Minimal Length, Measurable and Nonmeasurable Quantities, and Mathematical Apparatus of Quantum Theory I

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Abstract

In this paper it is supposed that the theory involves a minimal length. Within the scope of this supposition, for the case of a free particle, the notions of **measurability** and **measurable quantities** are used as a basis for a mathematical apparatus of a new theory that is a deformation of the conventional (nonrelativistic) quantum mechanics. Consideration is given to one example from gravity with two different but very close low-energy limits: (1) continuous limit based on the use of **nonmeasurable quantities**;(2) discrete limit based on the use of **measurable quantities**. In conclusion the main course for further studies is defined. The paper is a continuation of the earlier studies conducted by the author and of his latest publication devoted to the inferences concerning the introduction of a minimal length in a quantum theory and in gravity.

1 Introduction. Measurable and Nonmeasurable Quantities

One of the key problems of the modern fundamental physics (Quantum Theory (QT) and Gravity (GR)) is framing of a correct theory associated with the ultraviolet region, i. e. the region of the highest (apparently Planck's) energies approaching those of the Big Bang.

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However, it is well known that at high energies (on the order of the quantum gravity energies) the minimal length l_{min} to which the indicated energies are ≪sensitive≫, as distinct from the low ones, should inevitably become apparent in the theory. But if l_{min} is really present, it must be present at all the ≪Energy Levels≫ of the theory, low energies including.
What follows from the existence of the minimal length l_{min}? When the minimal length is involved, any nonzero **measurable** quantity having the dimensions of length should be a multiple of l_{min}. Otherwise, its **measurement** with the use of l_{min} would result in the **measurable** quantity ς, so that $\varsigma < l_{min}$, and this is impossible.
This suggests that the spatial-temporal quantities dx_μ are **nonmeasurable**quantities because the latter lead to the infinitely small length ds [1]

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu \tag{1}$$

nonmeasurable because of l_{min}.
And this has been indicated in my previous work [2].
Of course, as a mathematical notion, the quantities dx_μ, ds are naturally existent but one should realize that there is no way to express them in terms of the minimal possible measuring unit l_{min}.
So, trying to frame a theory (QT and GR) correct at all the energy levels using only the **measurable** quantities, one should realize that then the mathematical formalism of the theory should not involve any infinitesimal spatial-temporal quantities. Besides, proceeding from the acknowledged results associated with the Planck scales physics [3]–[11], one can infer that certain new parameters dependent on l_{min} should be involved.
What are the parameters of interest in the case under study? It is obvious that, as the quantum-gravitational effects will be revealed at very small (possibly Planck's) scales, these parameters should be dependent on some limiting values, e.g., $l_P \propto l_{min}$ and hence Planck's energy E_P.
This means that in high-energy QT and GR the energy- or, what is the same, measuring scales-dependent parameters should be necessarily introduced.
But, on the other hand, these parameters could hardly disappear totally at low energies both in QT and in GR.
But, provided l_{min} exists, it must be involved at all the energy levels, both

high and **low**.
The fact that l_{min} is omitted in the formulation of low-energy QT and GR and the theories give perfect results leads to two different inferences:

1.1. The influence of the above-mentioned new parameters associated with l_{min} in low-energy QT and GR is so small that it may be disregarded at the modern stage in evolution of the theory and of the experiment.

1.2. The modern mathematical apparatus of conventional QT and GR has been derived in terms of the infinitesimal spatial-temporal quantities dx_{μ} which, as noted above, are **nonmeasurable quantities** in the formalism of l_{min}.

This paper is just the first step in derivation of l_{min}-involving QT and GR with the use of only **measurable quantities**. It is a direct continuation of the previous author's work [2]. Sections 2 and 3 present definitions of **measurability** and initial mathematical formulations for the fundamental quantities (coordinates, momenta, and so on). In Section 4 the corresponding deformation of the nonrelativistic quantum mechanics is derived in terms of **measurable quantities** for the simplest case of the free massive particle m [12].
In Section 5, reasoning from the **measurability**, the author analyzes the low- and high-energy behavior of a very interesting gravitational model – Heuristic Markov's Model [13].
Finally, Conclusion presents the main course for further studies in this field.

2 Uncertainty Principle at All Scales Energies, Minimal Length and Measurability

We begin not with Heisenberg's Uncertainty Principle (HUP) [14]

$$\Delta x \geq \frac{\hbar}{\Delta p} \qquad (2)$$

but with its widely known high-energy generalization the Generalized Uncertainty Principle (GUP) [15]– [27]:

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_P^2 \frac{\Delta p}{\hbar}. \quad (3)$$

Here α' is the model-dependent dimensionless numerical factor and l_P is the Planckian length. (Note that the normalization $\Delta x \Delta p \geq \hbar$ is used rather than $\triangle x \Delta p \geq \hbar/2$.)
Note also that initially GUP (3) was derived within a string theory [15]– [18] and, subsequently, in a series of works far from this theory [19] – [25] it has been demonstrated that on going to high (Planck's) energies in the right-hand side of HUP (2) an additional ≪high-energy≫ term $\propto l_P^2 \frac{\triangle p}{\hbar}$ appears. Of particular interest is the work [19], where by means of a simple gedanken experiment it has been demonstrated that with regard to the gravitational interaction (3) is the case.
As (3) – quadratic inequality, then it naturally leads to the minimal length $l_{min} = \xi l_P = 2\sqrt{\alpha'} l_P$.
This means that the theory for the quantities with a particular dimension has a **minimal measurement unit**. At least, all the quantities with such a dimension should be ≪quantized≫, i. e. be measured by an integer number of these **minimal units** of measurement.
Specifically, if l_{min} – **minimal unit** of length, then for any length L we have the ≪**Integrality Condition**≫ **(IC)**

$$L = N_L l_{min}, \quad (4)$$

where $N_L > 0$ – integer.
What are the consequences for GUP (3) and HUP (2)?
Assuming that HUP is to a high accuracy derived from GUP on going to low energies or that HUP is a special case of GUP at low values of the momentum, we have

$$(GUP, \Delta p \rightarrow 0) = (HUP). \quad (5)$$

By the language of N_L from(4), (5) is nothing else but a change-over to the following:

$$(N_{\Delta x} \approx 1) \rightarrow (N_{\Delta x} \gg 1). \quad (6)$$

The assumed equalities in (2) and (3) may be conveniently rewritten in terms of l_{min} with the use of the deformation parameter α_a. This parameter has been introduced earlier in the papers [28]–[36] as a **deformation parameter** on going from the canonical quantum mechanics to the quantum mechanics at Planck's scales (early Universe) that is considered to be the quantum mechanics with the minimal length (QMML):

$$\alpha_a = l_{min}^2/a^2, \tag{7}$$

where a is the measuring scale.

Definition 1.
Deformation is understood as an extension of a particular theory by inclusion of one or several additional parameters in such a way that the initial theory appears in the limiting transition [37].

Then with the equality ($\Delta p \Delta x = \hbar$) (3) is of the form

$$\Delta x = \frac{\hbar}{\Delta p} + \frac{\alpha_{\Delta x}}{4}\Delta x. \tag{8}$$

In this case due to formulae (4) and (6) the equation (8) takes the following form:

$$N_{\Delta x} l_{min} = \frac{\hbar}{\Delta p} + \frac{1}{4N_{\Delta} x} l_{min} \tag{9}$$

or

$$(N_{\Delta x} - \frac{1}{4N_{\Delta} x}) l_{min} = \frac{\hbar}{\Delta p}. \tag{10}$$

That is

$$\Delta p = \frac{\hbar}{(N_{\Delta x} - \frac{1}{4N_{\Delta} x}) l_{min}}. \tag{11}$$

From (9)–(11) it is clear that HUP (2) in the case of the equality appears to a high accuracy in the limit $N_{\Delta} x \gg 1$ in conformity with (6).
It is easily seen that the parameter α_a from (7) is discrete as it is nothing else but

$$\alpha_a = l_{min}^2/a^2 = \frac{l_{min}^2}{N_a^2 l_{min}^2} = \frac{1}{N_a^2}. \tag{12}$$

At the same time, from (12) it is evident that α_a is irregularly discrete.
It is clear that from formula (11) at low energies ($N_\Delta x \gg 1$), up to a constant

$$\frac{\hbar^2}{l_{min}^2} = \frac{\hbar c^3}{4\alpha' G} \tag{13}$$

we have

$$\alpha_{\Delta x} = (\Delta p)^2. \tag{14}$$

But all the above computations are associated with the nonrelativistic case. As known, in the relativistic case, when the total energy of a particle with the mass m and with the momentum p equals [38]:

$$E = \sqrt{p^2c^2 + m^2c^4}, \tag{15}$$

a minimal value for Δx takes the form [39]:

$$\Delta x \approx \frac{c\hbar}{E}. \tag{16}$$

And in the **ultrarelativistic case**

$$E \approx pc \tag{17}$$

this means simply that

$$\Delta x \approx \frac{\hbar}{p}. \tag{18}$$

Provided the minimal length l_{min} is involved and considering the ≪**Integrality Condition**≫ **(IC)** (4), in the general case for (16) at the energies considerably lower than the Planck energies $E \ll E_P$ we obtain the following:

$$\Delta x = N_{\Delta x} l_{min} \approx \frac{c\hbar}{E}, \quad or \quad E \approx \frac{c\hbar}{N_{\Delta x}}. \tag{19}$$

Similarly, at the same energy scale in the ultrarelativistic case we have

$$p \approx \hbar / N_{\Delta x}. \tag{20}$$

Note that all the foregoing results associated with GUP and with its limiting transition to HUP for the pair $(\Delta x, \Delta p)$, as shown in [30], may be in **ultrarelativistic case** easily carried to the ≪energy - time≫ pair $(\Delta t, \Delta E)$. Indeed (3) gives [30]:

$$\frac{\Delta x}{c} \geq \frac{\hbar}{\Delta pc} + \alpha' l_P^2 \frac{\Delta p}{c\hbar}, \tag{21}$$

then

$$\Delta t \geq \frac{\hbar}{\Delta E} + \alpha' \frac{l_p^2}{c^2} \frac{\Delta pc}{\hbar} = \frac{\hbar}{\Delta E} + \alpha' t_p^2 \frac{\Delta E}{\hbar}. \tag{22}$$

where according to (17) the difference between ΔE and $\Delta(pc)$ can be neglected and t_P is the Planck time $t_P = L_P/c = \sqrt{G\hbar/c^5} \simeq 0,54\ 10^{-43} sec.$ From whence it follows that we have a maximum energy of the order of Planck's:

$$E_{max} \sim E_P$$

Then the foregoing formulae (2)–(10) are rewritten by substitution as follows:

$$\Delta x \rightarrow \Delta t; \Delta p \rightarrow \Delta E; l_{min} \rightarrow t_{min}; N_L \rightarrow N_{t=L/c} \tag{23}$$

Specifically, (10) takes the form

$$(N_{\Delta t} - \frac{1}{4N_{\Delta}t})t_{min} = \frac{\hbar}{\Delta E}. \tag{24}$$

As shown, for the ultrarelativistic case there is t_{min}.
Next we assume that for **all cases** there is a minimal measuring unit of time

$$t_{min} = l_{min}/v_{max} = l_{min}/c. \tag{25}$$

Then, similar to (4), we get the ≪**Integrality Condition**≫ **(IC)** for any time t:

$$t \equiv t(N_t) = N_t t_{min}, \tag{26}$$

for certain $|N_t| \geq 0$ – integer.
According to (24), let us define the corresponding energy E

$$E \equiv E(N_t) = \frac{\hbar}{|N_t - \frac{1}{4N_t}|t_{min}}. \tag{27}$$

Note that at low energies $E \ll E_P$, that is for $|N_t| \gg 1$, the formula (27) naturally takes the following form:

$$E \equiv E(N_t) = \frac{\hbar}{|N_t| t_{min}} = \frac{\hbar}{|t(N_t)|}. \qquad (28)$$

Definition 2.
1) Let us define the quantity having the dimensions of length L or time t **measurable***, when it satisfies the relation (4) (and respectively (26)).*

2) Let us define the quantity having the dimensions of momentum p or energy E **measurable***, when it satisfies in the corresponding cases (nonrelativistic and relativistic) the foregoing formulas (11),(19),(20),(27) for the momentums and energies. At low energies ($E \ll E_P$) this means that p and E , within the known multiplicative constants and sign, are coincident with $1/N_L$,$1/N_t$, where $|N_L| \gg 1$,$|N_t| \gg 1$ – integers.*

3) Let us define any physical quantity **measurable***, when its value is consistent with points 1) and 2) of this Definition.*

Thus, **measurable infinitesimal changes** in length (and hence in time) are **impossible** and any such changes are dependent on the existing energies.
In particular, a minimal possible **measurable** change of length is l_{min}. It corresponds to some maximal value of the energy E_{max} or momentum P_{max}, If $l_{min} \propto l_P$, then $E_{max} \propto E_P$,$P_{max} \propto P_{Pl}$, where $P_{max} \propto P_{Pl}$, where P_{Pl} is where the Planck momentum. Then denoting in **nonrelativistic** case with $\triangle_p(w)$ a **minimal measurable** change every spatial coordinate w corresponding to the energy E we obtain

$$\triangle_{P_{max}}(w) = \triangle_{E_{max}}(w) = l_{min}. \qquad (29)$$

Evidently, for lower energies (momentums) the corresponding values of $\triangle_p(w)$ are higher and, as the quantities having the dimensions of length are quantized (4), for $p \equiv p(N_p) < p_{max}$, $\triangle_p(w)$ is transformed to

$$|\triangle_{p(N_p)}(w)| = |N_p| l_{min}. \qquad (30)$$

where $|N_p| > 1$-integer so that we have

$$|N_p - \frac{1}{4N_p}|l_{min} = \frac{\hbar}{|p(N_p)|}. \qquad (31)$$

In the relativistic case the formula (29) holds, whereas (30) and (31) for $E \equiv E(N_E) < E_{max}$ are replaced by

$$|\triangle_{E(N_E)}(w)| = |N_E| l_{min}, \qquad (32)$$

where $|N_E| > 1$-integer.
Next we assume that at high energies $E \propto E_P$ there is a possibility only for the **nonrelativistic** case or **ultrarelativistic** case.
Then for the **ultrarelativistic** case, with regard to (17)–(24), formula (31) takes the form

$$|N_E - \frac{1}{4N_E}|l_{min} = \frac{\hbar c}{E(N_E)} = \frac{\hbar}{|p(N_p)|}, \qquad (33)$$

where $N_E = N_p$.
In the relativistic case at low energies we have

$$E \ll E_{max} \propto E_P. \qquad (34)$$

In accordance with (15),(16) formula (30) is of the form

$$|\triangle_{E(N_E)}(w)| = |N_E| l_{min} = \frac{\hbar c}{E(N_E)}, |N_E| \gg 1 - integer. \qquad (35)$$

In the nonrelativistic case at low energies (34) due to (31) we get

$$|\triangle_{p(N_p)}(w)| = |N_p| l_{min} = \frac{\hbar}{|p(N_p)|}, |N_p| \gg 1 - integer. \qquad (36)$$

In a similar way for the time coordinate t, by virtue of formulas (26)–(28), at the same conditions we have similar formulas (29),(30),(31)

$$\triangle_{E_{max}}(t) = t_{min}. \qquad (37)$$

For $E \equiv E(N_t) < E_{max}$

$$|\triangle_{E(N_t)}(t)| = |N_t| t_{min}, \tag{38}$$

where $|N_E| > 1$-integer, so that we obtain

$$|N_t - \frac{1}{4N_t}| t_{min} = \frac{\hbar c}{E(N_t)}. \tag{39}$$

In the relativistic case at low energies

$$E \ll E_{max} \propto E_P, \tag{40}$$

in accordance with (15),(16), formula (30) takes the form

$$|\triangle_{E(N_t)}(w)| = |N_t| l_{min} = \frac{\hbar c}{E(N_t)}, |N_t| \gg 1 - integer. \tag{41}$$

Now we consider a very simple but important example of the **nonmeasurable quantity** from [2]:
The infinitesimal increment of entropy dS of the spherically symmetric holographic screen $\mathcal{S}$ with the radius R and with the surface area A is a **nonmeasurable quantity**.
Really, it is obvious that infinitesimal variations of the screen surface area dA are possible only in a continuous theory involving no l_{min}.
When $l_{min} \propto l_P$ is involved, the minimal variation $\triangle A$ is evidently associated with a minimal variation in the radius R

$$R \rightarrow R \pm l_{min} = R \pm \triangle_{E_{max}}(R) \tag{42}$$

it is dependent on R and growing with $\sim R$ for $R \gg l_{min}$ (possible only at the maximum energy $E_{max} \propto E_P$)

$$\triangle_{\pm} A(R) = (A(R \pm l_{min}) - A(R)) \propto (\pm 2R l_{min} + l_{min}^2) \propto (\pm 2N_R + 1), \tag{43}$$

where $N_R = R/l_{min}$, as indicated above in (4).
But if $E \ll E_{max} \propto E_P$, then a minimal variation in the radius R is obviously greater than l_{min}

$$R \rightarrow R \pm \triangle_{E(N_E)}(R) = R \pm |N_E| l_{min}, \tag{44}$$

and in this case in the right-hand side of (43), within the constant l^2_{min}, we have the number quickly growing at low energies as well:

$$\triangle_{\pm}A(R) = (A(R \pm l_{min}) - A(R)) \propto (\pm 2RN_E l_{min} + N_E^2 l_{min}^2) \\ \propto N_E(\pm 2N_R + N_E). \qquad (45)$$

In any case from this it follows that dA has no chance to be a **measurable quantity**, as its measurability suggests measurability of the quantity dR, and this is impossible.
Since dS, within a multiplicative constant, equals dA [40],[41]: $dS \propto dA/4$, dS is also a **nonmeasurable quantity**.
Because of this, the ≪main instrument≫ in the well-known paper [42] that is the infinitesimal variation dN in the bit numbers N on the holographic screen $\mathcal{S}$ is also a **nonmeasurable quantity** [2] as $dN \propto dS$ to within an integer factor.
It is easily seen that the infinitesimal variation dV in the volume V of $\mathcal{S}$ is also a **nonmeasurable quantity**.
The following comments are of particular importance.

Comment 1
Obviously, when l_{min} is involved, the foregoing formulas for the momentums $p(N_p)$ and for the energies $E(N_E), E(N_t)$ may **certainly** give the highly accurate result that is close to the experimental one only at the verified low energies: $|N_p| \gg 1, |N_E| \gg 1, |N_t| \gg 1$.
In the case of high energies $E \propto E_{max} \propto E_P$ or, what is the same $|N_p| \rightarrow 1, |N_E| \rightarrow 1, |N_t| \rightarrow 1$, we have a certain, experimentally unverified, model with a correct low-energy limit

In what follows, within the scope of the above definitions, we consider, unless stated otherwise, **only measurable** increments (variations) of the space-time quantities and the corresponding momentums and energies.
Proceeding from all the above, this simply means that all minimal increments (variations) of the space-time quantities are dependent on the present energies and coincident with the corresponding **minimal uncertainties** from the **Uncertainty Principle at the All Scales Energies.**
In conclusion of this Section note the following.

Earlier HUP has been considered as a low-energy limit of GUP (5) with the minimal length attribute $l_{min} \propto l_P$. However, it is easily seen that even if we have no notion about the existence of GUP (3) (i. e. of the high-energy term $\propto l_P^2 \Delta p/\hbar$ in the right-hand side of (3)), still the use of **the infinitesimal quantities** dx_μ from the viewpoint of their **measurability** is problematic as at low energies, where HUP (2)) is valid, we have «great» Δx_μ, certainly higher than infinitesimal dx_μ. Because of this, to «measure» dx_μ we should go to high energies or to «small» Δx_μ.
At the same time, even at the ultimate (Planck's) energies a minimal «detected» (i. e. measurable) space-time volume is, within the known constants, restricted to

$$V_{min} \propto l_P^4. \tag{46}$$

Consequently, «detectability» of the infinitesimal space-time volume

$$V_{dx_\mu} = (dx_\mu)^4 \tag{47}$$

is impossible as this necessitates going to infinitely high energies

$$E \rightarrow \infty. \tag{48}$$

3 Space-Time Lattice of Measurable Quantities and Dual Lattice

So, provided the minimal length l_{min} exists, two lattices are naturally arising.

I.Lattice of the **space-time variation** – Lat_{S-T} representing, to within the known multiplicative constants, the sets of nonzero integers $N_w \neq 0$ and $N_t \neq 0$ in the corresponding formulas from the set (30)–(41) for each of the three space variables $w \doteq x; y; z$ and the time variable t

$$Lat_{S-T} \doteq (N_w, N_t), N_w \neq 0, N_t \neq 0 - integers. \tag{49}$$

Which restrictions should be initially imposed on these sets of nonzero integers?

It is clear that in every such set all the integers (N_w, N_t) should be sufficiently ≪close≫, because otherwise, for one and the same space-time point, variations in the values of its different coordinates are associated with principally different values of the energy E which are ≪far≫ from each other.
Note that the words ≪close≫ and ≪far≫ will be elucidated further in this text.
Thus, at the admittedly low energies (Low Energies) $E \ll E_{max} \propto E_P$ the low-energy part (sublattice) $Lat_{S-T}[LE]$ of Lat_{S-T} is as follows:

$$Lat_{S-T}[LE] = (N_w, N_t) \equiv (|N_x| \gg 1, |N_y| \gg, |N_z| \gg 1, |N_t| \gg 1). \quad (50)$$

At high energies (High Energies) $E \to E_{max} \propto E_P$ we, on the contrary, have the sublattice $Lat_{S-T}[HE]$ of Lat_{S-T}

$$Lat_{S-T}[HE] = (N_w, N_t) \equiv (|N_x| \approx 1, |N_y| \approx 1, |N_z| \approx 1, |N_t| \approx 1). \quad (51)$$

II. Next let us define the lattice **momentums-energies variation** Lat_{P-E} as a set to obtain
$(p_x(N_{x,p}), p_y(N_{y,p}), p_z(N_{z,p}), E(N_t))$ in the nonrelativistic and ultrarelativistic cases for all energies, and as a set to obtain
$(E_x(N_{x,E}), E_y(N_{y,E}), E_z(N_{z,E}), E(N_t))$ in the relativistic (but not ultrarelativistic) case for low energies $E \ll E_P$, where all the components of the above sets conform to the space coordinates (x, y, z) and time coordinate t and are given by the corresponding formulas(29)–(41) from the previous Section.
Note that, because of the suggestion made after formula (34) in the previous Section, we can state that the foregoing sets exhaust all the collections of momentums and energies possible for the lattice Lat_{S-T}.
From this it is inferred that, in analogy with point I of this Section, within the known multiplicative constants, we have

$$Lat_{P-E} \doteq \left(\frac{1}{N_w - \frac{1}{1/4N_w}}, \frac{1}{N_t - \frac{1}{1/4N_t}}\right), \quad (52)$$

where $N_w \neq 0, N_t \neq 0$-integers from (49). Similar to (50), we obtain the low-energy (Low Energy) part or the sublattice $Lat_{P-E}[LE]$ of Lat_{P-E}

$$Lat_{P-E}[LE] \approx (\frac{1}{N_w}, \frac{1}{N_t}), |N_w| \gg 1, |N_t| \gg 1. \tag{53}$$

In accordance with (51), the high-energy (High Energy) part (sublattice) $Lat_{P-E}[HE]$ of Lat_{P-E} takes the form

$$Lat_{P-E}[HE] \approx (\frac{1}{N_w - \frac{1}{1/4N_w}}, \frac{1}{N_t - \frac{1}{1/4N_t}}), |N_w| \to 1, |N_t| \to 1. \tag{54}$$

Considering **Comment 1** from the previous Section, it should be noted that, as currently the low energies $E \ll E_{max} \propto E_P$ are verified by numerous experiments and thoroughly studied, the sublattice $Lat_{P-E}[LE]$ (53) is correctly defined and rigorously determined by the sublattice $Lat_{S-T}[LE]$ (50).
However, at high energies $E \to E_{max} \propto E_P$ we can't be so confident ? the sublattice $Lat_{P-E}[HE]$ may be defined more exactly.
Specifically, α_a is obviously a small parameter. And, as demonstrated in [43],[44], in the case of GUP we have the following:

$$[\vec{x}, \vec{p}] = i\hbar(1 + a_1\alpha + a_2\alpha^2 + ...). \tag{55}$$

But, according to (12), $|1/N_a| = \sqrt{\alpha_a}$, then, due to (55), the denominators in the right-hand side of (54) may be also varied by adding the terms $\propto 1/N_w^2, \propto 1/N_w^3 ..., \propto 1/N_t^2, \propto 1/N_t^3 ...$, that is liable to influence the final result for $|N_w| \to 1, |N_t| \to 1$.
The notions «close» and «far» for Lat_{P-E} will be completely determined by the dual lattice $Lat_{S-T}[LE]$ and by formulas (30)– (41).
It is important to note the following.
In the low-energy sublattice $Lat_{P-E}[LE]$ all elements are varying very smoothly enabling the approximation of a continuous theory.
First, we consider this fact in terms of the mathematical instruments of this paper at the end of the following Section and then it will be considered, with the use of more convincing arguments, in Section 5.

4 Quantum Theory in Terms of Measurable Quantities. First Steps.

Let us begin with the nonrelativistic case.
4.1 We try to resolve Quantum Mechanics (QM) in terms of **measurable quantities** for the nonrelativistic case as a deformation of conventional (nonrelativistic) QM [12] within the scope of **Definition 1** of Section 2.
Let us term this deformation the energy-dependent deformation or ≪E-**deformation**≫.

4.2. As the instruments for the above-mentioned ≪E-**deformation**≫ we use formulas (30)for $\triangle_{p(N_p)}(w)$ and (38) for $\triangle_{E(N_t)}(t)$. In what follows instead of $\triangle_{E(N_t)}(t)$ we use $\triangle_{E(N_{E,t})}(t)$.

It is clear that the principal variations are mainly associated with the operators having in their representation the partial derivatives $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}$.
In accordance with the above arguments, specifically with formula (30), the first three of them should be replaced by the operators ≪inverse≫ to $\triangle_{p(N_p)}$:

$$\frac{\partial}{\partial w}\Psi(w) \mapsto [\triangle^{-1}_{p(N_p)}(w)]_{\pm}\Psi(w) \equiv \frac{1}{N_p l_{min}}(\Psi(w+N_p l_{min})-\Psi(w)), \quad |N_p|>1-integer, \tag{56}$$

where w – any of the space coordinates, i.e. $w \doteq x;y;z$, $\triangle w = \hbar/p(N_p) = |N_p|l_{min}$; the sign ≪+≫ in the left-hand side of (56) is for $N_p>0$, whereas ≪−≫ ? is for $N_p<0$. As Quantum Mechanics is a low-energy theory ($E \ll E_{max} \propto E_P$), we have $|N_p| \gg 1$.
Eliminating square brackets in the right-hand side of (56) and writing it for the function $\Psi(w)$ in a more customary form, we obtain

$$\frac{\partial \Psi(w)}{\partial w} \mapsto \triangle^{-1}_{p(N_p)}(w)_{\pm}\Psi = \frac{\Psi(w+N_p l_{min})-\Psi(w)}{N_p l_{min}}. \tag{57}$$

Similarly, for the time coordinate, $\frac{\partial}{\partial t}$ is replaced by the operator ≪inverse≫ to $\triangle_{E(N_{E,t})}$, i.e. we have $\triangle^{-1}_{E(N_{E,t})}$:

$$\frac{\partial}{\partial t}\Psi(t)\mapsto[\triangle^{-1}_{E(N_{E,t})}(t)]_{\pm}\Psi(t)\equiv\frac{1}{N_{E,t}t_{min}}(\Psi(t+N_{E,t}t_{min})-\Psi(t)),$$
$$|N_{E,t}|>1-integer. \quad (58)$$

$\triangle t=\hbar/E(N_{E,t})=|N_{E,t}|t_{min}$ and $|N_{E,t}|\gg 1$, and hence we obtain

$$\frac{\partial\Psi(t)}{\partial t}\mapsto\triangle^{-1}_{E(N_{E,t})}(t)_{\pm}\Psi=\frac{\Psi(t+N_{E,t}t_{min})-\Psi(t)}{N_{E,t}t_{min}}. \quad (59)$$

Obviously, with the set lower bound for the momentums $p\geq p_0>0$ (and respectively for the energies $E\geq E_0>0$), i.e. for $|N_p|\leq|N_{p_0}|<\infty,|N_{E,t}|\leq|N_{E_0,t}|<\infty$ in the limit $l_{min}\rightarrow 0,t_{min}\rightarrow 0$ we have:

$$\lim_{l_{min}\rightarrow 0}\triangle^{-1}_{p(N_p)}(w)_{\pm}=\frac{\partial}{\partial w},$$
$$\lim_{t_{min}\rightarrow 0}\triangle^{-1}_{E(N_{E,t})}(t)_{\pm}=\frac{\partial}{\partial t}. \quad (60)$$

Note that from formula (25) of Section 2 we derive the following:

$$(l_{min}\rightarrow 0)\Rightarrow(t_{min}\rightarrow 0). \quad (61)$$

Then, without loss of generality, we assume that hereinafter $N_p>0,N_{E,t}>0$, and in the low-energy case under consideration this means $N_p\gg 1,N_{E,t}\gg 1$ as the situation with negative N_p and $N_{E,t}$ is absolutely similar.
It is clear that all basic properties of the operators (paragraph 11, of Section 2 in [12]) for such ≪E**-deformation**≫ of Quantum Mechanics (QM) are retained.
In particular, for $\triangle^{-1}_{p(N_p)}(w)(\Psi_1(w)\Psi_2(w))$,$\Delta^{-1}_{E(N_{E,t}}(t)(\Psi_1(t)\Psi_2(t))$ we have:

$$\triangle^{-1}_{p(N_p)}(w)(\Psi_1(w)\Psi_2(w))=$$
$$=\frac{\Psi_1(w+N_pl_{min})\Psi_2(w+N_pl_{min})-\Psi_1(w)\Psi_2(w)}{N_pl_{min}}. \quad (62)$$

The numerator (62) denoted with $Numer'_w$ due to (57) is of the form

$$Numer'_w \equiv (N_p l_{min} \triangle^{-1}_{p(N_p)}(w)_\pm \Psi_1(w) + \Psi_1(w))$$
$$(N_p l_{min} \triangle^{-1}_{p(N_p)}(w)_\pm \Psi_2(w) + \Psi_2(w)) - \Psi_1(w)\Psi_2(w) =$$
$$N_p^2 l_{min}^2 (\triangle^{-1}_{p(N_p)}(w)_\pm \Psi_1)(\Delta^{-1}_{p(N_p)}(w)_\pm \Psi_2)$$
$$+N_p l_{min}(\triangle^{-1}_{p(N_p)}(w)_\pm \Psi_1)\Psi_2 + N_p l_{min}(\Delta^{-1}_{p(N_p)}(w)_\pm \Psi_2)\Psi_1. \quad (63)$$

It is easily seen that the second and the third terms in (63) make to(62) the contributions $(\triangle^{-1}_{p(N_p)}(w)_\pm \Psi_1)\Psi_2$ and $(\triangle^{-1}_{p(N_p)}(w)_\pm \Psi_2)\Psi_1$, respectively.
As regards the first term in (62), its contribution to (63) is equal to $N_p l_{min}(\triangle^{-1}_{p(N_p)}(w)_\pm \Psi_1)(\Delta^{-1}_{p(N_p)}$ and in limit $l_{min} \to 0$, $|N_p| \leq |N_{p_0}| < \infty$

$$\lim_{l_{min}\to 0} N_p l_{min}(\triangle^{-1}_{p(N_p)}(w)_\pm \Psi_1)(\triangle^{-1}_{p(N_p)}(w)_\pm \Psi_2) = 0. \quad (64)$$

Then in the limit

$$\lim_{l_{min}\to 0, |N_p|\leq|N_{p_0}|<\infty} \triangle^{-1}_{p(N_p)}(w)(\Psi_1(w)\Psi_2(w)) = \frac{\partial \Psi_1(w)}{\partial w}\Psi_2(w) + \frac{\partial \Psi_2(w)}{\partial w}\Psi_1(w). (65)$$

By the substitution of $w \mapsto t, l_{min} \mapsto t_{min}, N_E \mapsto N_{E,t}$ in all formulas (62)–(65) in a similar way we get

$$\lim_{t_{min}\to 0, |N_{E,t}|\leq|N_{E_0,t}|<\infty} \triangle^{-1}_{E(N_{E,t})}(t)(\Psi_1(t)\Psi_2(t)) = \frac{\partial \Psi_1(t)}{\partial t}\Psi_2(t) + \frac{\partial \Psi_2(t)}{\partial t}\Psi_1(t). (66)$$

This suggests that in the limiting transition $l_{min} \to 0$ we trivially obtain the expressions for the known commutators as follows:

$$\lim_{l_{min}\to 0, |N_p|\leq|N_{p_0}|<\infty} [\triangle^{-1}_{p(N_p)}(w), f(w)] = [\frac{\partial}{\partial w}, f(w)] = \frac{\partial f(w)}{\partial w};$$
$$f(w) = w, \lim_{l_{min}\to 0, |N_p|\leq|N_{p_0}|<\infty} [\triangle^{-1}_{p(N_p)}(w), w] = [\frac{\partial}{\partial w}, w] = 1. \quad (67)$$

Analogously, for $t_{min} \to 0$ we have

$$\lim_{t_{min}\to 0, |N_{E,t}|\leq|N_{E_0,t}|<\infty} [\triangle^{-1}_{E(N_{E,t})}(t), f(t)] = [\frac{\partial}{\partial t}, f(t)] = \frac{\partial f(t)}{\partial t};$$
$$f(t) = t, \lim_{t_{min}\to 0, |N_{E,t}|\leq|N_{E_0,t}|<\infty} [\triangle^{-1}_{E(N_{E,t})}(t), t] = [\frac{\partial}{\partial t}, t] = 1. \quad (68)$$

So, in the ≪continuous≫ limit $l_{min} \rightarrow 0, t_{min} \rightarrow 0$ the conventional QM [12] is involved.
The expressions for $\triangle^{-1}_{p(N_p)}(w), \triangle^{-1}_{E(N_{E,t})}(t)$ are dependent on ≪high numbers≫ $|N_p| \gg 1$,$|N_{E,t}| \gg 1$, respectively. But it is clear that in the above formalism the ordinary partial derivatives of a continuous theory (or of the conventional Quantum Mechanics) for the coordinate $\frac{\partial}{\partial w}$ and for the time $\frac{\partial}{\partial t}$ is most close to the case $|N_p| = 1$ in formulas (57) and $|N_{E,t}| = 1$ in (59), respectively.
As in the considered case $|\triangle w| \rightarrow l_{min}$ and $|\triangle t| \rightarrow t_{min}$, we obtain

$$\lim_{\triangle w \rightarrow 0} \frac{\Psi(w+\triangle w)-\Psi(w)}{\triangle w} = \lim_{|\triangle w| \rightarrow l_{min}} \frac{\Psi(w+\triangle w)-\Psi(w)}{\triangle w} = \\ = \frac{\Psi(w \pm l_{min})-\Psi(w)}{\pm l_{min}} \qquad (69)$$

and

$$\lim_{\triangle t \rightarrow 0} \frac{\Psi(t+\triangle t)-\Psi(t)}{\triangle t} = \lim_{|\triangle t| \rightarrow t_{min}} \frac{\Psi(t+\triangle t)-\Psi(t)}{\triangle t} = \\ = \frac{\Psi(t \pm t_{min})-\Psi(t)}{\pm t_{min}}. \qquad (70)$$

Paradox is in the fact that minimal increments l_{min} and t_{min} are associated with the maximal energy $E_{max} \propto E_P$. But in the case under study all the energies E are considerably lower than E_P:$E \ll E_P$.
At the same time, it is clear that for $|N_p| < \infty, |N_{E,t}| < \infty$ the limiting transitions (60) are independent of $N_p, N_{E,t}$.
Now let us proceed from the ≪coordinate≫ to the ≪momentum≫ representation.
Consider momentums at the point $p_w \doteq p_x, p_y, p_z$

$$[\triangle^{-1}_{p(N_p)}(p_w)]_{\pm}\Psi(p_w) = \frac{\Psi(p_w+\triangle p(N_p))-\Psi(p_w)}{\triangle p(N_p)}. \qquad (71)$$

From formula (36) in Section 2 it follows directly that, as in the considered case $|N_p| \gg 1$, we have $|\triangle p(N_p)| \approx \frac{\hbar}{|N_p| l_{min}}$ and (71) is of the form

$$[\triangle^{-1}_{p(N_p)}(p_w)]_{\pm}\Psi(p_w) = \frac{\Psi(p_w+\hbar/(N_p l_{min}))-\Psi(p_w))}{\hbar/(N_p l_{min})}. \qquad (72)$$

Then it is obvious that

$$\lim_{\triangle p(N_p)\rightarrow 0}[\triangle^{-1}_{p(N_p)}(p_w)]_{\pm}\Psi(p_w)=\lim_{|N_p|\rightarrow\infty}[\triangle^{-1}_{p(N_p)}(p_w)]_{\pm}\Psi(p_w)=\frac{\partial\Psi(p_w)}{\partial p_w}. \quad (73)$$

This means that, for the sufficiently high $|N_p|\gg 1$, we obtain

$$\triangle^{-1}_{p(N_p)}(p_w)]_{\pm}\Psi(p_w)\approx\frac{\partial\Psi(p_w)}{\partial p_w}. \quad (74)$$

In this way we have the following limiting transitions:

$$\lim_{l_{min}\rightarrow 0}\triangle^{-1}_{p(N_p)}(w)_{\pm}=\frac{\partial}{\partial w},$$
$$\lim_{t_{min}\rightarrow 0}\triangle^{-1}_{E(N_{E,t})}(t)_{\pm}=\frac{\partial}{\partial t},$$
$$\lim_{|N_p|\rightarrow\infty}\triangle^{-1}_{p(N_p)}(p_w)_{\pm}=\frac{\partial}{\partial p_w}. \quad (75)$$

And $\lim\limits_{|N_p|\rightarrow\infty}\triangle p(N_p)\rightarrow dp$ means that in the case under study, for the sufficiently high $|N_p|\gg 1$, we have

$$\triangle p(N_p)\approx dp. \quad (76)$$

In fact,(75) and (76) demonstrate that, for the sufficiently low energies, in the «momentum» representation (71)–(74) the resolved «E-**deformation**» of QM is practically continuous and approaching QM.
Then, for the wave function $\Psi(\mathbf{p},t)$ ([12], formula (11), paragraph 12 in Section II), where $(\mathbf{p},E)$ belongs to the lattice $Lat_{P-E}[LE]$ and $(\mathbf{r},t)$ belongs to the lattice $Lat_{S-T}[LE]$, by virtue of (76) we have

$$\widetilde{\Psi}_{N_0}(\mathbf{r},t)=\sum_{N_i,N_i\geq N_0}F(\mathbf{p})e^{i(\mathbf{p}r-Et)/\hbar}\triangle\mathbf{p}_{\mathbf{N_i}}\rightarrow\Psi(\mathbf{r},t)=\int F(\mathbf{p})e^{i(\mathbf{p}r-Et)/\hbar}d\mathbf{p} \quad (77)$$

or as a matter of fact

$$\widetilde{\Psi}_{N_0}(\mathbf{r},t)=\sum_{N_i,N_i\geq N_0}F(\mathbf{p})e^{i(\mathbf{p}r-Et)/\hbar}\triangle\mathbf{p}_{\mathbf{N_i}}\approx\Psi(\mathbf{r},t)=\int F(\mathbf{p})e^{i(\mathbf{p}r-Et)/\hbar}d\mathbf{p}, \quad (78)$$

because for sufficiently low N_0, i.e. for high momentums $\mathbf{p}$, the integral $\int F(\mathbf{p})e^{i(\mathbf{p}r-Et)/\hbar}d\mathbf{p}$ is simply undefined.
Next, as (77),(78) occur due to the existing limits (75), we can, with the use of (12)–(15) from paragraph 12 in Section II of [12], derive the Schrodinger equation for the free massive particle m as follows:

$$i\hbar \lim_{t_{min}\rightarrow 0} \Delta^{-1}_{E(N_{E,t})}(t)_{\pm}\widetilde{\Psi}_{N_0}(\mathbf{r},t) \approx i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\mathbf{\Delta}\Psi(\mathbf{r},t) \approx \approx -\frac{\hbar^2}{2m}\mathbf{\Delta}_{\mathbf{p(N_p)}}\widetilde{\Psi}_{N_0}(\mathbf{r},t). \quad (79)$$

Here, as usual, $\mathbf{\Delta}$ – Laplace operator

$$\nabla = \{\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\} = \lim_{l_{min}\rightarrow 0}\{\triangle^{-1}_{p(N_p)}(x)_{\pm},\Delta^{-1}_{p(N_p)}(y)_{\pm},\triangle^{-1}_{p(N_p)}(z)_{\pm}\} \equiv \lim_{l_{min}\rightarrow 0}\nabla_{p(N_p)} \quad (80)$$

$$\mathbf{\Delta}_{\mathbf{p(N_p)}} \equiv \nabla_{p(N_p)}\nabla_{p(N_p)}, \mathbf{\Delta} \equiv \nabla\nabla \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} = = \lim_{l_{min}\rightarrow 0}\nabla_{p(N_p)}\nabla_{p(N_p)} = \lim_{l_{min}\rightarrow 0}\mathbf{\Delta}_{\mathbf{p(N_p)}}. \quad (81)$$

Again using formula (75), or precisely its first two lines, we obtain

$$E \Rightarrow i\hbar \lim_{t_{min}\rightarrow 0}\triangle^{-1}_{E(N_{E,t})}(t)_{\pm} = i\hbar\frac{\partial}{\partial t},$$

$$\mathbf{p} \Rightarrow \frac{\hbar}{i}\{\lim_{l_{min}\rightarrow 0}\triangle^{-1}_{p(N_p)}(x)_{\pm},\lim_{l_{min}\rightarrow 0}\Delta^{-1}_{p(N_p)}(y)_{\pm},\lim_{l_{min}\rightarrow 0}\triangle^{-1}_{p(N_p)}(z)_{\pm}\} = \frac{\hbar}{i}\nabla, \quad (82)$$

where, as usual, $E = p^2/2m$.
Thus, the ≪E-**deformation**≫ defined above for small momentums $p \ll P_{max} \propto P_{pl}$ (or for $|N_p| \gg 1$) in the limit $l_{min} \rightarrow 0, t_{min} \rightarrow 0$ gives the conventional(nonrelativistic) QM [12]and hence is its deformation within the scope of **Definition 1** from Section 2.

4.3. Note that both of the above restrictions ($|N_p| \gg 1$ or $|N_p| \rightarrow \infty$ and $l_{min} \rightarrow 0, t_{min} \rightarrow 0$) of the limiting transition

$$\ll \mathbf{E-deformation} \gg \Rightarrow QM \quad (83)$$

are directly associated with the limiting transition $\alpha_a \to 0$, where α_a is the deformation parameter from formula (7) in Section 2.
Indeed, if a is a **nonmeasurable** quantity, we have

$$\lim_{l_{min}\to 0} \alpha_a = \lim_{l_{min}\to 0} l_{min}^2/a^2 = 0. \tag{84}$$

But if a is a **measurable** quantity and, according to (4), $a = N_a l_{min}$,where $|N_a| \geq 1$ – integer, then

$$\lim_{N_a^2\to\infty} \alpha_a = \lim_{N_a^2\to\infty} 1/N_a^2 = 0. \tag{85}$$

Formulas (84) and (85) point to the fact that α_a is the **deformation parameter** for the given ≪**E-deformation**≫, being the ≪**additional parameter**≫ mentioned in **Definition 1** of Section 2.
Of great importance is the following comment.
Comment 2 The above-mentioned limits for α_a (84) and (85) are practically indistinguishable but the first of them (84) leads to a continuous theory, whereas the second (85)– to a discrete theory, ≪**nearly continuous**≫ for small values of momentums and, at least experimentally, indistinguishable from the theory to which the limit (84) is leading.

In the following Section consideration is given to a key role of the parameter α_a in studies of a very interesting gravitational model –Heuristic Markov's Model, and also to the significance of **Comment 2** as applied to the low-energy limit of this model.

5 Heuristic Markov's Model

This heuristic model was introduced in the work [13] at the early eighties of the last century. This model already considered by the author in his previous paper [44] is treated from the standpoint of the above-mentioned arguments. In [13], it is assumed that ≪by the universal decree of nature a quantity of the material density ϱ is always bounded by its upper value given by the expression that is composed of fundamental constants≫ ([13], p.214):

$$\varrho \leq \varrho_p = \frac{c^5}{G^2\hbar}, \quad (86)$$

with ϱ_p as ≪Planck's density≫.
Then the quantity

$$\wp_\varrho = \varrho/\varrho_p \leq 1 \quad (87)$$

is the **deformation parameter** as it is used in [13] to construct the following **of Einstein's equations deformation or** $\wp_\varrho$**-deformation** ([13],formula (2)):

$$R_\mu^\nu - \frac{1}{2}R\delta_\mu^\nu = \frac{8\pi G}{c^4}T_\mu^\nu(1-\wp_\varrho^2)^n - \Lambda\wp_\varrho^{2n}\delta_\mu^\nu, \quad (88)$$

where $n \geq 1/2$, T_μ^ν–energy-momentum tensor, Λ– cosmological constant.
The case of the parameter $\wp_\varrho \ll 1$ or $\varrho \ll \varrho_p$ correlates with the classical Einstein equation, and the case when $\wp_\varrho = 1$ – with the de Sitter Universe.
In this way (88) may be considered as $\wp_\varrho$-deformation of the General Relativity.
As shown in [44], $\wp_\varrho$-of Einstein's equations deformation (88) is nothing else but α-deformation of GR for the parameter α_l at $x = l$ from (7).
If $\varrho = \varrho_l$ is the average material density for the Universe of the characteristic linear dimension l, i.e. of the volume $V \propto l^3$, we have

$$\wp_{l,\varrho} = \frac{\varrho_l}{\varrho_p} \propto \alpha_l^2 = \omega\alpha_l^2, \quad (89)$$

where ω is some computable factor.
Then it is clear that α_l-representation (88) is of the form

$$R_\mu^\nu - \frac{1}{2}R\delta_\mu^\nu = \frac{8\pi G}{c^4}T_\mu^\nu(1-\omega^2\alpha_l^4)^n - \Lambda\omega^{2n}\alpha_l^{4n}\delta_\mu^\nu, \quad (90)$$

or in the general form we have

$$R_\mu^\nu - \frac{1}{2}R\delta_\mu^\nu = \frac{8\pi G}{c^4}T_\mu^\nu(\alpha_l) - \Lambda(\alpha_l)\delta_\mu^\nu. \quad (91)$$

But, as r.h.s. of (91) is dependent on α_l of any value and particularly in the case $\alpha_l \ll 1$, i.e. at $l \gg l_{min}$, l.h.s of (91) is also dependent on α_l of any value and (91) may be written as

$$R_{\mu}^{\nu}(\alpha_l)-\frac{1}{2}R(\alpha_l)\delta_{\mu}^{\nu}=\frac{8\pi G}{c^4}T_{\mu}^{\nu}(\alpha_l)-\Lambda(\alpha_l)\delta_{\mu}^{\nu}. \qquad (92)$$

Thus, in this specific case we obtain the explicit dependence of GR on the available energies $E \sim 1/l$, that is insignificant at low energies or for $l \gg l_{min}$ and, on the contrary, significant at high energies, $l \rightarrow l_{min}$.

(5.1.1)**Low energies. Nonmeasurable case**. In this case at low energies, using formulas (7) and certainly (84)in the limit $l_{min}=0$ for $a=l$, we get a **continuous theory** coincident with the General Relativity.

(5.1.2)**Low energies. Measurable case**. In this case at low energies, using formulas (7), (12), and certainly (85) for $l_{min} \neq 0$, for $a=l$ (and hence for $N_l \gg 1$), we get a **discrete theory** which is a ≪**nearly continuous theory**≫, practically similar to the General Relativity with the slowly (smoothly) varying parameter $\alpha_{l(t)}$, where t – time.

So, due to low energies and momentums ($E \ll E_P, p \ll P_{Pl}$), the ≪**continuous case**≫ 5.1.1) (General Relativity) and the ≪**discrete case**≫ 5.1.2) that is actually a ≪**nearly continuous case**≫ are practically indistinguishable in line with **Comment 2** in the preceding Section.

(5.2)**At high energies we consider the measurable case** only. Then it is clear that at high energies the parameter $\alpha_{l(t)}$ is discrete and for the limiting value of $\alpha_{l(t)}=1$ we get a discrete series of equations of the form (91)(or a single equation of this form met by a discrete series of solutions) corresponding to $\alpha_{l(t)}=1;1/4;1/9;...$
As this takes place, $T_{\mu}^{\nu}(\alpha_l)\approx 0$, and in both cases 5.1.2) and 5.2) $\Lambda(\alpha_l)$ is not longer a cosmological constant, being a dynamical cosmological term.

Note that because of formula (14) in Section 2, $\sqrt{\alpha_{l(t)}}$ in cases (5.1.2) and (5.2) is an element of the lattice Lat_{P-E} from Section 3. And in case (5.1.2) it is an element of the sublattice $Lat_{P-E}[LE]$, whereas case 5.2) is associated with the sublattice $Lat_{P-E}[HE]$.

6 Conclusion

6.1. The main idea of the author is to demonstrate in Section 5 **the existence of the correct limiting high-energy transition**:

$$(5.1.2) \stackrel{High\ Energy}{\Rightarrow} (5.2) \tag{93}$$

and **the nonexistence of the correct limiting high-energy transition**:

$$(5.1.1) \stackrel{High\ Energy}{\Rightarrow} (5.2). \tag{94}$$

In the general case, based on the parameter α_a at the end of Section 4, this means that **there exists the correct limiting high-energy transition**:

$$\lim_{l_{min}\neq 0,|N_a|\gg 1} \alpha_a \stackrel{High\ Energy}{\Rightarrow} \lim_{l_{min}\neq 0,|N_a|\approx 1} \alpha_a \tag{95}$$

and **there is no correct limiting high-energy transition**

$$\lim_{l_{min}=0} \alpha_a \stackrel{High\ Energy}{\Rightarrow} \lim_{l_{min}\neq 0,|N_a|\approx 1} \alpha_a. \tag{96}$$

However, the whole theoretical physics, in which presently at low energies $E \ll E_P$ the minimal length l_{min}, is not involved (i. e. $l_{min} = 0$), is framed around the search for **nonexistent limits** (94) in a particular case of the model considered in Section 5 and gravity as a whole, and also in the general case (96) in terms of the parameter α_a, respectively.
6.2 Proceeding from the above, the program of further studies should be as follows.
6.2.1. To advance for the conventional and continuous theories with $l_{min} = 0$)– Quantum Theory (QT) and Gravity (General Relativity (GR)) at low energies — the corresponding low-energy theories (the so-called ≪**low-energy counterparts**≫) $QT[LE]^{l_{min}}$,$Grav[LE]^{l_{min}}$ based on the notion of the minimal length $l_{min} \neq 0$, **measurable quantities** in line with **Definition 2** and with the parameter α_a (7), (12) from Section 2, forming a ≪**nearly continuous theory**≫ in terms of the parameter α_a and being practically

indistinguishable from QT and GR, at least experimentally.
6.2.2. To frame for these ≪**low-energy counterparts**≫ the correct **limiting high-energy transition** (95):

$$QT[LE]^{l_{min}} \stackrel{N_a \to 1}{\Rightarrow} QT[HE]^{l_{min}},$$
$$Grav[LE]^{l_{min}} \stackrel{N_a \to 1}{\Rightarrow} Grav[HE]^{l_{min}}. \quad (97)$$

6.3 According to the hypothesis advanced by the author, some of the problems characteristics for the conventional theories (QT and GR), where $l_{min} = 0$, in particular the problem of ultraviolet and infrared divergence in a quantum field theory [45], will be lacking in their ≪**low-energy counterparts**≫ with $l_{min} \neq 0$: $QT[LE]^{l_{min}}$ and $Grav[LE]^{l_{min}}$.
Provided this hypothesis is true, a great work is required to study the structures $QT[LE]^{l_{min}}$ and $Grav[LE]^{l_{min}}$, specifically their symmetries and the like.
At the same time, such a study opens new possibilities. In particular, assuming the **measurable quantity** Compton wavelength $\bar{\lambda}_C$ a ≪**point object**≫ of the massive particle m [45]

$$\bar{\lambda}_C = \frac{\lambda_C}{2\pi} = \frac{\hbar}{mc}, \quad (98)$$

we can derive, according to (4) of Section 2, (98) of the following form:

$$\bar{\lambda}_C = N_{\bar{\lambda}_C} l_{min} = \frac{\hbar}{mc} \quad (99)$$

or

$$m = \frac{\hbar}{N_{\bar{\lambda}_C} l_{min} c} \propto \frac{1}{N_{\bar{\lambda}_C}}, \quad (100)$$

that is,to within the known multiplicative constant, m is an element of the sublattice $Lat_{P-E}[LE]$ from Section 3. Possibly this is associated with **the fermion masses hierarchy problem** [46].

Conflict of Interests
The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] Robert. M. Wald, *General Relativity*, The University Chicago Press Chicago and London (1984).

[2] A.E.Shalyt-Margolin, Minimal Length and the Existence of Some Infinitesimal Quantities in Quantum Theory and Gravity, Advances in High Energy Physics,Volume 2014 (2014), Article ID 195157, 8 pages

[3] G. Amelino-Camelia, Quantum Spacetime Phenomenology, *Living Rev.Rel.* ,vol.16,pp.5–129,2013;arxiv.org/abs/gr-qc/0806.0339.

[4] L. Garay, Quantum gravity and minimum length, *International Journal of Modern Physics A*, vol. 10, no. 2, pp. 145–146, 1995.

[5] G. Amelino-Camelia, L. Smolin,Prospects for constraining quantum gravity dispersion with near term observations, *Phys.Rev.D.* vol.80,084017,2009 .

[6] G.Gubitosi et al.,A constraint on planck-scale modifications to electrodynamics with CMB polarization data, *JCAP*, vol. 0908, pp.021-034,2009.

[7] G.Amelino-Camelia,Building a case for a planck-scale-deformed boost action: The planck-scale particle-localization limit. *Int.J.Mod.Phys.D.* vol.14,pp.2167–2180,2005.

[8] S. Hossenfelder et al.,Signatures in the Planck Regime, *Phys. Lett.B,* vol.575,pp.85–99,2003.

[9] S. Hossenfelder,Running Coupling with Minimal Length, *Phys.Rev.D,*vol.70,p.105003,2004;

[10] S. Hossenfelder,Self-consistency in Theories with a Minimal Length, *Class. Quant. Grav.,* vol.23,pp.1815–1821,2006.

[11] S. Hossenfelder, Minimal Length Scale Scenarios for Quantum Gravity, *Living Rev.Rel.,* vol.16,pp.2–91,2013.

[12] Albert Messiah, *Quantum Mechanics*, Volume I, North Holland Publishing Company, 1967

[13] M. A. Markov, Ultimate Matter Density as the Universal Low of Nature,*Pis'ma v ZHETF* vol. 36, p.p.214-216,1982.

[14] W. Heisenberg,Uber den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, *Zeitschrift fur Physik.*, vol.43,no 3-4,pp.172–198,1927; English translation: J. A. Wheeler and H. Zurek, in Quantum Theory and Measurement, Princeton Univ. Press, 1983, pp. 62-84.

[15] G. A. Veneziano, Stringy nature needs just two constants, *Europhysics Letters*, vol. 2, no. 3, pp. 199–211, 1986.

[16] D. Amati, M. Ciafaloni, and G. Veneziano, Can spacetime be probed below the string size? *Physics Letters B*, vol. 216, no. 1-2, pp. 41–47, 1989.

[17] E.Witten, Reflections on the fate of spacetime, *Physics Today*, vol. 49, no. 4, pp. 24–28, 1996.

[18] Joseph Polchinski,*String Theory*, Vol. I,II, Cambridge University Press,1998.

[19] R. J. Adler and D. I. Santiago, On gravity and the uncertainty principle, *Modern Physics Letters A*, vol. 14, no. 20, pp. 1371– 1378, 1999.

[20] D. V. Ahluwalia, Wave-particle duality at the Planck scale: freezing of neutrino oscillations, *Physics Letters A.*, vol. 275, no. 1-2,pp.31–35,2000.

[21] D. V. Ahluwalia, Interface of gravitational and quantum realms, *Modern Physics Letters A*, vol. 17, no. 15–17,pp.1135–1145,2002.

[22] M.Maggiore, The algebraic structure of the generalized uncertainty principle," *Physics Letters B*, vol. 319, no. 1–3, pp. 83–86, 1993.

[23] M.Maggiore,Black Hole Complementarity and the Physical Origin of the Stretched Horizon *Phys.Rev.* D49(1994) 2918–2921, [hep-th/9310157].

[24] M.Maggiore,A Generalized Uncertainty Principle in Quantum Gravity *Phys.Rev.* B304 (1993) 65–69, [hep-th/9301067].

[25] S.Capozziello,G.Lambiase and G.Scarpetta, The Generalized Uncertainty Principle from Quantum Geometry, *Int.J.Theor.Phys.* 39 (2000),15 [gr-qc/9910017]

[26] A. Kempf, G. Mangano, and R. B. Mann, Hilbert space representation of the minimal length uncertainty relation, *Physical Review D,* vol. 52, no. 2, pp. 1108–1118, 1995;

[27] Kourosh Nozari,Amir Etemadi, Minimal length, maximal momentum and Hilbert space representation of quantum mechanics, Physical Review D,vol.85,p.104029,2012.

[28] A. E. Shalyt-Margolin and J. G. Suarez, Quantum mechanics of the early universe and its limiting transition, http://arxiv.org/ abs/gr-qc/0302119.

[29] A. E. Shalyt-Margolin and J. G. Suarez, Quantum mechanics at Planck's scale and density matrix, *International Journal of Modern Physics D.,* vol. 12, no. 7, pp. 1265–1278, 2003.

[30] A. E. Shalyt-Margolin and A. Y. Tregubovich, Deformed density matrix and generalized uncertainty relation in thermodynamics, *Modern Physics Letters A.,* vol. 19, no. 1, pp. 71–82, 2004.

[31] A. E. Shalyt-Margolin, Non-unitary and unitary transitions in generalized quantum mechanics, new small parameter and information problemsolving, *Modern Physics Letters A.,* vol. 19, no. 5, pp. 391–403, 2004.

[32] A. E. Shalyt-Margolin, Pure states, mixed states and Hawking problem in generalized quantum mechanics, *Modern Physics Letters A.*, vol. 19, no. 27, pp. 2037–2045, 2004.

[33] A. E. Shalyt-Margolin, The universe as a nonuniform lattice in finite-volume hypercube: I. Fundamental definitions and particular features, *International Journal of Modern Physics D.,* vol. 13, no. 5, pp. 853–864, 2004.

[34] A. E. Shalyt-Margolin, The Universe as a nonuniformlattice in the finite-dimensional hypercube. II. Simple cases of symmetry breakdown and restoration, *International Journal of Modern Physics A,* vol. 20, no. 20-21, pp. 4951–4964, 2005.

[35] A. E. Shalyt-Margolin and V. I. Strazhev, The density matrix deformation in quantum and statistical mechanics of the early universe, in *Proceedings of the 6th International Symposium "Frontiers of Fundamental and Computational Physics"*, B. G. Sidharth, Ed., pp. 131–134, Springer, Berlin, Germany, 2006

[36] A. E. Shalyt-Margolin, The density matrix deformation in physics of the early universe and some of its implications, in *Quantum Cosmology Research Trends*, A. Reimer, Ed.,Horizons inWorld Physics no. 246, pp. 49–92, Nova Science, Hauppauge, NY, USA, 2005.

[37] L.Faddeev,Mathematical view of the evolution of physics,*Priroda,*vol.5,pp.11–16,1989.

[38] L.D.Landau and E.M.Lifshits,*Field Theory*,Theoretical Physics,vol.2,Moskow,Nauka,(1988).

[39] V.B. Berestetskii, E.M. Lifshitz and L.P. Pitaevskii,*Relativistic Quantum Theory.*,Theoretical Physics,vol.4,Moskow,Nauka,(1971).

[40] J. D. Bekenstein, Black holes and entropy, *Physical Review D.,* vol. 7, pp. 2333–2346, 1973.

[41] S. W. Hawking, Black holes and thermodynamics, *Physical Review D,* vol. 13, no. 2, pp.191–197,1976.

[42] E. Verlinde, On the origin of gravity and the laws of Newton, *Journal of High Energy Physics*, vol. 2011, no. 4, article 29, 2011.

[43] A. Shalyt-Margolin, Entropy in the present and early universe: new small parameters and dark energy problem, *Entropy*, vol. 12, no. 4, pp. 932–952, 2010.

[44] A. E. Shalyt-Margolin, Quantum theory at planck scale, limiting values, deformed gravity and dark energy problem, *International Journal of Modern Physics D,* vol. 21, no. 2, Article ID 1250013, 20 pages, 2012.

[45] M.E. Peskin, D.V. Schroeder, *An Introduction to Quantum Field Theory*,Addison-Wesley Publishing Company, 1995.

[46] S. V. Troitsky,Unsolved problems in particle physics,*Physics Uspekhi*, vol. 55, no. 1, pp. 72-95, 2012.