The arrangement field theory (AFT) Part 2

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Abstract

In this work we apply the formalism developed in the previous paper ("The arrangement field theory") to describe the content of standard model plus gravity. The resulting scheme finds an analogue in supersymmetric theories but now all quarks and leptons take the role of gauginos for $E6$ gauge fields. Moreover we discover a triality between *Arrangement Field Theory*, *String Theory* and *Loop Quantum Gravity*, which appear as different manifestations of the same theory. Finally we show as three families of fields arise naturally and we discover a new road toward unification of gravity with gauge and matter fields.

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Contents

1 Introduction

The arrangement field paradigm describes universe by means of a graph, ie an ensemble of vertices and edges. However there is a considerable difference between this framework and the usual modeling with spin-foams or spin-networks. The existence of an edge which connects two vertices is in fact probabilistic. In this framework the fundamental quantity is an invertible matrix M with dimension $n \times n$, where n is the number of vertices. In the entry ij of such matrix we have a quaternionic number which gives the probability amplitude for the existence of an edge connecting vertex i to vertex j. In the introductory work [1] we have developed a simple scalar field theory in this probabilistic graph (we call it "non-ordered space"). We have seen that a space-time metric emerges spontaneously when we fix an ensemble of edges. Moreover, the quantization of metric descends naturally from quantization of M in the non-ordered space. In section 2 we summarize these results.

In section 3 we express Ricci scalar as a simple quadratic function of M. We discover how the gravitational field emerges from diagonal components of M, in contrast to gauge fields which come out from non-diagonal components.

In section 4 we define a quartic function of M which develops a Gauss Bonnet term for gravity and the usual kinetic term for gauge fields.

In section 5 we discover a triality between *Arrangement Field Theory*, *String Theory* and *Loop Quantum Gravity* which appear as different manifestations of the same theory.

In section 6 we show that a grassmanian extension of M generates automatically all known fermionic fields, divided exactly in three families. We see how gravitational field exchanges homologous particles in different families. The resulting scheme finds an analogue in supersymmetric theories, with known fermionic fields which take the role of gauginos for known bosons.

In the subsequent sections we explore some practical implications of arrangement field theory, in connection to inflation, dark matter and quantum entanglement. Moreover we explain how deal with theory perturbatively by means of Feynman diagrams.

We warmly invite the reader to see introductory work [1] before proceeding.

2 Formalism

In paper [1] we have considered an euclidean 4-dimensional space represented by a graph with n vertices. In this section we retrace the fundamental results of that work, moving to Lorentzian spaces in the next section. Since now we assume the Einstein convention, summing over repeated indices.

In proof of **theorem 8** in [1] we have demonstrated the equivalence between the following actions:

$$S_1 = (M\varphi)^\dagger (M\varphi) \tag{1}$$

$$S_2 = \sum_{i=1}^{n} \sqrt{|h|}h^{\mu\nu}(x^i)(\nabla_\mu \varphi^i)^*(\nabla_\nu \varphi^i). \tag{2}$$

M is any invertible matrix $n \times n$ while the field φ is represented by a column array $1 \times n$, with an entry for every vertex in the graph:

$$\varphi = \begin{pmatrix} \varphi(x^1) \\ \varphi(x^2) \\ \varphi(x^3) \\ \vdots \\ \varphi(x^N) \end{pmatrix}. \tag{3}$$

The entries of both M and ϕ take values in the division ring of quaternions, usually indicated with $\mathbf{H}$. The first action considers the universe as an abstract ensemble of vertices, numbered from 1 to n, where n is the total number of space vertices. The entry (ij) in the matrix M represents the probability amplitude for the existence of an edge which connects the vertex number i to the vertex number j. We admit non-commutative geometries, which in this framework implies a possible inequivalence $|M^{ij}| \neq |M^{ji}|$. More, the first action is invariant under transformations $(U_1, U_2) \in U(n, \mathbf{H}) \otimes U(n, \mathbf{H})$ which send M in $U_2 M U_1^\dagger$.

In action (2) a covariant derivative for $U(n, \mathbf{H}) \otimes U(n, \mathbf{H})$ appears, represented by a skew hermitian matrix ∇ which expands according to $\nabla_\mu = \tilde{M}_\mu + A_\mu$. Here $\tilde{M}_\mu$ is a linear operator such that $lim_{\Delta \to 0} \tilde{M}_\mu = \partial_\mu$, where Δ is the graph step. If we number the space vertices along direction μ, $\tilde{M}_\mu$ becomes

$$\tilde{M}_\mu^{ij} = \frac{1}{2\Delta} \left[\delta^{(i+1)j} - \delta^{(i-1)j} \right] \tag{4}$$

$$\sum_j \tilde{M}^{ij} \varphi^j = \frac{1}{2\Delta} \sum_j \delta^{(i+1)j} \varphi^j - \delta^{(i-1)j} \varphi^j = \frac{\varphi(i+1) - \varphi(i-1)}{2\Delta}.$$

The gauge fields A act as skew hermitian matrices too:

$$A = (A^{ij}) = (A(x^i, x^j))$$

$$(A\phi)^i = A^{ij} \phi^j.$$

In proof of **theorem 5** we have discovered that for every normal matrix $\hat{M}$, which is neither hermitian nor skew hermitian, four couples (U_1, D^μ) exist, with U_1 unitary and D^μ diagonal, such that

$$U_1^\dagger D^\mu \nabla_\mu U_1 = \hat{M} \tag{5}$$

$$\sqrt{|h|} h^{\mu\nu}(x^i) = \frac{1}{2} d_i^{*\mu} d_i^\nu + c.c. \qquad D_\mu^{ij} = d_i^\mu \delta^{ij}. \tag{6}$$

Here h is a non degenerate metric while the first relation determines uniquely the values of gauge fields. The matrices ∇_μ, U_1, D^μ act on field arrays via matricial product and the ensemble of four couples (U_1, D^μ) is called "space arrangement".

Further, in proof of **theorem 6**, we have seen that for every invertible matrix M we can always find an unitary transformation U_M and a normal matrix $\hat{M}$, which is neither hermitian nor skew hermitian, such that $M = U_M \hat{M}$. If we define $U_2 = U_1 U_M^\dagger$, we have

$$M^\dagger M = \hat{M}^\dagger \hat{M} \tag{7}$$

$$U_2^\dagger D^\mu \nabla_\mu U_1 = M. \tag{8}$$

It's sufficient to substitute (8) in (1) to verify its equivalence with (2). We have called $\hat{M}$ the "associated normal matrix" of M.

The action of a transformation (U_1, U_2) on ∇ follows from its action on M. We can always use the invariance under $U(n, \mathbf{H}) \otimes U(n, \mathbf{H})$ to put M in the form $M = D^\mu \nabla_\mu$. Starting from this we have

$$U_2 M U_1^\dagger = U_2 D^\mu \nabla_\mu U_1^\dagger = U_2 D^\mu U_1^\dagger U_1 \nabla_\mu U_1^\dagger.$$

We define $\nabla' = U_1 \nabla_\mu U_1^\dagger$ the transformed of ∇ under (U_1, U_2) and $D'^\mu = U_2 D^\mu U_1^\dagger$ the transformed of D^μ. We assume that A_μ inside ∇_μ transforms correctly as a gauge field, so that

$$\nabla[A]_\mu \phi = \nabla[A] U_1^\dagger \phi' = U_1^\dagger \nabla[A_{U1}]_\mu \phi'$$

$$\phi' = U_1 \phi.$$

We want D'^μ remain diagonal and $h' = h[D'] = h[D]$. In this case there are two relevant possibilities:

1. D is a matrix made by blocks $m \times m$ with m integer divisor of n and every block proportional to identity. In this case the residual symmetry is $U(1, \mathbf{H})^n \times U(m, \mathbf{H})^{n/m}$ with elements (sV, V), s both diagonal and unitary, $V \in U(m, \mathbf{H})^{n/m}$;

2. h is any diagonal matrix. The symmetry reduces to $U(1, \mathbf{H})^n \otimes U(1, \mathbf{H})^n$ which is local $U(1, \mathbf{H}) \otimes U(1, \mathbf{H}) \sim SU(2) \otimes SU(2) \sim SO(4)$.

In this way, if we keep fixed the metric h and keep diagonal D, the action (2) will be invariant at least under $U(1,\mathbf{H})^n \otimes U(1,\mathbf{H})^n$ which doesn't modify h.

We have supposed that a potential for M breaks the $U(n,\mathbf{H}) \otimes U(n,\mathbf{H})$ symmetry in $U(1,\mathbf{H})^n \otimes U(m,\mathbf{H})^{n/m}$ where m is an integer divisor of n. We'll see in fact that the more natural potential has the form $tr\,(\alpha M^\dagger M - \beta M^\dagger M M^\dagger M)$, known as "mexican hat potential". This potential is a very typical potential for a spontaneous symmetry breaking. In this way all the vertices are grouped in n/m ensembles $\mathcal{U}^a$:

$$\mathcal{U}^a = \{x_1^a, x_2^a, x_3^a, \ldots, x_m^a\}$$

$$\varphi = (\varphi(x_i^a)) = \begin{pmatrix} \varphi(x_1^1) & \varphi(x_2^1) & \varphi(x_3^1) & \ldots & \varphi(x_m^1) \\ \varphi(x_1^2) & \varphi(x_2^2) & \varphi(x_3^2) & \ldots & \varphi(x_m^2) \\ \varphi(x_1^3) & \varphi(x_2^3) & \varphi(x_3^3) & \ldots & \varphi(x_m^3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi(x_1^{n/m}) & \varphi(x_2^{n/m}) & \varphi(x_3^{n/m}) & \ldots & \varphi(x_4^{n/m}) \end{pmatrix} \tag{9}$$

$$A = (A_{ij}^{ab}) = (A(x_i^a, x_j^b)).$$

Now the indices a, b of A act on the columns of φ, while the indices i, j act on the rows. The fields A_{ij}^{ab} with $a = b$ maintain null masses and then they continue to behave as gauge fields for $U(m,\mathbf{H})^{n/m}$. Every $U(m,\mathbf{H})$ term in $U(m,\mathbf{H})^{n/m}$ acts independently inside a single $\mathcal{U}^a$. So, if we consider the ensembles $\mathcal{U}^a$ as the "real" physical points, we can interpret $U(m,\mathbf{H})^{n/m}$ as a local $U(m,\mathbf{H})$.

It's simple to verify:

$$h^{\mu\nu}(x_i^a) = h^{\mu\nu}(x_j^a) \qquad \forall x_i^a, x_j^a \in \mathcal{U}^a$$

$$h^{\mu\nu}(x^a) \stackrel{!}{=} h^{\mu\nu}(\mathcal{U}^a) = h^{\mu\nu}(x_i^a) \qquad \forall x_i^a \in \mathcal{U}^a$$

$$A_{ij}(x^a) \stackrel{!}{=} Tr\,\left[A(x^a)T^{ij}\right], \quad where$$

$$A(x^a) = \sum_{ij} A(x_i^a, x_j^a) T^{ij}, \quad \text{with } T^{ij} \text{ generator of } U(m, \mathbf{H}) \tag{10}$$

3 Ricci scalar in the arrangement field paradigm

3.1 Hyperions

In this subsection we define an extension of $\mathbf{H}$ by inserting a new imaginary unit I. It satisfies:

$$I^2 = -1 \qquad I^\dagger = -I$$

$$[I, i] = [I, j] = [I, k] = 0$$

In this way a generic number assumes the form

$$v = a + Ib + ic + jd + ke + iIf + jIg + kIh, \qquad a, b, c, d, e, f, g, h \in \mathbf{R}$$

$$v = p + Iq, \qquad p, q \in \mathbf{R}$$

We call this numbers "Hyperions" and indicate their ensemble with Y. It's straightforward that such numbers are in one to one correspondence with even products of Gamma matrices. Explicitly:

$$1 \Leftrightarrow \gamma_0\gamma_0 = 1 \qquad I \Leftrightarrow \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$$

$$i \Leftrightarrow \gamma_2\gamma_1 \qquad iI \Leftrightarrow \gamma_0\gamma_3$$

$$j \Leftrightarrow \gamma_1\gamma_3 \qquad jI \Leftrightarrow \gamma_0\gamma_2$$

$$k \Leftrightarrow \gamma_3\gamma_2 \qquad kI \Leftrightarrow \gamma_0\gamma_1$$

Note that imaginary units i, j, k, iI, jI, kI satisfy the Lorentz algebra, with i, j, k which describe rotations and iI, jI, kI which describe boosts.

Definition 1 (bar-conjugation) *The bar-conjugation is an operation which exchanges I with $-I$ (or γ_0 with $-\gamma_0$ in the γ-representation). Explicitly, if $v = a + Ib + ic + jd + ke + iIf + jIg + kIh$ with $a, b, c, d, e, f, g, h \in \mathbf{R}$, then $\bar{v} = a - Ib + ic + jd + ke - iIf - jIg - kIh$.*

Definition 2 (pre-norm) *The pre-norm is a complex number with I as imaginary unit (we say "I-complex number"). Given an hyperion v, its pre-norm is $|v| = (\bar{v}^\dagger v)^{1/2}$. If $v \in \mathbf{H}$, its pre-norm coincides with usual norm $(v^\dagger v)^{1/2}$.*

Note that every hyperion v can be written in the polar form

$$v = |v|e^{ia+jb+kc+iId+jIe+kIf} \qquad a, b, c, d, e, f$$

$$|v|^2 = \bar{v}^\dagger v = |v|e^{-(ia+jb+kc+iId+jIe+kIf)}|v|e^{ia+jb+kc+iId+jIe+kIf} = |v|^2.$$

If M takes values in $\mathbf{Y}$, the probability for the existence of an edge (ij) can be defined as $||M^{ij}||$, which is the norm of pre-norm.

Remark 3 (Spectral theorem in Y)

The fundamental relation (5) descends uniquely from spectral theorem in $\mathbf{H}$. You can see from work of Yongge Tian [2] that spectral theorem is still valid in Y in the following form: "Every normal matrix M with entries in $\mathbf{Y}$ is diagonalizable by a transformation $U \in U(n, \mathbf{Y})$ which sends M in $UM\bar{U}^\dagger$". Here $U(n, \mathbf{Y})$ is the exponentiation of $u(n, \mathbf{Y}) = u(n, \mathbf{H}) \cup Iu(n, \mathbf{H})$ and M satisfies a generalized normality condition. Explicitly, $\bar{U}^\dagger = U^{-1}$ and $\bar{M}^\dagger M = M\bar{M}^\dagger$. This implies that (5) is valid too in the form

$$\bar{U}^\dagger D^\mu \nabla_\mu U = M$$

Matrix ∇ is now in $u(n, \mathbf{Y})$ and then it satisfies $\bar{\nabla}^\dagger = -\nabla$. Accordingly, its diagonal entries belong to Lorentz algebra (they don't comprise real and I-imaginary components).

To conclude, we don't know if an associate normal matrix exists for any invertible matrix with entries in $\mathbf{Y}$*. Fortunately, in lorentzian spaces there is no reason for using such machinery and we can start from the beginning with a normal arrangement matrix.*

Remark 4 (gauge fixing)

It follows from spectral theorem that eigenvalues λ *of* M *are equivalence classes*

$$\lambda \sim s\lambda\bar{s}^\dagger \qquad s \in \mathbf{Y}, \bar{s}^\dagger s = 1.$$

As a consequence, we can choose freely the diagonal matrix D *inside the equivalence class* $SD\bar{S}^\dagger$*, where* S *is both diagonal and unitary (*$\bar{S}^\dagger = S^{-1}$*). This choice does't affect the metric* $\sqrt{h}h^{\mu\nu} = Re\,(\bar{D}^{\dagger\mu} D^\nu)$*, granting for the persistence of a symmetry* $U(1,Y)^n = SO(1,3)^n$*, ie local* $SO(1,3)$*. Clearly this is a reworking of the usual gauge symmetry which acts on the tetrads, sending* e_a^μ *in* ${\Lambda_a}^b e_b^\mu$ *via the lorentz transformation* Λ*. In what follows we exploit* $SO(1,3)$*-symmetry to satisfy two conditions:*

$$\begin{aligned} &tr\,(\{\bar{\nabla}^\dagger_\mu, \nabla_\nu\}\bar{D}^{\dagger\mu} D^\nu) = 0 \\ &tr\,(D^\beta\{\nabla_\beta, \bar{\nabla}^\dagger_\mu\}\bar{D}^{\dagger\mu} D^\nu \nabla_\nu \bar{\nabla}^\dagger_\alpha \bar{D}^{\dagger\alpha}) = 0 \end{aligned} \tag{11}$$

Note that these are global conditions because operator tr *is analogous to a spacetime integration.*

3.2 Ricci scalar with hyperions

In this subsection we simplify the form of Ricci scalar by means of hyperions, in order to make it suitable for the arrangement field formalism. Given a gauge field ω_μ in $so(1,3)$ and a complex tetrad e^μ, we define

$$A_\mu = \omega_\mu^{ab}\gamma_a\gamma_b \qquad\qquad h^{\mu\nu} = Re\,(e_a^{\dagger\mu}e_b^\nu\eta^{ab}) \tag{12}$$

$$d^\mu = \sqrt{e}e^{\mu a}\gamma_0\gamma_a \qquad e = \left[det(-e_a^{\dagger\mu}e_b^\nu\eta^{ab})\right]^{-1/2} \in \mathbf{R}^+$$

$$\bar{d}^\mu = d^\mu(\gamma_0 \to -\gamma_0)$$

$$\Rightarrow \bar{d}^{\dagger\mu}d^\nu = ee^{\dagger\mu a}e^{\nu b}\gamma_a\gamma_b \quad \Rightarrow \sqrt{h}h^{\mu\nu} = \frac{1}{4}Re\,\left[tr(\bar{d}^{\dagger\mu}d^\nu)\right]$$

Note that our definitions are the same to require $\bar{A}^\dagger = -A$ in the hyperions framework. The Ricci scalar can be written as

$$\sqrt{h}R(x) = -\frac{1}{8}tr\left(\left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right)\bar{d}^{\dagger\mu}d^\nu\right)$$

To verify its correctness we expand first the commutator

$$\begin{aligned}
[A_\mu, A_\nu] &= \omega_\mu^{ab}\omega_\nu^{cd}\left(\gamma_a\gamma_b\gamma_c\gamma_d - \gamma_c\gamma_d\gamma_a\gamma_b\right) \\
&= \frac{1}{2}\omega_\mu^{ab}\omega_\nu^{cd}\left(\gamma_a\{\gamma_b,\gamma_c\}\gamma_d - \gamma_c\{\gamma_d,\gamma_a\}\gamma_b\right) + \\
&\quad + \frac{1}{2}\omega_\mu^{ab}\omega_\nu^{cd}\left(\gamma_a[\gamma_b,\gamma_c]\gamma_d - \gamma_c[\gamma_d,\gamma_a]\gamma_b\right) \\
&= \left(\omega_\mu^{ab}\omega_{b\nu}^{\ \ d} - \omega_\nu^{ab}\omega_{b\mu}^{\ \ d}\right)(\gamma_a\gamma_d) + \\
&\quad + \frac{1}{4!}\omega_\mu^{ab}\omega_\nu^{cd}\left(\varepsilon_{abcd}\varepsilon^{efgh}\gamma_e\gamma_f\gamma_g\gamma_h\right) \\
&= [\omega_\mu, \omega_\nu]^{ab}\gamma_a\gamma_b + \omega_\mu^{ab}\omega_{ab\nu}^{(D)}\gamma_5
\end{aligned} \tag{13}$$

In the last line we have defined $\omega_{ab\nu}^{(D)} = \varepsilon_{abcd}\omega_\nu^{cd}$. Hence

$$\begin{aligned}
R(x) &= -\frac{1}{8}tr(\gamma_a\gamma_b\gamma_c\gamma_d)\left(\partial_\mu\omega_\nu^{ab} - \partial_\nu\omega_\mu^{ab} + [\omega_\mu,\omega_\nu]^{ab}\right)e^{\dagger c\mu}e^{d\nu} - \\
&\quad - \frac{1}{8}tr(\gamma_5\gamma_b\gamma_c)\omega_\mu^{ab}\omega_{ab\nu}^{(D)}e^{\dagger c\mu}e^{d\nu}
\end{aligned} \tag{14}$$

Consider now the relations

$$\frac{1}{4}tr(\gamma_a\gamma_b\gamma_c\gamma_d) = \eta_{ab}\eta_{cd} - \eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc}$$

$$tr(\gamma_5\gamma_b\gamma_c) = 0$$

We obtain

$$R(x) = \left(\partial_\mu\omega_\nu^{ab} - \partial_\nu\omega_\mu^{ab} + [\omega_\mu, \omega_\nu]^{ab}\right) e_a^{\dagger\mu} e_b^\nu$$

which is the usual definition. We can move freely from matrices γ to hyperions, substituting tr with 4. In this way

$$\begin{aligned}
\sqrt{h}R(x) &= -\frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right) \bar{d}^{\dagger\mu} d^\nu \\
&= -\frac{1}{2}[\nabla_\mu, \nabla_\nu]\bar{d}^{\dagger\mu} d^\nu
\end{aligned}$$

$$\nabla_\mu = \partial_\mu + A_\mu \qquad\qquad A_\mu, d^\mu \in \mathbf{Y}$$

$$e_a^\mu = Re\, e_a^\mu + I\, Im\, e_a^\mu$$

$$\begin{aligned}
d^\mu &= Re\, e^{\mu 0} + iI\, Re\, e^{\mu 3} + jI\, Re\, e^{\mu 2} + kI\, Re\, e^{\mu 1} + \\
&\quad + I\, Im\, e^{\mu 0} - i\, Im\, e^{\mu 3} - j\, Im\, e^{\mu 2} - k\, Im\, e^{\mu 1}
\end{aligned}$$

3.3 Ricci scalar in the new paradigm

We try to define Hilbert-Einstein action as

$$S_{HE} = tr\,(\bar{M}^\dagger M).$$

We insert in S_{HE} the usual expansion $M = UD^\mu\nabla_\mu\bar{U}^\dagger$, obtaining

$$\begin{aligned}
S_{HE} &= tr\,[(\bar{U}\bar{D}^{\dagger\mu}\bar{\nabla}_\mu U^\dagger)^\dagger(UD^\nu\nabla_\nu\bar{U}^\dagger)] \\
&= tr\,[U\bar{\nabla}_\mu^\dagger\bar{D}^{\dagger\mu}\bar{U}^\dagger UD^\nu\nabla_\nu\bar{U}^\dagger] \\
&= tr\,[\nabla_\nu\bar{\nabla}_\mu^\dagger\bar{D}^{\dagger\mu}D^\nu]. \qquad (15)
\end{aligned}$$

Now we can impose the first condition in (11) which gives

$$S_{HE} = \frac{1}{2} tr\,\{[\nabla_\nu, \bar{\nabla}^\dagger_\mu]\bar{D}^{\dagger\mu} D^\nu\}. \tag{16}$$

Expanding the covariant derivatives we obtain

$$\begin{aligned} S_{HE} &= \frac{1}{2}\sum_{a,b,c}\{\partial^\dagger_\mu A_\nu(x^a,x^b) - \partial_\nu \bar{A}^\dagger_\mu(x^a,x^b) + \\ &\qquad +[\bar{A}^\dagger_\mu, A_\nu](x^a,x^b)\}\bar{d}^{\dagger\mu}(x^b)\delta^{bc}d^\nu(x^c)\delta^{ca} \\ &= \frac{1}{2}\sum_{a}\{\partial^\dagger_\mu A_\nu(x^a) - \partial_\nu \bar{A}^\dagger_\mu(x^a) + \\ &\qquad +[\bar{A}^\dagger_\mu, A_\nu](x^a,x^a)\}\bar{d}^{\dagger\mu}(x^a)d^\nu(x^a) \\ &= \frac{1}{2}\sum_{a,b\neq a}\{\partial^\dagger_\mu A_\nu(x^a) - \partial_\nu \bar{A}^\dagger_\mu(x^a) + [\bar{A}^\dagger_\mu(x^a), A_\nu(x^a)] + \\ &\qquad +[\bar{A}^\dagger_\mu(x^a,x^b), A_\nu(x^b,x^a)]\}\cdot \bar{d}^{\dagger\mu}(x^a)d^\nu(x^a) \end{aligned} \tag{17}$$

Consider now a symmetry breaking with residual group $U(m,\mathbf{Y})^{n/m}$ which regroups vertices in ensembles $\mathcal{U}^a = \{x^a_1, x^a_2, \ldots, x^a_m\}$. We assume that fields $A(x^a_i, x^b_j)$ with $a \neq b$ acquire big masses and thus we can neglect them. The symbol $\sum_a$ becomes $\sum_{a,i}$, while $\sum_{a,b\neq a}$ becomes $\sum_{a,i,b,j|(a,i)\neq(b,j)}$. After neglecting heavy fields, the last one is simply $\sum_{a,i,j\neq i}$.

$$\begin{aligned} S_{HE} &= \frac{1}{2}\sum_{a}\{\partial^\dagger_\mu tr\, A_\nu(x^a) - \partial_\nu tr\, \bar{A}^\dagger_\mu(x^a) + [tr\,\bar{A}^\dagger_\mu(x^a), tr\, A_\nu(x^a)] + \\ &\qquad + \sum_{i,j\neq i}[\bar{A}^{\dagger ij}_\mu(x^a)A^{ji}_\nu(x^a) - A^{ij}_\nu(x^a)\bar{A}^{\dagger ji}_\mu](x^a)\}\cdot \bar{d}^{\dagger\mu}(x^a)d^\nu(x^a) \end{aligned} \tag{18}$$

For what follows we write $S_{HE} = \frac{1}{2}\sum_a R^{ik}_{\mu\nu}\delta^{ik}\bar{d}^{\dagger\mu}d^\nu$ with

$$R^{ik}_{\mu\nu} = \partial^{\dagger}_{\mu} tr\, A_{\nu}(x^a) - \partial_{\nu} tr\, \bar{A}^{\dagger}_{\mu}(x^a) + [tr\, \bar{A}^{\dagger}_{\mu}(x^a), tr\, A_{\nu}(x^a)] + \\ + \sum_{i,j\neq i,k\neq j} [\bar{A}^{\dagger ij}_{\mu}(x^a) A^{jk}_{\nu}(x^a) - A^{ij}_{\nu}(x^a)\bar{A}^{\dagger jk}_{\mu}](x^a). \qquad (19)$$

$R^{ik}_{\mu\nu}$ is a generalization of curvature tensor. We have indicated with $tr\, A$ the track on ij, ie $\delta^{ij} A^{ij}(x^a) = \delta^{ij} A(x^a_i, x^a_j)$. Note that $[\bar{A}^{\dagger ii}, A^{jj}] = 0$ when $i \neq j$ and then $\sum_i [\tilde{A}^{\dagger ii}_{\mu}, A^{ii}_{\nu}] = \sum_{ij} [\bar{A}^{\dagger ii}_{\mu}, A^{jj}_{\nu}] = [tr\, \bar{A}^{\dagger}_{\mu}, tr\, A_{\nu}]$. Consider now any skew hermitian matrix W_{μ} with elements $W^{ij}_{\mu} = A^{ij}_{\mu}$ for $i \neq j$ and $W^{ij}_{\mu} = 0$ for $i = j$. It belongs to the subalgebra of $u(m, \mathbf{Y})$ made by all null track generators. This means that commutators between null track generators are null track generators too. In this way

$$\sum_{i,i\neq j} [\bar{A}^{\dagger}_{\mu}(x^i, x^j), A_{\nu}(x^j, x^i)] = tr[\bar{W}^{\dagger}_{\mu}, W_{\nu}] = 0.$$

Hence we can delete the mixed term in S_{EH}.

$$S_{HE} = \frac{1}{2} \sum_a \{\partial^{\dagger}_{\mu} tr\, A_{\nu}(x^a) - \partial_{\nu} tr\, \bar{A}^{\dagger}_{\mu}(x^a) + [tr\, \bar{A}^{\dagger}_{\mu}(x^a), tr\, A_{\nu}(x^a)]\} \cdot \\ \cdot \bar{d}^{\dagger\mu}(x^a) d^{\nu}(x^a)$$

In the arrangement field paradigm, the operator $\dagger$ transposes also rows with columns in matrices which represent ∂ and A. As we have seen, the fields A which intervene in R are only the diagonal ones, so the transposition of rows with columns is trivial. Note that ∇ satisfies a generalized condition of skew-hermiticity ($\bar{\nabla}^{\dagger} = -\nabla$) and then its diagonal components belong to lorentz algebra. This implies $tr\, \bar{A}^{\dagger} = -tr\, A$, matching exactly with our request in (12). Finally, if we consider the matrix which represents ∂ (we have called it $\tilde{M}$), we note that $\bar{\partial}^{\dagger} = \partial^T = -\partial$. Explicitly

$$\nabla^{\dagger}_{\nu} = (\partial_{\nu} + tr\, \bar{A}_{\nu})^{\dagger} = \partial^{\dagger}_{\nu} + tr\, \bar{A}^{\dagger}_{\nu} = -\partial_{\nu} - tr\, A_{\nu} = -\nabla_{\nu}.$$

Applying this to S_{HE},

$$\begin{aligned}
S_{HE} &= -\frac{1}{2}\sum_a \{\partial_\mu tr\, A_\nu(x^a) - \partial_\nu tr\, A_\mu(x^a) + [tr\, A_\mu(x^a), tr\, A_\nu(x^a)]\} \cdot \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \bar{d}^{\dagger\mu}(x^a) d^\nu(x^a) \\
&= -\frac{1}{2}[\overset{G}{\nabla}_\mu, \overset{G}{\nabla}_\nu]\bar{d}^{\dagger\mu}(x^a) d^\nu(x^a) \\
&= \sum_a \sqrt{h} R(x^a) \rightarrow \int d^4x\, \sqrt{h} R(x).
\end{aligned} \tag{20}$$

Here $\overset{G}{\nabla}$ is the gravitational covariant derivative $\overset{G}{\nabla} = \partial + tr\, A$. It's very remarkable that gauge fields in R are only the diagonal ones. First, this is the unique possibility to obtain $\overset{G}{\nabla}^\dagger{}_\nu = -\overset{G}{\nabla}_\nu$. Moreover, while gauge fields in R are tracks of matrices $(A_{ij})(x^a)$, we'll see as the other gauge fields in Standard Model correspond to non diagonal components.

4 The kinetic term

Until now we have obtained no terms which describe gauge interactions. In this section we find a such term, with the condition that it hasn't to change Einstein equations. One option is as follows:

$$\begin{aligned}
S_{GB} &= -tr\, (\bar{M}^\dagger M \bar{M}^\dagger M) \qquad (21)\\
&= -tr\, \left[U \bar{\nabla}^\dagger_\mu \bar{D}^{\dagger\mu} \bar{U}^\dagger U D^\nu \nabla_\nu \bar{U}^\dagger U \bar{\nabla}^\dagger_\alpha \bar{D}^{\dagger\alpha} \bar{U}^\dagger U D^\beta \nabla_\beta \bar{U}^\dagger\right] \\
&= -tr\, \left[\bar{\nabla}^\dagger_\mu \bar{D}^{\dagger\mu} D^\nu \nabla_\nu \bar{\nabla}^\dagger_\alpha \bar{D}^{\dagger\alpha} D^\beta \nabla_\beta\right]
\end{aligned}$$

We assume a residual symmetry under $U(m, \mathbf{Y})^{n/m}$. This means that D^μ are matrices made of blocks $m \times m$ where every block is a hyperionic multiple of identity. We use newly the correspondence between $(1, I, i, j, k, iI, jI, kI)$ and gamma matrices:

$$S_{GB} = -\frac{1}{4} tr(\gamma_a\gamma_b\gamma_c\gamma_d\gamma_e\gamma_f\gamma_g\gamma_h) \left[\nabla^a_\beta \bar{\nabla}^{b\dagger}_\mu \bar{D}^{\dagger c\mu} D^{d\nu} \nabla^e_\nu \bar{\nabla}^{f\dagger}_\alpha \bar{D}^{\dagger g\alpha} D^{h\beta}\right]$$

We use letters a, b, c, d for indices which run on Gamma matrices, α, β, μ, ν for spatial coordinates indices and ijk for gauge indices (ie indices which run inside a single $\mathcal{U}^a$). Pay attention to not confuse the index a in the first group with the index a which runs over the vertices like in x^a_i.

We will see that physical fields arise in three families, determined by the choice of a subspace inside Y. This is true both for fermionic and bosonic fields. Thus the indices with letters a, b, c, d run over the three families.

We proceed by imposing the second condition in (11), in such a way to ignore terms proportional to $\{\nabla_\beta, \bar{\nabla}^\dagger_\mu\}$ inside S_{GB}. We take

$$S_{GB} = \sum_a L_{GB}(x^a)$$

Then

$$\begin{aligned} L_{GB} &= R^{ij}_{ab\mu\beta} R^{abji}_{\nu\alpha} \bar{d}^{\dagger\mu}_c d^{c\nu} \bar{d}^{\dagger\alpha}_d d^{d\beta} - 4 R^{ij}_{ac\mu\beta} \bar{d}^{\dagger a\mu} R^{cbji}_{\nu\alpha} d^\alpha_b d^{d\beta} \bar{d}^{\dagger\alpha}_d + \\ &\quad + R^{ij}_{ac\mu\beta} \bar{d}^{\dagger a\mu} d^{c\beta} R^{cbji}_{\nu\alpha} \bar{d}^{\dagger\nu}_c d^\alpha_b \\ &= h R^{ij}_{ab\mu\beta} R^{abji\mu\beta} - 4h R^{ij}_{c\beta} R^{cji\beta} + h R^{ij} R^{ji} \end{aligned} \tag{22}$$

$R^{ij}_{\beta\mu}$ was defined in (19), while $\sqrt[4]{h} R^{ij}_\mu = R^{ij}_{\beta\mu} d^\beta$ and $\sqrt{h} R^{ij} = R^{ij}_{\beta\mu} d^\beta d^{*\mu}$. You understand in a moment that for $i \neq j$ we have $R^{ij}_{ac\beta\mu} R^{jiac}_{\nu\alpha} h^{\mu\alpha} h^{\nu\beta} = tr \sum_{(ac)} F^{(ac)}_{\mu\nu} F^{(ac)\mu\nu}$. The index (ac) runs over three fields families and $F_{(ac)\mu\nu}$ is a strength field tensor. In this way the terms $R^{ij\nu}_\beta R^{ji\beta}_\nu$ and $R^{ij} R_{ji}$ are terms which mix families.

The trouble with S_{GB} is that it generates a factor h instead of $\sqrt{h}$. However, we can solve the problem imposing the gauge condition $h = 1$. Note that for $i = j$ we have

$$L_{GB} = R_{ac\beta\mu} R^{ac\beta\mu} + R^2 - 4 R^\alpha_\mu R^\mu_\alpha$$

which is a topological term and it doesn't change the Einstein equations.

Remark 5 (symmetry breaking) *The combination of S_{HE} and S_{GB} gives to gravitational gauge field $\overset{G}{A}$ a potential with form*

$$\overset{G}{A}{}^2 - \overset{G}{A}{}^4.$$

This potential has non trivial minimums which imply a non-trivial expectation value for $\overset{G}{A}$. Moreover, inside S_{GB} we find the following kind of terms for other fields A:

$$\langle \overset{G}{A}{}^2 \rangle A^2 - A^4.$$

In this way we have a mass for gauge fields A and another potential with non-trivial minimums. Therefore, also gauge fields A have non-trivial expectation values. Finally, such expectation values give mass to fermionic fields via terms

$$\psi^\dagger \langle A \rangle \psi.$$

There is no need for a scalar Higgs boson.

5 Connections with Strings and Loop Gravity

We have seen in [1], at **Remark 13**, that some similarities exist between diagonal components of M (loops) and closed strings in string theory. Now we have discovered that such diagonal components describe a gravitational field. Is then a case that the lower energy state for closed string is the graviton? We think no. Moreover, we have seen that gauge fields correspond to non-diagonal components of M, ie open edge in the graph. This finds also a connection with open strings, whose lower energy states are gauge fields. We have shown that a symmetry $U(m, \mathbf{Y})$ arises when vertices are grouped in ensembles $\mathcal{U}^a$ containing m vertices. This seems to represent a superimposition of m universes or branes. Gauge fields for such symmetry correspond to open edge which connect vertices in the same $\mathcal{U}^a$. Is then a case that the same symmetry arises in open strings with endpoints in m superimposed branes? We still think no. Until now we have supposed that open edges between vertices in the same $\mathcal{U}^a$ have length zero, so that we haven't to introduce extra dimensions. However, by $T-duality$ such edges correspond to open strings with $U(m, \mathbf{Y})$ Chan-Paton which moves in an infinite extended extra dimension. This happens because an absente extra dimension is a compactified dimension with $R = 0$ and $T-duality$ sends R in $1/R$. Regarding edges between vertices in different $\mathcal{U}^a$, we see that they have a mass proportional to separation between endpoints. This is true both in our model and string theory.

The following two theorems emphasize a triality between *Arrangement Field Theory*, *String Theory* and *Loop Quantum Gravity*. We can see as they are different manifestations of the same theory.

Theorem 6 *Every element M^{ij} in the arrangement matrix can be written as a state in the Hilbert space of* Loop Quantum Gravity*, ie an holonomy for a $SO(1,3)$ gauge field*[1]*. In this way, every field (gauge or gravitational) becomes a manifestation of only gravitational field.*

Proof. An element M^{ij} can always be written in the following form:

$$M^{ij} = |M^{ij}| exp \left(\int_{x_i}^{x_j} A_\mu dx^\mu \right) \tag{23}$$

with $\mu = 1, 2, 3$ and

$$|M^{ij}| = exp \left(\int_{x_i}^{x_j} A_0 dx^0 \right).$$

Here A_μ is a $SO(1,3)$ connection and A_0 is an I-complex field. Obviously, we take A_μ hyperionic by using the usual correspondence with Gamma matrices. In

[1] In Loop Gravity the gauge field appears usually in the form iA with A hermitian. We incorporate the i inside A so that $A^{ab}\gamma_a\gamma_b$ corresponds to a hyperionic number.

this way A_μ is purely imaginary. The integration is intended over the edge which goes from vertex i to vertex j, parametrized by any $\tau \in [0,1]$. If you look (23), you see on the left a discrete space (the graph) with discrete derivatives and fields which are defined only on the vertices. On the right you find instead a Hausdorff space with continuous paths, continuous derivatives and fields which are defined everywhere. Applying eventually a transformation in $U(n, \mathbf{Y})$, we have

$$M^{ij} = D^{ik\mu} \nabla_\mu^{kj} = D^{ii\mu} \nabla_\mu^{ij} = d^\mu(x_i) \nabla_\mu^{ij}.$$

In the following we introduce a real constant λ, with length dimensions, in order to make M dimensionless:

$$M^{ij} = \lambda D^{ik\mu} \nabla_\mu^{kj} = \lambda D^{ii\mu} \nabla_\mu^{ij} = \lambda d^\mu(x_i) \nabla_\mu^{ij}. \tag{24}$$

In *Loop Quantum Gravity* we consider any space-time foliation defined by some temporary parameter and then we quantize the theory on a tridimensional slice. The simpler choice is a foliation along x_0: in this case the metric on the slice is simply the spatial block 3×3 inside the four dimensional metric when it's taken in temporary gauge. In such framework we have $d^0 = \mathbf{1}$ and $[d^\mu(x), A_\nu(x')] = G\delta^\mu_\nu \delta^3(x - x')$ with $\mu, \nu = 1, 2, 3$. We deduce the relation $d^\mu(x) = G\delta/\delta A_\mu(x)$ and apply it to (24) when vertices i and j sit on the same slice. We obtain

$$d^\mu(x_i) \nabla_\mu^{ij} = G \frac{\delta}{\delta A_\mu(x_i)} \nabla_\mu^{ij} = \frac{1}{\lambda} |M^{ij}| exp \left(\int_{x_i}^{x_j} A_\mu dx^\mu \right) \tag{25}$$

with $\mu = 1, 2, 3$. Note that $x_0(x_i) = x_0(x_j)$ when i and j sit on the same slice. Hence

$$|M^{ij}| = exp \left(\int_{x_i}^{x_j} A_0 dx^0 \right) = exp \left(\oint A_0 dx^0 \right).$$

Consider now the following relation:

$$exp \left(\int_{x_i}^{x_j} A_\mu dx^\mu \right) = \frac{\delta}{\delta A_\nu} \int_\Omega d^2 s\, n_\nu exp \left(\int_{x_i}^{x_j} A_\mu dx^\mu \right) \tag{26}$$

with

$$n_\nu = \frac{1}{2}\varepsilon_{\nu\mu\alpha}\frac{\partial x^\mu}{\partial s^a}\frac{\partial x^\alpha}{\partial s^b}\varepsilon^{ab}.$$

Ω is a two dimensional surface parametrized by coordinates s^a with $a = 1, 2$ and $\int_\Omega d^2s = G$. We assume that Ω contains the vertex x_i and no other point which is a vertex or sits along an edge. Substituting (26) in (25) we obtain

$$\frac{\delta}{\delta A^i_\nu}\nabla^{ij}_\nu = \frac{1}{\lambda G}\frac{\delta}{\delta A_\nu}\int_\Omega d^2s\, n_\nu |M^{ij}| exp\left(\int_{x_i}^{x_j} A_\mu dx^\mu\right)$$

and then

$$\begin{aligned}\nabla^{ij}_\nu &= \frac{1}{\lambda G}\int_\Omega d^2s\, n_\nu |M^{ij}| exp\left(\int_{x_i}^{x_j} A_\mu dx^\mu\right) + K_\nu(x_i, x_j) \\ &= \frac{1}{\lambda G}\int_\Omega d^2s\, n_\nu exp\left(\int_{x_i}^{x_j} A_\mu dx^\mu\right) + K_\nu(x_i, x_j).\end{aligned}$$

K_ν is any function of x_i and x_j independent from A_μ. In the second line we have taken $\mu = 0, 1, 2, 3$. For diagonal components this becomes

$$A^{ii}_\nu = \frac{1}{\lambda G}\int_\Omega d^2s\, n_\nu exp\left(\oint A_\mu dx^\mu\right) + K_\nu(x_i). \tag{27}$$

We have used $\partial^{ii} = 0$ because the matrix which represents the discrete derivative is null along diagonal. We choose loops and surfaces Ω in such a way to have

$$n_\nu \oint A_\mu dx^\mu = \lambda A_\nu(x_i) + O(\lambda^2).$$

Applying this into (27), it becomes

$$\begin{aligned}A^{ii}_\nu &= \frac{1}{\lambda G}\int_\Omega d^2s\, n_\nu \left(1 + \oint A_\mu dx^\mu + O(\lambda^2)\right) + K_\nu(x_i) \\ &= \frac{1}{\lambda G}(G n_\nu + G\lambda A_\nu(x_i) + G \cdot O(\lambda^2)) + K_\nu(x_i) \\ &= \frac{1}{\lambda}(n_\nu + \lambda A_\nu(x_i) + O(\lambda^2)) + K_\nu(x_i).\end{aligned} \tag{28}$$

If we set $K_\nu(x_i) = -n_\nu(x_i)/\lambda$, we obtain

$$A^{ii}_\nu = A_\nu(x_i) + O(\lambda).$$

This verifies the consistence of our definition and proves the theorem.

Note that λ could be taken equal to Δ because M contains a factor Δ^{-1} from definition (4) of $\tilde{M}$. In such case we obtain

$$A^{ii}_\nu = A_\nu(x_i)$$

in the continuous limit. ■

Remark 7 (third quantization) *Note that canonical quantization of gauge fields implies*

$$\left[\partial_0 A^{ij}_\alpha(x_a), A^{ij}_\nu(x_b)\right] = \left[\left(\int d^4x \partial_0 A_\mu(x) \frac{\delta \nabla^{ij}_\alpha}{\delta A_\mu(x)}\right)(x_a), \nabla^{ij}_\nu(x_b)\right] = \delta_{\alpha\nu}\delta^3(x_a - x_b).$$

Integration in the first factor is over continuous coordinates of Hausdorff space. Conversely, the argument x_a indicates simply to what ensemble $\mathcal{U}^a$ the edge (ij) belongs. Here we have used $\partial^{ij} = 0$, which holds not only for $i = j$ but also for x_i and x_j in the same ensemble $\mathcal{U}^a$. This implies $\nabla^{ij} = A^{ij}$. Moreover ∇^{ij} is a state in the Hilbert space of Loop Quantum Gravity and hence we have a sort of third quantization which applies on gravitational states and creates gauge fields:

$$\left[\left(\int d^4x \dot{A}_\mu(x) \frac{\delta \Psi[\Lambda, A]}{\delta A_\mu(x)}\right), \Psi^\dagger[\Lambda', A]\right] = \delta(\Lambda - \Lambda').$$

$$\left[\left(\int d^4x \dot{A}_\mu(x) \frac{\delta \Psi[\Lambda, A]}{\delta A_\mu(x)}\right), \Psi^\dagger[\Lambda, A']\right] = \delta(A - A').$$

This implies

$$\Psi[A] = \int D[d^\mu]\, a(d)\, exp\left(\frac{1}{G}\int d^4x\, d^\mu A_\mu\right) + b^\dagger(d)\, exp\left(\frac{1}{G}\int d^4x\, d^{\dagger\mu} A^\dagger_\mu\right)$$

$$\left[a(d), a^{\dagger}(d')\right] = \frac{1}{\int d^4x \dot{A}_\nu d^\nu} \delta(d - d^{\dagger\prime})$$

$$\left[b(d), b^{\dagger}(d')\right] = \frac{1}{\int d^4x \dot{A}_\nu d^\nu} \delta(d - d^{\dagger\prime})$$

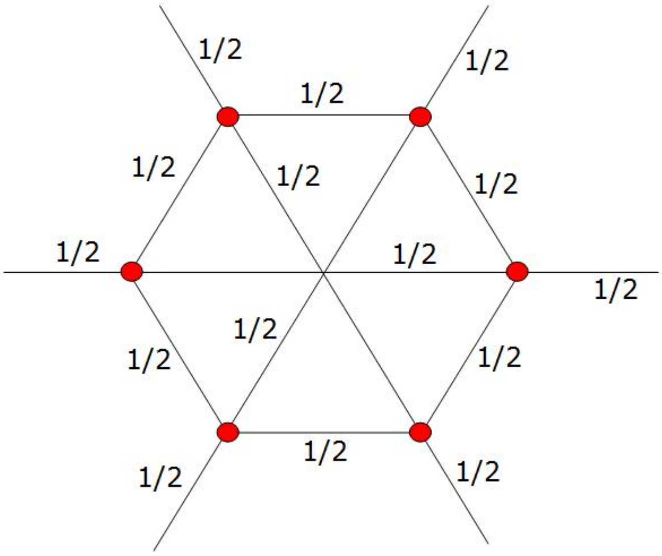

Figure 1: A spin network with symmetry $U(6, \mathbf{Y})$. The six vertices are assumed superimposed.

In figure 1 we see a spin network which defines a $U(6, \mathbf{Y})$ gauge field A^{ij} with $i, j = 1, 2, 3, 4, 5, 6$. The vertices are assumed superimposed. The symmetry group is bigger than $U(1, \mathbf{Y})^6 \sim SO(1, 3)^6$ which acts separately on the single vertices. The group grows in fact to $U(6, \mathbf{Y})$ because we can exchange the vertices without change the graph. We have the same situation with open strings: six strings with endpoints on six separated branes define a state with symmetry $U(1)^6$ but, if the branes are superimposed, the symmetry becomes $U(6)$.

Generators in $u(6, \mathbf{Y})$ are generators in $u(6, \mathbf{H})$ multiplied by 1 or I. In turn, generators in $u(6, \mathbf{H})$ can be divided in three families of generators in $u(6)$, one for every choice of imaginary unit (i, j or k). Note that commutation relations for $U(6)$ are satisfied if and only if

$$U^{ij}U^{jk} = U^{ik},$$

where U^{ij} is the holonomy from x_i to x_j. Hence

$$A_\mu = \partial_\mu \Gamma \qquad \text{with } \Gamma \text{ scalar.}$$

This means that gauge fields in $U(6)$ could exist without gravity, ie when A is a pure gauge. Otherwise, an holonomy with $A \neq \partial\Gamma$ exchanges gauge fields between different families.

Theorem 8 *The actions tr* $(M^\dagger M)$ *and tr* $(M^\dagger M M^\dagger M)$ *are sums of exponentiated string actions.*

Proof. We obtain from theorem 6:

$$\begin{aligned} M^{ij}M^{*jk}M^{kl}M^{*li} &= exp\left(\int_{\partial\Box} A_\mu dx^\mu\right) \\ &= exp\left(\int_{\Box} F_{\mu\nu} dx^\mu \wedge dx^\nu\right) \\ &= exp\left(\int_{\Box} \varepsilon^{ab} F_{\mu\nu} X^\mu_{,a} X^\nu_{,b}\, d^2 s\right) \end{aligned} \tag{29}$$

This is the exponential of an action for open strings whose worldsheet is a square made by edges (ij), (jk), (kl), (li). The strings move in a curved background with antisymmetric metric $F_{\mu\nu} = (d \wedge A)_{\mu\nu}$. In a similar manner

$$M^{ij}M^{*jk}M^{ki} = exp\left(\int_{\triangle} \varepsilon^{ab} F_{\mu\nu} X^\mu_{,a} X^\nu_{,b}\, d^2 s\right) \tag{30}$$

This is the exponential of an action for open strings whose worldsheet is a triangle.

$$M^{ij}M^{*ji} = exp\left(\int_{O} \varepsilon^{ab} F_{\mu\nu} X^\mu_{,a} X^\nu_{,b}\, d^2 s\right) \tag{31}$$

This is the exponential of an action for open strings whose worldsheet is a circle.

$$M^{ii} \quad = \quad exp\left(\int_O \varepsilon^{ab} F_{\mu\nu} X^{\mu}_{,a} X^{\nu}_{,b}\, d^2 s\right) \tag{32}$$

The same of above.

$$M^{ii} M^{jj} \quad = \quad exp\left(\int_{Cil} \varepsilon^{ab} F_{\mu\nu} X^{\mu}_{,a} X^{\nu}_{,b}\, d^2 s\right) \tag{33}$$

This is the exponential of an action for closed strings whose worldsheet is a cilinder. This concludes the proof. ■

6 Standard model interactions

We suppose that a residual symmetry for $U(6, \mathbf{Y})^{n/6}$ survives. If we consider the ensembles $\mathcal{U}^a = (x^a_1, x^a_2, x^a_3, x^a_4, x^a_5, x^a_6)$ as the real physical points, $U(6, \mathbf{Y})^{n/6}$ can be considered as a local $U(6, \mathbf{Y})$. We have defined $u(6, \mathbf{Y})$ as the complexified Lie algebra of $U(6, \mathbf{H})$, generated by all matrices in $u(6, \mathbf{H})$ and $Iu(6, \mathbf{H})$. By exponentiating $u(6, \mathbf{Y})$ we obtain a simple Lie group with complex dimension 78. According to Killing-Cartan classification, the only simple Lie group with complex dimension 78 is the group $E6$ and then $U(6, \mathbf{Y}) = E6$. This is remarkable because several works have already proposed $E6$ as the gauge group for grand unification theories. We consider the fields $A(x^a_i, x^b_j)$ with $a = b$ (we call them $A(x^a)$). They are 6×6 skew adjoint hyperionic matrices $\bar{A}^{\dagger} = -A$. These matrices form the $E6$ algebra which has 156 generators ω with $\bar{\omega}^{\dagger} = -\omega$.

$$\omega = \begin{pmatrix} \vec{y} & b+\vec{b} & c+\vec{c} & d+\vec{d} & e+\vec{e} & m+\vec{m} \\ -b+\vec{b} & \vec{a}_1 & f+\vec{f} & g+\vec{g} & h+\vec{h} & p+\vec{p} \\ -c+\vec{c} & -f+\vec{f} & \vec{a}_2 & s+\vec{s} & q+\vec{q} & r+\vec{r} \\ -d+\vec{d} & -g+\vec{g} & -s+\vec{s} & \vec{a}_3 & k+\vec{k} & t+\vec{t} \\ -e+\vec{e} & -h+\vec{h} & -q+\vec{q} & -k+\vec{k} & \vec{a}_4 & v+\vec{v} \\ -m+\vec{m} & -p+\vec{p} & -r+\vec{r} & -t+\vec{t} & -v+\vec{v} & \vec{a}_5 \end{pmatrix}$$

Consider now the subalgebra of the following form with complex (not hyperionic) components except for y which remains hyperionic:

$$\omega = \begin{pmatrix} \vec{y} & 0 & 0 & 0 & 0 & 0 \\ 0 & \vec{a}_1 & f+\vec{f} & g+\vec{g} & h+\vec{h} & p+\vec{p} \\ 0 & -f+\vec{f} & \vec{a}_2 & s+\vec{s} & q+\vec{q} & r+\vec{r} \\ 0 & -g+\vec{g} & -s+\vec{s} & \vec{a}_3 & k+\vec{k} & t+\vec{t} \\ 0 & -h+\vec{h} & -q+\vec{q} & -k+\vec{k} & \vec{a}_4 & v+\vec{v} \\ 0 & -p+\vec{p} & -r+\vec{r} & -t+\vec{t} & -v+\vec{v} & \vec{a}_5 \end{pmatrix}$$

Moreover we put the additional condition $\vec{a} = \sum_l \vec{a}_l = 0$. The field $y = tr\, \omega$ is the only one which contributes to Ricci scalar. Conversely, all other fields belong to a $SU(5)$ subgroup, which defines the Georgi - Glashow grand unification theory. The symmetry breaking in Georgi - Glashow model is induced by Higgs bosons in representations which contain triplets of color. These color triplet Higgs can mediate a proton decay that is suppressed by only two powers of GUT scale. However, our mechanism of symmetry breaking doesn't use such Higgs bosons, but descends from the expectation values of quadratic terms AA, which derive from non trivial minimums of a potential $AA - AAAA$. So we circumvent the problem.

Restrict now the attention to the $SO(1,3) \otimes SU(2) \otimes U(1) \otimes SU(3)$ generators, that are the generators of standard model plus gravity.

$$\omega = \begin{pmatrix} \vec{y} & 0 & 0 & 0 & 0 & 0 \\ 0 & \vec{a}_1 & f+\vec{f} & 0 & 0 & 0 \\ 0 & -f+\vec{f} & \vec{a}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \vec{a}_3 & k+\vec{k} & t+\vec{t} \\ 0 & 0 & 0 & -k+\vec{k} & \vec{a}_4 & v+\vec{v} \\ 0 & 0 & 0 & -t+\vec{t} & -v+\vec{v} & \vec{a}_5 \end{pmatrix}$$

We'll show in a moment that all standard model fields transform under this subgroup in the adjoint representation. In this way themselves are elements of $E6$ algebra, explicitly:

$$\psi = \psi^1 + I\psi^2 = \begin{pmatrix} 0 & e & -\nu & d^c_R & d^c_G & d^c_B \\ -e^* & 0 & e^c & -u_R & -u_G & -u_B \\ \nu^* & -e^{c*} & 0 & -d_R & -d_G & -d_B \\ -d^{c*}_R & u^*_R & d^*_R & 0 & u^c_B & -u^c_G \\ -d^{c*}_G & u^*_G & d^*_G & -u^{c*}_B & 0 & u^c_R \\ -d^{c*}_B & u^*_B & d^*_B & u^{c*}_G & -u^{c*}_R & 0 \end{pmatrix}$$

We have used the convention of Georgi - Glashow model, where the basic fields of ψ^1 are all left and the basic fields of $I\psi^2$ are all right. We have indicated with ^{c} the charge conjugation. The subscripts R, G, B indicates the color charge for the strong interacting particles (R=red, G=green, B=blue).

In Georgi - Glashow model the fermionic fields are divided in two families. The first one transforms in the representation $\bar{5}$ of $SU(5)$ (the fundamental representation). It is exactly the array (ω^{1j}) in the matrix above, with $j = 2, 3, 4, 5, 6$. This array transforms in fact in the fundamental representation for transformations in every $SU(5) \subset E6$ which acts on indices values $2 \div 6$.

The second family transforms in the representation 10 of $SU(5)$ (the skew symmetric representation). Unfortunately it isn't the sub matrix (ω^{ij}) with $i, j = 2, 3, 4, 5, 6$. This is in fact the skew adjoint representation of $U(5, \mathbf{Y})$, which is skew hermitian and not skew symmetric.

Do not lose heart. We'll see in a moment that such adjoint representation is a quaternionic combination of three skew symmetric representations, one for every fermionic family. This concept could appears cumbersome, but it will be clear along the following calculations.

Theorem 9 *The skew adjoint representation of $U(m, \mathbf{H})$ is a quaternionic combination of three skew symmetric representations of $U(m) = U(m, \mathbf{C})$ plus a real skew symmetric representation (which is also skew hermitian).*

Proof. Consider a fermionic matrix ψ which transforms in the adjoint representation of $U(m, \mathbf{H})$:

$$\psi \to U\psi U^\dagger$$

Take then a matrix ψ' with $\psi' k = \psi$. Its transformation law under $U(m) = U(m, \mathbf{C})$ is easily derived when this group is constructed by using imaginary unit i or j:

$$\psi' k \to U\psi' k U^\dagger = U\psi' U^T k.$$

Here we have used the relation $k\lambda = \lambda^* k$ for $\lambda \in \mathbf{H}$ without k component. We see that ψ' transforms in the skew symmetric representation:

$$\psi' \to U\psi' U^T$$

We obtain a complex matrix ψ' (with i as imaginary unit) when ψ has the form $Ak + Bj$ with A, B real matrices. Indeed:

$$\psi' = -\psi k = -Akk - Bjk = A - Bi$$

Sending ψ in ψ^* we bring ψ' to $-\psi'$ and so we satisfy the skew symmetry. Finally we can always write

$$\psi = \psi_0 + \psi_1 k + \psi_2 i + \psi_3 j$$

In this decomposition ψ_1, ψ_2, ψ_3 are complex matrices with complex unit respectively i, j, k. Explicitly:

$$\begin{aligned}
\psi_1 &= \phi_1 - i\xi_1 \qquad = \phi_1^1 - i\xi_1^1 + I(\phi_1^2 - i\xi_1^2) \\
\psi_2 &= \phi_2 - j\xi_2 \qquad = \phi_2^1 - j\xi_2^1 + I(\phi_2^2 - j\xi_2^2) \\
\psi_3 &= \phi_3 - k\xi_3 \qquad = \phi_3^1 - k\xi_3^1 + I(\phi_3^2 - k\xi_3^2).
\end{aligned}$$

Here all ϕ^1, ϕ^2 and ξ^1, ξ^2 are real fields. In this way $\psi_{1,2,3}$ transform in the skew symmetric representation of $U(m)$ when we construct this group by using the correspondent imaginary unit (i for ψ_1, j for ψ_2 and k for ψ_3). Hence they define the famous three fermionic families, relate each other by $U(1, \mathbf{H})$ transformations. Moreover ψ_0 is a real skew symmetric field. ■

Consider the following lagrangian

$$\begin{aligned}
tr(\psi^\dagger \nabla \psi) &= tr(k^* \psi_1^\dagger \nabla \psi_1 k) + tr(i^* \psi_2^\dagger \nabla \psi_2 i) + tr(j^* \psi_3^\dagger \nabla \psi_3 j) \\
&\quad - tr(i^* \phi_2^\dagger \nabla \xi_3 i) - tr(j^* \phi_3^\dagger \nabla \xi_1 j) - tr(k^* \phi_1^\dagger \nabla \xi_2 k) \\
&\quad - tr(\psi_0^\dagger \nabla \psi_0) \\
&= tr(\psi_1^\dagger \nabla \psi_1 k k^*) + tr(\psi_2^\dagger \nabla \psi_2 i i^*) + tr(\psi_3^\dagger \nabla \psi_3 j j^*) \\
&\quad - tr(\phi_2^\dagger \nabla \xi_3 i i^*) - tr(\phi_3^\dagger \nabla \xi_1 j j^*) - tr(\phi_1^\dagger \nabla \xi_2 k k^*) \\
&\quad - tr(\psi_0^\dagger \nabla \psi_0) \\
&= tr(\psi_1^\dagger \nabla \psi_1) + tr(\psi_2^\dagger \nabla \psi_2) + tr(\psi_3^\dagger \nabla \psi_3) \\
&\quad - tr(\phi_2^\dagger \nabla \xi_3) - tr(\phi_3^\dagger \nabla \xi_1) - tr(\phi_1^\dagger \nabla \xi_2) \\
&\quad - tr(\psi_0^\dagger \nabla \psi_0)
\end{aligned} \tag{34}$$

In the third last line we have the fermionic terms in Georgi-Glashow model for three families of fields in representation 10. In this way we can use the lagrangian $tr(\psi^\dagger\nabla\psi)$, with ψ in the adjoint representation, in place of Georgi-Glashow terms with $\psi_{1,2,3}$ in the skew symmetric representation. Mixed terms in the second last line give a reason to CKM and PMNS matrices which appear in standard model. Consider now the equivalence

$$tr(\psi^\dagger\psi\nabla) = tr(\psi\nabla\psi^\dagger) = tr((-\psi^\dagger)\nabla(-\psi)) = tr(\psi^\dagger\nabla\psi).$$

Hence

$$tr(\psi^\dagger\nabla\psi) = \frac{1}{2}tr(\psi^\dagger\{\nabla,\psi\}). \tag{35}$$

In this formalism, given $\omega \in su(3)\otimes su(2)\otimes u(1)$, the transformation $\delta\psi = [\omega,\psi]$ corresponds to the usual transformation $\delta\psi = \omega\psi$ in the standard model formalism. We see that the only fields which transform correctly under $SO(1,3)$ are e, ν and d^c. For now we do not care.

We note rather that, when we restrict the elements of ω from the hyperions to the complex numbers, we have 3 possibilities to do it. A complex number is not only in the form $a+ib$, with $a,b\in R$, but also $a+jb$ and $a+kb$. The same is true for a fixed linear combination $a+(ci+dj+fk)b$, where $c,d,f\in R$ and $c^2+d^2+f^2=1$.

The choice of j in place of i determines another set of (ω,ψ) isomorphic to the first one. In the same way we obtain a third set choosing k. The three sets are related by the group $SU(2)$ which rotates an unitary vector in R^3 with coordinates (c,d,f). Its generators are

$$\omega = \frac{\vec{y}}{6}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Their diagonal form suggests an identification between this group and the gravitational group $SU(2)^{\subset SO(1,3)}$. If the two groups coincided, all fields would transform correctly under $SU(2)^{\subset SO(1,3)}$. By extending this group to the entire $SO(1,3)$, we see that boosts exchange left fields with right fields.

Note that three families have to exist also for bosonic particles (photon, $W^{\pm}$, Z, gluons) although they are probably indistinguishable. Other interesting thing is that we have no warranty for the persistence of $E6$ in the entire universe. However we have surely at least the symmetry $U(1,\mathbf{Y}) = SO(1,3)$, which implies the secure existence of gravity.

6.1 Fermionic fields from a generalized arrangement matrix

We can introduce grassmann coordinate with derivatives ∂_g and $\bar{\partial}_g$, and covariant derivatives $\nabla_g = \partial_g + \psi$ and $\bar{\nabla}^{\dagger}_g = \bar{\partial}_g + \bar{\psi}^{\dagger}$. In the arrangement field formalism these descend from a grassmanian matrix M_g or $\bar{M}^{\dagger}_g$.

We can consider a unique generalized matrix $M_T = M_g + M$ that, up to a generalized $U(n,\mathbf{Y})$, becomes

$$\begin{aligned} M_T &= \theta\nabla_g + d^{\mu}\nabla_{\mu} = \theta\partial_g + \theta\psi + d^{\mu}\nabla_{\mu} \\ \bar{M}^{\dagger}_T &= \bar{\nabla}^{\dagger}_g\bar{\theta} + \bar{\nabla}^{\dagger}_{\mu}\bar{d}^{\dagger\mu} = \bar{\partial}_g\bar{\theta} + \bar{\psi}^{\dagger}\bar{\theta} + \bar{\nabla}^{\dagger}_{\mu}\bar{d}^{\dagger\mu}. \end{aligned} \tag{36}$$

In the same manner of d, θ is an ipercomplex sum of 4 grassmanian elements. Alternatively, we can consider θ and $\bar{\theta}$ as independent variables: in such case, θ and $\bar{\theta}$ are complex sums of 2 grassmanian elements (likewise in the usual supersymmetric theories). Expanding $tr\,(\bar{M}_T^\dagger M_T)$ we obtain

$$tr\,(\bar{M}_T^\dagger M_T) \;=\; tr\,\left(d^\nu \bar{d}^{\dagger\mu}\bar{\nabla}^\dagger_\mu \nabla_\nu\right) = \sum_a \sqrt{h}R(x^a). \tag{37}$$

To calculate $tr\,(\bar{M}_T^\dagger M_T \bar{M}_T^\dagger M_T)$ we write first $\bar{M}_T^{\dagger 2}$ and M_T^2.

$$\begin{aligned} M_T^2 &= \theta\partial_g + \theta\psi + \theta d^\mu\{\nabla_\mu, \psi\} + d^\mu\nabla_\mu d^\nu\nabla_\nu \\ \bar{M}_T^{\dagger 2} &= \bar{\partial}_g\bar{\theta} + \bar{\psi}^\dagger\bar{\theta} + \{\bar{\psi}^\dagger, \bar{\nabla}^\dagger_\alpha\}\bar{d}^{\dagger\alpha}\bar{\theta} + \bar{\nabla}^\dagger_\alpha\bar{d}^{\dagger\alpha}\bar{\nabla}^\dagger_\beta\bar{d}^{\dagger\beta} \end{aligned} \tag{38}$$

If M has the form (36), then $[M_T, \bar{M}_T^\dagger] = 0$. This implies $tr\,(\bar{M}_T^\dagger M_T \bar{M}_T^\dagger M_T) = tr\,(M_T^2 \bar{M}_T^{\dagger 2})$. We calculate its value starting from the following product

$$tr\,(\theta d^\mu\{\nabla_\mu, \psi\}\{\bar{\psi}^\dagger, \bar{\nabla}^\dagger_\alpha\}\bar{d}^{\dagger\alpha}\bar{\theta}) \;=\; tr\,(\theta\bar{\theta} d^\mu\{\nabla_\mu, \psi\}\{\bar{\psi}^\dagger, \bar{\nabla}^\dagger_\alpha\}\bar{d}^{\dagger\alpha}). \tag{39}$$

Remember that operator tr acts as a sum over vertices. Now every vertex is labeled by a couple (θ, x_i) and then

$$tr\,(\theta\bar{\theta}(***)) = \left(\int d\bar{\theta}d\theta\,\theta\bar{\theta}\right) tr\,(***) = tr\,(***)$$

Hence

$$\begin{aligned} tr\,(\theta d^\mu\{\nabla_\mu, \psi\}\{\bar{\psi}^\dagger, \bar{\nabla}^\dagger_\alpha\}\bar{d}^{\dagger\alpha}\bar{\theta}) &= tr\,(d^\mu\{\nabla_\mu, \psi\}\{\bar{\psi}^\dagger, \bar{\nabla}^\dagger_\alpha\}\bar{d}^{\dagger\alpha}) \\ &= tr\,(\bar{d}^{\dagger\alpha} d^\mu[\nabla_\mu, \bar{\nabla}^\dagger_\alpha]\psi\bar{\psi}^\dagger) \\ &= \sum_a \sqrt{h}R(x^a)\bar{\psi}^\dagger\psi \end{aligned} \tag{40}$$

In this way

$$\begin{aligned} tr\,(\bar{M}_T^\dagger M_T \bar{M}_T^\dagger M_T) &= tr\,\left(\bar{\psi}^\dagger d^\mu \{\nabla_\mu, \psi\} + \{\bar{\psi}^\dagger, \bar{\nabla}_\alpha^\dagger\}\bar{d}^{\dagger\alpha}\psi\right) + \\ &+ \sum_a \sqrt{h} R(x^a)\bar{\psi}^\dagger\psi + S_{GB} \end{aligned} \tag{41}$$

We have seen that every family distinguishes itself by the choice of complex unity. Inserting in ψ the definitions of $\psi_{1,2,3}$ we can write

$$\begin{aligned} \psi &= \psi_0^1 + i(\phi_2^1 + \xi_3^1) + j(\phi_3^1 + \xi_1^1) + k(\phi_1^1 + \xi_2^1) + \\ &+ I\psi_0^2 + iI(\phi_2^2 + \xi_3^2) + jI(\phi_3^2 + \xi_1^2) + kI(\phi_1^2 + \xi_2^2) \end{aligned}$$

Using the correspondence $(1, I, i, j, k, iI, jI, kI) \leftrightarrow \gamma\gamma$, the first term in (41) becomes

$$tr\,\left(\psi^{lm}\overline{(\gamma_l\gamma_m)^\dagger}\left(\gamma_0\gamma_s e^{\mu s}\overset{G}{\nabla}_\mu \psi^{np}(\gamma_n\gamma_p) + A_\mu\psi\right)\right) \tag{42}$$

$$\downdownarrows\!===\!\downdownarrows$$

$$tr\,\left(\psi^{lm}(\gamma_m\gamma_l)\left(\gamma_0\gamma_s e^{\mu s}\overset{G}{\nabla}_\mu, \psi^{np}(\gamma_n\gamma_p) + A_\mu\psi_0 + \sum_{q,q'=1}^{3} A_\mu^q \psi_{q'} i_{q'}\right)\right)$$

Here we have deleted the anticommutator by means of (35). In the covariant derivative we have included only the gravitational (track) contribution, while A_μ is intended to have null track. Moreover $i_1 = k$, $i_2 = i$ and $i_3 = j$.

In the second line we have divided the 75 generators A_μ in three families of 35 generators. Obviously, only two families are linearly independent. When they act on spinorial fields which belong to their own family, they behave exactly as the 35 generators of $SU(6)$ (which comprise the 24 generators of $SU(5)$). Conversely, when a generator A^q acts on a q'-field (with $q \neq q'$), it mimics the application of some generator $A^{q'}$ followed by a rotation in $SU(2)_{GRAVITY}$ which sends the family q' in the remaining family q''.

We explicit now one entry of $\psi = \psi^1 + I\psi^2$ by exploiting the correspondence with γ. We have

$$\psi =$$

$$\begin{pmatrix} \psi_0^1 + i(\phi_2^1 + \xi_3^1) & (\phi_3^1 + \xi_1^1) + i(\phi_1^1 + \xi_2^1) & i\psi_0^2 - (\phi_2^2 + \xi_3^2) & i(\phi_3^2 + \xi_1^2) + (\phi_1^2 + \xi_2^2) \\ -(\phi_3^1 + \xi_1^1) + i(\phi_1^1 + \xi_2^1) & \psi_0^1 - i(\phi_2^1 + \xi_3^1) & -i(\phi_3^2 + \xi_1^2) + (\phi_1^2 + \xi_2^2) & i\psi_0^2 + (\phi_2^2 + \xi_3^2) \\ i\psi_0^2 - (\phi_2^2 + \xi_3^2) & i(\phi_3^2 + \xi_1^2) + (\phi_1^2 + \xi_2^2) & \psi_0^1 + i(\phi_2^1 + \xi_3^1) & (\phi_3^1 + \xi_1^1) + i(\phi_1^1 + \xi_2^1) \\ -i(\phi_3^2 + \xi_1^2) + (\phi_1^2 + \xi_2^2) & i\psi_0^2 + (\phi_2^2 + \xi_3^2) & -(\phi_3^1 + \xi_1^1) + i(\phi_1^1 + \xi_2^1) & \psi_0^1 - i(\phi_2^1 + \xi_3^1) \end{pmatrix} \tag{43}$$

If we define the four components spinor

$$\hat{\psi} = \begin{pmatrix} \psi_0^1 + i(\phi_2^1 + \xi_3^1) \\ -(\phi_3^1 + \xi_1^1) + i(\phi_1^1 + \xi_2^1) \\ i\psi_0^2 - (\phi_2^2 + \xi_3^2) \\ -i(\phi_3^2 + \xi_1^2) + (\phi_1^2 + \xi_2^2) \end{pmatrix}$$

the derivative term can be rewritten as

$$\hat{\psi}^\dagger \, \gamma_0 \gamma_s e^{\mu s} \overset{G}{\nabla}_\mu \hat{\psi} \tag{44}$$

This is the Dirac action, although with a new interpretation of spinorial components. Adding the other terms

$$\begin{aligned} tr\,(\hat{M}^\dagger \hat{M} \hat{M}^\dagger \hat{M}) \quad &= \\ &= \int \left(\hat{\psi}^\dagger \, \gamma_0 \gamma_s e^{\mu s} \overset{G}{\nabla}_\mu \hat{\psi} + \hat{\psi} \sum_{q,q'} A_\mu^q \hat{\psi}_{q'} i_{q'} + \sqrt{h} R(x) \sum_q \hat{\psi}_q^\dagger \hat{\psi}_q \right) dx \end{aligned}$$

In this way we include all the contents of standard model as elements in the generalized $E6$ algebra. Terms which mix families can be used to calculate values in CKM and PMNS matrices. Masses for fermionic fields arise, as usual, from non null expectation values of $A_\mu(x_i^a, x_j^b)$ with $a \neq b$ in ∇_μ.

We obtain a contribute to Hilbert-Einstein action also from $\int d^4x\sqrt{h}R\bar{\psi}\psi$, due to a non null expectation value of $\sum_q \bar{\psi}_q\psi_q$. It contains in fact the chiral condensate, whose non null vacuum value breaks the chiral flavour symmetry of QCD Lagrangian.

Note that known fermionic fields fill up a matrix ψ with null track. However, only if $tr\,\psi \neq 0$ our action has an extra invariance under

$$\begin{aligned} A_\mu &\rightarrow d_\mu^{-1}\theta\psi \\ \psi &\rightarrow \overleftarrow{\partial}_g d^\mu A_\mu. \end{aligned} \tag{45}$$

Here $\overleftarrow{\partial}_g$ is a ∂_g which acts backwards. This means we have the same number of fermions and bosons, so that the vacuum energies erase each other.

Invariance (45) predicts the existence of a new colored fermionic sextuplet which sits on diagonal in ψ. Inside it we can include a conjugate neutrino (ν^c), a sterile neutrino (N) and a conjugate sterile neutrino (N^c). Explicitly

$$\psi = \begin{pmatrix} N & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu^c & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu^c & 0 & 0 & 0 \\ 0 & 0 & 0 & N^c & 0 & 0 \\ 0 & 0 & 0 & 0 & N^c & 0 \\ 0 & 0 & 0 & 0 & 0 & N^c \end{pmatrix}.$$

This field commutes with any gauge field in $U(1)\otimes SU(2)\otimes SU(3)$ and so it hasn't electromagnetic, weak or strong interactions. Moreover it gives a Dirac mass to neutrinos via the term

$$tr\,(\bar{\psi}^\dagger d^\mu A_\mu \psi) = \bar{\psi}^{\dagger ij} d^\mu A_\mu^{kl} \psi^{mn} f^{(ij)(kl)(mn)}.$$

Here $f^{(ij)(kl)(mn)}$ are structure constants for $SU(6)$ and masses for neutrinos are eigenvalues of $< d^\mu A_\mu >$.

6.2 The vector superfield

The invariance (45) suggests a connection with super-symmetric theories. We extend the generators of supersymmetric algebra by substituting $\gamma_0\gamma^\mu$ with d^μ, noting that $d^\mu = \gamma_0\gamma^\mu$ in a flat space.

$$Q = \partial_g + d^\mu \nabla_\mu \bar{\theta}$$
$$\bar{Q} = \bar{\partial}_g + \theta \nabla_\nu^\dagger \bar{d}^{\dagger\nu}$$

In the same manner we generalize the definition of vector superfield. In Wess-Zumino gauge it assumes the form

$$V = \theta\bar{\theta}\hat{V}$$
$$\hat{V} = d^\mu A_\mu + \theta\psi - \bar{\theta}\bar{\psi}^\dagger - \frac{1}{2}\theta\bar{\theta}D.$$

The field-strength superfield is then

$$W = \bar{\psi}^\dagger + \theta D + \frac{1}{2}\theta \bar{d}^{\dagger\nu} d^\mu [\nabla_\mu, \nabla_\nu] + \theta\theta d^\mu \nabla^\mu \psi$$

If we ignore the commutator and the non dynamical field D, we have

$$W = \bar{\psi}^\dagger + \theta \bar{M}_T^\dagger M_T + \theta\theta M_T \psi$$

It's easy to see that the usual term W^2 of supersymmetric theories generates the same terms we have found in $tr\,(\bar{M}_T^\dagger M_T) - tr\,(\bar{M}_T^\dagger M_T \bar{M}_T^\dagger M_T)$. This can mean that our theory includes supersymmetry, with the known fermionic fields which take the role of gauginos. In this way the right up quarks are gauginos for gluons, while right electrons are gauginos for W bosons.

7 Inflation

Our final action is

$$S = tr\left(\frac{\bar{M}^\dagger M}{16\pi G} - \bar{M}^\dagger M \bar{M}^\dagger M\right)$$

This is also an action for an $U(n, \mathbf{Y})$ gauge theory with coupling constant $1/G$ in a mono-vertex space-time. In these theories the scaling of coupling constant can be calculated exactly in the limit of large n. In several cases the coupling constant changes its sign for big values of scale: this has considerable consequences for the first times after Big Bang, when a measurement of G has sense only at very high energies (very small distances). What said suggests that such measurement can return a negative value of G, which implies a repulsive force of gravity. In turn, repulsive gravity implies an accelerate expansion for the universe.

Because the entries of M are probability amplitudes, we would be it was dimensionless. However, when we pass from M to ∇, we need a scale Δ to define the matrix ∂. This justify the inclusion of Δ^{-1} inside M. If we extract this factor, the Hilbert Einstein action becomes

$$\frac{\Delta^4}{16\pi G \Delta^2} tr\,(\bar{M}^\dagger M) = \frac{\Delta^2}{16\pi G} tr\,(\bar{M}^\dagger M)$$

where we have also added the correct volume form Δ^4. This seems a more natural formulation when M represents probability amplitudes. In this way we can take Δ very small but not zero. The most natural choice is $\Delta^2 \sim G$.

In this case, what does it mean that G is negative? Negative G implies negative $\Delta^2 = ds^2$. In lorentzian spaces $\Delta^2 = dt^2 - ds^2 < 0$. For purely temporal intervals we'll have $dt^2 < 0$, so the time becomes imaginary. An imaginary time is indistinguishable from space. This hypothesis of a "spatial" time had already been espoused by Hawking as a solution for eliminate the singularity in the Big Bang [7].

8 Classical solutions

We rewrite our action in the form

$$S = \frac{1}{2} tr\,(\bar{M}^\dagger M) - \frac{1}{4g} tr\,(\bar{M}^\dagger M \bar{M}^\dagger M)$$

where we have defined $g = \frac{\Delta^2}{32\pi G}$. We diagonalize M with a transformation in $U(n, \mathbf{Y})$ and define $M^{ii} \equiv \varphi(x_i)$, $\varphi(x) = a(x) + \vec{b}(x)$. The lagrangian becomes:

$$L = \frac{1}{2}\left[a(x_i)^2 + |\vec{b}(x_i)|^2\right] - \frac{1}{4g}\left[a(x_i)^4 + |\vec{b}(x_i)|^2 + 2a(x_i)^2|\vec{b}(x_i)|^2\right]$$

The motion equations are

$$g a(x) - a(x)^3 - a(x)|\vec{b}(x)|^2 = 0$$

$$g\vec{b}(x) - \vec{b}(x)|\vec{b}(x)|^2 - a(x)^2\vec{b}(x) = 0$$

There are two solutions:

$$(1) \qquad a(x) = \vec{b}(x) = 0$$

$$(2) \qquad a(x)^2 + |\vec{b}(x)|^2 = \bar{M}^\dagger M = g$$

The first one corresponds to the vacuum (all non-gravitational fields equal to zero) plus a solution of Einstein equations in the vacuum:

$$\psi = A_\mu = 0 \qquad R(x) = 0$$

The solution $\bar{M}^\dagger M = g$ corresponds to a vacuum expectation value for $\bar{M}^\dagger M$ equal to g. M contains a factor A, so that an expectation value for $\bar{M}^\dagger M$ corresponds to an expectation value for AA. This means that

$$AAAA =< AA > AA + \text{quantum perturbations}$$

$< AA >$ gives a mass for A. More precisely, for $A \in U(n, \mathbf{Y})/U(m, \mathbf{Y})^{n/m}$,

$$m_A^2 \sim \frac{< \bar{M}^\dagger M >}{\Delta^2} = \frac{g}{\Delta^2} = \frac{1}{32\pi G}$$

So the fields $A \in U(n, \mathbf{Y})/U(m, \mathbf{Y})^{n/m}$ have a mass in the order of Planck mass m_P. Moreover, in the primordial universe, when $k_B T \approx m_p$, all the fields behave like null mass fields. In that time the symmetry was then $U(n, \mathbf{Y})$ and no arrangement exists. Our conclusion is that Quantum Gravity cannot be treated as a quantum field theory in an ordinary space. In what follows we explain how overcome this trouble.

9 Quantum theory

Quantum theory is defined via the following path integral:

$$\int D[M(x,y)]D[\bar{M}^*(x,y)]Oe^{\int M(x,y)\bar{M}^*(x,y)dxdy - \int M(x,y)\bar{M}^*(x,y')M(x',y')\bar{M}^*(x',y)dxdydx'dy'}$$

with

$$\begin{aligned} Oe^{\int F(x,y)dxdy} &= 1 + \int F(x,y)dxdy + \frac{1}{2}\int F(x,y)F(x^1,y^1)dxdydx^1dy^1 + \\ &+ \ldots + \frac{1}{n!}\int F(x,y)F(x^1,y^1)\ldots F(x^{n-1},y^{n-1})dxdydx^1dy^1\ldots dx^{n-1}dy^{n-1} \end{aligned}$$

$$
\begin{aligned}
Oe^{\int F(x,x',y,y')dxdx'dydy'} &= 1+\int F(x,x',y,y')dxdydx'dy' + \\
&+\frac{1}{2}\int F(x,x',y,y')F(x^1,x'^1,y'^1,y^1)dxdydx'dy'dx^1dy^1dx'^1dy'^1 + \\
&+\ldots+\frac{1}{n!}\int F(x,x',y,y')F(x^1,x'^1,y^1,y'^1)\ldots \\
&\ldots F(x^{n-1},x'^{n-1},y^{n-1},y'^{n-1})dxdydx'dy'dx^1dy^1dx'^1dy'^1\ldots \\
&\qquad\qquad\qquad\ldots dx^{n-1}dy^{n-1}dx'^{n-1}dy'^{n-1} \\
&\overset{!}{=} \frac{1}{n!}F^n \qquad (46)
\end{aligned}
$$

The integration of F^n is very simple and gives

$$
\frac{1}{n!}\int D^2[M]e^{\int M^2dxdy}\; F^n = \frac{(4n)!}{n!2^{2n}(2n)!} = \frac{1}{n!}P(4n)
$$

Here $P(4n)$ gives the number of different ways to connect in couples $4n$ points.

It's clear that any universe configuration corresponds to an F^k inside which some connections have been fixed and the corresponding integrations have been removed. For example:

$$
\begin{aligned}
\hat{F}^k &= F(A,x',y,y')F(A,x'^1,B,y'^1)F(x^2,B,C,y'^2)\ldots \\
&\ldots F(x^{k-1},x'^{n-1},C,y'^{k-1})\cdot \\
&\cdot dxdydy'dx'^1dy'^1dx^2dy'^2\ldots dx^{k-1}dx'^{k-1}dy'^{k-1}dAdBdC
\end{aligned}
$$

If the fixed connections are m, then

$$
<\hat{F}^k>=\frac{\sum_n \frac{1}{n!}P(4(n+k)-2m)}{\sum_n \frac{1}{n!}P(4n)}
$$

Remark 10 *At relatively low energies we can tract* $\overset{G}{A}$ *as an ordinary gauge field. The arrangement field theory is then approximated with a common quantum theory on a curved background, determined by* $e^{\mu a}$.

10 Quantum Entanglement and Dark Matter

The elements of M which do not reside in or near the diagonal, describe connections between points that are not necessarily adjacent to each other, in the common sense. These connections construct discontinuous paths as in figure 2 and can be considered as quantum perturbations of the ordered space-time.

Such components permit us to describe the quantum entanglement effect, as it could be shown in detail in a complete coverage that goes beyond the purpose of the present paper.

It is remarkable that in this framework the discontinuity of paths is only apparent, and it is a consequence of imposing an arrangement to the space-time points. These discontinuous paths can be considered as continuous paths which cross wormholes. The trait of path inside a wormhole is described by a component of M far away from diagonal. The information seems to travel faster than light, but in reality it only takes a byway.

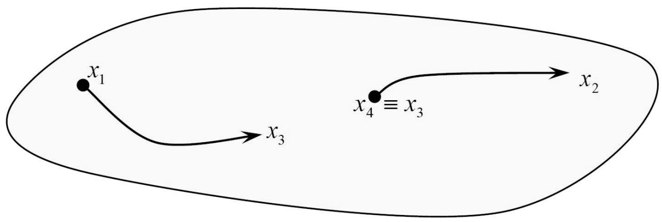

Figure 2: Discontinuous paths. The connection between x_3 and x_4 is done by a component of M far away from diagonal.

Imagine now a gravitational source with mass M_S which emits some gravitons with energy $\sim E_{PLANCK}$, directed to an orbiting body with mass M_B at distance

r. In this case (respect such gravitons) the fields $M(x^a, x^b)$ with $a \neq b$ would behave as they had null mass. This implies the probable existence of connections (practicable by such gravitons) between every couple of vertices in the path from the source to the orbiting body. This means that if $r = \Delta j$, $j \in \mathbf{N}$, the graviton could reach the orbiting body by traveling a shorter path $\Delta j'$, $j > j' \in \mathbf{N}$. The question is: what is the average gravitational force perceived by the orbiting body?

The probability for a graviton to reach a distance r passing through m vertices is

$$P_m = (1-a)^{m-1} a \qquad with \ \sum_{m=1}^{\infty} P_m = 1$$

where $a = 1/j$. These are the probabilities for extracting one determined object in a box with j objects at the m-th attempt. In this way the effective length traveled by the graviton will be Δm.

We use these probabilities to compute the average gravitational force in a semiclassical approximation.

$$\begin{aligned} < F > &= G \frac{M_B M_S}{\Delta^2} \frac{a}{1-a} \int_1^{\infty} \frac{(1-a)^m}{m^2} dm \\ &= G \frac{M_B M_S}{\Delta^2} \frac{a}{1-a} [log(1-a)] \int_{log(1-a)}^{-\infty} \frac{e^x}{x^2} dx \end{aligned} \tag{47}$$

The last integral gives

$$\int_{log(1-a)}^{-\infty} \frac{e^x}{x^2} dx = -Ei(log(1-a)) + \frac{1-a}{log(1-a)}$$

We expand $\langle F \rangle$ near $a = 0$ (which implies $j >> 1$), obtaining

$$\frac{a}{(1-a)} [log(1-a)] \int_{log(1-a)}^{-\infty} \frac{e^x}{x^2} dx \approx a + a^2(log(a) + \gamma) + O(a^3).$$

Here γ is the Eulero-Mascheroni constant. The dominant contribution is then

$$
\begin{aligned}
< F > &\approx G\frac{M_B M_S}{\Delta^2}\cdot a\cdot(1+a\,log(a)+a\gamma)\\
&\approx \frac{G}{\Delta}\frac{M_B M_S}{r}\left(1-\frac{\Delta}{r}\left(log\left(\frac{r}{\Delta}\right)-\gamma\right)\right)
\end{aligned} \tag{48}
$$

If the massive object orbits at a fix distance r, its centrifugal force has to be equal to the gravitational force. This gives

$$
< F > \approx \frac{G}{\Delta}\frac{M_B M_S}{r}\left(1-\frac{\Delta}{r}\left(log\left(\frac{r}{\Delta}\right)-\gamma\right)\right)=\frac{M_B v^2}{r}
$$

$$
v^2=\frac{G M_B M_S}{\Delta}\left(1-\frac{\Delta}{r}\left(log\left(\frac{r}{\Delta}\right)-\gamma\right)\right)
$$

We see that, varying the radius, the velocity remains more or less constant (It increases slightly with r). Can this explain the rotation curves of galaxies without introducing dark matter?

Surely not all gravitons have energy $> E_{PLANCK}$; at the same time we have to consider that G scales for small distances (hence for small m in (47)). It's possible that these factors reduces the extremely high value of r/Δ.

11 Conclusion

In this work we have applied the framework developed in [1] to describe the contents of our universe, ie forces and matter.

Doing this, we have discovered an unexpected road toward unification, which shows similarities with Loop Gravity, String Theory and Georgi - Glashow model. For the first time a natural symmetry justifies the existence of three particles families, not one more, not one less. Moreover a new version of supersymmetry seems to couple gauge fields with all known fermions, without necessity of imagining new particles never seen by experiments.

Clearly this fact closes the door to dark matter. To compensate this big absence, AFT proposes an explanation to galaxy rotation curves which doesn't make use of dark matter.

We don't say that this theory is exact. However there are several good signals which must be taken into account. We hope that a future teamwork can verify this theory in detail, deepening all its implications.

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