Minimal Length,Measurability,and Special Relativity

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Abstract

In this paper the author begins a study of Special Relativity on the basis of the **measurability** notion introduced in his previous works.

1 Introduction. Main Target

This paper is a continuation of the earlier works published by the author [1]–[10]. The main idea and target of these works is to construct a correct quantum theory and gravity in terms of the variations (increments) dependent on the existent energies.
Within such a theory, the small and infinitesimal variations $dx, \delta x, dp, \delta p...$ which, by definition, are independent of the existent energies should be withdrawn, being included only on passage to the particular limit. First of all, this holds true for the infinitesimal space-time variations dx_{μ} as the latter are at the basis of continuous space-time.
At the present time physics is using (not without success) the mathematical apparatus based on the infinitesimal space-time variations (increments)

$$dt, dx_i, i = 1, ..., 3 \qquad (1)$$

This mathematical apparatus comes from mathematical analysis [11], calculus of variations [12] and classical mechanics [13],[14]. Continuous space-time forms the base thereof. Then, this tool has been successfully applied

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in Quantum Theory (QT) [15], Special Relativity, and General Relativity (GR) [16]. But, due to the introduction of ultraviolet and infrared divergences into a Quantum Theory and also due to the correct passage to the high-energy (ultraviolet) region in Gravity, we are facing very serious problems.
By the author's opinion, these problems are solvable but beyond the paradigm of continuous space-time. The principal idea of the papers [1]–[10] is as follows:
(1.1) Within a discrete model for continuous space-time, at low energies (which are far from the Planck energies) the results, to a high accuracy, are identical to those obtained by a continuous model for space-time (and in this case may be called the quasi-continuous model). But at high (Planck's) energies the indicated model is fundamentally discrete, leading to principally new results.
(1.2) All variations in any physical system considered in such a discrete model should be dependent on the existent energies.
What possibilities are offered by the proposed approach? When studying the relationship mentioned in point **(1.2)** over the whole energy scales, we can combine low and high energies as a single unit and can solve particular problems including the following: problems of transition from low to high energies and vice versa; the ultraviolet (UV) and infra-red (IR) divergence problem in QT and GR.
In brief, as regards realization of points **(1.1)** and **(1.2)**, the author has obtained the following results.
The paper [7] shows that in a quantum theory, proceeding from the natural assumptions mentioned in [9] and refined in [10] **Principle of Bounded Space-Time Variations (Increments)**, the notion of continuous space-time can appear only in a certain limit. And this is related to the fact that measurement procedure and Heisenberg's Uncertainty Principle (HUP) [17] play a fundamental role in the quantum theory.
If **Principle of Bounded Space-Time Variations (Increments)** is correct, minimal length l_{min} and time $t_{min} = l_{min}/c$ appear in the nature, (where c is light speed). Then, based on l_{min} and t_{min} definitions of **measurability** and **measurable** quantities may be correctly input in theory. Some examples show, although in this case it becomes discrete, but in low

energies, E, far from Planck $E \ll E_P$, it is close to the initial theory in continuous space-time. Real discreteness of the theory is manifested only at high energies E close to Planck $E \approx E_P$ [1],[7],[9],[10].
Based on **measurable** quantities, the construction of *Classical Mechanics* is given in the paper [10]. It has been demonstrated how, in the limiting transition from **measurable** quantities, we can have the infinitesimal space-time variations (increments) (formula (1)) as fundamental ingredients of *Classical Mechanics*.
In the present paper the principles of Special Relativity are given in terms of the notion of **measurability** (**measurable** quantities).
The paper is structured as follows. Section 2 presents the results relevant for further interpretation. The presentation is rather detailed as (i) in subsequent sections there are many references to the basic notions from Section 2; and (ii) some results from the previous works (for example, *Comment 2*.* are made more specific by the author because they are important for Sections 3 and 4.
The original results are given in Section 3.
Finally, Section 4 presents concluding comments and explanations.

2 Initial Data and Necessary Information

This section gives the necessary initial data from [1]–[10]. Part of this information is presented in [7],[9],[10].

2.1 Minimal length, Primary and Generalized Measurability

The present study is based on the **Definition I.** [10] (it is improvement of the **Supposition I** in [7],[9]) and on the **Supposition II.** from [7],[9]:
Definition I. Let's call as **primarily measurable variation** any small variation (increment) $\widetilde{\Delta} x_{\mu}$ of any spatial coordinate x_{μ} of the arbitrary point $x_{\mu}, \mu = 1, ..., 3$ in some space-time system R if it may be realized in the form of the uncertainty (standard deviation) Δx_{μ} when this coordinate is measured within the scope of Heisenberg's Uncertainty Principle (HUP)

[17]

$$\widetilde{\Delta} x_\mu = \Delta x_\mu, \Delta x_\mu \simeq \frac{\hbar}{\Delta p_\mu}, \mu = 1, 2, 3 \tag{2}$$

for some $\Delta p_\mu \neq 0$.

Similarly, for $\mu = 0$ for pair "time-energy" (t, E), let's call any small variation (increment) by **primarily measurable variation** in the value of time $\widetilde{\Delta} x_0 = \widetilde{\Delta} t_0$ if it may be realized in the form of the uncertainty (standard deviation) $\Delta x_0 = \Delta t$ and then

$$\widetilde{\Delta} t = \Delta t, \Delta t \simeq \frac{\hbar}{\Delta E} \tag{3}$$

for some $\Delta E \neq 0$. Here HUP is given for the nonrelativistic case. In the relativistic case HUP has the distinctive features [18] which, however, are of no significance for the general formulation of **Definition I.**, being associated only with particular alterations in the right-hand side of the second relation Equation (3).

It is clear that at low energies $E \ll E_P$ (momenta $P \ll P_{pl}$) **Definition I.** sets a lower bound for the **primarily measurable variation** $\widetilde{\Delta} x_\mu$ of any space-time coordinate x_μ.

At high energies E (momenta P) this is not the case if E (P) have no upper limit. But, according to the modern knowledge, E (P) are bounded by some maximal quantities E_{max}, (P_{max})

$$E \leq E_{max}, P \leq P_{max}, \tag{4}$$

where in general E_{max}, P_{max} may be on the order of Planck quantities $E_{max} \propto E_P, P_{max} \propto P_{pl}$ and also may be the trans-Planck's quantities.

In any case the quantities P_{max} and E_{max} lead to the introduction of the minimal length l_{min} and of the minimal time t_{min}.

Supposition II. There is the minimal length l_{min} as *a minimal measurement unit* for all **primarily measurable variations** having the dimension of length, whereas the minimal time $t_{min} = l_{min}/c$ as *a minimal measurement unit* for all quantities or **primarily measurable variations (increments)** having the dimension of time, where c is the speed of light.

l_{min} and t_{min} are naturally introduced as $\Delta x_\mu, \mu = 1, 2, 3$ and Δt in Equations (2) and (3) for $\Delta p_\mu = P_{max}$ and $\Delta E = E_{max}$.

For definiteness, we consider that E_{max} and P_{max} are the quantities on the order of the Planck quantities, then l_{min} and t_{min} are also on the order of Planck quantities $l_{min} \propto l_P$, $t_{min} \propto t_P$.

Definition I. and **Supposition II.** are quite natural in the sense that there are no physical principles with which they are inconsistent.
The combination of **Definition I.** and **Supposition II.** will be called the **Principle of Bounded Primarily Measurable Space-Time Variations (Increments)** or for short **Principle of Bounded Space-Time Variations (Increments)** with abbreviation (PBSTV).
As the minimal unit of measurement l_{min} is available for all the **primarily measurable variations** ΔL having the dimensions of length, the "Integrality Condition" (IC) is the case

$$\Delta L = N_{\Delta L} l_{min}, \tag{5}$$

where $N_{\Delta L} > 0$ is an integer number.
In a like manner the same "Integrality Condition" (IC) is the case for all the **primarily measurable variations** Δt having the dimensions of time. And similar to Equation (5), we get the for any time Δt:

$$\Delta t \equiv \Delta t(N_t) = N_{\Delta t} t_{min}, \tag{6}$$

where similarly $N_{\Delta L} > 0$ is an integer number too.
Definition 1 (**Primary or Elementary Measurability.**)
(1) *In accordance with the PBSTV let us define the quantity having the dimensions of length or time as* **primarily (or elementarily) measurable**, *when it satisfies the relation Equation (5) (and respectively Equation (6)).*
(2)*Let us define any physical quantity* **primarily (or elementarily) measurable**, *when its value is consistent with points (1) of this Definition.*
Since in fact PBSTV introduce the minimal length l_{min} for **primarily measurable variations**, instead of HUP, we can consider its widely known high-energy generalization—the Generalized Uncertainty Principle (GUP) that naturally leads to the minimal length l_{min} [19]–[30]:

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_P^2 \frac{\Delta p}{\hbar}. \tag{7}$$

Here α' is the model-dependent dimensionless numerical factor and l_P is the Planckian length. As Equation (7) is a quadratic inequality, then it naturally leads to the minimal length $l_{min} = \xi l_P = 2\sqrt{\alpha'} l_P$.
Due to (5), we have

$$\Delta x = N_{\Delta x} l_{min}. \tag{8}$$

Then the transition from high to low energies in GUP, i.e. $(GUP, \Delta p \rightarrow 0) = (HUP)$, is nothing else but

$$(N_{\Delta x} \approx 1) \rightarrow (N_{\Delta x} \gg 1). \tag{9}$$

Substituting (8) into (7) and making the necessary calculations, we can see that in the general case

$$\Delta p \equiv \Delta p_{N_{\Delta x}} = \frac{\hbar}{(N_{\Delta x} - \frac{1}{4N_{\Delta x}}) l_{min}}. \tag{10}$$

Whereas at low energies $E \ll E_P$

$$\Delta p \equiv \Delta p_{N_{\Delta x}} = \frac{\hbar}{N_{\Delta x} l_{min}}. \tag{11}$$

At the same time, for the corresponding energy E we get

$$\Delta E \equiv \Delta E(N_t) = \frac{\hbar}{(N_t - \frac{1}{4N_t}) t_{min}} \tag{12}$$

or for low energies

$$\Delta E \equiv \Delta E(N_t) = \frac{\hbar}{N_t t_{min}}. \tag{13}$$

In the relativistic case the formulae corresponding to (10),(12) have been derived in [2],[7].
Note that the above-mentioned formulae may be conveniently rewritten in terms of l_{min} with the use of the deformation parameter α_a [7]. This parameter has been introduced earlier in the papers [31]–[38] as a *deformation parameter* (in terms of paper [39]) on going from the canonical quantum mechanics to the quantum mechanics at Planck's scales (early Universe)

that is considered to be the quantum mechanics with the minimal length (QMML):

$$\alpha_a = l_{min}^2/a^2, \tag{14}$$

where a is the measuring scale.
Actually, with the equality ($\Delta p \Delta x = \hbar$) Equation 7 is of the form

$$\Delta x = \frac{\hbar}{\Delta p} + \frac{\alpha_{\Delta x}}{4}\Delta x. \tag{15}$$

In this case due to Equations (5), (9) and (15) takes the following form:

$$N_{\Delta x} l_{min} = \frac{\hbar}{\Delta p} + \frac{1}{4N_{\Delta} x} l_{min} \tag{16}$$

or

$$(N_{\Delta x} - \frac{1}{4N_{\Delta} x}) l_{min} = \frac{\hbar}{\Delta p}. \tag{17}$$

That is

$$\Delta p = \frac{\hbar}{(N_{\Delta x} - \frac{1}{4N_{\Delta} x}) l_{min}}. \tag{18}$$

From Equations (16)–(10) it is clear that HUP Equation (2) appears to a high accuracy in the limit $N_{\Delta} x \gg 1$ in conformity with Equation 9.

It is easily seen that the parameter α_a from Equation (14) is discrete as it is nothing else but

$$\alpha_a = l_{min}^2/a^2 = \frac{l_{min}^2}{N_a^2 l_{min}^2} = \frac{1}{N_a^2}. \tag{19}$$

At the same time, from Equation (19) it is evident that α_a is irregularly discrete.

It is clear that from Equation (10) at low energies ($|N_{\Delta} x| \gg 1$), up to a constant

$$\frac{\hbar^2}{l_{min}^2} = \frac{\hbar c^3}{4\alpha' G} \tag{20}$$

we have

$$\alpha_{\Delta x} = (\Delta p)^2, (i.e. \alpha_{\Delta x} \propto (\Delta p)^2). \tag{21}$$

However, physical quantities complying with **Definition 1** won't be enough for the research of physical systems.
Indeed, such a variable as

$$\alpha_{Nl_{min}}(Nl_{min}) = l_{min}/N, \tag{22}$$

(where $\alpha_{Nl_{min}}$ is taken from formula (19) at $a = Nl_{min}$), is fully expressed in terms *only* **Primarily Measurable Quantities** of **Definition 1** and that's why it may appear at any stage of calculations, but apparently doesn't comply with **Definition 1**. That's why it's necessary to introduce the following definition generalizing **Definition 1**:

Definition 2. Generalized Measurability
We shall call any physical quantity as **generalized-measurable** or for simplicity **measurable** if any of its values may be obtained in terms of **Primarily Measurable Quantities** of **Definition 1**.

In what follows for simplicity we will use the term **Measurability** instead of **Generalized Measurability**.
It's evident that any **primarily measurable quantity (PMQ)** is **measurable**. Generally speaking, the contrary is not correct, as indicated by formula (22).

Naturally, of course that, a minimal possible **primarily measurable** and change of length is l_{min}. It corresponds to some maximal value of the energy E_{max} or momentum P_{max}, If $l_{min} \propto l_P$, then $E_{max} \propto E_P$,$P_{max} \propto P_{Pl}$, where $P_{max} \propto P_{Pl}$, where P_{Pl} is where the Planck momentum. Then denoting in *nonrelativistic* case with $\triangle_p(w)$ a minimal **primarily measurable** change every spatial coordinate w corresponding to the energy E we obtain

$$\triangle_{P_{max}}(w) = \triangle_{E_{max}}(w) = l_{min}. \tag{23}$$

Evidently, for lower energies (momenta) the corresponding values of $\triangle_p(w)$ are higher and, as the quantities having the dimensions of length are transformed to

$$|\triangle_{p(N_p)}(w)| = |N_p - \frac{1}{4N_p}|l_{min}. \tag{24}$$

where $|N_p| > 1$ is an integer number so that we have

$$|N_p - \frac{1}{4N_p}|l_{min} = \frac{\hbar}{|p(N_p)|}, \tag{25}$$

where $p(N_p)$ is already **generalized-measurable** value.
In the relativistic case for **primarily measurable** variations the Equation (23) holds, whereas Equation (24) for $E \equiv E(N_E) < E_{max}$ is replaced by

$$|\triangle_{E(N_E)}(w)| = |N_E|l_{min}, \tag{26}$$

where $|N_E| > 1$ is an integer.

Next we assume that at high energies $E \propto E_P$ there is a possibility only for the *nonrelativistic* case or *ultrarelativistic* case.

Then for the all **measurable** variations in *ultrarelativistic* case, formula (25) takes the form [7]:

$$|N_E - \frac{1}{4N_E}|l_{min} = \frac{\hbar c}{E(N_E)} = \frac{\hbar}{|p(N_p)|}, \tag{27}$$

where $N_E = N_p$ and similarly the formula (25) $E(N_E)$ $(p(N_p))$ are **generalized-measurable** values too.

In the relativistic case at low energies we have

$$E \ll E_{max} \propto E_P. \tag{28}$$

and formula (24) is of the form

$$|\triangle_{E(N_E)}(w)| = |N_E|l_{min} = \frac{\hbar c}{E(N_E)}, |N_E| \gg 1 - integer. \tag{29}$$

And the energy $E(N_E)$ becomes the **primary measured** value.

In the nonrelativistic case at low energies Equation (28) due to Equation (25) we get

$$|\triangle_{p(N_p)}(w)| = |N_p|l_{min} = \frac{\hbar}{|p(N_p)|}, |N_p| \gg 1 - integer. \tag{30}$$

where $p(N_p)$ is the **primary measured** value too.

In a similar way for the time coordinate t, by virtue of Equations (6)–(13), at the same conditions we have similar Equation (23) a minimal **primarily measurable** change

$$\triangle_{E_{max}}(t) = t_{min}. \tag{31}$$

For $E \equiv E(N_t) < E_{max}$

$$|\triangle_{E(N_t)}(t)| = |N_t - \frac{1}{4N_t}|t_{min}, \tag{32}$$

where $|N_{E(N_t)}| > 1$ is an integer, so that we obtain similarly (25) and (27) **generalized-measurable** value $E(N_t)$ from equation

$$|N_t - \frac{1}{4N_t}|t_{min} = \frac{\hbar c}{E(N_t)}. \tag{33}$$

In the relativistic case at low energies

$$E \ll E_{max} \propto E_P, \tag{34}$$

equation (24) takes the form [7]:

$$|\triangle_{E(N_t)}(w)| = |N_t| l_{min} = \frac{\hbar c}{E(N_t)}, |N_t| \gg 1 - integer, \tag{35}$$

where $E(N_t)$ is already **primary measured** value.

We shall make several important **Commentaries**:

Comment 2.*
From the above formulae it follows that, within GUP, the **primarily measurable** variations (quantities) are derived to a high accuracy from the **generalized-measurable** variations (quantities) *only* in the low-energy limit $E \ll E_P$, (formula (9))
Comment 2.1..
What is the main difference between **Definition 1** and **Definition 2**?

2.1.1.**Definition 1** defines variables which may be obtained as a result of an immediate experiment.

2.1.2. **Definition 2** defines the variables which may be *calculated* based on **primarily measurable quantities**, i.e. based on the data obtained in previous clause 2.1.1.

Comment 2.2.
It's evident that HUP-derived (2) $\Delta p_i \doteq \Delta p_{i,HUP}; i = 1, ..., 3$ are **primarily measurable** quantities:

$$\Delta p_i \simeq \frac{\hbar}{\Delta x_i} = \frac{\hbar}{N_{\Delta x_i} l_{min}} \tag{36}$$

However, variables $\Delta p_i \doteq \Delta p_{i,GUP}$ obtained from GUP (7) and defined by formula (10) are already obviously not the same, but only **measurable** quantities.
From formulae (20) and (21) follows that in case of correctness of HUP (2) i.e. in low energies $E \ll E_{max} \propto E_P$, in notations of formulae (24)–(35)

$$\alpha_{N_p l_{min}}(HUP) \doteq \alpha_{\Delta x} = p(N_p)^2 \frac{l_{min}^2}{\hbar^2} = \frac{1}{N_p^2} \tag{37}$$

where $\Delta x = N_p l_{min}$ and $p(N_p)$ is calculated from formula (30).
However, in high energies $E \approx E_P$, HUP is replaced with GUP, **primarily measurable quantity** $p(N_p)$ from formula (30) is replaced with **generalized measurable quantity** $\Delta p_i \doteq \Delta p_{i,GUP}$ from formula (25).
Then $\alpha_{N_p l_{min}}(HUP)$ may be replaced with $\alpha_{N_p l_{min}}(GUP)$:

$$\begin{aligned}\alpha_{N_p l_{min}}(GUP) &= p(N_p, GUP)^2 \frac{l_{min}^2}{\hbar^2} = \\ &= \frac{l_{min}^2}{(N_p - \frac{1}{4N_p})^2 l_{min}^2} = \frac{1}{(N_p - \frac{1}{4N_p})^2}\end{aligned} \tag{38}$$

When going over from high energies $E \approx E_P$ to low energies $E \ll E_P$ we have:

$$\alpha_{N_p l_{min}}(GUP) \stackrel{(|N_p| \approx 1) \to (|N_p| \gg 1)}{\longrightarrow} \alpha_{N_p l_{min}}(HUP) \tag{39}$$

In what follows all the considerations are given in terms of **measurable quantities** in the sense of **Definition 2** given in this Section. *Of course, this apply and to variations of space-time coordinates.*

2.2 Space-Time Lattice of Primarily Measurable Quantities, Dual Lattice and $\alpha - lattice$

For convenience, we denote the minimal length $l_{min} \neq 0$ by ℓ and $t_{min} \neq 0$ by $\tau = \ell/c$.
So, provided the minimal length ℓ exists, two lattices are naturally arising.
I. Lattice of the space-time variation—Lat_{S-T} representing, to within the known multiplicative constants, for sets of nonzero integers $N_w \neq 0$ and $N_t \neq 0$ in corresponding formulae from the set Equations 24 and (35) for each of the three space variables $w \doteq x; y; z$ and the time variable t

$$Lat_{S-T} \doteq (N_w \ell, N_t \tau). \quad (40)$$

Which restrictions should be initially imposed on these sets of nonzero integers?

It is clear that in every such set all the elements $(N_w \ell, N_t \tau)$ should be sufficiently "close", because otherwise, for one and the same space-time point, variations in the values of its different coordinates are associated with principally different values of the energy E which are "far" from each other.

Note that the words "close" and "far" will be elucidated further in this text.

Thus, at the admittedly low energies (Low Energies) $E \ll E_{max} \propto E_P$ the low-energy part (sublattice) $Lat_{S-T}[LE]$ of Lat_{S-T} is as follows:

$$Lat_{S-T}[LE] = (N_w \ell, N_t \tau); |N_x| \gg 1, |N_y| \gg 1, |N_z| \gg 1, |N_t| \gg 1. \quad (41)$$

At high energies (High Energies) $E \to E_{max} \propto E_P$ we, on the contrary, have the sublattice $Lat_{S-T}[HE]$ of Lat_{S-T}

$$Lat_{S-T}[HE] = (N_w \ell, N_t \tau); |N_x| \approx 1, |N_y| \approx 1, |N_z| \approx 1, |N_t| \approx 1. \quad (42)$$

We will call lattice Lat_{S-T} (40) as **primary (or primitive)** *lattice of the space-time variation.*

II. Next let us define the lattice momenta-energies variation Lat_{P-E} as a set to obtain $(p_x(N_{x,p}), p_y(N_{y,p}), p_z(N_{z,p}), E(N_t))$ in the nonrelativistic and ultrarelativistic cases for all energies, and as a set to obtain $(E_x(N_{x,E}), E_y(N_{y,E}), E_z(N_{z,E}), E(N_t))$ in the relativistic (but not ultrarelativistic) case for low energies $E \ll E_P$, where all the components of the above sets conform to the space coordinates (x, y, z) and time coordinate t and are given by corresponding formulae (23)–(35) from the previous Section.
Note that, because of the suggestion made after formula Equation (28) in the previous Section, we can state that the foregoing sets exhaust all the collections of momentums and energies possible for the lattice Lat_{S-T}.
From this it is inferred that, in analogy with point I of this Section, within the known multiplicative constants, we have

$$Lat_{P-E} \doteq (\frac{1}{N_w - \frac{1}{4N_w}}, \frac{1}{N_t - \frac{1}{4N_t}}), \tag{43}$$

where $N_w \neq 0, N_t \neq 0$ are integer numbers from Equation (40). Similar to Equation (41), we obtain the low-energy (Low Energy) part or the sublattice $Lat_{P-E}[LE]$ of Lat_{P-E}

$$Lat_{P-E}[LE] \approx (\frac{1}{N_w}, \frac{1}{N_t}), |N_w| \gg 1, |N_t| \gg 1. \tag{44}$$

In accordance with Equation (42), the high-energy (High Energy) part (sublattice) $Lat_{P-E}[HE]$ of Lat_{P-E} takes the form

$$Lat_{P-E}[HE] \approx (\frac{1}{N_w - \frac{1}{4N_w}}, \frac{1}{N_t - \frac{1}{4N_t}}), |N_w| \to 1, |N_t| \to 1. \tag{45}$$

It is important to note the following.

In the low-energy sublattice $Lat_{P-E}[LE]$ all elements are varying very smoothly enabling the approximation of a continuous theory.

We will preserve the lattice Lat_{P-E}, but **primary** lattice Lat_{S-T} will be replaced with "$\alpha - lattice$", **measurable space-time quantities**, which will be denoted by Lat^{α}_{S-T}:

$$Lat^{\alpha}_{S-T} \doteq (\alpha_{N_w\ell}N_w\ell, \alpha_{N_t\tau}N_t\tau) = (\frac{\ell}{N_w}, \frac{\tau}{N_t}). \tag{46}$$

In the last formula by the variable $\alpha_{N_t\tau}$ we mean the parameter α corresponding to the length $(N_t\tau)c$:

$$\alpha_{N_t\tau} \doteq \alpha_{(N_t\tau)c}. \tag{47}$$

As in this case low energies $E \ll E_P$ are discussed, $\alpha_{N_w\ell}$ in this formula is consistent with the corresponding parameter from formula (37):

$$\alpha_{N_w\ell} = \alpha_{N_w\ell}(HUP) \tag{48}$$

As it was mentioned in the previous section, in the low-energy $E \ll E_{max} \propto E_P$ all elements of sublattice $Lat_{P-E}[LE]$ are varying very smoothly enabling the approximation of a continuous theory.
It is similar to the low-energy part of the $Lat^{\alpha}_{S-T}[LE]$ of lattice Lat^{α}_{S-T} will vary very smoothly:

$$Lat^{\alpha}_{S-T}[LE] = (\frac{\ell}{N_w}, \frac{\tau}{N_t}); |N_x| \gg 1, |N_y| \gg 1, |N_z| \gg 1, |N_t| \gg 1. \tag{49}$$

In Section 5 of [7] three following cases were selected:

(a) *"Quantum Consideration, Low Energies"*:

$$1 \ll |N_w| \leq \widetilde{\mathbf{N}};$$

(b) *"Quantum Consideration, High Energies"*:

$$|N_w| \approx 1;$$

(c) *"Classical Picture"*:

$$1 \ll \widetilde{\mathbf{N}} \ll |N_w|.$$

Here $\widetilde{\mathbf{N}}$ is a cutoff parameter , defined by the current task [7].
It is assumed that there is a correct transition to the infinite limit in *"Classical Picture"* (c)

$$|N_w| \to \infty, |N_t| \to \infty \tag{50}$$

That's why, if for three space coordinates $x_i; i = 1, 2, 3$ we introduce the following notation:

$$\begin{gathered}\Delta(x_i) \doteq \widetilde{\Delta}[\alpha_{N_{\Delta x_i}}] = \alpha_{N_{\Delta x_i}\ell}(N_{\Delta x_i}\ell) = \ell/N_{\Delta x_i}; \\ \frac{\Delta[F(x_i)]}{\Delta(x_i)} \equiv \frac{F(x_i + \Delta(x_i)) - F(x_i)}{\Delta(x_i)},\end{gathered} \tag{51}$$

then it's evident that

$$\lim_{|N_{\Delta x_i}| \to \infty} \frac{\Delta[F(x_i)]}{\Delta(x_i)} = \lim_{\Delta(x_i) \to 0} \frac{\Delta[F(x_i)]}{\Delta(x_i)} = \frac{\partial F}{\partial x_i}. \tag{52}$$

Respectively, for time $x_0 = t$ we have:

$$\begin{gathered}\Delta(t) \doteq \widetilde{\Delta}[\alpha_{N_{\Delta t}}] = \alpha_{N_{\Delta t}\tau}(N_{\Delta t}\tau) = \tau/N_{\Delta t}; \\ \frac{\Delta[F(t)]}{\Delta(t)} \equiv \frac{F(t + \Delta(t)) - F(t)}{\Delta(t)},\end{gathered} \tag{53}$$

then

$$\lim_{|N_{\Delta t}| \to \infty} \frac{\Delta[F(t)]}{\Delta(t)} = \lim_{\Delta(t) \to 0} \frac{\Delta[F(t)]}{\Delta(t)} = \frac{dF}{dt}. \tag{54}$$

We shall designate for momenta $p_i; i = 1, 2, 3$

$$\begin{gathered}\Delta p_i = \frac{\hbar}{N_{\Delta x_i}\ell}; \\ \frac{\Delta_{p_i} F(p_i)}{\Delta p_i} \equiv \frac{F(p_i + \Delta p_i) - F(p_i)}{\Delta p_i} = \frac{F(p_i + \frac{\hbar}{N_{\Delta x_i}\ell}) - F(p_i)}{\frac{\hbar}{N_{\Delta x_i}\ell}}.\end{gathered} \tag{55}$$

From where similarly (52) we get

$$\begin{aligned}\lim_{|N_{\Delta x_i}| \to \infty} \frac{F(p_i + \Delta p_i) - F(p_i)}{\Delta p_i} &= \lim_{|N_{\Delta x_i}| \to \infty} \frac{F(p_i + \frac{\hbar}{N_{\Delta x_i}\ell}) - F(p_i)}{\frac{\hbar}{N_{\Delta x_i}\ell}} = \\ &= \lim_{\Delta p_i \to 0} \frac{F(p_i + \Delta p_i) - F(p_i)}{\Delta p_i} = \frac{\partial F}{\partial p_i}.\end{aligned} \tag{56}$$

Therefore, in low energies $E \ll E_P$, i.e. at $|N_{\Delta x_i}| \gg 1; i = 0, ..., 3$ in passages to the limit (52),(54),(56) it's possible to obtain known partial derivatives like in case of continuous space-time.
It should be noted that $\alpha - lattice$ Lat^{α}_{S-T} (formula (46)) is not introduced artificially. It appears, but with *"factor"* 1/4 from equation (15) written in the form

$$\Delta x - \frac{\hbar}{\Delta p} = \frac{1}{4}\alpha_{\Delta x}\Delta x. \tag{57}$$

It is evident that the factor 1/4 in the right part (57) is not significant in this case.
In [10] it has been shown that, using the limiting transition to low energies (i.e., at $(|N_{\Delta t}|, |N_{\Delta x_i}|) \to \infty$ formula (52)–(56)) from $\alpha - lattice$ Lat^{α}_{S-T} , we can get the *Classical Mechanics* in terms of the measurable quantities.
In this case the infinitesimal space-time variations (1) are appearing in the limit

$$(\alpha_{N_{\Delta t}\tau} N_{\Delta t}\tau = \frac{\tau}{N_{\Delta t}} = p_{N_{\Delta t}c}\frac{\ell^2}{c\hbar}) \stackrel{N_{\Delta t}\to\infty}{\longrightarrow} dt,$$
$$(\alpha_{N_{\Delta x_i}\ell} N_{\Delta x_i}\ell = \frac{\ell}{N_{\Delta x_i}} = p_{N_{\Delta x_i}}\frac{\ell^2}{\hbar}) \stackrel{N_{\Delta x_i}\to\infty}{\longrightarrow} dx_i, 1 = 1, ..., 3. \tag{58}$$

In what follows, $N_{\Delta t}$ is denoted by $N_{\Delta x_0}$ and the set $N_{\Delta x_i}, i = 0, ..., 3$ is denoted with $(N_{\Delta x_\mu})$.

3 Special Relativity in Terms of Measurable Quantities. Start

3.1 Basic Definitions and Tools

It is assumed that we are in the region of low energies $E \ll E_P$, and we start from the primarily-measurable momenta $(p_{N_{\Delta x_i}}, p_{N_{\Delta t}c})$ in the left-hand side of the formula (58) to have

$$|N_{\Delta x_\mu}| \gg 1 \tag{59}$$

for all the elements of the set $(N_{\Delta x_\mu})$.

Definition 3.1
Let us denote any of the **fixed** sets of momenta $(p_{N_{\Delta x_i}}, p_{N_{\Delta t}c}) \doteq (p_{N_{\Delta x_\mu}})$ meeting the condition (59) the *canonically measurable basic set* of space-time, and, as the *canonically measurable prototype* of the infinitesimal space-time interval square in the " flat case"

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{60}$$

with respect to $(p_{N_{\Delta x_\mu}})$, we take the expression

$$\Delta s^2_{(p_{N_{\Delta x_\mu}})} \doteq \Delta s^2_{(N_{\Delta x_\mu})} \doteq \eta_{\mu\nu} \frac{\ell^4}{\hbar^2} p_{N_{\Delta x_\mu}} p_{N_{\Delta x_\nu}} = \eta_{\mu\nu} \frac{\ell^2}{N_{\Delta x_\mu} N_{\Delta x_\nu}}, \tag{61}$$

where $\eta_{\mu\nu}$ is the Minkowskian metric

$$||\eta_{\mu\nu}|| = ||\eta^{\mu\nu}|| = Diag\,(1, -1, -1, -1)\,. \tag{62}$$

Next let us find the measurable prototype (analog) for Lorentz transformations. Then, in what follows, we assume that the speed of light $c = 1$.
It is interesting to consider the Lorentz transformations [40],[41] in terms of measurable quantities.
The Hyperbolic rotations

$$\begin{aligned} t' &= t \cosh\alpha + x \sinh\alpha, \\ x' &= t \sinh\alpha + x \cosh\alpha, \\ \alpha = const, &\quad y' = y, \quad z' = z \end{aligned} \tag{63}$$

in the infinitesimal form will be as follows:

$$\begin{aligned} dt' &= \Delta(\alpha) dt = dt \cosh\alpha + dx \sinh\alpha, \\ dx' &= \Delta(\alpha) dx = dt \sinh\alpha + dx \cosh\alpha, \\ \alpha = const, &\quad dy' = dy, \quad dz' = dz. \end{aligned} \tag{64}$$

We suppose that the effect of the Lorentz Group (LG) on the *canonically measurable basic set* $(p_{N_{\Delta x_\mu}})$ is the same as on (dx_μ) with the corresponding index μ. Specifically, formula (64) has a measurable analog that, up to the factor $\ell^2/\hbar$, will be of the form

$$\begin{aligned} p_t(\alpha) &\doteq \Delta(\alpha)p_{N_{\Delta t}} = p_{N_{\Delta t}}\cosh\alpha + p_{N_{\Delta x}}\sinh\alpha, \\ p_x(\alpha) &\doteq \Delta(\alpha)p_{N_{\Delta x}} = p_{N_{\Delta t}}\sinh\alpha + p_{N_{\Delta x}}\cosh\alpha, \\ \alpha &= const, \quad p_{N_{\Delta y'}} = p_{N_{\Delta y}}, \quad p_{N_{\Delta z'}} = p_{N_{\Delta z}}. \end{aligned} \tag{65}$$

Let (p_μ) denote some orbital element of LG generated in the four-dimensional space by the *canonically measurable basic set* $(p_{N_{\Delta x_\mu}})$, and we have

$$(p_\mu) \in (LG)(p_{N_{\Delta x_\mu}}) = \{g(p_{N_{\Delta x_\mu}})|g \in (LG)\}. \tag{66}$$

Then (p_μ) is termed as the *measurable basics set* of space-time, and the expression

$$\Delta s^2_{(p_\mu)} = \eta_{\mu\nu}\frac{\ell^4}{\hbar^2}p_\mu p_\nu \tag{67}$$

is identified as the *measurable prototype of the infinitesimal space-time interval square* (60) with respect to (p_μ).

It is easy to check out that, for the random canonical element

$$(p_\mu) \doteq (p_{N_{\Delta x_\mu}}), \tag{68}$$

the hyperbolic rotations (66)

$$(p_{N_{\Delta x_\mu}}) \to \Delta(\alpha)(p_{N_{\Delta x_\mu}}) \tag{69}$$

retain their quadratic form (61), and we have

$$\Delta s^2_{\Delta(\alpha)(p_{N_{\Delta x_\mu}})} = \Delta s^2_{(p_{N_{\Delta x_\mu}})}. \tag{70}$$

So, the operator $\Delta(\alpha) \in (LG)$ retains the Minkowskian metric in the "measurable form" (61). In a similar way, we can show that the orthogonal group $\mathbf{O(3)}$ in force in the subspace generated by $(p_{N_{\Delta x_i}}), i = 1, 2, 3$ and the representations about the axes retain their quadratic forms (61).
Thus, the Lorentz Group (LG) that is in force for $(p_{N_{\Delta x_\mu}})$ from (68) retains (61), and for all $g \in (LG)$ we have

$$\Delta s^2_{g(p_{N_{\Delta x_\mu}})} = \Delta s^2_{(p_\mu)}. \tag{71}$$

As usual, the Lorentz boost (66) may be written as

$$\begin{aligned}\Delta(\alpha)p_{N_{\Delta t}} &= \frac{p_{N_{\Delta t}} + p_{N_{\Delta x}}V}{\sqrt{1-V^2}},\\ \Delta(\alpha)p_{N_{\Delta x}} &= \frac{p_{N_{\Delta t}}V + p_{N_{\Delta x}}}{\sqrt{1-V^2}},\\ p_{N_{\Delta y'}} = p_{N_{\Delta y}}, &\quad p_{N_{\Delta z'}} = p_{N_{\Delta z}},\end{aligned} \tag{72}$$

where $\cosh\alpha = 1/\sqrt{1-V^2}, \sinh\alpha = V/\sqrt{1-V^2}$.

The *canonically measurable prototype* of the speed components $\upsilon_{x_i} = dx_i/dt$ in this case will be the quantities $\widetilde{\upsilon_{x_i}} = \frac{p_{N_{\Delta x_i}}}{p_{N_{\Delta x_0}}} = \frac{N_{\Delta x_0}}{N_{\Delta x_i}}$.

Consequently, in the measurable form for the speed components in the general case of (72) we get the following:

$$\begin{aligned}\frac{\Delta(\alpha)x'}{\Delta(\alpha)t'} &= \frac{V + \widetilde{\upsilon_x}}{1 + \widetilde{\upsilon_x}V},\\ \frac{\Delta(\alpha)y'}{\Delta(\alpha)t'} &= \frac{\widetilde{\upsilon_y}\sqrt{1-V^2}}{1 + \widetilde{\upsilon_x}V}\\ \frac{\Delta(\alpha)z'}{\Delta(\alpha)t'} &= \frac{\widetilde{\upsilon_z}\sqrt{1-V^2}}{1 + \widetilde{\upsilon_x}V}.\end{aligned} \tag{73}$$

Then it is assumed that all the quantities considered are **measurable** in the sense of **Definition 2. Generalized Measurability** from Section 2. This is true for all variations in the indicated quantities. Besides, it is assumed that infinitesimal increments of a continuous theory (dx_μ) are replaced by $(\frac{\ell^2}{\hbar}p_{N_{\Delta x_\mu}}) = (\frac{\ell}{N_{\Delta x_\mu}})$. (Of course, here for $\mu = 0$ it is assumed that $\frac{\ell}{cN_{\Delta x_0}} = \frac{t_{min}}{N_{\Delta x_0}}$. But, as in the above text it was denoted that $c = 1$, in this case we can use the above notation.)

This supposition is quite natural for the four-dimensional radius vector $(ct, x, y, z) \doteq (x_\mu)$.

For all other four-dimensional vectors, tensors, pseudotensors, and the like this means that their components are dependent *only* on **measurable** coordinates and **measurable** variations of these coordinates. It is easily seen, these quantities retain all their principal properties involved in tensor analysis (the corresponding LG representations in the space of these quantities,

convolution, etc.) because, by definition, **measurability** is not affecting these properties.
It is interesting to consider in this formalism a very important problem associated with differentiation and integration.
Let the function $\varphi(x_\mu)$ of **measurable** coordinates (x_μ) be scalar. (As noted above, LG retains the property of **measurability**. So, subsequently there is no need to qualify this specially.) In a continuous theory, from $\varphi(x_\mu)$ we can construct the 4-vector as follows:

$$\frac{\partial\varphi}{\partial x_\mu} = (\frac{\partial\varphi}{c\partial t}, \nabla\varphi). \tag{74}$$

Since it was assumed that $c = 1$ and hence $\frac{\ell}{N} = \frac{\ell}{cN} = \frac{t_{min}}{N}$, the analog of (74) in the formalism under study for the *canonical* basic set $(p_{N_{\Delta x_\mu}})$ will be of the form

$$\begin{aligned} \frac{\widehat{\triangle}}{\widehat{\triangle}_{(N_{\Delta x_\mu})} x_\mu}\varphi = \frac{\widehat{\triangle}\varphi}{\widehat{\triangle}_{(N_{\Delta x_\mu})} x_\mu} &\doteq (\frac{\varphi(x_\mu + \frac{\ell^2}{\hbar} p_{N_{\Delta x_\mu}}) - \varphi(x_\mu)}{\frac{\ell^2}{\hbar} p_{N_{\Delta x_\mu}}}) = \\ &= (\frac{\varphi(x_\mu + \ell/N_{\Delta x_\mu}) - \varphi(x_\mu)}{\ell/N_{\Delta x_\mu}}). \end{aligned} \tag{75}$$

This quantity, similar to the quantity (75) in the continuous case, is a 4-vector because all its components are transformed by LG as the corresponding components in a continuous theory.
Consequently, similarly to a continuous theory, the scalar product of two 4-vectors is also scalar and we have $\frac{\widehat{\triangle}\varphi}{\widehat{\triangle}_{(N_{\Delta x_\mu})} x_\mu}$, and $(p_{N_{\Delta x_\mu}})$, where

$$\begin{aligned} \widehat{\triangle}\varphi = \widehat{\triangle}_{(N_{\Delta x_\mu})}\varphi = \sum_{N_{\Delta x_\mu}} (\varphi(x_\mu + \frac{\ell^2}{\hbar} p_{N_{\Delta x_\mu}}) - \varphi(x_\mu)) = \\ = \sum_{N_{\Delta x_\mu}} (\phi(x_\mu + \ell/N_{\Delta x_\mu}) - \phi(x_\mu)). \end{aligned} \tag{76}$$

In fact, $\widehat{\triangle}_{(N_{\Delta x_\mu})}\varphi$ in formula (76) is a highly accurate lattice approximation for the differential $d\varphi = \frac{\partial\varphi}{\partial x_\mu} dx_\mu$ in the continuous case.

Since LG transforms the set $(p_{N_{\Delta x_\mu}})$ similarly to (dx_μ), all integral formulae for the continuous case in the four-dimensional space retain their form in the proposed **measurable** variant of a theory, with the corresponding substitution:

$$(dx_\mu) \Rightarrow \frac{\ell^2}{\hbar}(p_{N_{\Delta x_\mu}}); \frac{\partial}{\partial x_\mu} \Rightarrow \frac{\widehat{\triangle}}{\widehat{\triangle}_{(N_{\Delta x_\mu})} x_\mu}; \int \Rightarrow \sum. \tag{77}$$

In particular, the **measurable** analog of a scalar – element of integration with respect to the four-dimensional volume Ω in the continuous case

$$d\Omega \equiv \prod_\mu dx_\mu \tag{78}$$

is also scalar

$$\Delta_{(N_{\Delta x_\mu})}\Omega \equiv \frac{\ell^8}{\hbar^4} \prod_{N_{\Delta x_\mu}} p_{N_{\Delta x_\mu}}. \tag{79}$$

It is easily seen. Indeed, for LG acting in the continuous case we have the transformation of the coordinate system (x_μ) to the new variables (x'_μ)

$$d\Omega \Rightarrow J d\Omega' = J \prod_\mu dx'_\mu, \tag{80}$$

where J – Jacobian that is equal to 1, of the corresponding transformation $g \in LG, (dx_\mu) \to g(dx_\mu) = (dx'_\mu)$.
But it is obvious that, on going from the *canonical* basic set $(p_{N_{\Delta x_\mu}})$ to the randomly measurable basic set $(p'_\mu) = g(p_{N_{\Delta x_\mu}})$, we get the same Jacobian $J = 1$. In what follows all the calculations are performed in terms of some *canonically measurable* basic set $(p_{N_{\Delta x_\mu}})$.
In the present formalism we easily can find an analog for the 4-speed of a continuous theory

$$u_\mu = \frac{dx_\mu}{ds}, \tag{81}$$

where, due to $c = 1$, we have

$$ds = \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu} = dt\sqrt{1 - \frac{dx^2 + dy^2 + dz^2}{dt^2}} = dt\sqrt{1 - \mathrm{v}^2}. \tag{82}$$

In this case

$$(dx_\mu) \rightarrow \frac{\ell^2}{\hbar}(p_{N_{\Delta x_\mu}}); ds \rightarrow \Delta s_{(p_{N_{\Delta x_\mu}})} = \frac{\ell^2}{\hbar} p_{N_{\Delta x_0}} \sqrt{1-\widetilde{\upsilon}^2}, \quad (83)$$

where $|\widetilde{\upsilon}| = \sqrt{\sum_{i\neq 0} \widetilde{\upsilon}_{x_i^2}}$ – absolute value of the three-dimensional speed of a particle in terms of the measurable quantities.
If $\widetilde{\upsilon} = (\widetilde{\upsilon_{x_1}}, \widetilde{\upsilon_{x_2}}, \widetilde{\upsilon_{x_3}})$ – vector of the three-dimensional speed of a particle in terms of the measurable quantities, then, similar to the continuous case, we obtain the **measurable** 4–speed as follows:

$$\widetilde{u_\mu} = \frac{\frac{\ell^2}{\hbar}(p_{N_{\Delta x_\mu}})}{\Delta s_{(p_{N_{\Delta x_\mu}})}} = (\frac{1}{\sqrt{1-\widetilde{\upsilon}^2}}, \frac{\widetilde{\upsilon}}{\sqrt{1-\widetilde{\upsilon}^2}}). \quad (84)$$

As follows from (84), $\widetilde{u_\mu}$ is a function of $(p_{N_{\Delta x_\mu}})$ and of $\Delta s_{(p_{N_{\Delta x_\mu}})}$, i.e., we have

$$\widetilde{u_\mu} \equiv \widetilde{u_\mu}[(p_{N_{\Delta x_\mu}})] \equiv \widetilde{u_\mu}[\Delta s_{(p_{N_{\Delta x_\mu}})}]. \quad (85)$$

Then it is assumed that all measurable variations in $\Delta s_{(p_{N_{\Delta x_\mu}})}$ are generated by the measurable variations of $(p_{N_{\Delta x_\mu}})$. For any fixed set $N_{\Delta x_\mu}$ having the attribute of (59), we can find a set (possibly, not a single one) $N^{'}_{\Delta x_\mu}$ satisfying the same attribute and minimizing the following expression:

$$|\Delta s_{(p_{N^{'}_{\Delta x_\mu}})} - \Delta s_{(p_{N_{\Delta x_\mu}})}| = min|\Delta s_{(p_{N^{''}_{\Delta x_\mu}})} - \Delta s_{(p_{N_{\Delta x_\mu}})}| \doteq |\widetilde{\Delta} s_{(p_{N_{\Delta x_\mu}})}|,$$
$$|N^{'}_{\Delta x_\mu}| \gg 1, (\Delta s_{(p_{N^{''}_{\Delta x_\mu}})} \neq \Delta s_{(p_{N_{\Delta x_\mu}})}). \quad (86)$$

It is obvious that

$$\lim_{|N_{\Delta x_\mu}| \rightarrow \infty} \widetilde{\Delta} s_{(p_{N_{\Delta x_\mu}})} = 0. \quad (87)$$

Then we denote

$$\frac{\widetilde{\Delta}\widetilde{u_\mu}[\Delta s_{(p_{N_{\Delta x_\mu}})}]}{\widetilde{\Delta} s_{(p_{N_{\Delta x_\mu}})}} \doteq \frac{\widetilde{u_\mu}[\Delta s_{(p_{N^{'}_{\Delta x_\mu}})}] - \widetilde{u_\mu}[\Delta s_{(p_{N_{\Delta x_\mu}})}]}{\Delta s_{(p_{N^{'}_{\Delta x_\mu}})} - \Delta s_{(p_{N_{\Delta x_\mu}})}}. \quad (88)$$

Formula (88) is a **measurable** analog of the continuous quantity du_μ/ds representing the 4-acceleration of the canonical theory.
In this case the 4-acceleration du_μ/ds itself may be derived in passage to the limit as follows:

$$\lim_{(|N_{\Delta x_\mu}|)\to\infty} \frac{\widetilde{\Delta}\widetilde{u_\mu}[\Delta s_{(p_{N_{\Delta x_\mu}})}]}{\widetilde{\Delta} s_{(p_{N_{\Delta x_\mu}})}} \doteq \frac{\widetilde{\Delta u_\mu}}{\widetilde{\Delta} s} = \frac{du_\mu}{ds}. \tag{89}$$

3.2 Relativistic Mechanics in Terms of Measurable Quantities

Now we can readily obtain **measurable** analogs of all the known quantities in the continuous case.
Specifically, an analog of the operation for a free particle in the continuous case [41]

$$\mathbf{S} = -\beta \int_a^b ds \tag{90}$$

in the present formalism is replaced by the sum

$$\mathbf{S}_{(N_{\Delta x_\mu})} \doteq -\beta \sum_a^b \Delta s_{(p_{N_{\Delta x_\mu}})}, \tag{91}$$

where the summation in the left-hand side is performed by the steps $\Delta s_{(p_{N_{\Delta x_\mu}})}$ along the world line between the two specified events a and b for the particle in the initial and in the finite places in the particular instants of time t_1 and t_2; β – certain variable.
Similarly, for the same operation written in the form of the Lagrangian L in the continuous case

$$\mathbf{S} = \int_{t_1}^{t_1} Ldt, \tag{92}$$

in the present formalism we have

$$\mathbf{S}_{(N_{\Delta x_\mu})} = \sum_{t_1}^{t_2} L_{(N_{\Delta x_\mu})} \frac{\ell}{N_{\Delta x_0}}, \tag{93}$$

were in the general case all the variables, on which $L_{(N_{\Delta x_\mu})}$ is dependent, are measurable quantities in the sense of **Definition 2**. The sum in the right-hand side (93) is taken by the steps $\frac{\ell^2}{\hbar}p_{N_{\Delta x_0}} = \frac{\ell}{cN_{\Delta x_0}} = \frac{\ell}{N_{\Delta x_0}}$ due to the fact that $c = 1$.
In this case the three-dimensional speed υ in the initial Lagrangian L of a continuous theory

$$L = -\beta c\sqrt{1-\frac{\upsilon^2}{c^2}} = -\beta\sqrt{1-\upsilon^2}, c = 1 \tag{94}$$

should be replaced in $L_{(N_{\Delta x_\mu})}$ by the three-dimensional **measurable** speed $\widetilde{\upsilon}$ varying in the time t not *continuously* but *discretely* by the steps $\ell/N_{\Delta x_0}$.
All these definitions are easily extended to the case of a free particle having the mass m. In particular, formulae (91),(93) in this case are of the form

$$\mathbf{S}_{(N_{\Delta x_\mu})} = -m\sum_a^b \Delta s_{(p_{N_{\Delta x_\mu}})} \tag{95}$$

and/ respectively,

$$\mathbf{S}_{(N_{\Delta x_\mu})} = \sum_{t_1}^{t_2} L_{(N_{\Delta x_\mu})}\frac{\ell}{N_{\Delta x_0}}, \tag{96}$$

where the Lagrangian L due to $c = 1$ is equal to

$$L_{(N_{\Delta x_\mu})} = -mc\sqrt{1-\frac{\widetilde{\upsilon}^2}{c^2}} = -m\sqrt{1-\widetilde{\upsilon}^2}. \tag{97}$$

And $\widetilde{\upsilon}$ is varying in the time t discretely, as indicated in formula (94).
It is clear that in all the above-mentioned formulae there is a passage to the limit from the **measurable** operation $\mathbf{S}_{(N_{\Delta x_\mu})}$ to the corresponding continuous operation $\mathbf{S}$

$$\lim_{(|N_{\Delta x_\mu}|)\to\infty} \mathbf{S}_{(N_{\Delta x_\mu})} = \mathbf{S}. \tag{98}$$

Similarly, for the momentum of a particle in the continuous case

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{m\mathbf{v}}{\sqrt{1-\upsilon^2}} \tag{99}$$

we can easily find its measurable analog

$$\mathbf{p}_{(N_{\Delta x_\mu})} = \frac{m\widetilde{v}}{\sqrt{1-\widetilde{v}^2}}, \tag{100}$$

where $\widetilde{v}$–vector of the three-dimensional speed of a particle in terms of the measurable quantities (formula (84)).
Here, similar to the continuous case at low speeds, we have $|\widetilde{v}| \ll 1, (c=1)$, and then $\mathbf{p}_{(N_{\Delta x_\mu})} = m\widetilde{v}$.
In a similar way, for the fixed set $(N_{\Delta x_\mu})$, we can obtain **measurable** variants of all the quantities known in a continuous theory $\mathcal{E}, ...$ [41].
The corresponding quantities have the index $_{(N_{\Delta x_\mu})}$.
Specifically, for the energy $\mathcal{E}_{(N_{\Delta x_\mu})}$, we have

$$\mathcal{E}_{(N_{\Delta x_\mu})} = \mathbf{p}_{(N_{\Delta x_\mu})}\widetilde{v} - L_{(N_{\Delta x_\mu})} = \frac{m}{\sqrt{1-\widetilde{v}^2}}. \tag{101}$$

And hence, for the Hamiltonian, we have $\mathcal{H}_{(N_{\Delta x_\mu})}$

$$\mathcal{H}_{(N_{\Delta x_\mu})} = \sqrt{p^2_{(N_{\Delta x_\mu})} + m^2}, \tag{102}$$

with a limiting transition to a continuous theory

$$\lim_{(|N_{\Delta x_\mu}|)\to\infty} \mathbf{p}_{(N_{\Delta x_\mu})} = \mathbf{p};$$
$$\lim_{(|N_{\Delta x_\mu}|)\to\infty} \mathcal{E}_{(N_{\Delta x_\mu})} = \mathcal{E};$$
$$\lim_{(|N_{\Delta x_\mu}|)\to\infty} \mathcal{H}_{(N_{\Delta x_\mu})} = \mathcal{H}; ... \tag{103}$$

In this section all the limiting transitions from the **measurable** variant of a theory to the continuous variant may be derived using the results obtained in [10].
Actually, as the Lagrangian $L = L(v)$ may be represented, to a high accuracy, in the capacity of the function of **measurable** quantities, in this case of speed $\widetilde{v}$ (with v replaced by $\widetilde{v}$ and $L(v)$ replaced by $L_{meas}(\widetilde{v})$), the use of formulae (61)–(64) from [10] leads to

$$\lim_{(|N_{\Delta x_0}|)\to\infty} \frac{\Delta L_{meas}(\widetilde{v})}{\Delta\widetilde{v}} = \lim_{\widetilde{v}\to v, \Delta\widetilde{v}\to 0} \frac{\Delta L_{meas}(\widetilde{v})}{\Delta\widetilde{v}} = \frac{\partial L(v)}{\partial v}. \tag{104}$$

Also, the approach may be illustrated by the limiting transition from a **measurable** operation to the continuous operation $\lim\limits_{(|N_{\Delta x_\mu}|)\to\infty} \mathbf{S}_{(N_{\Delta x_\mu})} = \mathbf{S}$ (formula (98)). This transition follows directly from formulae (66)–(68) in [10].

4 Concluding Comments and Explanations

4.1. In the previous section we have proceeded from some fixed *canonically measurable basic set*$(p_{N_{\Delta x_\mu}})$. However, it is obvious that the orbit of LG (66)(retaining the quadratic form (61)), involves many *canonically measurable basic sets* rather than one. In particular, for the operation of a group of spatial rotations $\mathbf{O(3)}$, spatial components of the basic set $(p_{N_{\Delta x_i}}), i = 1, 2, 3$ may switch their positions, generating another *canonically measurable basic set* .

4.2. Let us denote the totality of all *canonically measurable basic sets* $(p_{N_{\Delta x_\mu}})$ as follows:

$$Bas_{(p_{N_{\Delta x_\mu}})} \doteq \{(p_{N_{\Delta x_\mu}}, \mu = 0, ..., 3), |N_{\Delta x_\mu}| \gg 1\}. \tag{105}$$

Then, due to the fact that, within the constant factor $\ell^2/\hbar$, we have the equality $p_{N_{\Delta x_\mu}} = \ell/N_{\Delta x_\mu}$, the set $Bas_{(p_{N_{\Delta x_\mu}})}$ is nothing else but the four-dimensional lattice

$$\begin{aligned} Bas_{(p_{N_{\Delta x_\mu}})} &= \frac{\ell}{N_{\Delta x_0}} \times \frac{\ell}{N_{\Delta x_1}} \times \frac{\ell}{N_{\Delta x_2}} \times \frac{\ell}{N_{\Delta x_3}} = \\ &= (\frac{\ell}{N_{\Delta x_\mu}})^4, |(N_{\Delta x_\mu})| \gg 1. \end{aligned} \tag{106}$$

It is clear that mapping τ_{x_μ} of any of the components $\ell/N_{\Delta x_\mu}$ of the lattice $\ell/(N_{\Delta x_\mu})^4$ into the real interval $\varsigma, |\varsigma| \ll 1$:

$$\tau_{x_\mu} : (\frac{\ell}{N_{\Delta x_\mu}}) \mapsto \frac{1}{N_{\Delta x_\mu}} \tag{107}$$

will be very close to the continuous mapping. For fairly high $|N_{\Delta x_\mu}|$, this mapping may be considered as continuous to any accuracy. In terms of the lattice $\ell/(N_{\Delta x_\mu})^4$ for $|N_{\Delta x_\mu}| \to \infty$ this fact reflects the essence of all the limiting transitions from a **measurable** variant of a theory to the continuous one.
As noted above, $Bas_{(p_{N_{\Delta x_\mu}})}$ is not retained by LG but any element of this set $(p_{N_{\Delta x_\mu}})$ is converted to some element $g(p_{N_{\Delta x_\mu}})$ (formula (66)) retaining the Minkowskian metric in the **measurable** form, i.e., to the quadratic form (61).

4.3. Clearly, for sufficiently high $|N_{\Delta x_\mu}| \gg 1$, all the calculations presented in this section are practically independent of the set $(N_{\Delta x_\mu})$. As $|(N_{\Delta x_\mu})|$ is growing, the transition from the fixed *canonically measurable basic set* $(p_{N_{\Delta x_\mu}})$ to the *canonically measurable basic set* $(p_{N'_{\Delta x_\mu}}), |N'_{\Delta x_\mu}| \geq |N_{\Delta x_\mu}|$ may be considered as the component-wise multiplication by a set of the factors $(\tau_\mu = N_{\Delta x_\mu}/N'_{\Delta x_\mu}), |\tau_\mu| \leq 1$ with one and the same operation of LG. But such a transition is impossible at high energies $E \propto E_P$, i.e., for $|N_{\Delta x_\mu}| \approx 1$. The explanation is as follows: (i) the presentation becomes **"appreciably discrete"** because in this case the difference $\frac{\hbar}{N'_{\Delta x_\mu}\ell} - \frac{\hbar}{N_{\Delta x_\mu}\ell}$ is great (due to $|N_{\Delta x_\mu}| \approx 1$) and there is no possibility to have nearly continuous mapping of ς from **4.2.**; (ii) due to formulae (10),(18) and so on from Section 2 of this paper, for $E \propto E_P$ the quantity $\frac{\hbar}{N_{\Delta x_\mu}\ell} \neq p_{N_{\Delta x_\mu}}$ and for small $|(N_{\Delta x_\mu})|$ momenta $p_{N_{\Delta x_\mu}}$ is of the form

$$p_{N_{\Delta x_\mu}} = p(N_{\Delta x_\mu}, GUP) = \frac{\hbar}{(N_{\Delta x_\mu} - \frac{1}{4N_{\Delta, x_\mu}})\ell} \tag{108}$$

where $p(N_{\Delta x_\mu}, GUP) = p(N_p, GUP)$ is taken from formula (38) for $N_p = N_{\Delta x_\mu}$.
Since, for high $|(N_{\Delta x_\mu})|$, LG has the same effect on any set $(p_{N_{\Delta x_\mu}})$ as on (dx_μ), for small $|(N_{\Delta x_\mu})|$, in accordance with the correspondence principle, LG must affect the set $(p(N_{\Delta x_\mu}, GUP))$ given by formula (108)in some other way.
Thus, in the proposed **"measurable"** presentation the Lorentz-invariance is from the very beginning violated at high Planck's energies. This means

that, unlike the continuous presentation, where violation of the Lorentz-invariance at Planck' energies is a subject of investigation [42]–[45], in the considered case this property is integrated (embedded) into the theory.
It should be noted that for high $|(N_{\Delta x_\mu})|$ we deal with **primarily measurable** variations, whereas for small $|(N_{\Delta x_\mu})|$ we have the **generalized-measurable** variations $p(N_{\Delta x_\mu}, GUP)$ from formula (108). Consequently, we can state the fact of the Lorentz-invariance violation on going from **primarily measurable** quantities to **generalized-measurable** quantities.

4.4. In this way, based on the formulae in this section, we can conclude that for a set of integers $(N_{\Delta x_\mu}), |(N_{\Delta x_\mu})| \gg 1$, with the use of the *canonically measurable basic set* $(p_{N_{\Delta x_\mu}})$ $(\frac{\ell^2}{\hbar}(p_{N_{\Delta x_\mu}}))$, we can construct a **measurable** variant of Special Relativity as a certain discrete approximation. In essence, this approximation may be called the *lattice* approximation due to formulae (106), (107).
Besides, as formula (61) may be given in the form

$$\Delta s^2_{(p_{N_{\Delta x_\mu}})} = \frac{\ell^4}{\hbar^2}\eta_{\mu\nu}p_{N_{\Delta x_\mu}}p_{N_{\Delta x_\nu}} = \eta_{\mu\nu}\ell^2(\alpha_{N_{\Delta x_\mu}}\alpha_{N_{\Delta x_\nu}})^{1/2}, \tag{109}$$

where $\alpha_{N_{\Delta x_\mu}}$ is a *deformation parameter* (formula (14),(19)...), the above-mentioned *discrete lattice* approximation may be called the Special Relativity **deformation** (in the sense of paper [39]).
For $|(N_{\Delta x_\mu})| \to \infty$ or the same $\alpha_{N_{\Delta x_\mu}} \to 0$, this **deformation** goes to the well-known (continuous) Special Relativity.
So, as $|(N_{\Delta x_\mu})|$ is growing, we can have more and more accurate approximation **measurable** towards a continuous theory.
By the authors opinion, for sufficiently high $|(N_{\Delta x_\mu})|$, the **measurable** variant of Special Relativity gives a more realistic description than the continuous **canonical** variant.
More precisely, it may be made the following assumption
Conjecture.
For any separate experiment in Special Relativity there is a set $(N_{\Delta x_\mu})$ so that the **measurable** variant of Special Relativity constructed with respect to this set can correspond to the results of this experiment with unimprovable accuracy.

4.5. Returning to the beginning of this paper (Section 1), it may be stated that in the suggested formalism of the **measurable** (discrete) variant of a theory, as compared to the continuous variant, the infinitesimal quantities dx_μ in essence are replaced (within the constant factor $\ell^2/\hbar$) by the quantities $p_{N_{\Delta x_\mu}}$ which are dependent on all the three fundamental constants $c, \hbar, G$, because the minimal length $\ell \propto l_P$ is depending on them. However, this dependence is not felt at all at low energies $E; E \ll E_P$ due to great numbers of $|N_{\Delta x_\mu}|$ or, similarly, low numbers of $1/|N_{\Delta x_\mu}|$ which are a measure of the energy scale.
The situation is changed drastically on going to high energies $E; E \approx E_P$. In this case $|N_{\Delta x_\mu}| \approx 1$ in accordance with $(1/|N_{\Delta x_\mu}| \gg 0)$, $p_{N_{\Delta x_\mu}}$ is replaced by $p(N_{\Delta x_\mu}, GUP)$ from formula (108), then the minimal length ℓ and hence all the fundamental constants $c, \hbar$, and G become important in a theory.
Thus, in the suggested formalism there are many **measurable** *variants of Special Relativity (at least we have one for every canonical set* $(p_{N_{\Delta x_\mu}}), |N_{\Delta x_\mu}| \gg 1$ *but some of them are coincident (item* **4.1.***)). Nevertheless, due to the above given formulae, the difference between these* **measurable** *variants is insignificant.*

Afterword

A **Measurable** variant of Special Relativity is constructed only in terms of the **primarily measurable** variations $p_{N_{\Delta x_\mu}}, |N_{\Delta x_\mu}| \gg 1$ by virtue of the fact that in the **"flat case"** of the Minkowskian space the existent energies E are considerably lower that the Planck energies $E \ll E_P$.
Still it is obvious that, to construct a **measurable** variant of General Relativity (GR) at all the energy scales, we need both the **primarily measurable** variations $p_{N_{\Delta x_\mu}}, |N_{\Delta x_\mu}| \gg 1$ and **generalized-measurable** variations $p(N_{\Delta x_\mu}, GUP), |N_{\Delta x_\mu}| \approx 1$ from formula (108). In author's opinion, such construction should be realized jointly with a construction of a **measurable** variant for Quantum Theory (QT).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] A.E.Shalyt-Margolin, Minimal Length and the Existence of Some Infinitesimal Quantities in Quantum Theory and Gravity, *Adv. High Energy Phys.,* **2014** (2014), 8.
http://dx.doi.org/10.1155/2014/195157

[2] A.E.Shalyt-Margolin, Holographic Principle, Minimal Length and Measurability, *J. Adv. Phys.,* **5(3)** (2016), 263–275.
http://dx.doi.org/10.1166/jap.2016.1274

[3] Alexander Shalyt-Margolin, Minimal Length, Measurability, Continuous and Discrete Theories. Chapter 7 in *Horizons in World Physics. Volume 284*, Reimer, A., Ed.,Nova Science, Hauppauge, NY, USA,2015, pp.213–229.

[4] Alexander Shalyt-Margolin, Chapter 5 in *Advances in Dark Energy Research*, Ortiz,Miranda L., Ed.;Nova Science, Hauppauge, NY, USA,2015, pp.103–124

[5] Alexander Shalyt-Margolin, Minimal Length at All Energy Scales and Measurability,*Nonlinear Phenomena in Complex Systems,* **19(1)** (2016),30–40.

[6] A.E. Shalyt-Margolin, Uncertainty Principle at All Energies Scales and Measurability Conception for Quantum Theory and Gravity,*Nonlinear Phenomena in Complex Systems,* **19(2)** (2016),166–181.

[7] Alexander Shalyt-Margolin, Minimal Length, Measurability and Gravity,*Entropy,* **18(3)** (2016), 80.
http://dx.doi.org/10.3390/e18030080

[8] A.E.Shalyt-Margolin, Space-Time Fluctuations, Quantum Field Theory with UV-cutoff and Einstein Equations,*Nonlinear Phenomena in Complex Systems,* **17(2)** (2014), 138–146.

[9] Alexander Shalyt-Margolin,The Uncertainty Principle, Spacetime Fluctuations and Measurability Notion in Quantum Theory and Gravity,*Advanced Studies in Theoretical Physics,* **10(5)** (2016), 201–222. http://dx.doi.org/10.12988/astp.2016.6312

[10] Alexander Shalyt-Margolin,Minimal Length, Primary and Generalized Measurability and Classical Mechanics,*Advanced Studies in Theoretical Physics,* **10(8)** (2016), 361 - 384. http://dx.doi.org/10.12988/astp.2016.6725

[11] Hans Grauert and Ingo Lieb,*Differential Und Integralrechnung*,(In German) Springer-Verlag,Berlin,Geidelberg,New York,1967(In German). http://dx.doi.org/10.1007/978-3-662-36708-7

[12] H. Sagan, *Introduction to the calculus of variations*, Dover publications, Inc., N.Y., 1993.

[13] Haret C. Rosu, *Classical Mechanics*, graduate course, Guanajuato, Mexico : September 1999,arXiv:physics/9909035 [physics.ed-ph].

[14] D. Ter Haar,*Elements of Hamiltonian Mechanics*,(In German) University Reader in Theoretical Physics Oxford,Pergamon Press.

[15] M.E. Peskin, D.V. Schroeder, *An Introduction to Quantum Field Theory*,Addison-Wesley Publishing Company, 1995.

[16] R.M.Wald,*General Relativity*,University of Chicago Press, Chicago, Ill,USA,1984. http://dx.doi.org/10.7208/chicago/9780226870373.001.0001

[17] W. Heisenberg, Uber den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Z. Phys.*, **43** (1927), 172–198. (In German) http://dx.doi.org/10.1007/bf01397280

[18] V.B. Berestetskii,E.M. Lifshitz,L.P.Pitaevskii, *Relativistic Quantum Theory*, Pergamon, Oxford, UK, 1971.

[19] G. A. Veneziano, Stringy nature needs just two constants, *Europhys. Lett.*, **2** (1986), 199–211. http://dx.doi.org/10.1209/0295-5075/2/3/006

[20] D.Amati, M. Ciafaloni and G. A. Veneziano, Can spacetime be probed below the string size? *Phys. Lett. B*, **216** (1989), 41–47. http://dx.doi.org/10.1016/0370-2693(89)91366-x

[21] E.Witten, Reflections on the fate of spacetime, *Phys. Today* **49** (1996), 24–28.http://dx.doi.org/10.1063/1.881493

[22] R. J. Adler and D. I. Santiago, On gravity and the uncertainty principle, *Mod. Phys. Lett. A*, **14** (1999), 1371–1378. http://dx.doi.org/10.1142/s0217732399001462

[23] D.V.Ahluwalia, Wave-particle duality at the Planck scale: Freezing of neutrino oscillations, *Phys. Lett. A*, **A275** (2000), 31–35. http://dx.doi.org/10.1016/s0375-9601(00)00578-8

[24] D.V.Ahluwalia, Interface of gravitational and quantum realms, *Mod. Phys. Lett. A*, **A17** (2002), 1135–1145. http://dx.doi.org/10.1142/s021773230200765x

[25] M. Maggiore, The algebraic structure of the generalized uncertainty principle, *Phys. Lett. B*, **319** (1993), 83–86. http://dx.doi.org/10.1016/0370-2693(93)90785-g

[26] M. Maggiore, Black Hole Complementarity and the Physical Origin of the Stretched Horizon, *Phys. Rev. D*, **49** (1994), 2918–2921. http://dx.doi.org/10.1103/physrevd.49.2918

[27] M. Maggiore, Generalized Uncertainty Principle in Quantum Gravity. *Phys. Rev. D*,**304** (1993), 65–69. http://dx.doi.org/10.1016/0370-2693(93)91401-8

[28] S.Capozziello,G.Lambiase and G.Scarpetta, The Generalized Uncertainty Principle from Quantum Geometry, *Int. J. Theor. Phys.*, **39** (2000), 15–22. http://dx.doi.org/10.1023/a:1003634814685

[29] A. Kempf, G. Mangano and R.B. Mann, Hilbert space representation of the minimal length uncertainty relation, *Phys. Rev. D*, **52** (1995), 1108–1118. http://dx.doi.org/10.1103/physrevd.52.1108

[30] K.Nozari,A.Etemadi, Minimal length, maximal momentum and Hilbert space representation of quantum mechanics, *Phys. Rev. D*, **85** (2012), 104029. 1118. http://dx.doi.org/10.1103/physrevd.85.104029

[31] A.E.Shalyt-Margolin, J.G. Suarez, Quantum Mechanics of the Early Universe and Its Limiting Transition. Available online: http://arxiv.org/abs/gr-qc/0302119 (accessed on 30 August 2003).

[32] A.E.Shalyt-Margolin, J.G. Suarez, Quantum mechanics at Planck scale and density matrix, *Int. J. Mod. Phys. D*, **12** (2003), 1265–1278. http://dx.doi.org/10.1142/s0218271803003700

[33] A.E. Shalyt-Margolin and A.Ya. Tregubovich, Deformed density matrix and generalized uncertainty relation in thermodynamics, *Mod. Phys. Lett. A*, **19** (2004), 71–82.http://dx.doi.org/10.1142/s0217732304012812

[34] A.E.Shalyt-Margolin, Non-unitary and unitary transitions in generalized quantum mechanics, new small parameter and information problem-solving, *Mod. Phys. Lett. A*, **19** (2004), 391–403. http://dx.doi.org/10.1142/s0217732304013155

[35] A.E.Shalyt-Margolin, Pure states, mixed states and Hawking problem in generalized quantum mechanics, *Mod. Phys. Lett. A*,**19** (2004), 2037–2045. http://dx.doi.org/10.1142/s0217732304015312

[36] A.E.Shalyt-Margolin, The universe as a nonuniform lattice in finite-volume hypercube: I. Fundamental definitions and particular features,*Int. J. Mod. Phys. D*, **13** (2004), 853–864. http://dx.doi.org/10.1142/s0218271804004918

[37] A.E.Shalyt-Margolin, The Universe as a nonuniformlattice in the finite-dimensional hypercube. II. Simple cases of symmetry breakdown and

restoration, *Int. J. Mod. Phys. A*, **20** (2005), 4951–4964. http://dx.doi.org/10.1142/s0217751x05022895

[38] A.E.Shalyt-Margolin, The density matrix deformation in physics of the early universe and some of its implications. In *Quantum Cosmology Research Trends*; Reimer, A., Ed.; Nova Science: Hauppauge, NY, USA, 2005; pp. 49–92.

[39] L.Faddeev, Mathematical view of the evolution of physics, *Priroda*, **5** (1989), 11–16.

[40] Emil T.Akhmedov,Lectures on General Theory of Relativity,arXiv:1601.04996 [gr-qc].

[41] Landau, L.D.; Lifshits, E.M. *Field Theory*; Theoretical Physics: Moskow, Russia, 1988; Volume 2

[42] G. Amelino-Camelia, Quantum Spacetime Phenomenology, *Living Rev. Relativ.*, **16** (2013), 5–129. http://dx.doi.org/10.12942/lrr-2013-5

[43] Cheng-Gang Shao, Yu-Jie Tan, Wen-Hai Tan, Shan-Qing Yang, Jun Luo, Michael Edmund Tobar, Quentin G. Bailey, J.C. Long, E. Weisman, Rui Xu, Alan Kostelecky, Combined search for Lorentz violation in short-range gravity *Phys.Rev.Lett.* **117** (2016),071102.

[44] Silke Weinfurtner, Stefano Liberati, Matt Visser,Modelling Planck-scale Lorentz violation via analogue models, *J.Phys.Conf.Ser.* **33** (2006) 373–385

[45] Daniel Sudarsky, Perspectives on Quantum Gravity Phenomenology, *Int.J.Mod.Phys.* **D14** (2005) 2069–2094