QUASI-NEARLY SUBHARMONICITY AND SEPARATELY QUASI-NEARLY SUBHARMONIC FUNCTIONS

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Abstract: Wiegerinck has shown that a separately subharmonic function need not be subharmonic. Improving previous results of Lelong, of Avanissian, of Arsove and of us, Armitage and Gardiner gave an almost sharp integrability condition which ensures a separately subharmonic function to be subharmonic. Completing now our recent counterparts to the cited results of Lelong, Avanissian and Arsove for so called quasi-nearly subharmonic functions, we present a counterpart to the cited result of Armitage and Gardiner for separately quasi-nearly subharmonic function. This counterpart enables us to slightly improve Armitage's and Gardiner's original result, too. The method we use is a rather straightforward and technical, but still by no means easy, modification of Armitage's and Gardiner's argument combined with an old argument of Domar.

Key words: Subharmonic, quasi-nearly subharmonic, separately subharmonic, separately quasi-nearly subharmonic, integrability condition.

1. Introduction

1.1. Solving a long standing problem, Wiegerinck [Wi88, Theorem, p. 770], see also Wiegerinck and Zeinstra [WZ91, Theorem 1, p. 246], showed that a separately subharmonic function need not be subharmonic. On the other hand, Armitage and Gardiner [AG93, Theorem 1, p. 256] showed that a separately subharmonic function u in a domain Ω in $\mathbb{R}^{m+n}$, $m \geq n \geq 2$, is subharmonic provided $\phi(\log^+ u^+)$ is locally integrable in Ω, where $\phi : [0, +\infty) \to [0, +\infty)$ is an increasing function such that

$$\int\limits_1^{+\infty} s^{(n-1)/(m-1)} (\phi(s))^{-1/(m-1)} \, ds < +\infty.$$

Armitage's and Gardiner's result includes the previous results of Lelong [Le45, Théorème 1 bis, p. 315], of Avanissian [Av61, Théorème 9, p. 140], see also [He71, Theorem, p. 31], of Arsove [Ar66, Theorem 1, p. 622], and of us [Ri89, Theorem 1, p. 69]. Though Armitage's and Gardiner's result is almost sharp, it is, nevertheless, based on Avanissian's result, or, alternatively, on the more general results of Arsove and us. See [Ri07₃].

In [Ri07₃, Proposition 3; Theorem 1, Corollary 1, Corollary 2, Corollary 3; Theorem 2, Corollary; Theorem 3 (Proposition 3.1, p. 57; Theorem 3.1, Corollary 3.1, Corollary 3.2, Corollary 3.3, pp. 58-63; Theorem 4.1, Corollary 4.1, pp. 64-65; Theorem 4.2, p. 65)] we have extended the cited result of Lelong, Avanissian, Arsove, and us to the so called quasi-nearly subharmonic functions. The purpose of this paper is to extend also Armitage's and Gardiner's result to this more general setup. Our result will, at the same time, give a slight refinement to Armitage's and Gardiner's result. The method of proof will be a rather straightforward and technical, but still by no means easy, modification of Domar's and Armitage's and Gardiner's argument, see [Do57, Lemma 1, pp. 431-432 and 430] and [AG93, proof of Proposition 2, pp. 257-259, proof of Theorem 1, pp. 258-259].

1.2. Notation. Our notation is rather standard, see e.g. [Ri06$_1$], [Ri07$_3$] and [He71]. m_N is the Lebesgue measure in the Euclidean space $\mathbb{R}^N$, $N \geq 2$. We write ν_N for the Lebesgue measure of the unit ball $B^N(0,1)$ in $\mathbb{R}^N$, thus $\nu_N = m_N(B^N(0,1))$. D is a domain of $\mathbb{R}^N$. The complex space $\mathbb{C}^n$ is identified with the real space $\mathbb{R}^{2n}$, $n \geq 1$. Constants will be denoted by C and K. They will be nonnegative and may vary from line to line.

2. Quasi-nearly subharmonic functions

2.1. Nearly subharmonic functions. We recall that an upper semicontinuous function $u : D \to [-\infty, +\infty)$ is *subharmonic* if for all $\overline{B^N(x,r)} \subset D$,

$$u(x) \leq \frac{1}{\nu_N \, r^N} \int\limits_{B^N(x,r)} u(y) \, dm_N(y).$$

The function $u \equiv -\infty$ is considered subharmonic.

We say that a function $u : D \to [-\infty, +\infty)$ is *nearly subharmonic*, if u is Lebesgue measurable, $u^+ \in \mathcal{L}^1_{\mathrm{loc}}(D)$, and for all $\overline{B^N(x,r)} \subset D$,

$$u(x) \leq \frac{1}{\nu_N \, r^N} \int\limits_{B^N(x,r)} u(y) \, dm_N(y).$$

Observe that in the standard definition of nearly subharmonic functions one uses the slightly stronger assumption that $u \in \mathcal{L}^1_{\mathrm{loc}}(D)$, see e.g. [He71, p. 14]. However, our above, slightly more general definition seems to be more useful, see [Ri07$_3$, Proposition 1 (iii) and Proposition 2 (vi) and (vii) (Proposition 2.1 (iii) and Proposition 2.2 (vi), (vii), pp. 54-55)].

2.2. Quasi-nearly subharmonic functions. A Lebesgue measurable function $u : D \to [-\infty, +\infty)$ is *K-quasi-nearly subharmonic*, if $u^+ \in \mathcal{L}^1_{\mathrm{loc}}(D)$ and if there is a constant $K = K(N, u, D) \geq 1$ such that for all $\overline{B^N(x,r)} \subset D$,

$$(1) \qquad\qquad u_M(x) \leq \frac{K}{\nu_N \, r^N} \int\limits_{B^N(x,r)} u_M(y) \, dm_N(y)$$

for all $M \geq 0$, where $u_M := \max\{u, -M\} + M$. A function $u : D \to [-\infty, +\infty)$ is *quasi-nearly subharmonic*, if u is K-quasi-nearly subharmonic for some $K \geq 1$.

A Lebesgue measurable function $u : D \to [-\infty, +\infty)$ is *K-quasi-nearly subharmonic n.s. (in the narrow sense)*, if $u^+ \in \mathcal{L}^1_{\mathrm{loc}}(D)$ and if there is a constant $K = K(N, u, D) \geq 1$ such that for all $\overline{B^N(x,r)} \subset D$,

$$(2) \qquad\qquad u(x) \leq \frac{K}{\nu_N \, r^N} \int\limits_{B^N(x,r)} u(y) \, dm_N(y).$$

A function $u : D \to [-\infty, +\infty)$ is *quasi-nearly subharmonic n.s.*, if u is K-quasi-nearly subharmonic n.s. for some $K \geq 1$.

Quasi-nearly subharmonic functions (perhaps with a different termonology, and sometimes in certain special cases, or just the corresponding generalized mean value inequality (2)) have previously been considered at least in [FS72], [Ku74], [To86], [Ri89], [Pa94], [Mi96], [Ri00], [Ri03], [Ri04], [PR08], [Ri06$_1$], [Ri06$_2$], [Ri07$_3$], [Ko07] and [DP07]. We recall here only that this function class includes, among others, subharmonic functions, and, more generally, quasisubharmonic (see e.g. [Le45, p. 309], [Av61, p. 136], [He71, p. 26]) and also nearly subharmonic functions (see e.g. [He71, p. 14]), also functions satisfying certain natural growth conditions, especially certain eigenfunctions, and polyharmonic functions. Also, the class of Harnack functions is included, thus, among others, nonnegative harmonic functions as well as

nonnegative solutions of some elliptic equations. In particular, the partial differential equations associated with quasiregular mappings belong to this family of elliptic equations, see Vuorinen [Vu82]. Observe that already Domar [Do57, p. 430] has pointed out the relevance of the class of (nonnegative) quasi-nearly subharmonic functions. For, at least partly, an even more general function class, see [Do88].

For examples and basic properties of quasi-nearly subharmonic functions, see the above references, especially [PR08] and [Ri07$_3$]. For the sake of convenience of the reader we recall the following:

(i) A K-quasi-nearly subharmonic function n.s. is K-quasi-nearly subharmonic, but not necessarily conversely.

(ii) A nonnegative Lebesgue measurable function is K-quasi-nearly subharmonic if and only if it is K-quasi-nearly subharmonic n.s.

(iii) A Lebesgue measurable function is 1-quasi-nearly subharmonic if and only if it is 1-quasi-nearly subharmonic n.s. and if and only if it is nearly subharmonic (in the sense defined above).

(iv) If $u : D \to [-\infty, +\infty)$ is K_1-quasi-nearly subharmonic and $v : D \to [-\infty, +\infty)$ is K_2-quasi-nearly subharmonic, then $\sup\{u, v\}$ is $\sup\{K_1, K_2\}$-quasi-nearly subharmonic in D. Especially, $u^+ := \max\{u, 0\}$ is K_1-quasi-nearly subharmonic in D.

(v) Let $\mathcal{F}$ be a family of K-quasi-nearly subharmonic (resp. K-quasi-nearly subharmonic n.s.) functions in D and let $w := \sup_{u \in \mathcal{F}} u$. If w is Lebesgue measurable and $w^+ \in \mathcal{L}^1_{\text{loc}}(D)$, then w is K-quasi-nearly subharmonic (resp. K-quasi-nearly subharmonic n.s.) in D.

(vi) If $u : D \to [-\infty, +\infty)$ is quasi-nearly subharmonic n.s., then either $u \equiv -\infty$ or u is finite almost everywhere in D, and $u \in \mathcal{L}^1_{\text{loc}}(D)$.

3. Lemmas

3.1. The following result and its proof is essentially due to Domar [Do57, Lemma 1, pp. 431-432 and 430]. We state the result, however, in a more general form, at least seemingly. See also [AG93, p. 258].

3.2. Lemma. Let $K \geq 1$. Let $\varphi : [0, +\infty) \to [0, +\infty)$ and $\psi : [0, +\infty) \to [0, +\infty)$ be increasing (strictly or not) functions such that there are s_0, $s_1 \in \mathbb{N}$, $s_0 < s_1$, such that

(i) the inverse functions φ^{-1} and ψ^{-1} are defined on $[\inf\{\varphi(s_1 - s_0), \psi(s_1 - s_0)\}, +\infty)$,

(ii) $2K(\psi^{-1} \circ \varphi)(s - s_0) \leq (\psi^{-1} \circ \varphi)(s)$ for all $s \geq s_1$.

Let $u : D \to [0, +\infty)$ be a K-quasi-nearly subharmonic function. Suppose that

$$u(x_j) \geq (\psi^{-1} \circ \varphi)(j)$$

for some $x_j \in D$, $j \geq s_1$. If

$$R_j \geq \left(\frac{2K}{\nu_N}\right)^{1/N} \left[\frac{(\psi^{-1} \circ \varphi)(j+1)}{(\psi^{-1} \circ \varphi)(j)} \, m_N(A_j)\right]^{1/N}$$

where

$$A_j := \{\, x \in D \, : \, (\psi^{-1} \circ \varphi)(j - s_0) \leq u(x) < (\psi^{-1} \circ \varphi)(j+1) \,\},$$

then either $B^N(x_j, R_j) \cap (\mathbb{R}^N \setminus D) \neq \emptyset$ or there is $x_{j+1} \in B^N(x_j, R_j)$ such that

$$u(x_{j+1}) \geq (\psi^{-1} \circ \varphi)(j+1).$$

Proof. Choose

$$R_j \geq \left(\frac{2K}{\nu_N}\right)^{1/N} \left[\frac{(\psi^{-1} \circ \varphi)(j+1)}{(\psi^{-1} \circ \varphi)(j)} \, m_N(A_j)\right]^{1/N},$$

and suppose that $B^N(x_j, R_j) \subset D$. Suppose on the contrary that $u(x) < (\psi^{-1} \circ \varphi)(j+1)$ for all $x \in B^N(x_j, R_j)$. Using the assumption (1) (or (2)) we see that

$$(\psi^{-1} \circ \varphi)(j) \leq u(x_j) \leq \frac{K}{\nu_N R_j^N} \int\limits_{B^N(x_j, R_j)} u(x)\, dm_N(x)$$

$$\leq \frac{K}{\nu_N R_j^N} \int\limits_{B^N(x_j, R_j) \cap A_j} u(x)\, dm_N(x) + \frac{K}{\nu_N R_j^N} \int\limits_{B^N(x_j, R_j) \setminus A_j} u(x)\, dm_N(x)$$

$$< \left[\frac{K m_N(B^N(x_j, R_j) \cap A_j)}{\nu_N R_j^N} \frac{(\psi^{-1} \circ \varphi)(j+1)}{(\psi^{-1} \circ \varphi)(j)} + \frac{K m_N(B^N(x_j, R_j) \setminus A_j)}{\nu_N R_j^N} \frac{(\psi^{-1} \circ \varphi)(j - s_0)}{(\psi^{-1} \circ \varphi)(j)} \right] (\psi^{-1} \circ \varphi)(j)$$

$$< (\psi^{-1} \circ \varphi)(j),$$

a contradiction. $\square$

3.3. The next lemma is a slightly generalized version of Armitage's and Gardiner's result [AG93, Proposition 2, p. 257]. The proof of our refinement is − as already pointed out − a rather straightforward modification of Armitage's and Gardiner's argument [AG93, proof of Proposition 2, pp. 257-259].

3.4. Lemma. *Let $K \geq 1$. Let $\varphi : [0, +\infty) \to [0, +\infty)$ and $\psi : [0, +\infty) \to [0, +\infty)$ be increasing functions such that there are $s_0, s_1 \in \mathbb{N}$, $s_0 < s_1$, such that*

(i) *the inverse functions φ^{-1} and ψ^{-1} are defined on $[\inf\{\, \varphi(s_1 - s_0), \psi(s_1 - s_0)\,\}, +\infty)$,*
(ii) *$2K(\psi^{-1} \circ \varphi)(s - s_0) \leq (\psi^{-1} \circ \varphi)(s)$ for all $s \geq s_1$,*
(iii) *$\sum_{j=s_1+1}^{+\infty} \left[\frac{(\psi^{-1} \circ \varphi)(j+1)}{(\psi^{-1} \circ \varphi)(j)} \frac{1}{\varphi(j - s_0)} \right]^{1/(N-1)} < +\infty.$*

Let $u : D \to [0, +\infty)$ be a K-quasi-nearly subharmonic function. Let $\tilde{s}_1 \in \mathbb{N}$, $\tilde{s}_1 \geq s_1$, be arbitrary. Then for each $x \in D$ and $r > 0$ such that $\overline{B^N(x, r)} \subset D$ either

$$u(x) \leq (\psi^{-1} \circ \varphi)(\tilde{s}_1 + 1)$$

or

$$\Phi(u(x)) \leq \frac{C}{r^N} \int\limits_{B^N(x, r)} \psi(u(y))\, dm_N(y)$$

where $C = C(N, K, s_0)$ and $\Phi : [s_2, +\infty) \to [0, +\infty)$,

$$\Phi(t) := \left(\sum_{j=j_0}^{+\infty} \left[\frac{(\psi^{-1} \circ \varphi)(j+1)}{(\psi^{-1} \circ \varphi)(j)} \frac{1}{\varphi(j - s_0)} \right]^{1/(N-1)} \right)^{1-N},$$

and $j_0 \in \{s_1 + 1, s_1 + 2, \dots\}$ is such that

$$(\psi^{-1} \circ \varphi)(j_0) \leq t < (\psi^{-1} \circ \varphi)(j_0 + 1),$$

and $s_2 := (\psi^{-1} \circ \varphi)(s_1 + 1)$.

Proof. Take $x \in D$ and $r > 0$ arbitrarily such that $\overline{B^N(x, r)} \subset D$. We may suppose that $u(x) > (\psi^{-1} \circ \varphi)(\tilde{s}_1 + 1)$. Since φ and ψ are increasing, there is an integer $j_0 \geq \tilde{s}_1 + 1$ such that

$$(\psi^{-1} \circ \varphi)(j_0) \leq u(x) < (\psi^{-1} \circ \varphi)(j_0 + 1).$$

write $x_{j_0} := x$, $D_0 := B^N(x_{j_0}, r)$ and for each $j \geq j_0$,

$$A_j := \{\, y \in D_0 \,:\, (\psi^{-1} \circ \varphi)(j - s_0) \leq u(y) < (\psi^{-1} \circ \varphi)(j+1) \,\},$$

$$R_j := \left(\frac{2K}{\nu_N} \right)^{1/N} \left[\frac{(\psi^{-1} \circ \varphi)(j+1)}{(\psi^{-1} \circ \varphi)(j)} m_N(A_j) \right]^{1/N}.$$

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If $B^N(x_{j_0}, R_{j_0}) \cap (\mathbb{R}^N \setminus D_0) \neq \emptyset$, then clearly

$$r < R_{j_0} \leq \sum_{k=j_0}^{+\infty} R_k.$$

On the other hand, if $B^N(x_{j_0}, R_{j_0}) \subset D_0$, then by Lemma 3.2 there is $x_{j_0+1} \in B^N(x_{j_0}, R_{j_0})$ such that $u(x_{j_0+1}) \geq (\psi^{-1} \circ \varphi)(j_0 + 1)$. Suppose now that for $k = j_0, j_0 + 1, \ldots, j$,

$B^N(x_k, R_k) \subset D_0, \ x_{k+1} \in B^N(x_k, R_k)$ (this for $k = j_0, j_0+1, \ldots, j-1$), and $u(x_k) \geq (\psi^{-1} \circ \varphi)(k)$.

By Lemma 3.2 there then is $x_{j+1} \in B^N(x_j, R_j)$ such that $u(x_{j+1}) \geq (\psi^{-1} \circ \varphi)(j + 1)$. Since u is locally bounded above and $(\psi^{-1} \circ \varphi)(k) \to +\infty$ as $k \to +\infty$, we may suppose that $B^N(x_{j+1}, R_{j+1}) \cap (\mathbb{R}^N \setminus D_0) \neq \emptyset$. But then

$$r < \text{dist}(x_{j_0}, x_{j_0+1}) + \text{dist}(x_{j_0+1}, x_{j_0+2}) + \cdots + \text{dist}(x_j, x_{j+1}) + \text{dist}(x_{j+1}, \mathbb{R}^N \setminus D_0),$$

thus

(3) $$r < R_{j_0} + R_{j_0+1} + \cdots + R_j + R_{j+1} \leq \sum_{k=j_0}^{+\infty} R_k.$$

Using then the notation

$$a_k := \{\, y \in D_0 \,:\, (\psi^{-1} \circ \varphi)(k) \leq u(y) < (\psi^{-1} \circ \varphi)(k + 1)\,\},$$

$k = j_0 - s_0, j_0 + 1 - s_0, \ldots$, we get from (3):

$$r < \sum_{k=j_0}^{+\infty} \left(\frac{2K}{\nu_N}\right)^{1/N} \left[\frac{(\psi^{-1} \circ \varphi)(k + 1)}{(\psi^{-1} \circ \varphi)(k)} m_N(A_k)\right]^{1/N}$$

$$< \left(\frac{2K}{\nu_N}\right)^{1/N} \sum_{k=j_0}^{+\infty} \left(\left[\frac{(\psi^{-1} \circ \varphi)(k + 1)}{(\psi^{-1} \circ \varphi)(k)} \frac{1}{\varphi(k - s_0)}\right]^{1/N} [\varphi(k - s_0) m_N(A_k)]^{1/N}\right)$$

$$< \left(\frac{2K}{\nu_N}\right)^{1/N} \left(\sum_{k=j_0}^{+\infty} \left[\frac{(\psi^{-1} \circ \varphi)(k + 1)}{(\psi^{-1} \circ \varphi)(k)} \frac{1}{\varphi(k - s_0)}\right]^{1/(N-1)}\right)^{(N-1)/N} \left[\sum_{k=j_0}^{+\infty} \varphi(k - s_0) m_N(A_k)\right]^{1/N}$$

$$< \left(\frac{2K}{\nu_N}\right)^{1/N} \left(\sum_{k=j_0}^{+\infty} \left[\frac{(\psi^{-1} \circ \varphi)(k + 1)}{(\psi^{-1} \circ \varphi)(k)} \frac{1}{\varphi(k - s_0)}\right]^{1/(N-1)}\right)^{(N-1)/N} \left[\sum_{k=j_0}^{+\infty} \int_{A_k} \psi(u(y)) dm_N(y)\right]^{1/N}$$

$$< \left(\frac{2K}{\nu_N}\right)^{1/N} \left(\sum_{k=j_0}^{+\infty} \left[\frac{(\psi^{-1} \circ \varphi)(k + 1)}{(\psi^{-1} \circ \varphi)(k)} \frac{1}{\varphi(k - s_0)}\right]^{1/(N-1)}\right)^{(N-1)/N} \left(\sum_{k=j_0}^{+\infty} \left[\sum_{j=k-s_0}^{k} \int_{a_j} \psi(u(y)) dm_N(y)\right]\right)^{1/N}$$

$$< \left[\frac{2(s_0 + 1)K}{\nu_N}\right]^{1/N} \left(\sum_{k=j_0}^{+\infty} \left[\frac{(\psi^{-1} \circ \varphi)(k + 1)}{(\psi^{-1} \circ \varphi)(k)} \frac{1}{\varphi(k - s_0)}\right]^{1/(N-1)}\right)^{(N-1)/N} \left[\int_{D_0} \psi(u(y)) dm_N(y)\right]^{1/N}.$$

Thus

$$\Phi(u(x)) \leq \frac{C}{r^N} \int_{D_0} \psi(u(y)) dm_N(y),$$

where $C = C(N, K, s_0)$ and $\Phi : [s_2, +\infty) \to [0, +\infty)$,

$$\Phi(t) := \left(\sum_{k=j_0}^{+\infty} \left[\frac{(\psi^{-1} \circ \varphi)(k + 1)}{(\psi^{-1} \circ \varphi)(k)} \frac{1}{\varphi(k - s_0)}\right]^{1/(N-1)}\right)^{1-N},$$

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where $j_0 \in \{\, s_1 + 1, s_1 + 2, \dots \}$ is such that

$$(\psi^{-1} \circ \varphi)(j_0) \leq t < (\psi^{-1} \circ \varphi)(j_0 + 1),$$

and $s_2 = (\psi^{-1} \circ \varphi)(s_1 + 1)$.

The function Φ may be extended to the whole interval $[0, +\infty)$, for example as follows:

$$\Phi(t) := \begin{cases} \Phi(t), & \text{when } t \geq s_2, \\ \frac{t}{s_2}\Phi(s_2), & \text{when } 0 \leq t < s_2. \end{cases}$$

$\square$

3.5. Remark. Write $s_3 := \sup\{\, s_1 + 3, (\psi^{-1} \circ \varphi)(s_1 + 3)\,\}$, say. (We may suppose that s_3 is an integer.) Suppose that, in addition to the assumptions (i), (ii), (iii) of Lemma 3.4, also the following assumption is satisfied:

(iv) *the function*

$$[s_1 + 1, +\infty) \ni s \mapsto \frac{(\psi^{-1} \circ \varphi)(s+1)}{(\psi^{-1} \circ \varphi)(s)} \frac{1}{\varphi(t - s_0)} \in \mathbb{R}$$

is decreasing.

Then one can replace the function $\Phi \mid [s_3, +\infty)$ by the function $\Phi_1 \mid [s_3, +\infty)$, where $\Phi_1(= \Phi_1^{\varphi,\psi}) : [0, +\infty) \to [0, +\infty)$,

$$\Phi_1(t)(= \Phi_1^{\varphi,\psi}(t)) := \begin{cases} \left(\int\limits_{(\varphi^{-1}\circ\psi)(t)-2}^{+\infty} \left[\frac{(\psi^{-1}\circ\varphi)(s+1)}{(\psi^{-1}\circ\varphi)(s)} \frac{1}{\varphi(s-s_0)} \right]^{1/(N-1)} ds \right)^{1-N}, & \text{when } t \geq s_3, \\ \frac{t}{s_3}\Phi_1(s_3), & \text{when } 0 \leq t < s_3. \end{cases}$$

Similarly, if the function

$$[s_1 + 1, +\infty) \ni s \mapsto \frac{(\psi^{-1} \circ \varphi)(s+1)}{(\psi^{-1} \circ \varphi)(s)} \in \mathbb{R}$$

is bounded, then in Lemma 3.4 one can replace the function $\Phi \mid [s_3, +\infty)$ by the function $\Phi_2 \mid [s_3, +\infty)$, where $\Phi_2(= \Phi_2^{\varphi,\psi}) : [0, +\infty) \to [0, +\infty)$,

$$\Phi_2(t)(= \Phi_2^{\varphi,\psi}(t)) := \begin{cases} \left[\int\limits_{(\varphi^{-1}\circ\psi)(t)-2}^{+\infty} \frac{ds}{\varphi(s-s_0)^{1/(N-1)}} \right]^{1-N}, & \text{when } t \geq s_3, \\ \frac{t}{s_3}\Phi_2(s_3), & \text{when } 0 \leq t < s_3. \end{cases}$$

4. The condition

4.1. Next we propose a counterpart to Armitage's and Gardiner's result [AG93, Theorem 1, p. 256] for quasi-nearly subharmonic functions. The proof below follows Armitage's and Gardiner's argument [AG93, proof of Theorem 1, pp. 258-259], but is, at least formally, more general. Compare also Corollary 4.7 below.

4.2. Theorem. *Let Ω be a domain in $\mathbb{R}^{m+n}$, $m \geq n \geq 2$, and let $K \geq 1$. Let $u : \Omega \to [-\infty, +\infty)$ be a Lebesgue measurable function. Suppose that the following conditions are satisfied:*

(a) *For each $y \in \mathbb{R}^n$ the function*

$$\Omega(y) \ni x \mapsto u(x, y) \in [-\infty, +\infty)$$

is K-quasi-nearly subharmonic.

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(b) *For each $x \in \mathbb{R}^m$ the function*

$$\Omega(x) \ni y \mapsto u(x,y) \in [-\infty, +\infty)$$

is K-quasi-nearly subharmonic.

(c) *There are increasing functions $\varphi : [0, +\infty) \to [0, +\infty)$ and $\psi : [0, +\infty) \to [0, +\infty)$ and $s_0, s_1 \in \mathbb{N}$, $s_0 < s_1$, such that*

(c1) *the inverse functions φ^{-1} and ψ^{-1} are defined on $[\inf\{\,\varphi(s_1 - s_0), \psi(s_1 - s_0)\,\}, +\infty)$,*

(c2) $2K(\psi^{-1} \circ \varphi)(s - s_0) \le (\psi^{-1} \circ \varphi)(s)$ *for all $s \ge s_1$,*

(c3) *the function*

$$[s_1 + 1, +\infty) \ni s \mapsto \frac{(\psi^{-1} \circ \varphi)(s+1)}{(\psi^{-1} \circ \varphi)(s)} \in \mathbb{R}$$

is bounded,

(c4) $\displaystyle \int\limits_{s_1}^{+\infty} \frac{s^{(n-1)/(m-1)}}{\varphi(s-s_0)^{1/(m-1)}} ds < +\infty,$

(c5) $\psi \circ u^+ \in \mathcal{L}^1_{\mathrm{loc}}(\Omega).$

Then u is quasi-nearly subharmonic in Ω.

Proof. Recall that $s_3 = \sup\{\, s_1 + 3, (\psi^{-1} \circ \varphi)(s_1 + 3) \,\}$. Write $s_4 := \sup\{\, s_3 + s_0, (\varphi^{-1} \circ \psi)(s_1 + 3) \,\}$ and $s_5 := s_4 + s_0$, say. Clearly, $s_0 < s_1 < s_2 < s_3 < s_4 < s_5$. (We may suppose that s_3, s_4 and s_5 are integers.) One may replace u by $u_M := \sup\{\, u^+, M \,\}$, where $M = \sup\{\, s_5 + 3, (\psi^{-1} \circ \varphi)(s_4 + 3), (\varphi^{-1} \circ \psi)(s_4 + 3) \,\}$, say. We continue to denote u_M by u.

Take $(x_0, y_0) \in \Omega$ and $r > 0$ arbitrarily such that $\overline{B^m(x_0, 2r) \times B^n(y_0, 2r)} \subset \Omega$. By [Ri07$_3$, Proposition 3 (Proposition 3.1, p. 57)] (that is, by a counterpart to [Ri89, Theorem 1, p. 69], say) it is sufficient to show that u is bounded above in $B^m(x_0, r) \times B^n(y_0, r)$.

Take $(\xi, \eta) \in B^m(x_0, r) \times B^n(y_0, r)$ arbitrarily. In order to apply Lemma 3.4 to the K-quasi-nearly subharmonic function $u(\cdot, \eta)$ in $B^m(\xi, r)$, check that the assumptions are satisfied. Since (i) and (ii) are clearly satisfied, it remains to show that

$$\sum_{j=s_1+1}^{+\infty} \left[\frac{(\psi^{-1} \circ \varphi)(j+1)}{(\psi^{-1} \circ \varphi)(j)} \frac{1}{\varphi(j-s_0)} \right]^{1/(m-1)} < +\infty.$$

Because of the assumption (c3), it is sufficient to show that

$$\sum_{j=s_1+1}^{+\infty} \frac{1}{\varphi(j-s_0)^{1/(m-1)}} < +\infty.$$

This is seen as follows. Observe first that

$$\sum_{j=s_1+1}^{+\infty} \frac{1}{\varphi(j-s_0)^{1/(m-1)}} \le \int_{s_1}^{+\infty} \frac{ds}{\varphi(s-s_0)^{1/(m-1)}} \le \int_{s_1}^{+\infty} \frac{s^{(n-1)/(m-1)}}{\varphi(s-s_0)^{1/(m-1)}} ds < +\infty.$$

We know that $u(\xi, \eta) \ge s_4$. Therefore it follows from Lemma 3.4 and Remark 3.5 that

$$(4) \quad \Phi_2(u(\xi, \eta)) = \left[\int\limits_{(\varphi^{-1} \circ \psi)(u(\xi,\eta))-2}^{+\infty} \frac{ds}{\varphi(s-s_0)^{1/(m-1)}} \right]^{1-m} \le \frac{C}{r^m} \int\limits_{B^m(\xi, r)} \psi(u(x, \eta)) dm_m(x).$$

Recall that here $\Phi_2 (= \Phi_2^{\varphi, \psi}) : [0, +\infty) \to [0, +\infty)$,

$$\Phi_2(t)(= \Phi_2^{\varphi, \psi}(t)) := \begin{cases} \left[\displaystyle\int\limits_{(\varphi^{-1} \circ \psi)(t)-2}^{+\infty} \frac{ds}{\varphi(s-s_0)^{1/(m-1)}} \right]^{1-m}, & \text{when } t \ge s_3, \\[3ex] \frac{t}{s_3} \Phi_2(s_3), & \text{when } 0 \le t < s_3. \end{cases}$$

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Take then the integral mean values of both sides of (4) over $B^n(\eta, r)$:

$$\frac{C}{r^n} \int\limits_{B^n(\eta,r)} \Phi_2(u(\xi,y))dm_n(y) \leq \frac{C}{r^n} \int\limits_{B^n(\eta,r)} [\frac{C}{r^m} \int\limits_{B^m(\xi,r)} \psi(u(x,y))dm_m(x)]dm_n(y)$$

$$(5) \qquad\qquad \leq \frac{C}{r^{m+n}} \int\limits_{B^m(\xi,r)\times B^n(\eta,r)} \psi(u(x,y))dm_{m+n}(x,y)$$

$$\leq \frac{C}{r^{m+n}} \int\limits_{B^m(x_0,2r)\times B^n(y_0,2r)} \psi(u(x,y))dm_{m+n}(x,y).$$

In order to apply Lemma 3.4 (and Remark 3.5) once more, define $\psi_1 : [0,+\infty) \to [0,+\infty)$, $\psi_1(t) := \Phi_2(t)$, where $\Phi_2 : [0,+\infty) \to [0,+\infty)$ is as above. Define $\varphi_1 : [0,+\infty) \to [0,+\infty)$,

$$\varphi_1(t) := \begin{cases} \frac{t}{s_3}\psi_1((\psi^{-1}\circ\varphi)(s_3)) = \frac{t}{s_3}\Phi_2(\psi^{-1}(\varphi(s_3))), & \text{when } 0 \leq t < s_3, \\ \psi_1((\psi^{-1}\circ\varphi)(t)) = \Phi_2(\psi^{-1}(\varphi(t))), & \text{when } t \geq s_3. \end{cases}$$

It is straightforward to see that both ψ_1 and φ_1 are strictly increasing and continuous. Observe also that for $t \geq s_4$, say,

$$(6) \qquad \varphi_1(t) = \Phi_2((\psi^{-1}\circ\varphi)(t)) = \left[\int\limits_{(\varphi^{-1}\circ\psi)((\psi^{-1}\circ\varphi)(t))-2}^{+\infty} \frac{ds}{\varphi(s-s_0)^{1/(m-1)}} \right]^{1-m}$$

$$= \left[\int\limits_{t-2}^{+\infty} \frac{ds}{\varphi(s-s_0)^{1/(m-1)}} \right]^{1-m}.$$

Check then that the assumptions of Lemma 3.4 (and Remark 3.5) are fullfilled. Write $\tilde{s}_0 := s_0$ and $\tilde{s}_1 := s_4$. The assumption (i) is clearly satisfied for ψ_1 and φ_1. Then for all $s \geq \tilde{s}_1$,

$$\varphi_1(t) = \psi_1((\psi^{-1}\circ\varphi)(t)) \Leftrightarrow (\psi_1^{-1}\circ\varphi_1)(t) = (\psi^{-1}\circ\varphi)(t).$$

Thus also the assumption (ii) is satisfied. It remains to show that

$$\sum_{j=s_4+1}^{+\infty} \left[\frac{(\psi_1^{-1}\circ\varphi_1)(j+1)}{(\psi_1^{-1}\circ\varphi_1)(j)} \frac{1}{\varphi_1(j-s_0)} \right]^{1/(n-1)} < +\infty,$$

say. It is surely sufficient to show that

$$(7) \qquad \int\limits_{s_5+s_0+2}^{+\infty} \frac{ds}{\varphi_1(s-s_0)^{1/(n-1)}} < +\infty.$$

Define $F : [s_5,+\infty) \times [s_5+s_0+2,+\infty) \to [0,+\infty)$,

$$F(s,t) := \begin{cases} 0, & \text{when } s_5 \leq s < t-s_0-2, \\ \varphi(s-s_0)^{-1/(m-1)}, & \text{when } s_5+s_0+2 \leq t-s_0-2 \leq s. \end{cases}$$

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Suppose that $m > n$ and write $p := (m-1)/(n-1)$. Using Minkowski's Inequality, see e.g. [Ga07, p. 14], one obtains, with the aid of (6),

$$\left[\int_{s_5+s_0+2}^{+\infty} \frac{dt}{\varphi_1(t-s_0)^{1/(n-1)}} \right]^{(n-1)/(m-1)} = \left[\int_{s_5+s_0+2}^{+\infty} \left(\left[\int_{t-s_0-2}^{+\infty} \frac{ds}{\varphi(s-s_0)^{1/(m-1)}} \right]^{1-m} \right)^{-1/(n-1)} dt \right]^{(n-1)/(m-1)}$$

$$= \left(\int_{s_5+s_0+2}^{+\infty} \left[\int_{t-s_0-2}^{+\infty} \frac{ds}{\varphi(s-s_0)^{1/(m-1)}} \right]^{(m-1)/(n-1)} dt \right)^{(n-1)/(m-1)}$$

$$= \left(\int_{s_5+s_0+2}^{+\infty} \left[\int_{s_5}^{+\infty} F(s,t)ds \right]^{(m-1)/(n-1)} dt \right)^{(n-1)/(m-1)} \leq \int_{s_5}^{+\infty} \left[\int_{s_5+s_0+2}^{+\infty} F(s,t)^{(m-1)/(n-1)} dt \right]^{(n-1)/(m-1)} ds$$

$$\leq \int_{s_5}^{+\infty} \left[\int_{s_5+s_0+2}^{s+s_0+2} \frac{dt}{\varphi(s-s_0)^{1/(n-1)}} \right]^{(n-1)/(m-1)} ds \leq \int_{s_5}^{+\infty} \frac{[(s+s_0+2)-(s_5+s_0+2)]^{(n-1)/(m-1)}}{\varphi(s-s_0)^{1/(m-1)}} ds$$

$$\leq \int_{s_5}^{+\infty} \frac{(s-s_5)^{(n-1)/(m-1)}}{\varphi(s-s_0)^{1/(m-1)}} ds \leq \int_{s_5}^{+\infty} \frac{s^{(n-1)/(m-1)}}{\varphi(s-s_0)^{1/(m-1)}} ds < +\infty.$$

The case $m = n$ is considered similarly, just replacing Minkowski's Inequality with Fubini's Theorem.

Now we can apply Lemma 3.4 (and Remark 3.5) to the left hand side of (5). Recall that $\tilde{s}_0 = s_0$, $\tilde{s}_1 = s_4$, $\tilde{s}_3 := \sup\{\tilde{s}_1+3, (\psi_1^{-1}\circ\varphi_1)(\tilde{s}_1+3)\}$, and $\tilde{s}_4 := \sup\{\tilde{s}_3+\tilde{s}_0, (\varphi_1^{-1}\circ\psi_1)(\tilde{s}_1+3)\}$. (Here and below, in the previous definitions just replace the functions φ and ψ with the functions φ_1 and ψ_1, respectively.) Write moreover $s_4^* := \sup\{\tilde{s}_4, (\psi^{-1}\circ\varphi)(s_4)\}$, say. Since $u(\xi,\eta) \geq M \geq s_4^* \geq \tilde{s}_4$ for all $(\xi,\eta) \in B^m(x_0,r) \times B^n(y_0,r)$, we obtain, using (5),

$$\Psi(u(\xi,\eta)) = \left[\int_{(\varphi_1^{-1}\circ\psi_1)(u(\xi,\eta))-2}^{+\infty} \frac{ds}{\varphi_1(s-s_0)^{1/(n-1)}} \right]^{1-n}$$

(8)
$$\leq \frac{C}{r^n} \int_{B^n(\eta,r)} \Phi_2(u(\xi,y))dm_n(y)$$

$$\leq \frac{C}{r^{m+n}} \int_{B^m(x_0,2r)\times B^n(y_0,2r)} \psi(u(x,y))dm_{m+n}(x,y).$$

Here now $\Psi(= \Phi_2^{\varphi_1,\psi_1}) : [0,+\infty) \to [0,+\infty)$,

$$\Psi(t)(= \Phi_2^{\varphi_1,\psi_1}(t)) := \begin{cases} \left[\int_{(\varphi_1^{-1}\circ\psi_1)(t)-2}^{+\infty} \frac{ds}{\varphi_1(s-s_0)^{1/(n-1)}} \right]^{1-n} & , \text{ when } t \geq \tilde{s}_3, \\ \frac{t}{\tilde{s}_3}\Psi(\tilde{s}_3), & \text{ when } 0 \leq t < \tilde{s}_3, \end{cases}$$

see Remark 3.5 above. From (8), from the facts that $(\varphi_1^{-1}\circ\psi_1)(t) = (\varphi^{-1}\circ\psi)(t) \to +\infty$ as $t \to +\infty$, from (7), and from the fact that

$$\int_{B^m(x_0,2r)\times B^n(y_0,2r)} \psi(u(x,y))dm_{m+n}(x,y) < +\infty,$$

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one sees that u must be bounded above in $B^m(x_0, r) \times B^n(y_0, r)$, concluding the proof. □

4.3. Corollary. *Let Ω be a domain in $\mathbb{R}^{m+n}$, $m \geq n \geq 2$, and let $K \geq 1$. Let $u : \Omega \to [-\infty, +\infty)$ be a Lebesgue measurable function. Suppose that the following conditions are satisfied:*

(a) *For each $y \in \mathbb{R}^n$ the function*
$$\Omega(y) \ni x \mapsto u(x, y) \in [-\infty, +\infty)$$
is K-quasi-nearly subharmonic.

(b) *For each $x \in \mathbb{R}^m$ the function*
$$\Omega(x) \ni y \mapsto u(x, y) \in [-\infty, +\infty)$$
is K-quasi-nearly subharmonic.

(c) *There is a strictly increasing surjection $\varphi : [0, +\infty) \to [0, +\infty)$ such that*

(c1) $\displaystyle\int\limits_{s_0+1}^{+\infty} \frac{s^{(n-1)/(m-1)}}{\varphi(s-s_0)^{1/(m-1)}} ds < +\infty$ *for some $s_0 \in \mathbb{N}$,*

(c2) $\varphi(\log^+ u^+) \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.

Then u is quasi-nearly subharmonic in Ω.

Proof. Just choose $\psi = \varphi \circ \log^+$ and apply Theorem 4.2. □

4.4. Remark. One sees easily that the condition (c1) can be replaced by the condition

(c1') $\displaystyle\int\limits_{1}^{+\infty} \frac{s^{(n-1)/(m-1)}}{\varphi(s)^{1/(m-1)}} ds < +\infty$.

4.5. Corollary. *Let Ω be a domain in $\mathbb{R}^{m+n}$, $m \geq n \geq 2$, and let $K \geq 1$. Let $u : \Omega \to [-\infty, +\infty)$ be a Lebesgue measurable function. Suppose that the following conditions are satisfied:*

(a) *For each $y \in \mathbb{R}^n$ the function*
$$\Omega(y) \ni x \mapsto u(x, y) \in [-\infty, +\infty)$$
is K-quasi-nearly subharmonic.

(b) *For each $x \in \mathbb{R}^m$ the function*
$$\Omega(x) \ni y \mapsto u(x, y) \in [-\infty, +\infty)$$
is K-quasi-nearly subharmonic.

(c) *There is a strictly increasing surjection $\varphi : [0, +\infty) \to [0, +\infty)$ such that*

(c1) $\displaystyle\int\limits_{s_0+1}^{+\infty} \frac{s^{(n-1)/(m-1)}}{\varphi(s-s_0)^{1/(m-1)}} ds < +\infty$ *for some $s_0 \in \mathbb{N}$,*

(c2) $\varphi(\log(1 + (u^+)^r)) \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$ *for some $r > 0$.*

Then u is quasi-nearly subharmonic in Ω.

Proof. It is easy to see that the assumptions of Theorem 4.2 are satisfied. We leave the details to the reader. □

4.6. Next our slight improvement to Armitage's and Gardiner's original result:

4.7. Corollary. *Let Ω be a domain in $\mathbb{R}^{m+n}$, $m \geq n \geq 2$. Let $u : \Omega \to [-\infty, +\infty)$ be such that the following conditions are satisfied:*

(a) *For each $y \in \mathbb{R}^n$ the function*
$$\Omega(y) \ni x \mapsto u(x, y) \in [-\infty, +\infty)$$
is subharmonic.

(b) *For each $x \in \mathbb{R}^m$ the function*

$$\Omega(x) \ni y \mapsto u(x,y) \in [-\infty, +\infty)$$

is subharmonic.

(c) *There is a strictly increasing surjection $\varphi : [0, +\infty) \to [0, +\infty)$ such that*

(c1) $\displaystyle\int\limits_1^{+\infty} \frac{s^{(n-1)/(m-1)}}{\varphi(s)^{1/(m-1)}} ds < +\infty,$

(c2) $\varphi(\log^+[(u^+)^r]) \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$ *for some $r > 0$.*

Then u is subharmonic in Ω.

Proof. By [Ri07$_3$, Proposition 2 (v), (vi), and Proposition 1 (iv) (Proposition 2.2 (v), (vi), and Proposition 2.1 (iv), p. 55)], see also [Ri06$_1$, Lemma 2.1, p. 32], $(u^+)^r$ satisfies the assumptions of Corollary 4.3. Thus $(u^+)^r$ is quasi-nearly subharmonic in Ω, and therefore e.g. by [Ri07$_3$, Proposition 2 (iii) (Proposition 2.2 (iii), p. 55)] locally bounded above. Hence also u is locally bounded above, and thus subharmonic in Ω, by [Ri89, Theorem 1, p. 69], say. $\qquad\square$

Added 16 Oct 2008: For a further, still slightly improved version of the above Corollary 4.7, see [Ri08, Corollary 3.3.3].

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