0 < \infty$, then \mathcal{L}_B is the domain of the selfadjoint strictly positive definite square root of some selfadjoint strictly positive definite operator $B(\partial/\partial x)$ acting in $(L^2(\mathbb{R}^n))^m$. For any given $\omega_1 \in]\omega_0, \infty[$ the operator $B(\partial/\partial x)$ may be constructed as a matricial partial differential operator with constant coefficients whose symbol \tilde{B} has the following property:

for every $\xi \in \mathbb{R}^n$ the matrix $\tilde{B}(\xi) \in M_m$ is hermitian such that

$$\tilde{B}(\xi) \geq \mathbb{1} \quad \text{and} \quad \tilde{B}(\xi)\tilde{P}(\xi) + \tilde{P}(\xi)^*\tilde{B}(\xi) \leq 2\omega_1\tilde{B}(\xi). \quad (4.2)$$

Let $\mathcal{N}(\xi)$ be the hermitian strictly positive definite square root of $\tilde{B}(\xi)$. Then $\mathcal{N}(\xi) = \frac{1}{2\pi i} \int_C z^{1/2}(z\mathbb{1} - \tilde{B}(\xi))^{-1} dz$ where C is a closed rectifiable curve contained in $\{z \in \mathbb{C} : \operatorname{Re} z \geq 1/2\}$ and winding once about $\sigma(\tilde{B}(\xi))$, which is a finite subset of $[1, \infty[$. It follows that $\mathcal{N}(\xi)$ is a C^∞ -function of ξ . By (4.2) for every $\xi \in \mathbb{R}^n$ one has

$$\|\mathcal{N}(\xi)^{-1}\|_{L(\mathbb{C}^m)} \leq 1 \quad \text{and} \quad \mathcal{N}(\xi)\tilde{P}(\xi)\mathcal{N}(\xi)^{-1} + (\mathcal{N}(\xi)\tilde{P}(\xi)\mathcal{N}(\xi)^{-1})^* \leq 2\omega_1\mathbb{1}.$$

The norm in \mathcal{L}_B is defined by the formula

$$\begin{aligned} \|u\|_{\mathcal{L}_B} &= \left\| \left(B \left(\frac{\partial}{\partial x} \right) \right)^{1/2} u \right\|_{(L^2(\mathbb{R}^n))^m} \\ &= (2\pi)^n \left(\int_{\mathbb{R}^n} \|\mathcal{N}(\xi)(\mathcal{F}^{-1}u)(\xi)\|_{\mathbb{C}^m}^2 d\xi \right)^{1/2}. \end{aligned} \quad (4.3)$$

\mathcal{L}_B is a Hilbert space continuously imbedded in $(L^2(\mathbb{R}^n))^m$.

Theorem 2.

- (I) Suppose that $p \in [1, \infty[$, \mathcal{N} is a Baker function for \tilde{P} , and $\mathcal{N} \in L^p_{\text{loc}}(\mathbb{R}^n; M_m)$. Then the same statements as in Theorem 1 are true for $E = \mathcal{B}_{\mathcal{N}, p}$ provided $P(\partial/\partial x)|_E$ is replaced by the restriction of $P(\partial/\partial x)$ to the set $\{u \in \mathcal{B}_{\mathcal{N}, p} : P(\partial/\partial x)u \in \mathcal{B}_{\mathcal{N}, p}\}$.
- (II) If $\omega_0 < \infty$, $\omega_1 \in]\omega_0, \infty[$, $E = \mathcal{L}_B$ is constructed so that (4.2) is satisfied, and $P(\partial/\partial x)|_E$ is replaced by the restriction of $P(\partial/\partial x)$ to the set $\{u \in \mathcal{L}_B : P(\partial/\partial x)u \in \mathcal{L}_B\}$, then (b) holds and $\omega_0 \leq \omega_E \leq \omega_1$.

The implication (a) \Rightarrow (b) \wedge ($\omega_0 = \omega_{\mathcal{B}_{\mathcal{N}, p}}$) of item (I) was proved by G. Birkhoff [B] without the assumption that the Baker function is locally bounded or locally integrable. The statement (II), except the inequality $\omega_0 \leq \omega_{\mathcal{L}_B}$, goes back to S. D. Eidelman and S. G. Krein. The proof is given in Section I.8 of S. G. Krein's monograph [Kr]. Apart from the results proved in [B] and [Kr], Theorem 2 contains information that in both cases, $E = \mathcal{L}_B$ and $E = \mathcal{B}_{\mathcal{N}, p}$ with some special \mathcal{N} , condition (b) implies that $\omega_0 \leq \omega_E$.

In order to prove this last statement suppose that (b) holds, and let $(S_t)_{t \geq 0} \subset L(E)$ be the (unique) semigroup of class (C_0) occurring in (b). The equicontinuity index ω_E of this semigroup is finite and equal to the characteristic exponent $\lim_{t \rightarrow \infty} \frac{1}{t} \log \|S_t\|_{L(E)}$ of the function $t \mapsto \|S_t\|_{L(E)}$. Consequently, by (4.1) and (4.3), for every $\varepsilon > 0$ there is $K_\varepsilon \in [1, \infty[$ such that

$$\left(\int_{\mathbb{R}^n} \|\mathcal{N}(\eta)(\mathcal{F}^{-1}S_t\mathcal{F}\varphi)(\eta)\|_{\mathbb{C}^m}^p d\eta \right)^{1/p} \leq K_\varepsilon e^{(\omega_E + \varepsilon)t} \left(\int_{\mathbb{R}^n} \|\mathcal{N}(\eta)\varphi(\eta)\|_{\mathbb{C}^m}^p d\eta \right)^{1/p}$$

for every $t \in [0, \infty[$ and $\varphi \in (L^\infty_c(\mathbb{R}^n))^m$, i.e. M_m -valued φ which is bounded and measurable on \mathbb{R}^n and has compact support. By the uniqueness theorem from Section 2, one has $S_t = G_t|_E$, so that $(\mathcal{F}^{-1}S_t\mathcal{F}\varphi)(\eta) = e^{t\tilde{P}(\eta)}\varphi(\eta)$ and

hence

$$\begin{aligned} & \left(\int_{\mathbb{R}^n} \|\mathcal{N}(\eta) e^{t\tilde{P}(\eta)} \varphi(\eta)\|_{\mathbb{C}^m}^p d\eta \right)^{1/p} \\ & \leq K_\varepsilon e^{(\omega_E + \varepsilon)t} \left(\int_{\mathbb{R}^n} \|\mathcal{N}(\eta) \varphi(\eta)\|_{\mathbb{C}^m}^p d\eta \right)^{1/p} \end{aligned} \quad (4.4)$$

whenever $t \in [0, \infty[$ and $\varphi \in (L_c^\infty(\mathbb{R}^n))^m$.

For any $t \in [0, \infty[$ and $\xi \in \mathbb{R}^n$ pick a $z_{t,\xi} \in \mathbb{C}^m$ such that $\|z_{t,\xi}\|_{\mathbb{C}^m} = 1$ and

$$\|\mathcal{N}(\xi) e^{t\tilde{P}(\xi)}\|_{L(\mathbb{C}^m)} = \|\mathcal{N}(\xi) e^{t\tilde{P}(\xi)} z_{t,\xi}\|_{\mathbb{C}^m}. \quad (4.5)$$

For every $\xi \in \mathbb{R}^n$ and $r > 0$ let $\phi_{\xi,r} \in L_c^\infty(\mathbb{R}^n)$ be a non-negative function such that $\int_{\mathbb{R}^n} \phi_{\xi,r}(\eta) d\eta = 1$, the support of $\phi_{\xi,r}$ is equal to the ball with center at ξ and radius r , and $\phi_{\xi,r}$ is constant in this ball. Since, in case (I), $\mathcal{N} \in L_{\text{loc}}^p(\mathbb{R}^n; M_m)$, it follows that there is a set $Z_t \subset \mathbb{R}^n$ of n -dimensional Lebesgue measure zero such that if $\xi \in \mathbb{R}^n \setminus Z_t$, then ξ is a Lebesgue point of the locally integrable function $\|\mathcal{N}(\cdot) e^{t\tilde{P}(\cdot)}\|^p$, i.e.

$$\lim_{r \downarrow 0} \int_{\mathbb{R}^n} \|\mathcal{N}(\eta) e^{t\tilde{P}(\eta)}\|^p - \|\mathcal{N}(\xi) e^{t\tilde{P}(\xi)}\|^p |\phi_{\xi,r}(\eta)| d\eta = 0. \quad (4.6)$$

See [S-K], Theorem 5.3, p. 164. In case (II) this difficult theorem need not be used because (4.6) with $Z_t = \emptyset$ holds by virtue of continuity of \mathcal{N} . From (4.5) and (4.6) it follows that whenever $t \in [0, \infty[$ and $\xi \in \mathbb{R}^n \setminus Z_t$, then

$$\|\mathcal{N}(\xi) e^{t\tilde{P}(\xi)}\| = \lim_{r \downarrow 0} \left(\int_{\mathbb{R}^n} \|\mathcal{N}(\eta) e^{t\tilde{P}(\eta)} (\phi_{\eta,r}(\eta))^{1/p} z_{t,\xi}\|^p d\eta \right)^{1/p}.$$

Hence, applying (4.4) to $\varphi(\eta) = (\phi_{\xi,r}(\eta))^{1/p} z_{t,\xi}$, one concludes that whenever $t \in [0, \infty[$ and $\xi \in \mathbb{R}^n \setminus (Z_0 \cup Z_t)$, then

$$\begin{aligned} & \|\mathcal{N}(\xi) e^{t\tilde{P}(\xi)}\| \\ & \leq K_\varepsilon e^{(\omega_E + \varepsilon)t} \lim_{r \downarrow 0} \left(\int_{\mathbb{R}^n} \|\mathcal{N}(\eta) (\phi_{\xi,r}(\eta))^{1/p} z_{t,\xi}\|^p d\eta \right)^{1/p} \leq K_\varepsilon e^{(\omega_E + \varepsilon)t} \|\mathcal{N}(\xi)\| \end{aligned}$$

and consequently

$$\|e^{t\tilde{P}(\xi)}\| \leq K_\varepsilon \|\mathcal{N}(\xi)\| \|\mathcal{N}(\xi)^{-1}\| e^{(\omega_E + \varepsilon)t}.$$

By Proposition 2.2, p. 251, and Corollary 2.4, p. 252, in [E-N] it follows that

$$\max\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi))\} = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|e^{t\tilde{P}(\xi)}\| \leq \omega_E + \varepsilon$$

for every $\xi \in \mathbb{R}^n \setminus \bigcup_{k \in \mathbb{N}_0} Z_k$. Since $\bigcup_{k \in \mathbb{N}_0} Z_k$ has measure zero and $\max\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi))\}$ depends continuously on ξ , one concludes that $\omega_0 \leq \omega_E + \varepsilon$. This implies the inequality $\omega_0 \leq \omega_E$ because $\varepsilon > 0$ is arbitrary.

5. Gårding's lemma

Theorem (L. Gårding). *Consider a polynomial of $n+1$ variables with complex coefficients*

$$p(X_0, X_1, \dots, X_n) = \sum_{k=1}^m p_k(X_1, \dots, X_n) X_0^k + p_0(X_1, \dots, X_n) \in \mathbb{C}[X_0, \dots, X_n].$$

Suppose that $p_m(\xi_1, \dots, \xi_n) > 0$ for every $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. For every $r \in [0, \infty[$ define

$$\Lambda(r) = \sup\{\operatorname{Re} \lambda : \lambda \in \mathbb{C}, (\xi_1, \dots, \xi_n) \in \mathbb{R}^n, \|(\xi_1, \dots, \xi_n)\| \leq r, p(\lambda, \xi_1, \dots, \xi_n) = 0\}.$$

Then there is a real A and a rational α such that

$$\Lambda(r) = Ar^\alpha(1 + o(1)) \quad \text{as } r \rightarrow \infty.$$

This theorem was formulated by L. Gårding in [G] as the Lemma on p. 11. The argument consisted in

- (A) explaining the existence of a polynomial $q(Y_1, Y_2) \in \mathbb{C}[Y_1, Y_2]$ such that $q(r, \Lambda(r)) = 0$ for every $r \in [0, \infty[$,
- (B) applying the Puiseux series expansions of algebraic functions \mathcal{R} of one complex variable z satisfying the equation $q(z, \mathcal{R}(z)) = 0$.

L. Hörmander [H1], proof of Lemma 3.9, noticed that stage (A) may be realized by an application of a theorem asserting that the projection onto \mathbb{R}^d of a semi-algebraic subset of \mathbb{R}^{d+n} is a semi-algebraic subset of \mathbb{R}^d . This projection theorem may be proved by an argument similar to that from A. Seidenberg's proof [Se] of the decision theorem of A. Tarski (belonging to

mathematical logic). Detailed presentations of Seidenberg's proof in the case of the projection theorem are given in [G2] and [F]. L. Hörmander's proof of the projection theorem presented in the Appendix to [H2] is based on an argument resembling that from P. J. Cohen's proof [C1, 2] of the decision theorem. In [Ki] the reasonings of stage (B) of Gårding's proof are presented with exact references to the theory of algebraic functions of one complex variable presented in [S-Z].

The above theorem of Gårding yields at once

Corollary 1. *Let $P(X_1, \dots, X_n) \in M_m[X_1, \dots, X_n]$ and suppose that there is $C \in]0, \infty[$ such that*

$$\max\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi))\} \leq C + C \log(1 + |\xi|) \quad \text{for every } \xi \in \mathbb{R}^n.$$

Then

$$\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi)), \xi \in \mathbb{R}^n\} < \infty.$$

This corollary was formulated as a conjecture by I. G. Petrovskiĭ in a footnote on p. 24 of [P].

6. Interpolation polynomials and estimations of $e^{t\tilde{P}(\xi)}$

Assumption (A). Let $\lambda_1, \dots, \lambda_m \in \mathbb{C}$. Denote by S the set $\{\lambda_1, \dots, \lambda_m\}$. For every $\lambda \in S$ denote by $m(\lambda)$ the number of occurrences of λ in the sequence $\lambda_1, \dots, \lambda_m$. Let $p(\lambda) = p_0 + p_1\lambda + \dots + p_d\lambda^d$ be a polynomial of degree d with complex coefficients. Let f be a function holomorphic in an open set $O \subset \mathbb{C}$ containing S . Choose $r > 0$ such that $K := \bigcup_{\lambda \in S} \{z \in \mathbb{C} : |z - \lambda| \leq r\} \subset O$ and let \mathcal{C} be the boundary of K oriented so that $\operatorname{Index}(\mathcal{C}, \lambda) = 1$ for every $\lambda \in S$.

Theorem I. *Under assumption (A) the following two conditions are equivalent:*

- (i) $p^{(k)}(\lambda) = f^{(k)}(\lambda)$ for every $\lambda \in S$ and $k = 0, \dots, m(\lambda) - 1$,
- (ii) $p(A) := p_0\mathbb{1} + \sum_{k=1}^d p_k A^k = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(z\mathbb{1} - A)^{-1} dz$ for every matrix $A \in M_m$ such that $\sigma(A) = S$ and for every $\lambda \in \sigma(A)$ the spectral multiplicity of λ is equal to $m(\lambda)$.

There is exactly one polynomial p of order no greater than $m-1$ satisfying (i).

For a polynomial p of arbitrary degree d the equivalence (i) \Leftrightarrow (ii) may be deduced either from Theorems 5, 8 and 10 of Section VII.1 of [D-S], or from

Theorems 134, 138 and 234 of [G-L]. For $d \leq m - 1$ the equivalence (i) \Leftrightarrow (ii) is a part of Fact 1 stated in [Hig]. In connection with (i), interpolation terminology is used: p is called the *interpolation polynomial* for f , the numbers $\lambda \in S$ are called the *nodes of interpolation*, and $m(\lambda)$ is the *multiplicity of the node* λ .

Theorem II. *For a polynomial p of degree no greater than $m - 1$ conditions (i) and (ii) are equivalent to either of the following conditions:*

$$(iii) \quad p(\lambda) = c_0 + c_1(\lambda - \lambda_1) + c_2(\lambda - \lambda_1)(\lambda - \lambda_2) + \cdots \\ + c_{m-1}(\lambda - \lambda_1) \cdots (\lambda - \lambda_{m-1}) \quad (6.1)$$

where

$$c_k = c_k(f; \lambda_1, \dots, \lambda_{k+1}) \\ = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(z - \lambda_1)^{-1}(z - \lambda_2)^{-1} \cdots (z - \lambda_{k+1})^{-1} dz \quad (6.2)$$

$$\text{for } k = 0, \dots, m - 1, \\ (iv) \quad p(\lambda) = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z) \sum_{\mu=1}^m \frac{1}{\mu!} \frac{Q^{(\mu)}(z)}{Q(z)} (\lambda - z)^{\mu-1} dz \quad \text{where } Q(z) = \prod_{k=1}^m (z - \lambda_k).$$

If $O \supset \text{conv}S$, then

$$c_k(f; \lambda_1, \dots, \lambda_{k+1}) = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{k-1}} f^{(k)}((1 - t_1)\lambda_1 + (t_1 - t_2)\lambda_2 + \cdots \\ + (t_{k-1} - t_k)\lambda_k + t_k\lambda_{k+1}) dt_k dt_{k-1} \cdots dt_1 \quad (6.3)$$

for $k = 0, \dots, m - 1$. Furthermore,

$$\frac{1}{\mu!} \frac{Q^{(\mu)}(z)}{Q(z)} = \tau_\mu \left(\frac{1}{z - \lambda_1}, \dots, \frac{1}{z - \lambda_m} \right) \quad \text{for } \mu = 1, \dots, m \quad (6.4)$$

where $\tau_\mu(x_1, \dots, x_m) = \sum_{1 \leq i_1 < \dots < i_\mu \leq m} x_{i_1} \cdots x_{i_\mu}$, $\mu = 1, \dots, m$, are the elementary symmetric polynomials of m variables x_1, \dots, x_m . Consequently, condition (iv) may be written in the equivalent form:

$$(iv)' \quad p(\lambda) = a_0 + a_1\lambda + \cdots + a_{m-1}\lambda^{m-1} \quad (6.5)$$

where

$$a_k = a_k(f; \lambda_1, \dots, \lambda_m) = \sum_{l=0}^{m-k-1} \binom{k+l}{k} I_{k+l+1}^l \quad \text{for } k = 0, \dots, m-1 \quad (6.6)$$

and

$$I_\mu^l = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(-z)^l \tau_\mu \left(\frac{1}{z - \lambda_1}, \dots, \frac{1}{z - \lambda_m} \right) dz \quad (6.7)$$

for $\mu = 1$ and $l = 0, \dots, \mu - 1$.

The explicit formulas (6.2) and (6.3) for the coefficients c_k in the Newton form (6.1) of the interpolation polynomial of degree no greater than $m - 1$ are deduced in Section I.4.2 and I.4.3 of A. O. Gelfond's book [Ge]. E. A. Gorin [G1] inferred from (6.2) that the coefficients a_0, \dots, a_{m-1} of the interpolation polynomial in the form (6.5) are linear combinations of the integrals

$$I_{i_1, \dots, i_k}^l = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(-z)^l (z - \lambda_{i_1})^{-1} \dots (z - \lambda_{i_k})^{-1} dz. \quad (6.8)$$

The exact computation of these linear combinations by Gorin's method is possible but troublesome. We will prove (6.6)–(6.7) by another method, based on Theorem I. Notice that, in connection with Cauchy's problem for a system of PDE with constant coefficients, formulas similar to (6.6)–(6.7) were used by E. A. Gorin [G1] and T. Ushijima [U].

Proof of (iv) and (6.4)–(6.7). Take a matrix $A \in M_m$ such that $\sigma(A) = S$ and for every $\lambda \in \sigma(A)$ the spectral multiplicity of λ is equal to $m(\lambda)$. By Taylor's formula and the Cayley–Hamilton theorem,

$$Q(z)\mathbb{1} + \sum_{\mu=1}^m \frac{1}{\mu!} Q^{(\mu)}(z)(A - z\mathbb{1})^\mu = Q(A) = 0,$$

whence

$$(z\mathbb{1} - A)^{-1} = \sum_{\mu=1}^m \frac{1}{\mu!} \frac{Q^{(\mu)}(z)}{Q(z)} (A - z\mathbb{1})^{\mu-1} \quad \text{for every } z \in \mathbb{C} \setminus S,$$

and so the polynomial $p(\lambda)$ occurring in (iv) satisfies (ii). Moreover, its degree is no greater than $m - 1$, so that, by Theorem I, it is the unique polynomial satisfying (ii). This proves the equivalence (ii) \Leftrightarrow (iv) in the class of polynomials of degree no greater than $m - 1$.

In order to prove (6.4) notice that $Q(z) = \tau_m(z - \lambda_1, \dots, z - \lambda_m)$ and

$$\frac{d}{dz} \tau_\mu(z - \lambda_1, \dots, z - \lambda_m) = (m - \mu + 1) \tau_{\mu-1}(z - \lambda_1, \dots, z - \lambda_m)$$

for $z \in \mathbb{C}$ and $\mu = 1, \dots, m$ where $\tau_0 \equiv 1$. Consequently,

$$Q^{(\mu)}(z) = \left(\frac{d}{dz}\right)^\mu \tau_m(z - \lambda_1, \dots, z - \lambda_m) = \mu! \tau_{m-\mu}(z - \lambda_1, \dots, z - \lambda_m),$$

and so

$$\frac{1}{\mu!} \frac{Q^{(\mu)}(z)}{Q(z)} = \frac{\tau_{m-\mu}(z - \lambda_1, \dots, z - \lambda_m)}{\tau_m(z - \lambda_1, \dots, z - \lambda_m)} = \tau_\mu\left(\frac{1}{z - \lambda_1}, \dots, \frac{1}{z - \lambda_m}\right)$$

for $z \in \mathbb{C} \setminus S$ and $\mu = 1, \dots, m$. Therefore the polynomial occurring in (iv) may be written in the form

$$p(\lambda) = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z) \left[\sum_{\mu=1}^m \tau_\mu\left(\frac{1}{z - \lambda_1}, \dots, \frac{1}{z - \lambda_m}\right) (\lambda - z)^{\mu-1} \right] dz,$$

whence the formulas (6.6)–(6.7) for the coefficients a_k , $k = 0, 1, \dots, m-1$, occurring in (6.5) follow by applying the binomial formula to $(\lambda - z)^{\mu-1}$.

I. M. Gelfand and G. E. Shilov in Sec. II.6.1 of their book [G-S3] have reproduced the proof of A. O. Gelfond's formula (6.3) and observed that this formula implies at once the important inequality

$$\|e^{tA}\|_{L(\mathbb{C}^m)} \leq e^{\omega t} \left(1 + \sum_{k=1}^{m-1} \frac{(2t)^k}{k!} \|A\|_{L(\mathbb{C}^m)}^k \right), \quad (6.9)$$

$$\omega = \max\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\},$$

for every $A \in M_m$ and $t \in [0, \infty[$. The proof of (6.3) and (6.9) is also presented in Sec. 7.2 of A. Friedman's book [F].

The inequality (6.9) is crucial for the proofs of our Theorem 1 from Section 3 in cases (i)–(iv). In case (v) we follow T. Ushijima [U] and instead of using (6.9) we base on estimation of some Gorin's integrals. This method yields the following

Proposition. *Let $\tilde{P} \in M_m[\xi_1, \dots, \xi_m]$ be the symbol of a matricial differential operator $P(\partial/\partial x)$ with constant coefficients described in Section 1. Suppose that the Petrovskiĭ correctness condition is satisfied:*

$$\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi)), \xi \in \mathbb{R}^n\} = \omega_0 < \infty. \quad (6.10)$$

Then there are functions $p_k \in C^\infty(\mathbb{R}^{1+n}; \mathbb{C})$, $k = 0, 1, \dots, 2m$, such that

$$e^{t\tilde{P}(\xi)} = p_0(t, \xi)\mathbb{1} + \sum_{k=1}^{2m} p_k(t, \xi)\tilde{P}(\xi)^k \quad \text{for every } (t, \xi) \in \mathbb{R}^{1+n} \quad (6.11)$$

and

$$\sup\{e^{-(\omega_0+\varepsilon)t}|p_k(t, \xi)| : t \in [0, \infty[, \xi \in \mathbb{R}^n\} < \infty \quad (6.12)$$

for every $\varepsilon > 0$ and $k = 0, 1, \dots, 2m$.

Proof. By Theorems 5 and 10 in Sec. VII.1 of [D-S], or by Facts 1 and 8 in Sec. 1 of [Hig], for every $(t, \xi) \in \mathbb{R}^{1+n}$ one has

$$\begin{aligned} e^{t\tilde{P}(\xi)} &= \lim_{N \rightarrow \infty} \left(\mathbb{1} + t\tilde{P}(\xi) + \dots + \frac{t^N}{N!}\tilde{P}(\xi)^N \right) \\ &= \lim_{N \rightarrow \infty} \frac{1}{2\pi i} \int_{\mathcal{C}_\xi} \left(1 + tz + \dots + \frac{t^N}{N!}z^N \right) (z\mathbb{1} - \tilde{P}(\xi))^{-1} dz \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}_\xi} e^{tz} (z\mathbb{1} - \tilde{P}(\xi))^{-1} dz \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}_\xi} (z - z_0)^{m+1} (z\mathbb{1} - \tilde{P}(\xi))^{-1} dz \\ &\quad \times \frac{1}{2\pi i} \int_{\mathcal{C}_\xi} (z - z_0)^{-m-1} e^{tz} (z\mathbb{1} - \tilde{P}(\xi))^{-1} dz \\ &= (\tilde{P}(\xi) - z_0\mathbb{1})^{m+1} \cdot \frac{1}{2\pi i} \int_{\mathcal{C}_\xi} (z - z_0)^{-m-1} e^{tz} (z\mathbb{1} - \tilde{P}(\xi))^{-1} dz \end{aligned}$$

where $z_0 \in \mathbb{C}$ is a point such that $\operatorname{Re} z_0 > \omega_0$ and \mathcal{C}_ξ is a rectifiable closed path contained in $\{z \in \mathbb{C} \setminus \sigma(\tilde{P}(\xi)) : \operatorname{Re} z < \operatorname{Re} z_0\}$ and winding once about $\sigma(\tilde{P}(\xi))$. By Theorem I and Theorem II(iv) and (iv)', it follows that

$$\begin{aligned} e^{t\tilde{P}(\xi)} &= (\tilde{P}(\xi) - z_0\mathbb{1})^{m+1} (a_0(t, \xi)\mathbb{1} \\ &\quad + a_1(t, \xi)\tilde{P}(\xi) + \dots + a_{m-1}(t, \xi)\tilde{P}(\xi)^{m-1}) \end{aligned} \quad (6.13)$$

for every $(t, \xi) \in \mathbb{R}^{1+n}$ where

$$\begin{aligned} a_k(t, \xi) &= \sum_{l=0}^{m-1-k} \binom{k+l}{k} \frac{1}{2\pi i} \int_{\mathcal{C}_\xi} (z - z_0)^{-m-1} e^{tz} (-z)^l \\ &\quad \times \frac{1}{(k+l+1)!} \frac{\left(\frac{d}{dz}\right)^{k+l+1} Q(z, \xi)}{Q(z, \xi)} dz \end{aligned} \quad (6.14)$$

$$\begin{aligned}
&= \sum_{l=0}^{m-1-k} \binom{k+l}{k} \frac{1}{2\pi i} \int_{\mathcal{C}_\xi} (z-z_0)^{-m-1} e^{tz} (-z)^l \\
&\quad \times \tau_{k+l+1} \left(\frac{1}{z-\lambda_1(\xi)}, \dots, \frac{1}{z-\lambda_m(\xi)} \right) dz. \tag{6.15}
\end{aligned}$$

Above, $\lambda_1(\xi), \dots, \lambda_m(\xi)$ is the sequence of eigenvalues of $\tilde{P}(\xi)$ in which the number of occurrences of each eigenvalue is equal to its spectral multiplicity, and

$$Q(z, \xi) = \det(z\mathbb{1} - \tilde{P}(\xi)) = \prod_{k=1}^m (z - \lambda_k(\xi))$$

is the characteristic polynomial of $\tilde{P}(\xi)$.

By (6.13) the proposition follows once it is proved that

$$a_k \in C^\infty(\mathbb{R}^{1+n}; \mathbb{C}) \tag{6.16}$$

and

$$\sup\{e^{-(\omega_0+\varepsilon)t} |a_k(t, \xi)| : t \in [0, \infty[, \xi \in \mathbb{R}^n\} < \infty \tag{6.17}$$

whenever $k = 0, \dots, m-1$ and $\varepsilon > 0$.

In order to prove (6.16) notice that every $\xi_0 \in \mathbb{R}^n$ has an open neighbourhood U such that \mathcal{C}_{ξ_0} winds once about $\sigma(\tilde{P}(\xi))$ whenever $\xi \in U$. (This follows from a theorem of Hurwitz. See [S-Z], Sec. III.11.) Consequently, for every $\xi \in U$ one can replace \mathcal{C}_ξ by \mathcal{C}_{ξ_0} without changing the values of the integrals in (6.14), and then (6.16) follows because $Q(z, \xi)$ is a C^∞ -function of (z, ξ) non-vanishing on the open set $\{(z, \xi) \in \mathbb{C} \times \mathbb{R}^n : z \notin \sigma(\tilde{P}(\xi))\}$ which contains $\{(z, \xi) \in \mathbb{C} \times \mathbb{R}^n : z \in \mathcal{C}_{\xi_0}\}$ if $\xi \in U$.

It remains to prove (6.17). To this end, fix $\varepsilon > 0$ and take $\delta \in]0, \min(\varepsilon, \frac{1}{2}(\operatorname{Re} z_0 - \omega_0))]$. Let $\xi \in \mathbb{R}^n$. Since $\sigma(\tilde{P}(\xi)) \subset \{z \in \mathbb{C} : \operatorname{Re} z \leq \omega_0\}$, without changing the values of the integrals in (6.15) one can choose a closed rectifiable path \mathcal{C}_ξ winding once about $\sigma(\tilde{P}(\xi))$ such that

$$\mathcal{C}_\xi \subset D_{\xi, \delta} := \{z \in \mathbb{C} : \operatorname{Re} z - \omega_0 \leq \delta \leq \operatorname{dist}(z, \sigma(\tilde{P}(\xi)))\}.$$

For every $\xi \in \mathbb{R}^n$ the straight line

$$L = \{z \in \mathbb{C} : \operatorname{Re} z = \omega_0 + \delta\}$$

is contained in $D_{\xi, \delta}$. Furthermore, whenever $t \in [0, \infty[, \xi \in \mathbb{R}^n, z \in D_{\xi, \delta}$ and $k+l = 0, \dots, m-1$, then $|z - \lambda_i(\xi)| \geq \delta$, $|z - z_0| \geq \operatorname{Re} z_0 - \operatorname{Re} z \geq \omega_0 + 2\delta - (\omega_0 + \delta) = \delta$ and

$$\left| \frac{z}{z - z_0} \right| = \left| 1 + \frac{z_0}{z - z_0} \right| \leq 1 + \frac{|z_0|}{\delta},$$

so that

$$\begin{aligned} & |(z - z_0)^{-m-1} e^{tz} (-z)^l (z - \lambda_{i_1}(\xi))^{-1} \cdots (z - \lambda_{i_{k+l+1}}(\xi))^{-1}| \\ & \leq \left| \frac{z}{z - z_0} \right|^l |z - z_0|^{l+1-m} \delta^{-k-l-1} |z - z_0|^{-2} e^{(\omega_0 + \delta)t} \\ & \leq \left(1 + \frac{|z_0|}{\delta} \right)^l \delta^{-m-k} |z - z_0|^{-2} e^{(\omega_0 + \delta)t}, \end{aligned}$$

and hence

$$\left| (z - z_0)^{-m-1} e^{tz} (-z)^l \tau_{k+l+1} \left(\frac{1}{z - \lambda_1(\xi)}, \dots, \frac{1}{z - \lambda_m(\xi)} \right) \right| \leq K_\delta |z - z_0|^{-2} e^{(\omega_0 + \delta)t}$$

where K_δ is a finite number depending only on δ . Therefore, by Cauchy's integral theorem, the integration contour \mathcal{C}_ξ in (6.15) may be replaced by the straight line L . Since $\delta \leq \varepsilon$, one obtains the estimate

$$|a_k(t, \xi)| \leq \left(\sum_{l=0}^{m-1-k} \binom{k+l}{k} \right) \frac{K_\delta}{2\pi} \left(\int_L |z - z_0|^{-2} dz \right) e^{(\omega_0 + \varepsilon)t}$$

for $k = 0, \dots, m-1$, $t \in [0, \infty[$ and $\xi \in \mathbb{R}^n$, proving (6.17).

7. Proof of Theorem 1 in cases (iv) and (v)

Theorem 1 is the conjunction of three implications: (a) \Rightarrow (b) \cap ($\omega_E \leq \omega_0$), (b) \Rightarrow (a), and (b) \wedge ($\omega_E < \infty$) \Rightarrow ($\omega_0 \leq \omega_E$), the proofs of which differ in particular cases (i)–(v). The present section is devoted to Theorem 1(iv) and 1(v). The proofs in these cases are independent of the general theory of l.c.v.s. and the advanced theory of distributions.

Let either $E = (H^\infty(\mathbb{R}^n))^m$ and $\|u\|_j = (\sum_{0 \leq |\alpha| \leq j} \|(\partial/\partial x)^\alpha u\|_{(L^2(\mathbb{R}^n))^m}^2)^{1/2}$ for $j = 0, 1, \dots$, or $E = \{u \in (L^2(\mathbb{R}^n))^m : (P(\partial/\partial x))^l u \in (L^2(\mathbb{R}^n))^m \text{ for } l = 1, 2, \dots\}$ and $\|u\|_j = \sum_{l=0}^j \|(P(\partial/\partial x))^l u\|_{(L^2(\mathbb{R}^n))^m}$ for $j = 0, 1, \dots$.

Proof of (a) \Rightarrow (b) \wedge ($\omega_E \leq \omega_0$). Suppose that (a) holds, i.e. $\omega_0 \leq \infty$. Let $(G_t)_{t \in \mathbb{R}}$ be the one-parameter group (2.1). Whenever $E = (H^\infty(\mathbb{R}^n))^m$,

$\varphi \in (C_c^\infty(\mathbb{R}^n))^m$, $u = \mathcal{F}\varphi \in (Z_n)^m$, $j = 0, 1, \dots$ and $t \in \mathbb{R}$, then, by the Remark at the end of Section 2, $(\partial/\partial x)^\alpha G_t u = G_t (\partial/\partial x)^\alpha u \in (Z_n)^m$ and, by Plancherel's theorem,

$$\begin{aligned} \|G_t u\|_j &= \left(\sum_{0 \leq |\alpha| \leq j} \left\| G_t \left(\frac{\partial}{\partial x} \right)^\alpha u \right\|_0^2 \right)^{1/2} \\ &= (2\pi)^{n/2} \left(\sum_{0 \leq |\alpha| \leq j} \int_{\mathbb{R}^n} \|e^{t\tilde{P}(\xi)} \xi^\alpha \varphi(\xi)\|_{\mathbb{C}^m}^2 d\xi \right)^{1/2} \\ &\leq (2\pi)^{n/2} \left(\int_{\mathbb{R}^n} \|e^{t\tilde{P}(\xi)}\|_{L\mathbb{C}^m}^2 \left(\sum_{0 \leq |\alpha| \leq j} |\xi^\alpha|^2 \right) \|\varphi(\xi)\|_{\mathbb{C}^m}^2 d\xi \right)^{1/2}. \end{aligned}$$

Hence, by the Gelfand–Shilov inequality (6.9), for every $\varepsilon > 0$ there are $K_\varepsilon \in]0, \infty[$ and $K'_\varepsilon \in]0, \infty[$ such that if $t \in [0, \infty[$, $j = 0, 1, \dots$ and $u \in (Z_n)^m$, then

$$\begin{aligned} \|G_t u\|_j &\leq (2\pi)^{n/2} K_\varepsilon e^{(\omega_0 + \varepsilon)t} \left(\int_{\mathbb{R}^n} (1 + \|\tilde{P}(\xi)\|_{L(\mathbb{C}^m)}^{m-1})^2 \left(\sum_{0 \leq |\alpha| \leq j} |\xi^\alpha|^2 \right) \|\varphi(\xi)\|_{\mathbb{C}^m}^2 d\xi \right)^{1/2} \\ &\leq (2\pi)^{n/2} K'_\varepsilon e^{(\omega_0 + \varepsilon)t} \left(\int_{\mathbb{R}^n} \left(\sum_{0 \leq |\alpha| \leq j + (m-1)d} |\xi^\alpha|^2 \right) \|\varphi(\xi)\|_{\mathbb{C}^m}^2 d\xi \right)^{1/2} \\ &\leq (2\pi)^{n/2} K'_\varepsilon e^{(\omega_0 + \varepsilon)t} \left(\sum_{0 \leq |\alpha| \leq j + (m-1)d} \left\| \mathcal{F}^{-1} \left(\frac{\partial}{\partial x} \right)^\alpha u \right\|_{(L^2(\mathbb{R}^n))^m}^2 \right)^{1/2}. \end{aligned}$$

Again by Plancherel's theorem, it follows that

$$\begin{aligned} \|G_t u\|_j &\leq K'_\varepsilon e^{(\omega_0 + \varepsilon)t} \left(\sum_{0 \leq |\alpha| \leq j + (m-1)d} \left\| \left(\frac{\partial}{\partial x} \right)^\alpha u \right\|_{(L^2(\mathbb{R}^n))^m}^2 \right)^{1/2} \\ &= K'_\varepsilon e^{(\omega_0 + \varepsilon)t} \|u\|_{j + (m-1)d} \end{aligned} \quad (7.1)$$

whenever $t \in [0, \infty[$, $u \in (Z_n)^m$ and $j = 0, 1, \dots$. Since $(Z_n)^m$ is dense in E , one concludes that $G_t E \subset E$ for every $t \in [0, \infty[$, and that

$$\text{the operators } S_t = G_t|_E, \quad t \in [0, \infty[, \text{ constitute a one-parameter semigroup } (S_t)_{t \geq 0} \subset L(E) \text{ such that } \omega_E \leq \omega_0. \quad (7.2)$$

Now we are going to prove that (7.2) holds also in case (v), i.e. when $E = \{u \in (L^2(\mathbb{R}^n))^m : (P(\partial/\partial x))^l u \in (L^2(\mathbb{R}^n))^m \text{ for } l = 1, 2, \dots\}$. In this case for every $\varphi \in (C_c(\mathbb{R}^n))^m$, $u \in \mathcal{F}\varphi \in (Z_n)^m$, $j, l = 0, 1, \dots$ and $t \in \mathbb{R}$ one has $(P(\partial/\partial x))^l G_t u = G_t (P(\partial/\partial x))^l u \in (Z_n)^m$ and, by Plancherel's theorem,

$$\begin{aligned} \|G_t u\|_j &= \sum_{l=0}^j \left\| \left(P \left(\frac{\partial}{\partial x} \right) \right)^l G_t u \right\|_{(L^2(\mathbb{R}^n))^m} \\ &= \sum_{l=0}^j \left\| \left(P \left(\frac{\partial}{\partial x} \right) \right)^l \mathcal{F} e^{t\tilde{P}} \varphi \right\|_{(L^2(\mathbb{R}^n))^m} \\ &= \sum_{l=0}^j \|\mathcal{F} \tilde{P}^l e^{t\tilde{P}} \varphi\|_{(L^2(\mathbb{R}^n))^m} = (2\pi)^{n/2} \sum_{l=0}^j \|\tilde{P}^l e^{t\tilde{P}} \varphi\|_{(L^2(\mathbb{R}^n))^m}. \end{aligned}$$

Hence, by the Proposition at the end of Section 6, for every $\varepsilon > 0$ there is $K_\varepsilon \in]0, \infty[$ such that whenever $\varepsilon \in (C_c(\mathbb{R}^n))^m$, $u = \mathcal{F}\varphi \in (Z_n)^m$, $j = 0, 1, \dots$ and $t \in [0, \infty[$, then

$$\begin{aligned} \|G_t u\|_j &= (2\pi)^{n/2} \sum_{l=0}^j \left\| \sum_{k=0}^{2m} p_k(t, \cdot) \tilde{P}^{k+l} \varphi \right\|_{(L^2(\mathbb{R}^n))^m} \\ &\leq (2\pi)^{n/2} K_\varepsilon e^{(\omega_0 + \varepsilon)t} \sum_{l=0}^{j+2m} \|\tilde{P}^l \varphi\|_{(L^2(\mathbb{R}^n))^m} \\ &= K_\varepsilon e^{(\omega_0 + \varepsilon)t} \sum_{l=0}^{j+2m} (2\pi)^{n/2} \left\| \mathcal{F}^{-1} \left(P \left(\frac{\partial}{\partial x} \right) \right)^l u \right\|_{(L^2(\mathbb{R}^n))^m}. \end{aligned}$$

Again by Plancherel's theorem, it follows that

$$\begin{aligned} \|G_t u\| &\leq K_\varepsilon e^{(\omega_0 + \varepsilon)t} \sum_{l=0}^{j+2m} \left\| \left(P \left(\frac{\partial}{\partial x} \right) \right)^l u \right\|_{(L^2(\mathbb{R}^n))^m} \\ &= K_\varepsilon e^{(\omega_0 + \varepsilon)t} \|u\|_{j+2m} \end{aligned} \tag{7.3}$$

whenever $t \in [0, \infty[$, $u \in (Z_n)^m$ and $j = 1, 2, \dots$. Since, also in case (v), $(Z_n)^m$ is dense in E , one concludes that (7.2) holds.

It is easy to see that for both the Fréchet spaces E considered in the present section, $(C_c^\infty(\mathbb{R}^n))^m$ is continuously imbedded in $\mathcal{F}^{-1}E$ and is sequentially dense in $\mathcal{F}^{-1}E$. Hence $(Z_n)^m = (\mathcal{F}C_c^\infty(\mathbb{R}^n))^m$ is continuously imbedded in E and $(Z_n)^m$ is sequentially dense in E . Therefore in order to complete the

proof of the implication (a) \Rightarrow (b) \wedge ($\omega_E \leq \omega_0$) it remains to apply to the one-parameter semigroup satisfying (7.2) the following

Lemma. *Suppose that (7.2) holds for some l.c.v.s. E imbedded in $(Z'_n)^m$ such that $\frac{\partial}{\partial x_\nu}|_E \in L(E)$ for every $\nu = 1, \dots, n$. Suppose moreover that $(Z_n)^m$ is continuously imbedded in E and sequentially dense in E . Then $(S_t)_{t \geq 0} \subset L(E)$ is a (C_0) -semigroup with infinitesimal generator $P(\partial/\partial x)|_E$.*

Proof. Pick $u \in E$ and let $(u_\nu)_{\nu=1,2,\dots} \subset (Z_n)^m$ be a sequence converging to u in the topology of E . If $\nu \in \mathbb{N}$, $t \in]0, \infty[$ and $\tau \in [0, \infty[$, then $G_t u_\nu - G_\tau u_\nu = \int_\tau^t G_s P(\partial/\partial x) u_\nu ds$ and

$$\begin{aligned} \frac{1}{t}(G_t u_\nu - u_\nu) - P\left(\frac{\partial}{\partial x}\right)u_\nu &= \frac{1}{t} \int_0^t \left[G_s P\left(\frac{\partial}{\partial x}\right)u_\nu - P\left(\frac{\partial}{\partial x}\right)u_\nu \right] ds \\ &= \frac{1}{t} \int_0^t \int_0^s G_\tau \left(P\left(\frac{\partial}{\partial x}\right) \right)^2 u_\nu d\tau ds = \frac{1}{t} \int_0^t s G_{t-s} \left(P\left(\frac{\partial}{\partial x}\right) \right)^2 u_\nu ds, \end{aligned}$$

so that

$$S_t u - S_\tau u = \lim_{\nu \rightarrow 0} \int_\tau^t S_s P\left(\frac{\partial}{\partial x}\right)u_\nu ds$$

and

$$\frac{1}{t}(S_t u - u) - P\left(\frac{\partial}{\partial x}\right)u = \lim_{\nu \rightarrow \infty} \frac{1}{t} \int_0^t s S_{t-s} \left(P\left(\frac{\partial}{\partial x}\right) \right)^2 u_\nu ds.$$

Here the integrals are Riemann integrals of continuous functions taking values in the complete l.c.v.s. $(Z_n)^m$ (isomorphic to $(C_c^\infty(\mathbb{R}^n))^m$ which is complete; see [S2], p. 66, Theorem I). Let p be any continuous seminorm on E . The restriction $p|_{(Z_n)^m}$ is a continuous seminorm on $(Z_n)^m$, so that

$$\begin{aligned} p(S_t u - S_\tau u) &= \lim_{\nu \rightarrow \infty} p\left(\int_\tau^t S_s P\left(\frac{\partial}{\partial x}\right)u_\nu ds \right) \leq a|t - \tau|, \\ a &= \sup\{p(S_s P(\partial/\partial x)u_\nu) : s \in [0, T], \nu \in \mathbb{N}\}, \end{aligned}$$

and

$$\begin{aligned} p\left(\frac{1}{t}(S_t u - u) - P\left(\frac{\partial}{\partial x}\right)u\right) &= \lim_{\nu \rightarrow \infty} p\left(\frac{1}{t} \int_0^t s S_{t-s} \left(P\left(\frac{\partial}{\partial x}\right) \right)^2 u_\nu ds\right) \leq \frac{1}{2}bt, \\ b &= \sup\{p(S_s (P(\partial/\partial x))^2 u_\nu) : s \in [0, T], \nu \in \mathbb{N}\}. \end{aligned}$$

By (7.2) the semigroup $(S_t)_{t \geq 0} \subset L(E)$ is locally equicontinuous, whence $\{S_s(P(\partial/\partial x))^k u_\nu : s \in [0, T], \nu \in \mathbb{N}\}$, $k = 1, 2, \dots$, are bounded subsets of E , and so a and b are finite. Consequently, the function $]0, \infty[\ni t \mapsto S_t u \in E$ is continuous, and $\lim_{t \downarrow 0} \frac{1}{t}(S_t u - u) = P(\partial/\partial x)u$ in the topology of E . This proves that $(S_t)_{t \geq 0} \subset L(E)$ is a (C_0) -semigroup with infinitesimal generator $P(\partial/\partial x)|_E$.

Proof of (b) \Rightarrow (a) and (b) \wedge ($\omega_E < \infty$) \Rightarrow ($\omega_0 \leq \omega_E$). If (b) holds, then, by the Uniqueness Theorem from Section 2, for every $t \in [0, \infty[$ there are $C(t) \in]0, \infty[$ and $j(t) \in \mathbb{N}_0$ such that $\|G_t u\|_0 = \|S_t u\|_0 \leq C(t)\|u\|_{j(t)}$ for every $u \in E$. If $\varphi \in (C_c^\infty(\mathbb{R}^n))^m$ and $u = \mathcal{F}\varphi \in (Z_n)^m$, then $G_t u = \mathcal{F}e^{t\tilde{P}}\varphi$, so that

$$\begin{aligned} \|e^{t\tilde{P}}\varphi\|_0 &= \|\mathcal{F}^{-1}G_t u\|_0 \leq (2\pi)^{-n/2}\|G_t u\|_0 \leq (2\pi)^{-n/2}C(t)\|u\|_{j(t)} \\ &= (2\pi)^{-n/2}C(t)\|\mathcal{F}\varphi\|_{j(t)} \leq \left(\int_{\mathbb{R}^n} Q_t(\eta)^2 \|\varphi(\eta)\|_{\mathbb{C}^m}^2 d\eta \right)^{1/2} \end{aligned}$$

where Q_t is a real polynomial on \mathbb{R}^n . Consequently, for every $t \in [0, \infty[$ there are $K(t) \in]0, \infty[$ and $k(t) \in \mathbb{N}$ such that

$$\left(\int_{\mathbb{R}^n} \|e^{t\tilde{P}}\varphi(\eta)\|_{\mathbb{C}^m}^2 d\eta \right)^{1/2} \leq K(t) \left(\int_{\mathbb{R}^n} (1 + |\eta|^2)^{k(t)} \|\varphi(\eta)\|_{\mathbb{C}^m}^2 d\eta \right)^{1/2} \quad (7.4)$$

whenever $\varphi \in (C_c^\infty(\mathbb{R}^n))^m$. Furthermore, if (b) \wedge ($\omega_E < \infty$) holds, then for every $\varepsilon > 0$ there are $K_\varepsilon \in]0, \infty[$ and $j_\varepsilon \in \mathbb{N}$ such that $(2\pi)^{-n/2}C(t) \leq K_\varepsilon e^{(\omega_E + \varepsilon)t}$ and $j(t) \leq j_\varepsilon$ for every $t \in [0, \infty[$, whence it follows that for some $k_\varepsilon \in \mathbb{N}$ one has

$$\left(\int_{\mathbb{R}^n} \|e^{t\tilde{P}}\varphi(\eta)\|_{\mathbb{C}^m}^2 d\eta \right)^{1/2} \leq K_\varepsilon e^{(\omega_E + \varepsilon)t} \left(\int_{\mathbb{R}^n} (1 + |\eta|^2)^{k_\varepsilon} \|\varphi(\eta)\|_{\mathbb{C}^m}^2 d\eta \right)^{1/2} \quad (7.5)$$

whenever $t \in [0, \infty[$ and $\varphi \in (C_c^\infty(\mathbb{R}^n))^m$. Inequalities (7.4) and (7.5) resemble (4.4), and our subsequent arguments resemble that from the proof of Theorem 2 of Section 4. All this is similar to the argument used by T. Ushijima in the Correction to [U].

For every $t \in [0, \infty[$ and $\xi \in \mathbb{R}^n$ choose $z_{t,\xi} \in \mathbb{C}^m$ such that $\|z_{t,\xi}\|_{\mathbb{C}^m} = 1$ and $\|e^{t\tilde{P}(\xi)}\|_{L(\mathbb{C}^m)} = \|e^{t\tilde{P}(\xi)}z_{t,\xi}\|_{\mathbb{C}^m}$. Let $(\phi_{\xi,\nu})_{\nu=1,2,\dots} \subset C_c^\infty(\mathbb{R}^n)$ be a sequence of non-negative functions such that for every $\nu = 1, 2, \dots$ the support of $\phi_{\xi,\nu}$ is contained in the ball with centre at ξ and radius $1/\nu$, and $\int_{\mathbb{R}^n} \phi_{\xi,\nu}(\eta)^2 d\eta = 1$.

If (b) holds, then applying (7.4) to $\varphi(\eta) = \phi_{\xi,\nu}(\eta)z_{t,\xi}$ one concludes that whenever $t \in [0, \infty[$ and $\xi \in \mathbb{R}^n$, then

$$\begin{aligned} \|e^{t\tilde{P}(\xi)}\|_{L(\mathbb{C}^m)} &= \|e^{t\tilde{P}(\xi)}z_{t,\xi}\|_{\mathbb{C}^m} = \lim_{\nu \rightarrow \infty} \left(\int_{\mathbb{R}^n} \|e^{t\tilde{P}(\eta)}\phi_{\xi,\nu}(\eta)z_{t,\xi}\|_{\mathbb{C}^m}^2 d\eta \right)^{1/2} \\ &\leq K(t) \lim_{\nu \rightarrow \infty} \left(\int_{\mathbb{R}^n} (1 + |\eta|^2)^{k(t)} \phi_{\xi,\nu}(\eta)^2 d\eta \right)^{1/2} \\ &= K(t)(1 + |\xi|^2)^{k(t)/2}. \end{aligned} \quad (7.6)$$

Let $\rho(e^{\tilde{P}(\xi)})$ denote the spectral radius of the matrix $e^{\tilde{P}(\xi)}$. By Corollary 2.4 on p. 252 of [E-N], (7.6) implies that

$$\begin{aligned} \max\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi))\} &= \log \rho(e^{\tilde{P}(\xi)}) \leq \log \|e^{\tilde{P}(\xi)}\|_{L(\mathbb{C}^m)} \\ &\leq \log K(1) + \frac{1}{2}k(1) \log(1 + |\xi|^2) \end{aligned}$$

for every $\xi \in \mathbb{R}^n$. By the Corollary at the end of our Section 5, it follows that

$$\omega_0 = \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi)), \xi \in \mathbb{R}^n\} < \infty.$$

If (b) \wedge ($\omega_E < \omega$) holds, then the difficult results quoted in Section 5 need not be used. Applying (7.5) to $\varphi(\eta) = \phi_{\xi,\nu}(\eta)z_{t,\xi}$ one concludes that whenever $t \in [0, \infty[$ and $\xi \in \mathbb{R}^n$, then

$$\|e^{t\tilde{P}(\xi)}\|_{L(\mathbb{C}^m)} \leq K_\varepsilon e^{(\omega_\varepsilon + \varepsilon)t} (1 + |\xi|^2)^{k_\varepsilon/2},$$

whence, by Proposition 2.2, p. 251, and Corollary 2.4, p. 252, in [E-N],

$$\max\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi))\} = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|e^{t\tilde{P}(\xi)}\|_{L(\mathbb{C}^m)} \leq \omega_E + \varepsilon.$$

This implies that $\omega_0 \leq \omega_E$ because $\varepsilon > 0$ is arbitrary.

8. Conditions on $e^{t\tilde{P}(\xi)}$ equivalent to the Petrovskii correctness

Let $\tilde{P}(\xi)$ be the symbol of the matricial differential operator $P(\partial/\partial x)$ defined in Section 1. For any $\omega \in \mathbb{R}$ consider the conditions:

$$\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi)), \xi \in \mathbb{R}^n\} \leq \omega \quad (\text{the Petrovskii correctness}); \quad (8.1)$$

there is $k \in \mathbb{N}$ such that

$$\sup\{e^{-(\omega + \varepsilon)t} (1 + |\xi|)^{-k} \|e^{t\tilde{P}(\xi)}\|_{M_m} : t \in [0, \infty[, \xi \in \mathbb{R}^n\} < \infty \quad (8.2)$$

for every $\varepsilon > 0$;

for every multiindex $\alpha \in \mathbb{N}_0^n$ there is $k_\alpha \in \mathbb{N}$ such that

$$\sup\{e^{-(\omega+\varepsilon)t}(1+|\xi|)^{-k_\alpha}\|(\partial/\partial\xi)^\alpha e^{t\tilde{P}(\xi)}\|_{M_m} : t \in [0, \infty[, \xi \in \mathbb{R}^n\} < \infty \quad (8.3)$$

for every $\varepsilon > 0$.

Condition (8.1) implies (8.2) by the Gelfand–Shilov inequality (6.9), and (8.2) implies (8.1) because $\max\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi))\} = t^{-1} \log \rho(e^{t\tilde{P}(\xi)}) \leq t^{-1} \log \|e^{t\tilde{P}(\xi)}\|_{L(\mathbb{C}^m)}$ for every $t \in]0, \infty[$ where ρ stands for the spectral radius. See [E–N], p. 252. The partial derivatives occurring in (8.3) make sense because the function $\mathbb{R}^{1+n} \ni (t, \xi) \mapsto e^{t\tilde{P}(\xi)} \in M_m$ is infinitely differentiable, by arguments mentioned at the beginning of Section 2. Obviously (8.3) implies (8.2), and the proof of the converse implication will be given shortly. Therefore for any fixed $\omega \in \mathbb{R}$ the conditions (8.1), (8.2) and (8.3) are equivalent.

I. G. Petrovskii considered in [P] the following conditions:

$$\sup\{(1 + \log(1 + |\xi|))^{-1} \operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi)), \xi \in \mathbb{R}^n\} < \infty; \quad (8.4)$$

for every $T \in]0, \infty[$ there is $k_T \in \mathbb{N}$ such that

$$\sup\{(1 + |\xi|)^{-k_T} \|e^{t\tilde{P}(\xi)}\|_{M_m} : t \in [0, T], \xi \in \mathbb{R}^n\} < \infty; \quad (8.5)$$

for every multiindex $\alpha \in \mathbb{N}_0^n$ and every $T \in]0, \infty[$

there is $k_{\alpha, T} \in \mathbb{N}$ such that

$$\sup\{(1 + |\xi|)^{-k_{\alpha, T}} \|(\partial/\partial\xi)^\alpha e^{t\tilde{P}(\xi)}\|_{M_m} : t \in [0, T], \xi \in \mathbb{R}^n\} < \infty. \quad (8.6)$$

The three conditions (8.4)–(8.6) are equivalent to each other, and each is equivalent to the existence of an $\omega \in \mathbb{R}$ for which the conditions (8.1)–(8.3) are satisfied. This follows from the Corollary at the end of Section 5 and arguments similar to those proving the mutual equivalence of (8.1), (8.2) and (8.3).

Proof of the implication (8.2) \Rightarrow (8.3). For every $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, $\xi \in \mathbb{R}^n$ and $t \in [0, \infty[$ put

$$\begin{aligned} \tilde{P}_\alpha(\xi) &= \left(\frac{\partial}{\partial\xi_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial\xi_n}\right)^{\alpha_n} \tilde{P}(\xi), \\ U_\alpha(t, \xi) &= \left(\frac{\partial}{\partial\xi_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial\xi_n}\right)^{\alpha_n} e^{t\tilde{P}(\xi)}. \end{aligned}$$

If $\alpha, \beta \in \mathbb{N}_0^n$, then let $\beta \leq \alpha$ mean that $\beta_\nu \leq \alpha_\nu$ for every $\nu = 1, \dots, n$. If $\beta \leq \alpha$, then $\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$ where $\binom{\alpha_\nu}{\beta_\nu} = \frac{\alpha_\nu!}{\beta_\nu!(\alpha_\nu - \beta_\nu)!}$. Condition (8.3) means that whenever $\alpha \in \mathbb{N}_0^n$ then

there is $k \in \mathbb{N}$ such that

$$\sup\{e^{-(\omega+\varepsilon)t}(1+|\xi|)^{-k}\|U_\alpha(t, \xi)\| : t \in [0, \infty[, \xi \in \mathbb{R}^n\} < \infty \quad (8.7)_\alpha$$

for every $\varepsilon > 0$.

Condition (8.2) is identical with (8.7)₀. Hence the implication (8.2) \Rightarrow (8.3) will follow once we prove that if $l \in \mathbb{N}_0$ and (8.7) _{β} holds for every $\beta \in \mathbb{N}_0^n$ such that $|\beta| = \beta_1 + \cdots + \beta_n \leq l$, then (8.7) _{α} holds for every $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| = l + 1$. So, pick any α such that $|\alpha| = l + 1$. Then

$$\begin{aligned} \frac{d}{dt}U_\alpha(t, \xi) &= \left(\frac{\partial}{\partial \xi}\right)^\alpha \frac{d}{dt}U_0(t, \xi) = \left(\frac{\partial}{\partial \xi}\right)^\alpha (\tilde{P}(\xi)U_0(t, \xi)) \\ &= \tilde{P}(\xi)U_\alpha(t, \xi) + V_\alpha(t, \xi) \end{aligned} \quad (8.8)$$

where

$$V_\alpha(t, \xi) = \sum_{\beta \leq \alpha, |\beta| \leq l} \binom{\alpha}{\beta} \tilde{P}_{\alpha-\beta}(\xi)U_\beta(t, \xi).$$

Since (8.7) _{β} holds whenever $|\beta| \leq l$, it follows that

there is $k \in \mathbb{N}$ such that

$$\sup\{e^{-(\omega+\varepsilon)t}(1+|\xi|)^{-k}\|V_\alpha(t, \xi)\| : t \in [0, \infty[, \xi \in \mathbb{R}^n\} < \infty \quad (8.9)$$

for every $\varepsilon > 0$.

By (8.8) one has

$$U_\alpha(t, \xi) = \int_0^t U_0(t - \tau, \xi)V_\alpha(\tau, \xi) d\tau, \quad t \in [0, \infty[, \xi \in \mathbb{R}^n. \quad (8.10)$$

Conditions (8.7)₀ and (8.9) imply (8.7) _{α} , by (8.10).

Remark. The above proof is similar to the proof of Lemma 2 in Sec. 2 of Chap. 1 of [P]. By (6.9), condition (8.1) implies (8.2) with $k = (m - 1)d$. From this last, by the induction procedure used above, one obtains (8.3) with $k_\alpha = (md - 1)(|\alpha| + 1)$.

9. The space $\mathcal{O}_M(\mathbb{R}^n; M_m)$

The paper [P] of I. G. Petrovskii makes evident the fundamental role of smooth slowly increasing functions in the theory of Cauchy's problem for systems of PDE with constant coefficients. A continuous function ϕ defined on \mathbb{R}^n is called *slowly increasing* if there is $k \in \mathbb{N}_0$ such that $\sup\{(1 + |\xi|)^{-k} |\phi(\xi)| : \xi \in \mathbb{R}^n\} < \infty$. The space $\mathcal{O}_M(\mathbb{R}^n; M_m)$ of M_m -valued slowly increasing infinitely differentiable functions on \mathbb{R}^n consists of M_m -valued C^∞ -functions ϕ on \mathbb{R}^n such that ϕ and all its partial derivatives are slowly increasing. The present section is devoted to the properties of $\mathcal{O}_M(\mathbb{R}^n; M_m)$ as the space of multipliers for $(\mathcal{S}(\mathbb{R}^n))^m$ and $(\mathcal{S}'(\mathbb{R}^n))^m$.

An M_m -valued function ϕ defined on \mathbb{R}^n is called a *multiplier for $(\mathcal{S}(\mathbb{R}^n))^m$* if $\phi \cdot \varphi \in (\mathcal{S}(\mathbb{R}^n))^m$ for every $\varphi \in (\mathcal{S}(\mathbb{R}^n))^m$ and the multiplication operator $\phi \cdot : \varphi \mapsto \phi \cdot \varphi$ belongs to $L((\mathcal{S}(\mathbb{R}^n))^m)$. The set of multipliers for $(\mathcal{S}(\mathbb{R}^n))^m$ will be denoted by $\mathcal{M}((\mathcal{S}(\mathbb{R}^n))^m)$. Obviously $\mathcal{M}((\mathcal{S}(\mathbb{R}^n))^m) \subset C^\infty(\mathbb{R}^n; M_m)$.

Let $\mathcal{S}'(\mathbb{R}^n)$ be the L. Schwartz space of tempered distributions on \mathbb{R}^n , and let $\mathcal{S}'_b(\mathbb{R}^n)$ denote $\mathcal{S}'(\mathbb{R}^n)$ endowed with topology of uniform convergence on bounded subsets of $\mathcal{S}(\mathbb{R}^n)$. A function $\phi \in C^\infty(\mathbb{R}^n; M_m)$ is called a *multiplier for $(\mathcal{S}'_b(\mathbb{R}^n))^m$* if $\phi \cdot T \in (\mathcal{S}'(\mathbb{R}^n))^m$ for every $T = (T_1, \dots, T_m) \in (\mathcal{S}'(\mathbb{R}^n))^m$ and the multiplication operator $\phi \cdot : T \mapsto \phi \cdot T$ belongs to $L((\mathcal{S}'_b(\mathbb{R}^n))^m)$. The set of multipliers for $(\mathcal{S}'_b(\mathbb{R}^n))^m$ will be denoted by $\mathcal{M}((\mathcal{S}'_b(\mathbb{R}^n))^m)$.

Proposition. $\mathcal{M}((\mathcal{S}'_b(\mathbb{R}^n))^m) = \mathcal{M}((\mathcal{S}(\mathbb{R}^n))^m)$.

Proof. If $\phi(t) = (\phi_{i,j}(t))_{i,j=1,\dots,m}$, then $\phi \in \mathcal{M}((\mathcal{S}(\mathbb{R}^n))^m)$ if and only if $\phi_{i,j} \in \mathcal{M}(\mathcal{S}(\mathbb{R}^n))$ for every $i, j = 1, \dots, m$. A similar equivalence holds for $\mathcal{S}'_b(\mathbb{R}^n)$. Therefore it is sufficient to prove the Proposition for $m = 1$. It is clear that $\mathcal{M}(\mathcal{S}(\mathbb{R}^n)) \subset \mathcal{M}(\mathcal{S}'_b(\mathbb{R}^n))$. To prove the opposite inclusion notice that the pair of l.c.v.s. $(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'_b(\mathbb{R}^n))$ is reflexive with respect to the duality form $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}'_b(\mathbb{R}^n) \ni (u, T) \mapsto T(u) \in \mathbb{C}$. See [S], Sec. VII.4, p. 238, remarks after Theorem IV; [Y], p. 140, Theorem 2.

Henceforth fix any $\phi \in \mathcal{M}(\mathcal{S}'(\mathbb{R}^n))$. We have to prove that $\phi \in \mathcal{M}(\mathcal{S}(\mathbb{R}^n))$. Whenever $u \in \mathcal{S}(\mathbb{R}^n)$, then $\mathcal{S}'_b(\mathbb{R}^n) \ni T \mapsto \langle \phi \cdot T, u \rangle \in \mathbb{C}$ is a linear functional continuous on $\mathcal{S}'_b(\mathbb{R}^n)$ and therefore, by reflexivity, there is a unique $v \in \mathcal{S}(\mathbb{R}^n)$ such that $\langle \phi \cdot T, u \rangle = \langle T, v \rangle$ for every $T \in \mathcal{S}'(\mathbb{R}^n)$. The map $A : \mathcal{S}(\mathbb{R}^n) \ni u \mapsto v \in \mathcal{S}(\mathbb{R}^n)$ is algebraic linear. Let $(u_k)_{k=1,2,\dots} \subset \mathcal{S}(\mathbb{R}^n)$ be a sequence converging to zero in the Fréchet topology in $\mathcal{S}(\mathbb{R}^n)$, and let B be a bounded subset of $\mathcal{S}'_b(\mathbb{R}^n)$. Then $\phi \cdot B$ is again bounded, so that

$$\lim_{k \rightarrow \infty} \sup\{|\langle T, Au_k \rangle| : T \in B\} = \lim_{k \rightarrow \infty} \sup\{|\langle \phi \cdot T, u_k \rangle| : T \in B\} = 0.$$

This means that $\lim_{k \rightarrow \infty} Au_k = 0$ in the topology of the strong dual of $\mathcal{S}'(\mathbb{R}^n)$, and hence, again by reflexivity, $\lim_{k \rightarrow \infty} Au_k = 0$ in the original Fréchet topology of $\mathcal{S}(\mathbb{R}^n)$. Thus

$$\langle \phi \cdot T, u \rangle = \langle T, Au \rangle \quad \text{for every } u \in \mathcal{S}(\mathbb{R}^n) \text{ and } T \in \mathcal{S}'(\mathbb{R}^n) \quad (9.1)$$

where $A = L(\mathcal{S}(\mathbb{R}^n))$. Fix any $u \in \mathcal{S}(\mathbb{R}^n)$. The Proposition will follow once we show that $\phi \cdot u = Au$. To prove this, let $(u_k)_{k=1,2,\dots} \subset C_c^\infty(\mathbb{R}^n)$ be a sequence that converges to u in the Fréchet topology of $\mathcal{S}(\mathbb{R}^n)$. By the definition of multiplication of a distribution by a C^∞ -function, and by (9.1),

$$\begin{aligned} \lim_{k \rightarrow \infty} \sup\{|\langle T, \phi \cdot u_k - Au \rangle| : T \in B\} \\ &= \lim_{k \rightarrow \infty} \sup\{|\langle \phi \cdot T, u_k \rangle - \langle T, Au \rangle| : T \in B\} \\ &= \lim_{k \rightarrow \infty} \sup\{|\langle T, Au_k - Au \rangle| : T \in B\} = 0 \end{aligned}$$

for every bounded subset B of $\mathcal{S}'_b(\mathbb{R}^n)$. By reflexivity, this implies that $\lim_{k \rightarrow \infty} \phi \cdot u_k = Au \in \mathcal{S}(\mathbb{R}^n)$ in the Fréchet topology of $\mathcal{S}(\mathbb{R}^n)$. On the other hand, $\lim_{k \rightarrow \infty} \phi \cdot u_k = \phi \cdot u$ pointwise on \mathbb{R}^n . Therefore $\phi \cdot u = Au \in \mathcal{S}(\mathbb{R}^n)$ for every $u \in \mathcal{S}(\mathbb{R}^n)$, which means that $\phi \in \mathcal{M}(\mathcal{S}(\mathbb{R}^n))$.

We now formulate the main results of the present section.

Theorem A. $\mathcal{O}_M(\mathbb{R}^n; M_m) = \mathcal{M}((\mathcal{S}(\mathbb{R}^n))^m)$.

This Theorem and the preceding Proposition imply at once

Corollary. $\mathcal{O}_M(\mathbb{R}^n; M_m) = \mathcal{M}((\mathcal{S}'(\mathbb{R}^n))^m)$.

This Corollary (for $m = 1$) is mentioned without proof on p. 246 of [S]. The Corollary and Theorem A (for $m = 1$) are formulated simultaneously in Theorem 25.5 stated without proof on p. 275 of [T].

Theorem B. *For any subset B of $\mathcal{O}_M(\mathbb{R}^n; M_m)$ the following five conditions are equivalent:*

$$\begin{aligned} &\text{for every } \alpha \in \mathbb{N}_0^n \text{ there is } k_\alpha \in \mathbb{N}_0^n \text{ such that} \\ &\sup\{(1 + |\xi|)^{-k} \|(\partial/\partial\xi)^\alpha \phi(\xi)\|_{M_m} : \xi \in \mathbb{R}^n, \phi \in B\} < \infty; \end{aligned} \quad (9.2)$$

the family of multiplication operators

$$\{\phi \cdot : \phi \in B\} \subset L((\mathcal{S}(\mathbb{R}^n))^m) \quad \text{is equicontinuous;} \quad (9.3)$$

$B \cdot C$ is a bounded subset of $(\mathcal{S}(\mathbb{R}^n))^m$

$$\text{whenever } C \text{ is a bounded subset of } (\mathcal{S}(\mathbb{R}^n))^m; \quad (9.4)$$

the family of multiplication operators

$$\{\phi \cdot : \phi \in B\} \subset L((\mathcal{S}'(\mathbb{R}^n))^m) \text{ is equicontinuous;} \quad (9.5)$$

$B \cdot C'$ is a bounded subset of $(\mathcal{S}'(\mathbb{R}^n))^m$

$$\text{whenever } C' \text{ is a bounded subset of } (\mathcal{S}'(\mathbb{R}^n))^m. \quad (9.6)$$

In Theorem A the inclusion $\mathcal{O}_M(\mathbb{R}^n; M_m) \subset \mathcal{M}((\mathcal{S}(\mathbb{R}^n))^m)$ is obvious and the difficult opposite inclusion is a particular case of the implication (9.3) \Rightarrow (9.2). The implication (9.2) \Rightarrow (9.3) is obvious. For $q = 2, \dots, 6$ denote by $(9.q)^\dagger$ the condition obtained from (9.q) by replacing B by $B^\dagger = \{\phi^\dagger : \phi \in B\}$ where ϕ^\dagger denotes the transpose of the matrix ϕ . It is obvious that $(9.2)^\dagger \Leftrightarrow (9.2)$. Furthermore, $(9.3) \Rightarrow (9.4) \Rightarrow (9.5)^\dagger \Rightarrow (9.6)^\dagger \Rightarrow (9.3)$. Indeed, the implications $(9.3) \Rightarrow (9.4)$ and $(9.5)^\dagger \Rightarrow (9.6)^\dagger$ are trivial. The implication $(9.4) \Rightarrow (9.5)^\dagger$ follows from the fact that the polars of bounded subsets of $(\mathcal{S}(\mathbb{R}^n))^m$ constitute a basis of the topology of $(\mathcal{S}'_b(\mathbb{R}^n))^m$. The implication $(9.6)^\dagger \Rightarrow (9.3)$ follows from the fact that $\{(\mathcal{S}(\mathbb{R}^n))^m, (\mathcal{S}'_b(\mathbb{R}^n))^m\}$ is a reflexive pair of l.c.v.s. in duality, so that the polars of bounded subsets of $(\mathcal{S}'(\mathbb{R}^n))^m$ constitute a basis for the original Fréchet topology of $(\mathcal{S}(\mathbb{R}^n))^m$. Hence the equivalence $(9.2) \Rightarrow (9.3)$ implies that $(9.q) \Rightarrow (9.q)^\dagger$ for $q = 2, \dots, 6$. Therefore all what remains to do is to prove the implication $(9.3) \Rightarrow (9.2)$.

In order to simplify the proof of $(9.3) \Rightarrow (9.2)$, notice the following equality concerning elements of $L((\mathcal{S}(\mathbb{R}^n))^m)$:

$$\left(\left(\frac{\partial}{\partial \xi} \right)^\alpha \phi \right) \cdot = \left(\frac{\partial}{\partial \xi} \right)^\alpha (\phi \cdot) - \sum_{\beta \leq \alpha, |\beta| < |\alpha|} \binom{\alpha}{\beta} \left(\left(\frac{\partial}{\partial \xi} \right)^\beta \phi \right) \cdot \left(\frac{\partial}{\partial \xi} \right)^{\alpha - \beta} \cdot$$

By induction on $|\alpha| = \alpha_1 + \dots + \alpha_n$ this equality implies that if $\phi \in \mathcal{M}((\mathcal{S}(\mathbb{R}^n))^m)$, then $(\partial/\partial \xi)^\alpha \phi \in \mathcal{M}((\mathcal{S}(\mathbb{R}^n))^m)$ for every $\alpha \in \mathbb{N}_0^n$, and if (9.3) holds then for every $\alpha \in \mathbb{N}_0^n$ the family of multiplication operators $\{((\partial/\partial \xi)^\alpha \phi) \cdot : \phi \in B\} \subset L((\mathcal{S}(\mathbb{R}^n))^m)$ is equicontinuous. Therefore the implication $(9.3) \Rightarrow (9.2)$ will follow once we prove that (9.3) implies the condition

there is $k \in \mathbb{N}_0$ such that

$$\sup\{(1 + |\xi|)^{-k} \|\phi(\xi)\|_{M_m} : \xi \in \mathbb{R}^n, \phi \in B\} < \infty. \quad (9.7)$$

In the proof of $(9.3) \Rightarrow (9.7)$ we will use

Lemma. $\mathcal{M}((\mathcal{S}(\mathbb{R}^n))^m) \subset \mathcal{S}'(\mathbb{R}^n; M_m)$.

Proof. Let $\phi \in \mathcal{M}((\mathcal{S}(\mathbb{R}^n))^m)$. Then $\mathcal{F}^{-1}(\phi \cdot) \mathcal{F} \in L((\mathcal{S}(\mathbb{R}^n))^m)$ and $\mathcal{F}^{-1}(\phi \cdot) \mathcal{F}$ commutes with translations. Therefore, by a variant of a theorem of L. Schwartz ([S], p. 162; [Y], p. 158) there is a unique $T \in \mathcal{S}'(\mathbb{R}^n; M_m)$ such that $\mathcal{F}^{-1}(\phi \cdot \mathcal{F}\varphi) = T * \varphi$ for every $\varphi \in (\mathcal{S}(\mathbb{R}^n))^m$. Consequently, $\phi \cdot \mathcal{F}\varphi = \mathcal{F}(T * \varphi) = \mathcal{F}T \cdot \mathcal{F}\varphi$ for every $\varphi \in (\mathcal{S}(\mathbb{R}^n))^m$, and hence

$$\phi = \mathcal{F}T \in \mathcal{S}'(\mathbb{R}^n; M_m).$$

Proof of (9.3) \Rightarrow (9.7). This proof, or rather the proof of a Fourier precursor of (9.8) \Rightarrow (9.9), resembles that of Theorem 3.1 in [Ch], pp. 82–83, and that of a part of Theorem XXV in Sec. VI.8 of [S].

For notational convenience we introduce a set J of indices and a one-to-one map of $J \ni \iota \mapsto \phi_\iota \in B$ of J onto B . We assume that $\phi_\iota \in \mathcal{M}((\mathcal{S}(\mathbb{R}^n))^m)$ for every $\iota \in J$ and that

the family of multiplication operators

$$\{\phi_\iota \cdot : \iota \in J\} \subset L((\mathcal{S}(\mathbb{R}^n))^m) \text{ is equicontinuous.} \quad (9.8)$$

We have to prove that

there is $k \in \mathbb{N}_0$ such that

$$\sup\{(1 + |\xi|)^{-k} \|\phi_\iota(\xi)\|_{M_m} : \xi \in \mathbb{R}^n, \iota \in J\} < \infty. \quad (9.9)$$

This last condition will follow once we prove that there is a scalar polynomial Q and for every $\iota \in J$ there are $f_\iota, g_\iota \in L^1(\mathbb{R}^n; M_m)$ such that

$$\sup_{\iota \in J} \|f_\iota\|_{L^1(\mathbb{R}^n; M_m)} < \infty, \quad \sup_{\iota \in J} \|g_\iota\|_{L^1(\mathbb{R}^n; M_m)} < \infty \quad (9.10)$$

and

$$T_\iota := \mathcal{F}\phi_\iota = Q\left(\frac{\partial}{\partial x}\right) f_\iota + g_\iota \quad \text{for every } \iota \in J \quad (9.11)$$

where, in accordance with the Lemma, $T_\iota = \mathcal{F}\phi_\iota \in \mathcal{S}'(\mathbb{R}^n; M_m)$, and $Q(\partial/\partial x)$ acts on f_ι in the sense of distributions. Indeed, if (9.10) and (9.11) hold, then

$$\phi_\iota(\xi) = \frac{1}{(2\pi)^n} [Q(-i\xi) \widehat{f}_\iota(-\xi) + \widehat{g}_\iota(-\xi)],$$

where $\widehat{f}_\iota, \widehat{g}_\iota$ are continuous and bounded on \mathbb{R}^n , and

$$\sup\{\|\widehat{f}_\iota(\xi)\|_{M_m}, \|\widehat{g}_\iota(\xi)\|_{M_m} : \xi \in \mathbb{R}^n, \iota \in J\} < \infty,$$

so that (9.9) is satisfied.

Construction of Q , f_l and g_l . We will construct Q , f_l and g_l in the form

$$Q = \Delta^k, \quad 2k \geq l + n + 2, \quad f_l = T_l * u \quad \text{and} \quad g_l = T_l * v \quad (9.12)$$

where $u \in C_K^l$, $v \in C_K^\infty$, $K = \{x \in \mathbb{R}^n : |x| \leq 1\}$, and l is sufficiently large. Since $T_l * \varphi = \mathcal{F}(\phi_l \cdot \mathcal{F}^{-1}\varphi)$ for every $\varphi \in (\mathcal{S}(\mathbb{R}^n))^m$, from (9.8) it follows that the family of convolution operators $\{T_l * : l \in J\} \subset L((\mathcal{S}(\mathbb{R}^n))^m)$ is equicontinuous. This implies that the family of convolution operators

$$\{(T_l *)|_{C_K^\infty} : l \in J\} \subset L(C_K^\infty; L^1(\mathbb{R}^n; M_m))$$

is also equicontinuous. Since $L^1(\mathbb{R}^n; M_m)$ is a Banach space, there are $l \in \mathbb{N}_0$ and $C \in]0, \infty[$ such that

$$\|T_l * w\|_{L^1(\mathbb{R}^n; M_m)} \leq C \|w\|_{C_K^l} \quad \text{for every } l \in J \text{ and } w \in C_K^\infty.$$

Since C_K^∞ is dense in C_K^l , it follows that for every $l \in J$ the convolution $T_l * w$ of the distribution $T_l \in \mathcal{S}(\mathbb{R}^n; M_m)$ and the scalar distribution represented by a function $w \in C_K^l$ is a function $T_l * w \in L^1(\mathbb{R}^n; M_m)$ such that

$$\|T_l * w\|_{L^1(\mathbb{R}^n; M_m)} \leq C \|w\|_{C_K^l} \quad (9.13)$$

where $C \in]0, \infty[$ is independent of $l \in J$ and $w \in C_K^l$.

The formulas determining u and v . From (9.13) it follows that if $u \in C_K^l$ and $v \in C_K^l$ are fixed, then for f_l and g_l defined by (9.12) the condition (9.10) is satisfied. Therefore the only problem which remains to be solved consists in choosing $u \in C_K^l$ and $v \in C_K^l$ so that

$$T_l = \Delta^k f_l + g_l = \Delta^k (T_l * u) + T_l * v = T_l * (\Delta^k u + v).$$

To this end it is sufficient to choose $u \in C_K^l$, $v \in C_K^l$ and $k \in \mathbb{N}$ so that

$$\Delta^k u + v = \delta.$$

From the formula (II, 3;19) on p. 47 of [S] (or from Theorem 5.1 of [Ch], p. 99, or from [G-S1], the example at the end of Sec. III.2.1 of the 1958 ed. or Sec. I.6.1 of the 1959 ed.) it follows that if $k \in \mathbb{N}$ and $2k \geq l + n + 1$, then the differential operator Δ^k on \mathbb{R}^n has a fundamental solution \mathcal{E} which is a function belonging to $C^l(\mathbb{R}^n)$ such that $\mathcal{E}|_{\mathbb{R}^n \setminus \{0\}} \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Let $\gamma \in C_K^\infty$ be equal to 1 in a neighbourhood of 0. Define

$$u = \gamma \mathcal{E}, \quad v = \Delta^k (1 - \gamma) \mathcal{E}.$$

Then $u \in C_K^l$, $v \in C_K^\infty$ and $\Delta^k u + v = \Delta^k (\gamma \mathcal{E} + (1 - \gamma) \mathcal{E}) = \Delta^k \mathcal{E} = \delta$.

10. Proof of Theorem 1(i)

Proof of (a) \Rightarrow (b) \wedge ($\omega_E \leq \omega_0$). Let $E = (\mathcal{S}(\mathbb{R}^n))^m$. Suppose that (a) holds, i.e. $\omega_0 < \infty$. Then, by the implications (8.1) \Rightarrow (8.3) and (9.2) \Rightarrow (9.3), for every $\varepsilon > 0$ the family of multiplication operators

$$\{e^{-(\omega+\varepsilon)t}e^{t\tilde{P}} : t \in [0, \infty[\} \subset L(E)$$

is equicontinuous. Let $(G_t)_{t \in \mathbb{R}} \subset L((Z'_n)^m)$ be the one-parameter group (2.1). By invariance of E with respect to the Fourier transformation it follows that $S_t := G_t|_E = \mathcal{F}e^{t\tilde{P}}\mathcal{F}^{-1} \subset E$ for every $t \in [0, \infty[$, and the family of operators $(S_t)_{t \geq 0} \subset L(E)$ is a one-parameter group for which $\omega_E \leq \omega_0$. The Lemma from Section 7 shows that this is a (C_0) -semigroup with infinitesimal generator $P(\partial/\partial x)|_{(\mathcal{S}(\mathbb{R}^n))^m}$.

Proof of (b) \Rightarrow (a) and (b) \wedge ($\omega_E < \infty$) \Rightarrow ($\omega_0 \leq \omega_E$). Suppose that (b) holds, i.e. $P(\partial/\partial x)|_{(\mathcal{S}(\mathbb{R}^n))^m}$ is the infinitesimal generator of a (C_0) -semigroup $(S_t)_{t \geq 0} \subset L((\mathcal{S}(\mathbb{R}^n))^m)$. Then, by the Uniqueness Theorem from Section 2, $S_t = G_t|_{(\mathcal{S}(\mathbb{R}^n))^m}$ for every $t \in [0, \infty[$ where the operators $G_t = \mathcal{F}e^{t\tilde{P}}\mathcal{F}^{-1} \in L((Z'_n)^m)$, $t \in \mathbb{R}$, constitute the one-parameter (C_0) -group $(G_t)_{t \in \mathbb{R}} \subset L((Z'_n)^m)$ considered in Section 2. Consequently, if (b) holds, then

$$e^{t\tilde{P}}|_{(\mathcal{S}(\mathbb{R}^n))^m} = \mathcal{F}^{-1}S_t\mathcal{F}|_{(\mathcal{S}(\mathbb{R}^n))^m} \in L((\mathcal{S}(\mathbb{R}^n))^m) \quad (10.1)$$

for every $t \in [0, \infty[$. By Theorem A of Section 9 it follows that

$$e^{t\tilde{P}} \in \mathcal{M}((\mathcal{S}(\mathbb{R}^n))^m) = \mathcal{O}_M(\mathbb{R}^n; M_m) \quad \text{for every } t \in [0, \infty[.$$

In particular, $e^{\tilde{P}} \in \mathcal{O}_M(\mathbb{R}^n; M_m)$, and hence there is $k \in \mathbb{N}$ such that

$$\sup\{(1 + |\xi|)^{-k} \|e^{\tilde{P}(\xi)}\|_{L(\mathbb{C}^m)} : \xi \in \mathbb{R}^n\} = K < \infty.$$

By Corollary 2.4 on p. 252 of [E-N], this implies that

$$\begin{aligned} \max\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi))\} &\leq \log \rho(e^{\tilde{P}(\xi)}) \leq \log \|e^{\tilde{P}(\xi)}\|_{L(\mathbb{C}^m)} \\ &\leq \log K + k \log(1 + |\xi|) \end{aligned}$$

for every $\xi \in \mathbb{R}^n$ where ρ stands for the spectral radius. By the Corollary at the end of Section 5, it follows that $\omega_0 = \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi)), \xi \in \mathbb{R}^n\} < \infty$.

If (b) holds and $\omega_E < \infty$, then, in addition to (10.1), for every $\varepsilon > 0$ the family of multiplication operators

$$\{e^{-(\omega_E+\varepsilon)t}e^{t\tilde{P}}|_{(\mathcal{S}(\mathbb{R}^n))^m} : t \in [0, \infty[\} = \{\mathcal{F}^{-1}e^{-(\omega_E+\varepsilon)t}S_t\mathcal{F}|_{(\mathcal{S}(\mathbb{R}^n))^m} : t \in [0, \infty[\} \\ \subset L((\mathcal{S}(\mathbb{R}^n))^m)$$

is equicontinuous, and hence, by the implication (9.3) \Rightarrow (9.2), the condition (8.3) is satisfied for $\omega = \omega_E$. By the implication (8.3) \Rightarrow (8.1), it follows that $\omega_0 \leq \omega_E$.

11. Proof of Theorem 1(ii)

The proof of Theorem 1(ii) is analogous to that of Theorem 1(i). The Lemma from Section 7 applies to $E = (\mathcal{S}'_b(\mathbb{R}^n))^m = (\mathcal{F}\mathcal{S}'_b(\mathbb{R}^n))^m$ because $(Z_n)^m = (\mathcal{F}C_c^\infty(\mathbb{R}^n))^m$ and $C_c^\infty(\mathbb{R}^n)$ is sequentially dense in $\mathcal{S}'_b(\mathbb{R}^n)$. This last may be proved by approximation of distributions in $\mathcal{S}'(\mathbb{R}^n)$ by cutting and regularizing. See [R], p. 253, Proposition 4; [T], Sec. 28.

12. Proof of Theorem 1(iii)

The topology in $(C_b^\infty(\mathbb{R}^n))^m$ is determined by the sequence of norms

$$\|u\|_j = \sup\{\|(\partial/\partial x)^\alpha u(x)\|_{\mathbb{C}^m} : \alpha \in \mathbb{N}_0^n, |\alpha| \leq j, x \in \mathbb{R}^n\}, \\ j = 0, 1, \dots, \quad (12.1)$$

where $u \in (C_b^\infty(\mathbb{R}^n))^m$. The implication (a) \Rightarrow (b) \wedge ($\omega_E \leq \omega_0$) for $E = (C_b^\infty(\mathbb{R}^n))^m$ will be proved by using some variants of estimates proved by I. G. Petrovskii. These variants are uniform in $t \in [0, \infty[$. The non-uniform estimates used in [P] permit one only to prove (a) \Rightarrow (b) without showing that $\omega_E < \infty$ and $\omega_E \leq \omega_0$.

Suppose that $\omega_0 < \infty$ and let $S_t^\circ := G_t|_{(\mathcal{S}(\mathbb{R}^n))^m}$ for every $t \in [0, \infty[$ where $(G_t)_{t \in \mathbb{R}} \subset L((Z'_n)^m)$ is the one-parameter group (2.1). Then, by Theorem 1(i), $(S_t^\circ)_{t \geq 0} \subset L((\mathcal{S}(\mathbb{R}^n))^m)$ is a one-parameter (C_0) -semigroup with infinitesimal generator $P(\partial/\partial x)|_{(\mathcal{S}(\mathbb{R}^n))^m}$.

Lemma. *If $\omega_0 < \infty$, then for every $\varepsilon > 0$ there is $M_\varepsilon \in]0, \infty[$ such that*

$$\|(S_t^\circ u)(x)\|_{\mathbb{C}^m} \leq M_\varepsilon \|u\|_{k_0} e^{(\omega_0+\varepsilon)t} \prod_{\nu=1}^n (1 + |x_\nu|)^{-2} \quad (12.2)$$

for every $u \in (C_I^\infty(\mathbb{R}^n))^m$, $t \in [0, \infty[$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ where $I = [-1/2, 1/2]^n$ and $k_0 = n((dm - 1)(2n + 1) + 2)$.

Proof. We follow I. G. Petrovskii [P], pp. 12–17, but instead of analogues of (8.4) and (8.6) we use (8.1) and (8.3). Notice that

$$\min(1, a^{-k}) \leq 2^k(1 + a)^{-k} \quad \text{for every } \alpha \in]0, \infty[\text{ and } k \in \mathbb{N}. \quad (12.3)$$

Whenever $u \in (C_c^\infty(\mathbb{R}^n))^m$, $t \in [0, \infty[$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $x_\nu \neq 0$ for $\nu = 1, \dots, n$, then $\mathcal{F}^{-1}u \in (\mathcal{S}(\mathbb{R}^n))^m$, $e^{t\tilde{P}} \in \mathcal{O}_M(\mathbb{R}^n; M_m)$, $e^{t\tilde{P}}\mathcal{F}^{-1}u \in (\mathcal{S}(\mathbb{R}^n))^m$, and hence

$$\begin{aligned} (S_t^\circ u)(x) &= (\mathcal{F}e^{t\tilde{P}}\mathcal{F}^{-1}u)(x) = \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} e^{t\tilde{P}(\xi)} (\mathcal{F}^{-1}u)(\xi) d\xi \\ &= (-1)^n (x_1 \cdots x_n)^{-2} \\ &\quad \times \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \left(\frac{\partial^n}{\partial \xi_1 \cdots \partial \xi_n} \right)^2 [e^{t\tilde{P}(\xi)} (\mathcal{F}^{-1}u)(\xi)] d\xi. \end{aligned} \quad (12.4)$$

By the Remark at the end of Section 8, the inequality $\omega_0 < \infty$ is equivalent to (8.3) with $\omega = \omega_0$ and $k_\alpha = (md - 1)(|\alpha| + 1)$. Therefore (12.3) and (12.4) imply that for every $\varepsilon > 0$ there is $K_\varepsilon \in]0, \infty[$ such that

$$\begin{aligned} \|(S_t^\circ u)(x)\|_{\mathbb{C}^m} &\leq K_\varepsilon e^{(\omega_0 + \varepsilon)t} \prod_{\nu=1}^n (1 + |x_\nu|)^{-2} \\ &\quad \times \int_{\mathbb{R}^n} \prod_{\nu=1}^n (1 + |\xi_\nu|)^{(md-1)(2n+1)} \sup_{|\alpha| \leq 2n} \|(\partial/\partial \xi)^\alpha (\mathcal{F}^{-1}u)(\xi)\|_{\mathbb{C}^m} d\xi \end{aligned} \quad (12.5)$$

whenever $u \in (C_c^\infty(\mathbb{R}^n))^m$, $t \in [0, \infty[$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Furthermore,

$$\begin{aligned} \left(\frac{\partial}{\partial \xi} \right)^\alpha (\mathcal{F}^{-1}u)(\xi) &= \frac{1}{(2\pi)^n} (-i)^{|\alpha|} \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} x^\alpha u(x) dx \\ &= \frac{1}{(2\pi)^n} (-i)^{|\alpha|} (\xi^\beta)^{-1} \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} \left(\frac{\partial}{\partial x} \right)^\beta (x^\alpha u(x)) dx \end{aligned}$$

whenever $u \in (C_c^\infty(\mathbb{R}^n))^m$, $\alpha, \beta \in \mathbb{N}_0^n$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and $\xi_\nu \neq 0$ for $\nu = 1, \dots, n$. Consequently, by (12.3), for every $k \in \mathbb{N}$ there is $C_k \in]0, \infty[$

such that whenever $u \in (C_I^\infty(\mathbb{R}^n))^m$, $\xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$, then

$$\begin{aligned} & \left\| \left(\frac{\partial}{\partial \xi} \right)^\alpha (\mathcal{F}^{-1}u)(\xi) \right\|_{\mathbb{C}^m} \\ & \leq C_k \prod_{\nu=1}^n (1 + |\xi_\nu|)^{-k} \sup\{ \|(\partial/\partial x)^\gamma u(x)\|_{\mathbb{C}^m} : \gamma \in \mathbb{N}_0^n, |\gamma| \leq kn, x \in I \}. \end{aligned} \quad (12.6)$$

From (12.5) and (12.6) with $k = (md - 1)(2n + 1) + 2$ one obtains (12.2).

Proof of (a) \Rightarrow (b) \wedge ($\omega_E \leq \omega_0$) for $E = (C_b^\infty(\mathbb{R}^n))^m$. Let \mathbb{Z} be the set of all integers (positive or non-positive). For any $z = (z_1, \dots, z_n) \in \mathbb{Z}^n$ denote by τ_z the operator of translation: $(\tau_z f)(x) := f(x_1 + \frac{1}{2}z_1, \dots, x_n + \frac{1}{2}z_n)$ for every function f defined on \mathbb{R}^n and every $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Let $I = [-1/2, 1/2]^n$. Following [P], fix a function $v \in C_I^\infty(\mathbb{R}^n)$ with values in $[0, 1]$ such that $\sum_{z \in \mathbb{Z}^n} \tau_z v \equiv 1$ on \mathbb{R}^n . Assume that $\omega_0 < \infty$. Since the operators S_t° , τ_z and $(\partial/\partial x)^\alpha$ commute, and

$$\frac{\partial}{\partial t} S_t^\circ(u\tau_z v) = P\left(\frac{\partial}{\partial x}\right) S_t^\circ(u\tau_z v),$$

one has

$$\left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial t}\right)^\beta S_t^\circ(u\tau_z v) = \tau_z S_t^\circ \left(\frac{\partial}{\partial x}\right)^\alpha \left(P\left(\frac{\partial}{\partial x}\right)\right)^\beta (v\tau_{-z}u) \quad (12.7)$$

for every $u \in (C_b^\infty(\mathbb{R}^n))^m$, $t \in [0, \infty[$, $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0$. Since the norms (12.1) are translation-invariant, from (12.2) and (12.7) it follows that

whenever $u \in (C_b^\infty(\mathbb{R}^n))^m$, $t \in [0, \infty[$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $z = (z_1, \dots, z_n) \in \mathbb{Z}^n$, $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathbb{N}_0$ and $\varepsilon > 0$, then

$$\begin{aligned} & \left\| \left[\left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial t}\right)^\beta S_t^\circ(u\tau_z v) \right] (x) \right\|_{\mathbb{C}^m} \\ & \leq M_{\varepsilon, \alpha, \beta} \|u\|_{k_0 + |\alpha| + \beta d} e^{(\omega_0 + \varepsilon)t} \prod_{\nu=1}^n (1 + |x_\nu + \frac{1}{2}z_\nu|)^{-2} \end{aligned} \quad (12.8)$$

where $k_0 = n((dm - 1)(2n + 1) + 2)$ and $M_{\varepsilon, \alpha, \beta} \in]0, \infty[$ depends only on ε , α and β .

The series $\sum_{z \in \mathbb{Z}^n} \prod_{\nu=1}^n (1 + |x_\nu + \frac{1}{2}z_\nu|)^{-2}$ of continuous functions of $x = (x_1, \dots, x_n)$ is uniformly convergent on every bounded subset of \mathbb{R}^n , so that

its sum

$$s(x) = \sum_{z \in \mathbb{Z}^n} \prod_{\nu=1}^n (1 + |x_\nu + \frac{1}{2}z_\nu|)^{-2}$$

is a continuous function of x . Since s is periodic, it is bounded on \mathbb{R}^n . Consequently, if $u \in (C_b^\infty(\mathbb{R}^n))^m$ and $u_z(t, x) = [S_t^\circ(u\tau_z v)](x)$, then, by (12.8), $\sum_{z \in \mathbb{Z}^n} u_z(t, x)$ is a series of functions of (t, x) belonging to $C^\infty([0, \infty[; (\mathcal{S}(\mathbb{R}^n))^m)$, and it converges uniformly on every bounded subset of $[0, \infty[\times \mathbb{R}^n$ together with all partial derivatives in t and x_1, \dots, x_n . Therefore, by the theorem on term by term differentiation, whenever $u \in (C_b^\infty(\mathbb{R}^n))^m$ and

$$u(t, x) := \sum_{z \in \mathbb{Z}^n} u_z(t, x) \quad \text{for } (t, x) \in [0, \infty[\times \mathbb{R}^n, \quad (12.9)$$

then $u(\cdot, \cdot) \in (C^\infty([0, \infty[\times \mathbb{R}^n))^m$, $u(\cdot, \cdot)$ satisfies the PDE

$$\frac{\partial}{\partial t} u(t, x) = P\left(\frac{\partial}{\partial x}\right) u(t, x) \quad \text{for } (t, x) \in [0, \infty[\times \mathbb{R}^n,$$

and $u(0, x) = u(x)$ for $x \in \mathbb{R}^n$. Inequality (12.8) and boundedness of s imply that

for every $\varepsilon > 0$, $j \in \mathbb{N}_0$, and $\beta \in \mathbb{N}_0$ there is

$M_{\varepsilon, j, \beta} \in]0, \infty[$ such that

$$\left\| \left(\frac{\partial}{\partial t} \right)^\beta u(t, \cdot) \right\|_j \leq M_{\varepsilon, j, \beta} \|u\|_{j+\beta d+k_0} e^{(\omega_0+\varepsilon)t} \quad (12.10)$$

whenever $u \in (C_b^\infty(\mathbb{R}^n))^m$ and $t \in [0, \infty[$.

Consequently,

$$\text{if } u \in (C_b^\infty(\mathbb{R}^n))^m, \text{ then } u(\cdot, \cdot) \in C^\infty([0, \infty[; (C_b^\infty(\mathbb{R}^n))^m) \quad (12.11)$$

and

if $u \in (C_b^\infty(\mathbb{R}^n))^m$ and $t_0 \in [0, \infty[$, then

$$\lim_{[0, \infty[\ni t \rightarrow t_0} \frac{1}{t - t_0} (u(t, \cdot) - u(t_0, \cdot)) = P\left(\frac{\partial}{\partial x}\right) u(t_0, \cdot) \quad (12.12)$$

in the topology of $(C_b^\infty(\mathbb{R}^n))^m$.

Let $(G_t)_{t \in \mathbb{R}} \subset L((Z'_n)^m)$ be the one-parameter group (2.1). By the Uniqueness Theorem from Section 2, from (12.11), (12.12) and the continuous imbeddings $(C_b^\infty(\mathbb{R}^n))^m \subset (\mathcal{S}'(\mathbb{R}^n))^m \subset (Z'_n)^m$ it follows that

$$u(t, \cdot) = G_t u \quad \text{for every } t \in [0, \infty[\text{ and } u \in (C_b^\infty(\mathbb{R}^n))^m. \quad (12.13)$$

Consequently, by (12.10), one has $G_t|_{(C_b^\infty(\mathbb{R}^n))^m} \in L((C_b^\infty(\mathbb{R}^n))^m)$, and, by (12.11)–(12.13), $(S_t)_{t \geq 0} := (G_t|_{(C_b^\infty(\mathbb{R}^n))^m})_{t \geq 0} \subset L((C_b^\infty(\mathbb{R}^n))^m)$ is a (C_0) -semigroup with infinitesimal generator $P(\partial/\partial x)|_{(C_b^\infty(\mathbb{R}^n))^m}$.

Proof of (a) \Rightarrow (b) and (b) \wedge $(\omega_E < \infty) \Rightarrow (\omega_0 \leq \omega_E)$ for $E = (C_b^\infty(\mathbb{R}^n))^m$. We adapt an argument due to I. G. Petrovskiĭ in [P], pp. 7–9, to the semi-group-theoretical formulation. For every $\xi \in \mathbb{R}^n$ let χ_ξ be the character of \mathbb{R}^n such that $\chi_\xi(x) = e^{i(x, \xi)}$ for $x \in \mathbb{R}^n$. The Fourier transformation \mathcal{F} is an isomorphism of $\mathcal{D}'(\mathbb{R}^n)$ onto Z'_n , and since it acts on $(\mathcal{D}'(\mathbb{R}^n))^m$ coordinatewise, it is also an isomorphism of $(\mathcal{D}'(\mathbb{R}^n))^m$ onto $(Z'_n)^m$. One has $\chi_\xi \in C_b^\infty(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset Z'_n$ and $\chi_\xi = \mathcal{F}\delta_\xi$. Whenever $t \in \mathbb{R}$ is fixed, then $e^{t\tilde{P}} \in C^\infty(\mathbb{R}^n; M_m)$ and the multiplication operator $e^{t\tilde{P}} \cdot$ maps $(\mathcal{D}'(\mathbb{R}^n))^m$ into $(\mathcal{D}'(\mathbb{R}^n))^m$. Consequently, whenever $t \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ and $z \in \mathbb{C}^m$, then

$$\begin{aligned} [\mathcal{F}(e^{t\tilde{P}} \cdot) \mathcal{F}^{-1}](\chi_\xi \otimes z) &= \mathcal{F}(e^{t\tilde{P}} \cdot (\delta_\xi \otimes z)) \\ &= \mathcal{F}(\delta_\xi \otimes (e^{t\tilde{P}(\xi)} z)) = \chi_\xi \otimes (e^{t\tilde{P}(\xi)} z). \end{aligned}$$

This means that

$$G_t(\chi_\xi \otimes z) = \chi_\xi \otimes (e^{t\tilde{P}(\xi)} z) \quad \text{for } t \in \mathbb{R}^n, \xi \in \mathbb{R}^n \text{ and } z \in \mathbb{C}^m \quad (12.14)$$

where $(G_t)_{t \in \mathbb{R}} \subset L((Z'_n)^m)$ is the one-parameter (C_0) -group (2.1).

Suppose now that (b) holds for $E = (C_b^\infty(\mathbb{R}^n))^m$, and let $(S_t)_{t \geq 0} \subset L((C_b^\infty(\mathbb{R}^n))^m)$ be the one-parameter (C_0) -semigroup with infinitesimal generator $P(\partial/\partial x)|_{(C_b^\infty(\mathbb{R}^n))^m}$. Then, by the Uniqueness Theorem from Section 2, $G_t|_{(C_b^\infty(\mathbb{R}^n))^m} = S_t$ for every $t \in [0, \infty[$. Consequently if $t \in [0, \infty[$ and $u \in (C_b^\infty(\mathbb{R}^n))^m$, then $G_t u \in (C_b^\infty(\mathbb{R}^n))^m$, and for every $t \in [0, \infty[$ there are $K_t \in]0, \infty[$ and $k_t \in \mathbb{N}_0$ such that

$$\|G_t u\|_0 \leq K_t \|u\|_{k_t} \quad \text{whenever } u \in (C_b^\infty(\mathbb{R}^n))^m. \quad (12.15)$$

From (12.14) and (12.15) it follows that if $t \in [0, \infty[$ and $\xi \in \mathbb{R}^n$, then

$$\begin{aligned}
\|e^{t\tilde{P}(\xi)}\|_{L(\mathbb{C}^m)} &= \sup\{\|\chi_\xi(x) \otimes (e^{t\tilde{P}(\xi)}z)\|_{\mathbb{C}^m} : x \in \mathbb{R}^n, z \in \mathbb{C}^m, \|z\|_{\mathbb{C}^m} \leq 1\} \\
&= \sup_{\|z\|_{\mathbb{C}^m} \leq 1} \|\chi_\xi \otimes (e^{t\tilde{P}(\xi)}z)\|_0 = \sup_{\|z\|_{\mathbb{C}^m} \leq 1} \|G_t(\chi_\xi \otimes z)\| \\
&\leq \sup_{\|z\|_{\mathbb{C}^m} \leq 1} K_t \|\chi_\xi \otimes z\|_{k_t} \\
&\leq K_t \sup\{|\partial/\partial x^\alpha \chi_\xi(x)| : x \in \mathbb{R}^n, \alpha \in \mathbb{N}_0^n, |\alpha| \leq k_t\} \\
&\leq K_t \sup\{|\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}| : \alpha_\nu \in \mathbb{N}_0 \text{ for } \nu = 1, \dots, n, \\
&\hspace{15em} \alpha_1 + \cdots + \alpha_n \leq k_t\} \\
&\leq K_t(1 + |\xi_1| + \cdots + |\xi_n|)^{k_t} \leq K_t(1 + n|\xi|)^{k_t} \leq K_t(1 + |\xi|)^{nk_t}.
\end{aligned}$$

By Corollary 2.4, p. 252 of [E-N] it follows that whenever $t \in]0, \infty[$ and $\xi \in \mathbb{R}^n$, then

$$\begin{aligned}
\max\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi))\} &= t^{-1} \log \rho(e^{t\tilde{P}(\xi)}) \leq t^{-1} \log \|e^{t\tilde{P}(\xi)}\|_{L(\mathbb{C}^m)} \\
&\leq t^{-1} \log K_t + t^{-1} nk_t \log(1 + |\xi|) \quad (12.16)
\end{aligned}$$

where $\rho(e^{t\tilde{P}(\xi)})$ denotes the spectral radius of the matrix $e^{t\tilde{P}(\xi)}$. From (12.16), by the Corollary at the end of Section 5, it follows that $\omega_0 < \infty$, where

$$\omega_0 := \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi)), \xi \in \mathbb{R}^n\} < \infty,$$

proving (a).

If (b) $\wedge (\omega_E < \infty)$ holds for $E = (C_b^\infty(\mathbb{R}^n))^m$, then there is a $k \in \mathbb{N}$, and for every $\varepsilon > 0$ there is an $M_\varepsilon \in]0, \infty[$ such that for k_t and K_t occurring in (12.15) and (12.16) one has

$$k_t \leq k \quad \text{and} \quad K_t \leq M_\varepsilon e^{(\omega_E + \varepsilon)t} \quad \text{for every } t \in [0, \infty[.$$

Consequently, by (12.16),

$$\max\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi))\} \leq \omega_E + \varepsilon + t^{-1}[\log M_\varepsilon + nk \log(1 + |\xi|)]$$

for every $\xi \in \mathbb{R}^n$ and $t > 0$. Since $\varepsilon > 0$ and $t > 0$ are arbitrary, it follows that $\max\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{P}(\xi))\} \leq \omega_E$ for every $\xi \in \mathbb{R}^n$, proving that $\omega_0 \leq \omega_E$.

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