The Rational Number $\frac{\mathbf{n}}{\mathbf{p}}$ as a sum of two unit fractions

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1 Introduction

In a 2011 paper in the journal *Asian Journal of Algebra* (see [1]), the authors consider, among other equations, the diophantine equations

$$2xy = n(x+y) \text{ and } 3xy = n(x+y).$$

For the first equation, with n being an odd positive integer, they give the solution (in positive integers x and y) $\dfrac{n+1}{2} = x = \dfrac{(n-1)}{2} + 1$, $y = n\left(\dfrac{(n-1)}{2} + 1\right) = n\left(\dfrac{n+1}{2}\right)$.

For the second equation, with $n \equiv 2(\text{mod}3)$, they present the particular solution,

$$\frac{n+1}{3} = x = \frac{(n-2)}{3} + 1, \quad y = n\left(\frac{(n-2)}{3} + 1\right) = n\left(\frac{n+1}{3}\right).$$

If in the above equations we assume n to be prime, then these two equations become special cases of the diophantine equation, $nxy = p(x+y)$, with p being a prime and n a positive integer with $n \geq 2$.

This two-variable symmetric diophantine equation is the subject matter of this article; with the added condition that the integer n is not divisible by the prime p. Observe that this equation can be written equivalently in fraction form:

$$\frac{n}{p} = \frac{1}{x} + \frac{1}{y}.$$

This problem then can be approached from the point of view of decomposing a positive rational number into a sum of two unit fractions (i.e., two rational numbers whose numerators are equal to 1). The ancient Egyptians left behind an entire body of work involving the decomposition of a given fraction into a sum of two or more unit fractions. They did so by creating tables containing the decomposition of specific fractions into sums of unit fractions. An excellent source on the subject of the work of the ancient Egyptians on unit fractions is the book by David M. Burton, "The History of Mathematics, An Introduction" (see [2]). Note that thanks to the identity $\frac{1}{k} = \frac{1}{k+1} + \frac{1}{k(k+1)}$, a unit fraction can always be written as a sum of two unit fractions.

We state our theorem.

Theorem 1. *Let p be a prime, n a positive integer, $n \geq 2$. Also, assume that $gcd(p,n) = 1$ (equivalently, n is not divisible by p). Consider the two-variable symmetric diophantine equation,*

$$nxy = p(x+y) \tag{1}$$

with the two variables x and y taking values from the set $\mathbb{Z}^+$ of positive integers. Then, *(i) If $n = 2$ and $p \geq 3$, equation (1) has exactly three distinct solutions, the following positive integer pairs:*

$$(x, y) = (p, p),\ (x, y) = \left(p\left(\frac{p+1}{2}\right),\ \frac{p+1}{2}\right),$$

and its symmetric counterpart

$$(x, y) = \left(\frac{p+1}{2},\ p\left(\frac{p+1}{2}\right)\right).$$

(ii) If $n \geq 3$, and n is a divisor of $p+1$. Then equation (1) has exactly two distinct solutions:

$$(x, y) = \left(p\left(\frac{p+1}{n}\right),\ \frac{p+1}{n}\right) \text{ and } (x, y) = \left(\frac{p+1}{n},\ p\left(\frac{p+1}{n}\right)\right).$$

(iii) If n is not a divisor of $p+1$, Equation (1) has no solution.

2 A lemma from number theory

The following lemma, commonly referred to as Euclid's lemma, is of great significance in number theory.

Lemma 1. *(Euclid's lemma): Suppose that a, b, c are positive integers such that a is a divisor of the product bc; and $gcd(a, b) = 1$ (i.e., a and b are relatively prime), then a must be a divisor of c.*

Typically, this lemma and its proof can be found in an introductory number theory book. For example, see reference [3].

3 Proof of Theorem 1

First we show that the positive integer pairs listed in Theorem 1 are indeed solutions to Equation (1).

If $n=2$ and $p \geq 3$, then for $(x,y)=(p,p)$, a straightforward calculation shows both sides of (1) are equal to $2p^2$; and for $(x,y)=\left(\frac{p(p+1)}{2}, \frac{p+1}{2}\right)$, a calculation shows that both sides of (1) are equal to $\frac{p(p+1)^2}{2}$.

If $n \geq 3$ and n is a divisor of $p+1$, then for $(x,y)=\left(p\left(\frac{p+1}{n}\right), \frac{p+1}{n}\right)$, a calculation shows that both sides of equation (1) are equal to $\frac{p(p+1)^2}{n}$.

In the second part of this proof, we show that there are no other solutions to equation (1). To do so, we will demonstrate that if (t_1, t_2) is a solution to (1), then it must be one of the solutions listed in Theorem 1. So, let (t_1, t_2) be a positive integer solution to equation (1).

We have,

$$\left\{ \begin{array}{c} p(t_1+t_2) = nt_1t_2 \\ \\ t_1, t_2 \in \mathbb{Z}^+ \end{array} \right\} \tag{2}$$

Let d be the greatest common divisor of t_1 and t_2. Then

$$\left\{ \begin{array}{l} t_1 = du_1,\ t_2 = du_2; \\ \text{for relatively prime positive integers } u_1 \text{ and } u_2; \\ \gcd(u_1, u_2) = 1 \end{array} \right\} \tag{3}$$

From (2) and (3) we obtain,

$$p(u_1+u_2) = nd\,u_1u_2 \tag{4}$$

Since the prime p is relatively prime to n. By (4) and Lemma 1, it follows that p must divide the product du_1u_2. Since p is a prime number, it must divide at least one of d and u_1u_2. We distinguish between two cases: The case wherein p divides the product u_1u_2; and the case in which p is a divisor of d.

Case 1: *p is a divisor of u_1u_2.*

Since p is a prime, and the integers u_1 and u_2 are relatively prime by (3), and also in view of the fact that p divides the product u_1u_2, it follows that p must divide exactly one of u_1, u_2. It must divide one but not the other. Thus, there are two subcases in Case 1.

Subcase 1a being the one with $p|u_1$ (i.e., p divides u_1);
Subcase 1b: p divides u_2.

But these two subcases are symmetric since equation (4) is symmetric in u_1 and u_2. Thus, without loss of generality, we need only consider the subcase $p|u_1$. So we set

$$(u_1 = pv_1,\ v_1 \text{ a positive integer}) \tag{5}$$

Combining (5) with (4) we get,

$$\left\{ \begin{array}{c} pv_1 + u_2 = nd\, v_1 u_2 \\ \\ \text{or equivalently, } \ u_2 = v_1 \cdot (ndu_2 - p) \end{array} \right\} \tag{6}$$

According to (6), the positive integer v_1 is a divisor of u_2. But, by (5) v_1 is also a divisor of u_1. Since u_1 and u_2 are relatively prime by (3), it follows that

$$v_1 = 1 \tag{7}$$

Hence, by (7) and (6), we further obtain,

$$p = u_2(nd - 1) \tag{8}$$

According to (8), u_2 is a divisor of p, and since p is a prime it follows that either $u_2 = 1$ or $u_2 = p$. If $u_2 = 1$, then (8) yields $p + 1 = nd$ which implies that n is a divisor of $p+1$. Using $d = \dfrac{p+1}{n}$, $v_1 = 1, u_2 = 1$, we also get $u_1 = p$ (by (5)). So, by (3) we obtain the solution $t_1 = p\left(\dfrac{p+1}{n}\right), t_2 = \dfrac{p+1}{n}$ (already a verified solution in the first part of the proof). Now, if $u_2 = p$ in (8), then $2 = nd$ which implies either $n = 2$ and $d = 1$, or $n = 1$ and $d = 2$. But $n \geq 2$, so the latter possibility is ruled out. Thus, $u_2 = p$, $n = 2$, and $d = 1$. Also, by (7) we have $v_1 = 1$ and so $u_1 = p$ by (5).

Hence, (3) yields $t_1 = p = t_2$; (p, p) with $n = 2$ being a solution verified in the first part of the proof.

Case 2: *p is a divisor of d*

We set

$$(d = p\delta,\ \delta \text{ is a positive integer}) \tag{9}$$

by (9) and (4) we have,

$$u_1 + u_2 = n\delta u_1 u_2 \tag{10}$$

Clearly, by inspection, we see that equation (10) implies that the positive integers u_1 and u_2 must divide each other. Since they are relatively prime, it follows that

$$u_1 = u_2 = 1 \tag{11}$$

Equations (10) and (11) yield

$$2 = n\delta \tag{12}$$

Due to the fact $n \geq 2$, (12) implies that $n = 2$ and $\delta = 1$. So, by (11), (9), and (3), it is clear that (since $d = p$) $u_1 = u_2 = p$. This produces $(u_1, u_2) = (p, p)$, with $n = 2$. An already verified solution. The proof is complete. $\square$

References

[1] Kishan, Hari, Rani, Megha and Agarwal, Smiti, The Diophantine Equations of Second and Higher Degree of the Form $3xy = n(x + y)$ and $3xyz = n(xy + yz + zx)$, etc., *Asian Journal of Algebra* 4(1), (2011), pp. 31-37.

[2] Burton, David M., "The History of Mathematics, An Introduction", Sixth Edition, McGraw Hill, (2007), p. 40.

[3] Rose, Kenneth H., "Elementary Number Theory and Its Applications", 5th Edition, Pearson, Addison Wesley, (2005), p. 109.