Backward stochastic Volterra integral equations associated with a Lévy process and applications*

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Abstract

In this paper, we study a class of backward stochastic Volterra integral equations driven by Teugels martingales associated with an independent Lévy process and an independent Brownian motion (BSVIELs). We prove the existence and uniqueness as well as stability of the adapted M-solutions for those equations. Moreover, a duality principle and then a comparison theorem are established. As an application, we derive a class of dynamic risk measures by means of M-solutions of certain BSVIELs.

Keywords: Backward stochastic Volterra integral equation, Teugels martingales, duality principle, comparison theorem, dynamic coherent risk measure.

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1 Introduction

The general nonlinear case backward stochastic differential equations (BSDEs), i.e., equations in form

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s)ds - \int_t^T Z(s)dW_s, \quad t \in [0, T], \tag{1}$$

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was first introduced by Pardoux and Peng [12] in 1990, they proved the existence and uniqueness of solutions for BSDEs under Lipschitz conditions. Since then, a lot of work have been devoted to the study of the theory of BSDEs as well as to their applications. This is due to the connections of BSDEs with mathematical finance as well as stochastic optimal control and stochastic games (see e.g., [7], [14], [8]).

In Nualart and Schoutens [10], the authors gave a martingale representation theorem associated to Teugels martingales corresponding to a Lévy process. Furthermore, Nualart and Schoutens [11] studied the corresponding BSDEs associated to a Lévy process. The results were important from a pure mathematical point of view as well as in the world of finance. It could be used for the purpose of option pricing in a Lévy market and the partial differential equation which provided an analogue of the famous Black-Scholes partial differential equation. Following that, Bahlali et al. [3] considered the BSDEs driven by a Brownian motion and the martingales of Teugels associated with an independent Lévy process, having a Lipschitz or a locally Lipschitz coefficient.

On the other hand, stochastic Volterra equation had been investigated by Berger and Mizel in [4] and [5], Protter in [15] and Pardoux and Protter in [13]. As a natural generalization of the BSDE theory, Lin [9] firstly considered the solvability of the adapted solution for backward stochastic Volterra integral equations (BSVIEs) with uniform Lipschtz coefficient of the form

$$\begin{aligned} Y(t) &= \xi + \int_t^T f(s, Y(s), Z(t,s))ds \\ &\quad - \int_t^T [g(t,s,Y(s)) + Z(t,s)]dW_s, \quad t \in [0,T]. \end{aligned} \tag{2}$$

Following it, Aman and N'Zi [1] considered the same equation and weakened the uniform Lipschtz condition on the coefficient to a local one. Thereafter, Yong [18] extended the equations (2) to a generalized form. For the more general cases of BSVIEs (2), Anh and Yong [2] and Yong ([19], [20]) studied them and gave its applications in stochastic optimal control, mathematics finance and risk management, where the notion of M-solution was introduced to ensure the unique solvability of the adapted solution. Recently, Ren [17] established the well-posedness of adapted M-solutions for BSVIEs driven by both Brownian motion and a Poisson random measure.

Motivated by above works, it is natural and necessary to consider the backward stochastic Volterra integral equations driven by a standard Brownian motion and the Teugels martingales associated with an independent Lévy process (BSVIELs). We first show the existence and uniqueness of M-solutions for those equations. Then, a duality principle between the linear BSVIELs and the linear forward stochastic Volterra integral equations driven by the same Brownian motion and the Teugels martingales associated with an independent Lévy process (FSVIELs) is presented. Further, as an important application of the duality principle, we establish a comparison theorem for M-solutions of BSVIELs. Finally, a class of dynamic risk measures are derived by means of M-solutions of one kind of BSVIELs. We would like to point that we adopt the similar method in Anh and Yong [2], but the dynamic system is different from [2].

The rest of the paper is organized as follows. In Section 2, we introduce some preliminaries, then we prove the existence and uniqueness of the adapted M-solutions for BSVIELs in Section 3. In Section 4, we establish a duality principle between linear BSVIELs and linear FSVILs as well as a comparison theorem for M-solutions of BSVIELs. In Section 5, a class of dynamic coherent risk measures be derived by means of M-solutions of certain BSVIELs.

2 Preliminaries

Given $T > 0$ a fixed real number. Let's first introduce the following two mutually independent processes:

- $\{W_t : t \in [0, T]\}$: a standard Brownian motion in $\mathbb{R}^d$;
- A $\mathbb{R}$-valued Lévy process $(L_t)_{0 \le t \le T}$ corresponding to a standard Lévy measure ν satisfying the following conditions:

  (i) $\int_R (1 \wedge y^2)\nu(dy) < \infty$,

  (ii) $\int_{]-\varepsilon,\varepsilon[^c} \mathrm{e}^{\lambda|y|}\nu(dy) < \infty$, for every $\varepsilon > 0$ and for some $\lambda > 0$.

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space, the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \le t \le T}$ is generated by the two processes given above, i.e.,

$$\mathcal{F}_t = \sigma\{W_s, 0 \le s \le t\} \vee \sigma\{L_s, 0 \le s \le t\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the set of all P-null subsets of $\mathcal{F}$.

We define:

- $\mathbb{L}^2_{\mathcal{F}_T}(0,T;\mathbb{R}^n) = \{\psi : [0,T]\times\Omega \to \mathbb{R}^n \mid \psi(\cdot)$ is $\mathcal{B}([0,T])\otimes\mathcal{F}_T$-measurable such that $\mathbf{E}\int_0^T |\psi(t)|^2 dt < \infty\}$;

- $\mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^n) = \{\psi(\cdot) \in \mathbb{L}^2_{\mathcal{F}_T}(0,T;\mathbb{R}^n) | \psi(\cdot)$ is $\mathbb{F}$-adapted$\}$;

- $\ell^2 = \{x = (x^{(i)})_{i\geq 1} \mid \|x\| = \left[\sum_{i=1}^{\infty}(x^{(i)})^2\right]^{\frac{1}{2}} < \infty\}$.

Remark 2.1 *In all the definitions of the relevant spaces in this paper,* $[0,T]$ *can be replaced by any* $[R,S]$ *with* $0 \leq R < S \leq T$.

In what follows, for any $0 \leq R < S \leq T$, we denote

$$\Delta[R,S] = \{(t,s) \in [R,S]^2 \mid R \leq s \leq t \leq S\},$$

$$\Delta^c[R,S] = \{(t,s) \in [R,S]^2 \mid R \leq t < s \leq S\}.$$

For simplicity, we denote $\Delta[0,T] = \Delta,\ \Delta^c[0,T] = \Delta^c$.

We denote by $(H^{(i)})_{i\geq 1}$ the Teugels martingales associated with the Lévy process $\{L_t :\ t \in [0,T]\}$. More precisely

$$H_t^{(i)} = c_{i,i}Y_t^{(i)} + c_{i,i-1}Y_t^{(i-1)} + \cdots + c_{i,1}Y_t^{(1)},$$

where $Y_t^{(i)} = L_t^{(i)} - \mathbb{E}[L_t^{(i)}] = L_t^{(i)} - t\mathbb{E}[L_1^{(i)}]$ for all $i \geq 1$ and $L_t^{(i)}$ are so called power-jump processes, i.e., $L_t^{(1)} = L_t$ and $L_t^{(i)} = \sum_{0\leq s\leq t}(\Delta L_t)^i$ for $i \geq 2$. Here, for any process $x(t)$, we denote by $x(t-) = \lim_{s\to t-} x(s)$ and $\Delta x_t = x(t) - x(t-)$.

It was shown in [10] that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $\mu(dx) = x^2\nu(dx) + \sigma^2\delta_0(dx)$:

$$q_{i-1} = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + \cdots + c_{i,1}.$$

We set

$$p_i(x) = xq_{i-1}(x) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + \cdots + c_{i,1}x.$$

The martingales $(H^{(i)})_{i\geq 1}$ can be chosen to be pairwise strongly orthonormal martingales. Furthermore, $[H^{(i)}, H^{(j)}], i \neq j$, and $\{[H^{(i)}, H^{(j)}]_t - t\}_{t\geq 0}$ are uniformly integrable martingales with initial value 0, i.e., $\langle H^{(i)}, H^{(j)}\rangle_t = \delta_{ij}t$.

Throughout this paper, we consider the following BSVIEL:

$$\begin{aligned} Y(t) &= \psi(t) + \int_t^T f(t,s,Y(s-),Z(t,s),Z(s,t),U(t,s),U(s,t))ds \\ &\quad + \int_t^T Z(t,s)dW_s - \sum_{i=1}^{\infty}\int_t^T U^{(i)}(t,s)dH_s^{(i)},\ 0 \le t \le T, \end{aligned} \tag{3}$$

where $\psi(\cdot) \in \mathbb{L}^2_{\mathcal{F}_T}(0,T;\mathbb{R})$ and $f : \Delta^c \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \ell^2 \times \ell^2 \times \Omega \to \mathbb{R}$ is a given map.

We denote by $\mathbb{L}^2(0,T;\mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^d))$ the space of all processes $Z : [0,T]^2 \times \Omega \to \mathbb{R}^d$ such that for almost all $t \in [0,T]$, $Z(t,\cdot) \in \mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$ satisfying $\mathbf{E}\int_0^T\int_0^T |Z(t,s)|^2 dsdt < \infty$. We denote by $\ell^2(0,T;\mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}))$ the space of processes $U : [0,T]^2 \times \Omega \to \ell^2$ such that for each $i \ge 1$ and almost all $t \in [0,T]$, $U^{(i)}(t,\cdot) \in \mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R})$ satisfying $\|U\|^2 = \sum_{i=1}^{\infty}\mathbb{E}\int_0^T\int_0^T |U^{(i)}(t,s)|^2 dsdt < \infty$.

We make the following assumptions:

(**H1**) Let $f : \Delta^c \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \ell^2 \times \ell^2 \times \Omega \to \mathbb{R}$ be $\mathcal{B}(\Delta^c \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \ell^2 \times \ell^2) \otimes \mathcal{F}_T$-measurable such that $s \to f(t,s,y,z,\eta,u,\zeta)$ is $\mathbb{F}-$adapted for all $(t,y,z,\eta,u,\zeta) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \ell^2 \times \ell^2$ and

$$E\int_0^T \left(\int_t^T |f_0(t,s)|ds\right)^2 dt < \infty,$$

where $f_0(t,s) \equiv f(t,s,0,0,0,0,0)$. Moreover, for any $(y_1,z_1,\eta_1,u_1,\zeta_1)$ and $(y_2,z_2,\eta_2,u_2,\zeta_2) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \ell^2 \times \ell^2$, it holds

$$\begin{aligned} & |f(t,s,y_1,z_1,\eta_1,u_1,\zeta_1) - f(t,s,y_2,z_2,\eta_2,u_2,\zeta_2)| \\ \le\ & L_y(t,s)|y_1-y_2| + L_z(t,s)|z_1-z_2| + L_\eta(t,s)|\eta_1-\eta_2| \\ & + L_u(t,s)\|u_1-u_2\| + L_\zeta(t,s)\|\zeta_1-\zeta_2\|, \end{aligned} \tag{4}$$

where the coefficients $L_y(t,s), L_z(t,s), L_\eta(t,s), L_u(t,s)$ and $L_\zeta(t,s)$ are determined functions from Δ^c to $\mathbb{R}$ such that

$$\sup_{t\in[0,T]}\int_t^T [L_y^2(t,s) + L_\eta^2(t,s) + L_\zeta^2(t,s)]ds < \infty, \tag{5}$$

$$\sup_{t\in[0,T]}\int_t^T [L_z^2(t,s) + L_u^2(t,s)]ds < 1. \tag{6}$$

Let's give the notion of M-solution of BSVIEL (3).

Definition 2.2 *A triple of processes* $(Y(\cdot), Z(\cdot,\cdot), U(\cdot,\cdot)) \in \mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}) \times \mathbb{L}^2(0,T;\mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^d)) \times \ell^2(0,T;\mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}))$ *is called an adapted M-solution of BSVIEL (3) if (3) is satisfied in the Itô's sense for almost all* $0 \le t \le T$, *and it holds that*

$$Y(t) = EY(t) + \int_0^t Z(t,s)dW_s + \sum_{i=1}^{\infty} \int_0^t U^{(i)}(t,s)dH_s^{(i)}. \tag{7}$$

3 Existence and uniqueness of solution

Theorem 3.1 *Suppose (**H1**) holds. Then for any* $\psi(\cdot) \in \mathbb{L}^2_{\mathcal{F}_T}(0,T;\mathbb{R})$, *the BSVIEL (3) has a unique M-solution* $(Y(\cdot), Z(\cdot,\cdot), U(\cdot,\cdot)) \in \mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}) \times \mathbb{L}^2(0,T;\mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^d)) \times \ell^2(0,T;L^2_{\mathbb{F}}(0,T;\mathbb{R}))$.

Proof. From (5) and (6), we know that there exists a sequence $0 = T_0 < T_1 < \cdots < T_{k-1} < T_k = T$ and $\delta \in (0,1)$ satisfying:

$$\sup_{t\in[T_{i-1},T_i]} \int_t^{T_i} [L_y^2(t,s) + L_\eta^2(t,s) + L_\zeta^2(t,s)]ds \le \frac{1-\delta}{4},\ 1 \le i \le k, \tag{8}$$

$$\sup_{t\in[T_{i-1},T_i]} \int_t^{T_i} [L_z^2(t,s) + L_u^2(t,s)]ds \le \frac{1-\delta}{4},\ 1 \le i \le k. \tag{9}$$

We split the rest of the proof into several steps.

Step 1. The existence and uniqueness of M-solution for BSVIELs (3) on $[T_{k-1}, T]$.

Let $\mathcal{M}^2[T_{k-1},T]$ be the subspace of $(y(\cdot), z(\cdot,\cdot), u(\cdot,\cdot)) \in \mathbb{L}^2_{\mathbb{F}}(T_{k-1},T;\mathbb{R}) \times \mathbb{L}^2(T_{k-1},T;\mathbb{L}^2_{\mathbb{F}}(T_{k-1},T;\mathbb{R}^d)) \times \ell^2(T_{k-1},T;\mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}))$ such that

$$y(t) = Ey(t) + \int_0^t z(t,s)dW_s + \sum_{i=1}^{\infty} \int_0^t u^{(i)}(t,s)dH_s^{(i)}, \quad t \in [T_{k-1},T]. \tag{10}$$

Furthermore, for any $(y(\cdot), z(\cdot,\cdot), u(\cdot,\cdot)) \in \mathcal{M}^2[T_{k-1},T]$, $(t,r) \in \Delta$ and $t \in [T_{k-1},T]$, we have

$$\begin{aligned} & E\int_r^t |z(t,s)|^2 ds + E\int_r^t \|u(t,s)\|^2 ds \\ \le\ & E\int_0^t |z(t,s)|^2 ds + E\int_0^t \|u(t,s)\|^2 ds \\ \le\ & E|y(t) - Ey(t)|^2 \le E|y(t)|^2. \end{aligned} \tag{11}$$

For every $(y(\cdot), z(\cdot,\cdot), u(\cdot,\cdot)) \in \mathcal{M}^2[T_{k-1}, T]$ and $t \in [T_{k-1}, T]$, we denote

$$\overline{\varphi}(t) = \varphi(t) + \int_t^T f(t, s, Y(s-), z(t,s), z(s,t), u(t,s), u(s,t))ds. \tag{12}$$

By (8) and (9), from (**H1**) and Cauchy-Schwartz inequality, for any $t \in [T_{k-1}, T]$, we have

$$\begin{aligned} |\overline{\varphi}(t)|^2 \;\; \leq \;\; & C\left[|\varphi(t)|^2 + \left(\int_t^T f_0(t,s)ds\right)^2 + \int_t^T |y(s)|^2 ds\right] \\ & +(1-\delta^2)\left[\int_t^T |z(t,s)|^2 ds + \int_t^T |z(s,t)|^2 ds \right. \\ & \left. + \int_t^T \|u(t,s)\|^2 ds + \int_t^T \|u(s,t)\|^2 ds\right]. \end{aligned} \tag{13}$$

Hereafter C is a generic positive constant which may be different from line to line.

Noting (11), for any $r \in [T_{k-1}, T]$, we have

$$\begin{aligned} & E\int_r^T |\overline{\varphi}(t)|^2 dt \\ \leq \;\; & CE\left[\int_r^T |\varphi(t)|^2 dt + \int_r^T \left(\int_t^T |f_0(t,s)|ds\right)^2 dt + \int_r^T \int_t^T |y(s)|^2 ds\right] dt \\ & +(1-\delta^2)E\left[\int_r^T |y(t)|^2 dt + \int_r^T \int_t^T |z(t,s)|^2 dsdt \right. \\ & \left. + \int_r^T \int_t^T \|u(t,s)\|^2 dsdt\right]. \end{aligned}$$

which implies that $\overline{\varphi}(\cdot) \in \mathbb{L}^2_{\mathcal{F}_T}(T_{k-1}, T; \mathbb{R})$. Then, for any $t \in [T_{k-1}, T]$, by the martingale representation theorem in Bahlari et al. [3], there exists a unique pair of processes $(Z(\cdot,\cdot), U(\cdot,\cdot)) \in \mathbb{L}^2(0,T; \mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^d)) \times \ell^2(0,T;\mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}))$ such that

$$\overline{\varphi}(t) \;\; = \;\; E\overline{\varphi}(t) + \int_0^T Z(t,s)dW_s + \sum_{i=1}^{\infty}\int_0^T U^{(i)}(t,s)dH_s^{(i)}. \tag{14}$$

Let

$$Y(t) \;\; = \;\; E\overline{\varphi}(t) + \int_0^t Z(t,s)dW_s + \sum_{i=1}^{\infty}\int_0^T U^{(i)}(t,s)dH_s^{(i)}, \tag{15}$$

we then get

$$
\begin{aligned}
Y(t) &= \overline{\varphi}(t) - \int_t^T Z(t,s)dW_s - \sum_{i=1}^{\infty}\int_t^T U^{(i)}(t,s)dH_s^{(i)} \\
&= \varphi(t) + \int_t^T f(t,s,Y(s-),z(t,s),z(s,t),u(t,s),u(s,t))ds \\
&\quad - \int_t^T Z(t,s)dW_s - \sum_{i=1}^{\infty}\int_t^T U^{(i)}(t,s)dH_s^{(i)}.
\end{aligned}
$$

Thus, we get a unique adapted M-solution $(Y(\cdot),Z(\cdot,\cdot),U(\cdot,\cdot))$ for BSVIEL (3) on $[T_{k-1},T]$. Clearly, $(Y(\cdot),Z(\cdot,\cdot),U(\cdot,\cdot)) \in \mathcal{M}^2[T_{k-1},T]$.

Next, we prove the uniqueness of adapted M-solution.

Let $(Y(\cdot),Z(\cdot,\cdot),U(\cdot,\cdot))$ and $(\overline{Y}(\cdot),\overline{Z}(\cdot,\cdot),\overline{U}(\cdot,\cdot))$ be two adapted M-solutions of BSVIEL (3) on $[T_{k-1},T]$. For all $t \in [T_{k-1},T]$, by (**H1**) and Cauchy-Schwartz inequality, we're able to obtain

$$
\begin{aligned}
& E|Y(t)-\overline{Y}(t)|^2 + E\int_t^T |Z(t,s)-\overline{Z}(t,s)|^2 ds \\
& +E\int_t^T \|U(t,s)-\overline{U}(t,s)\|^2 ds \\
\leq\ & CE\int_t^T |Y(s)-\overline{Y}(s)|^2 ds + (1-\delta^2)E\left[\int_t^T |Z(t,s)-\overline{Z}(t,s)|^2 ds\right. \\
& +E\int_t^T |Z(s,t)-\overline{Z}(s,t)|^2 ds + E\int_t^T \|U(t,s)-\overline{U}(t,s)\|^2 ds \\
& \left.+E\int_t^T \|U(s,t)-\overline{U}(s,t)\|^2 ds\right] \qquad (16)
\end{aligned}
$$

Similar to (11), we have

$$
\begin{aligned}
& E\int_r^t |Z(t,s)-\overline{Z}(t,s)|^2 ds + E\int_r^t \|U(t,s)-\overline{U}(t,s)\|^2 ds \\
& \leq E|Y(t)-\overline{Y}(t)|^2,\ (t,r)\in\Delta \qquad (17)
\end{aligned}
$$

Hence, for $r \in [T_{k-1}, T)$, we have

$$
\begin{aligned}
& E\int_r^T |Y(t)-\overline{Y}(t)|^2 dt + E\int_r^T\int_t^T |Z(t,s)-\overline{Z}(t,s)|^2 dsdt \\
& +E\int_r^T\int_t^T \|U(t,s)-\overline{U}(t,s)\|^2 dsdt \\
\leq\ & CE\int_r^T\int_t^T |Y(s)-\overline{Y}(s)|^2 dsdt \\
& +(1-\delta^2)\left[E\int_r^T |Y(t)-\overline{Y}(t)|^2 dt + E\int_r^T\int_t^T |Z(t,s)-\overline{Z}(t,s)|^2 dsdt \right.\\
& \left. +E\int_r^T\int_t^T \|U(t,s)-\overline{U}(t,s)\|^2 dsdt\right]. \qquad (18)
\end{aligned}
$$

Then the uniqueness is an immediate consequence of Gronwall's inequality.

Step 2. Solvability of a stochastic integral equation on $[T_{k-1}, T]$.

For $(t,s) \in [T_{k-2}, T_{k-1}] \times [T_{k-1}, T]$, by Step 1, we know that the values $Y(s), Z(s,t)$ and $U(s,t)$ are already determined. Hence, for $(t,s,z,u) \in [T_{k-2}, T_{k-1}] \times [T_{k-1}, T] \times \mathbb{R}^d \times \ell^2$, we can define

$$
f^{k-1}(t,s,z,u) = f(t,s,Y(s-),z,Z(s,t),u,U(s,t)). \qquad (19)
$$

We now consider the following stochastic integral equation :

$$
\begin{aligned}
\varphi^{k-1}(t) &= \varphi(t) + \int_{T_{k-1}}^T f^{k-1}(t,s,Z(t,s),U(t,s))ds - \int_{T_{k-1}}^T Z(t,s)dW_s \\
&\quad -\sum_{i=1}^{\infty}\int_{T_{k-1}}^T U^{(i)}(t,s)dH_s^{(i)}, \quad t \in [T_{k-2}, T_{k-1}], \qquad (20)
\end{aligned}
$$

By (**H1**), the above equation admits a unique solution $(\varphi^{k-1}(\cdot), Z(\cdot,\cdot), U(\cdot,\cdot))$ such that $\varphi^{k-1}(t)$ being $\mathcal{F}_{T_{k-1}}$-adapted. This uniquely determines the values $Z(t,s)$ and $U(t,s)$ for $(t,s) \in [T_{k-2}, T_{k-1}] \times [T_{k-1}, T]$.

Step 3. Complete the proof by induction.

By the previous two steps, we have determine the values $Y(t)$ for $t \in [T_{k-1}, T]$, and the values $Z(t,s)$ and $U(t,s)$ for $(t,s) \in ([T_{k-1}, T] \times [0,T]) \cup ([T_{k-2}, T_{k-1}] \times [T_{k-1}, T])$. From the definition of $f^{k-1}(t,s,z,u)$, one can see that $(\varphi^{k-1}(\cdot), Z(\cdot,\cdot), U(\cdot,\cdot))$ satisfies

$$
\begin{aligned}
\varphi^{k-1}(t) &= \varphi(t) + \int_{T_{k-1}}^{T} f(t,s,Y(s-),Z(t,s),Z(s,t),U(t,s),U(s,t))ds \\
&\quad - \int_{T_{k-1}}^{T} Z(t,s)dW_s - \sum_{i=1}^{\infty} \int_{T_{k-1}}^{T} U^{(i)}(t,s)dH_s^{(i)}, t \in [T_{k-2}, T_{k-1}].
\end{aligned}
$$

For $t \in [0, T_{k-1}]$, we consider the following equation:

$$
\begin{aligned}
Y(t) &= \varphi^{k-1}(t) + \int_{t}^{T_{k-1}} f(t,s,Y(s-),Z(t,s),Z(s,t),U(t,s),U(s,t))ds \\
&\quad - \int_{t}^{T_{k-1}} Z(t,s)dW_s - \sum_{i=1}^{\infty} \int_{t}^{T_{k-1}} U^{(i)}(t,s)dH_s^{(i)}. \qquad (21)
\end{aligned}
$$

Note that $\varphi^{k-1}(t)$ is $\mathcal{F}_{T_{k-1}}$-adapted. From Step 1, we are able to prove that (21) is solvable on $[T_{k-2}, T_{k-1}]$, then the values $Y(t)$ for $t \in [T_{k-2}, T_{k-1}]$, the values $Z(t,s)$ and $U(t,s)$ for $(t,s) \in [T_{k-2}, T_{k-1}] \times [0, T_{k-1}]$ are determined. Therefore, we obtain the values $Y(t)$ for $t \in [T_{k-2}, T]$, and the values $Z(t,s)$ and $U(t,s)$ for $(t,s) \in [T_{k-2}, T] \times [0, T]$. Moreover, for $t \in [T_{k-2}, T_{k-1}]$, we have

$$
\begin{aligned}
Y(t) &= \varphi^{k-1}(t) + \int_{t}^{T_{k-1}} f(t,s,Y(s-),Z(t,s),Z(s,t),U(t,s),U(s,t))ds \\
&\quad - \int_{t}^{T_{k-1}} Z(t,s)dW_s - \sum_{i=1}^{\infty} \int_{t}^{T_{k-1}} U^{(i)}(t,s)dH_s^{(i)} \\
&= \varphi(t) + \int_{t}^{T} f(t,s,Y(s-),Z(t,s),Z(s,t),U(t,s),U(s,t))ds \\
&\quad - \int_{t}^{T} Z(t,s)dW_s - \sum_{i=1}^{\infty} \int_{t}^{T} U^{(i)}(t,s)dH_s^{(i)}. \qquad (22)
\end{aligned}
$$

Thus, the equation is solvable on $[T_{k-2}, T]$. We then complete the proof by induction. □

Further, we have the following stable result.

Theorem 3.2 *Let $\overline{f} : \Delta^c \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \ell^2 \times \ell^2 \times \Omega \to \mathbb{R}$ satisfy (H1) and $\overline{\psi}(\cdot) \in \mathbb{L}^2_{\mathcal{F}}(0,T;\mathbb{R})$. Let $(\overline{Y}(\cdot), \overline{Z}(\cdot,\cdot), \overline{U}(\cdot,\cdot)) \in \mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}) \times \mathbb{L}^2(0,T;\mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^d)) \times$*

$\ell^2(0,T;\mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}))$ *be the unique M-solution of BSVIEL (3) corresponding to* $(\overline{\psi}(\cdot),\overline{f})$. *Then, for all* $r\in[0,T]$, *we have*

$$\begin{aligned}
&E\int_r^T|Y(t)-\overline{Y}(t)|^2dt+E\int_r^T\int_r^T|Z(t,s)-\overline{Z}(t,s)|^2dsdt\\
&+E\int_r^T\int_r^T\|U(t,s)-\overline{U}(t,s)\|^2dsdt\\
\leq\ & C\Bigg[E\int_r^T|\psi(t)-\overline{\psi}(t)|^2dt\\
&+\int_r^T\bigg(\int_r^T|f(t,s,Z(t,s),Z(s,t),U(t,s),U(s,t))\\
&\qquad-\overline{f}(t,s,Z(t,s),Z(s,t),U(t,s),U(s,t))|ds\bigg)^2dt\Bigg]. \qquad (23)
\end{aligned}$$

Proof. The proof is similar to Step 1 of the proof of Theorem 3.1. $\square$

4 Duality principle and comparison theorem

In this section, we establish a duality principle between linear BSVIELs and linear FSVIELs. As an application of the duality principle, a comparison theorem for M-solutions of certain BSVIELs is given.

Consider the following BSVIEL:

$$\begin{aligned}
Y(t)\ =\ &\varphi(t)+\int_t^T[B_0(t,s)Y(s-)+B(t,s)Z(s,t)+\sum_{i=1}^{\infty}C^{(i)}(t,s)U^{(i)}(s,t)]ds\\
&-\int_t^T Z(t,s)dW_s-\sum_{i=1}^{\infty}\int_t^T U^{(i)}(t,s)dH_s^{(i)}. \qquad (24)
\end{aligned}$$

Here $B_0(\cdot,\cdot)$, $B(\cdot,\cdot)=(B_1(\cdot,\cdot),\cdots,B_d(\cdot,\cdot))^T$ and $\big(C^{(i)}(\cdot,\cdot)\big)_{i\geq1}$ satisfying the following assumption:
(**H2**) For each $j=0,1,\dots,d$, the process $B_j:\Delta^c\times\Omega\to\mathbb{R}$ such that for each $t\in[0,T]$, $B_j(t,s)$ is F-adapted, and $\sup_{(t,s)\in\Delta^c}\operatorname{esssup}_{\omega\in\Omega}|B_j(t,s)|<\infty$. For each $i\geq1$, the process $C^{(i)}:\Delta^c\times\Omega\to\mathbb{R}$ such that for each $t\in[0,T]$, $C^{(i)}(t,s)$ is F-adapted, and $\sup_{(t,s)\in\Delta^c}\operatorname{esssup}_{\omega\in\Omega}|C^{(i)}(t,s)|<\infty$.

We now state the duality principle.

Theorem 4.1 *Suppose (H2) hold. Let $X(t)$ be the solution of the following $\mathbb{R}$-valued forward stochastic Volterra integral equation:*

$$\begin{aligned} X(t) &= \psi(t) + \int_0^t X(s)B_0(s,t)ds + \int_0^t X(s)B(s,t)dW_s \\ &\quad + \sum_{i=1}^{\infty} \int_0^t X(s)C^{(i)}(s,t)dH_s^{(i)}. \end{aligned} \tag{25}$$

Then, we have the following duality principle:

$$E\int_0^T Y(t)\psi(t)dt = E\int_0^T X(t)\varphi(t)dt. \tag{26}$$

Proof. From the definition of M-solution, we have

$$
\begin{aligned}
& E\int_0^T Y(t)\psi(t)dt \\
= \;& E\int_0^T Y(t)\left(X(t) - \int_0^t X(s)B_0(s,t)ds - \int_0^t X(s)B(s,t)dW_s \right.\\
& \qquad\qquad \left. - \sum_{i=1}^{\infty}\int_0^t X(s)C^{(i)}(s,t)dH_s^{(i)}\right) dt \\
= \;& E\int_0^T Y(t)X(t)dt - E\int_0^T\int_s^T B_0(s,t)X(s)Y(t)dtds \\
& -E\int_0^T \left(\int_0^t X(s)B(s,t)dW_s + \sum_{i=1}^{\infty}\int_0^t X(s)C^{(i)}(s,t)dH_s^{(i)}\right)\times \\
& \qquad \left(EY(t) + \int_0^t Z(t,s)dW_s + \sum_{i=1}^{\infty}\int_0^t U^{(i)}(t,s)dH_s^{(i)}\right) dt \\
= \;& E\int_0^T Y(t)X(t)dt - E\int_0^T\int_t^T B_0(t,s)X(t)Y(s)dsdt \\
& -E\int_0^T\int_0^t X(s)B(s,t)Z(t,s)dsdt - \sum_{i=1}^{\infty} E\int_0^T\int_0^t X(s)C^{(i)}(s,t)U^{(i)}(t,s)dtds \\
= \;& E\int_0^T Y(t)X(t)dt - E\int_0^T\int_t^T B_0(t,s)X(t)Y(s)dsdt \\
& -E\int_0^T\int_s^T X(s)B(s,t)Z(t,s)dtds - \sum_{i=1}^{\infty} E\int_0^T\int_s^T X(s)C^{(i)}(s,t)U^{(i)}(t,s)dtds \\
= \;& E\int_0^T Y(t)X(t)dt - E\int_0^T\int_t^T B_0(t,s)X(t)Y(s)dsdt \\
& -E\int_0^T\int_t^T X(t)B(t,s)Z(s,t)dsdt - \sum_{i=1}^{\infty} E\int_0^T\int_t^T X(t)C^{(i)}(t,s)U^{(i)}(s,t)dsdt \\
= \;& E\int_0^T \left[X(t)\left(Y(t) - \int_t^T B_0(t,s)Y(s)ds - \int_t^T B(t,s)Z(s,t)ds \right.\right.\\
& \qquad\qquad \left.\left. - \sum_{i=1}^{\infty}\int_t^T C^{(i)}(t,s)U^{(i)}(s,t)ds\right)\right] dt \\
= \;& E\int_0^T X(t)\psi(t)dt.
\end{aligned}
$$

The proof is complete. $\Box$

With the help of the duality principle given above, we're able to establish a comparison theorem for M-solutions of certain BSVIELs. Before we state the main result, we show the following Lemma.

Lemma 4.1 *Consider the following FSVIEL:*

$$\begin{aligned} X(t) &= g(t) + \int_0^t a(s,t)X(s)ds - \int_0^t b(s,t)X(s)dW_s \\ &\quad - \sum_{i=1}^{\infty} \int_0^t c^{(i)}(s,t)X(s)dH_s^i, \quad t \in [0,T], \end{aligned} \tag{27}$$

where $a : [0,T]^2 \times \Omega \to R$, $b : [0,T]^2 \times \Omega \to \mathbb{R}^d$ *and* $c : [0,T]^2 \times \Omega \to \ell^2$ *are three* $\mathcal{B}([0,T]^2) \otimes \mathcal{F}_T$*-measurable and uniformly bounded processes, and for almost all* $t \in [0,T]$, $a(t,\cdot), b(t,\cdot)$ *and* $c(t,\cdot)$ *are* $\mathbb{F}$*-adapted. Moreover, for all* $(t,s,\omega) \in [0,T]^2 \times \Omega$, $\sum_{i=1}^{\infty} c^{(i)}(s)\Delta H_s^{(i)} > -1$.

Then for any $g(\cdot) \in \mathbb{L}^2_{\mathcal{F}_T}(0,T;R)$ *with* $g(t) \geq 0$, *we have*

$$X(t) \geq 0, \quad t \in [0,T], a.s. \tag{28}$$

Proof. The proof follows the ideas in [18]. We first consider a special case of FSVIEL (27). More precisely, let $0 = \tau_0 < \tau_1 < \cdots$ being a sequence of $\mathbb{F}$-stopping times, and

$$\begin{aligned} a(s,t) &= \sum_{k\geq 0} a_k(s)\mathbf{1}_{[\tau_k,\tau_{k+1}]}(t),\ b(s,t) = \sum_{k\geq 0} b_k(s)\mathbf{1}_{[\tau_k,\tau_{k+1}]}(t), \\ c(s,t) &= \sum_{k\geq 0} c_k(s)\mathbf{1}_{[\tau_k,\tau_{k+1}]}(t), \qquad g(t) = \sum_{k\geq 0} g_k\mathbf{1}_{[\tau_k,\tau_{k+1}]}(t), \end{aligned}$$

where for all $k \geq 0$, $a_k(\cdot), b_k(\cdot)$ and $c_k(\cdot)$ being some $\mathbb{F}$-adapted and bounded processes such that $\sum_{i=1}^{\infty} c_k^{(i)}(s)\Delta H_s^{(i)} > -1$, and each $g_k \geq 0$ is $\mathcal{F}_{\tau_k}$-measurable. As a result, on $[0,\tau_1]$, the equation (27) is equivalent to

$$\begin{aligned} X(t) &= g_0 + \int_0^t a_0(s)X(s)ds - \int_0^t b_0(s)X(s)dW_s \\ &\quad - \sum_{i=1}^{\infty} \int_0^t c_0^{(i)}(s)X(s)dH_s^{(i)}, \end{aligned} \tag{29}$$

From Protter [16], the unique solution to equation (29) takes the form

$$\begin{aligned} X(t) &= g_0 \exp\left(\int_0^t ((a_0(r) - b_0(r)^2/2)dr + dM_r)\right) \\ &\quad \prod_{t<r\leq s} (1 + \Delta M_r) \exp(-\Delta M_r) \geq 0, \end{aligned}$$

where

$$M_r = \int_0^r b_0(s)dW_s + \sum_{i=1}^{\infty} \int_0^r c_0^{(i)}(s)dH_s^{(i)}.$$

By induction, we can prove that (28) holds on $[\tau_i, \tau_{i+1}]$. The general case can be proved by approximation. $\square$

Next, we consider the following BSVIEL:

$$\begin{aligned} Y(t) &= -\varphi(t) + \int_t^T f(t, s, Y(s-), Z(s,t), U(s,t))ds \\ &\quad - \int_t^T Z(t,s)dW_s - \sum_{i=1}^{\infty} \int_t^T U^{(i)}(t,s)dH_s^i,\ t \in [0,T], \end{aligned} \tag{30}$$

where the function $f : [0,T]^2 \times \mathbb{R} \times \mathbb{R}^d \times \ell^2 \to \mathbb{R}$ satisfies assumption (**H1**) in a simplified way.

As we know, the comparison theorem is not always hold for BSDEs with jump. One can see Barles et al. [6] for a counterexample. In our frame, we need the following extra assumption on the coefficient f:
(**H3**) the function $f(t,s,y,z,u)$ is nondecreasing in u.

Theorem 4.2 *Let $f, \overline{f} : [0,T]^2 \times \mathbb{R} \times \mathbb{R}^d \times \ell^2 \to \mathbb{R}$ satisfying (H1) and (H3) and let $\varphi(\cdot), \overline{\varphi}(\cdot) \in \mathbb{L}^2_{\mathcal{F}_T}(0,T;\mathbb{R})$ such that*

$$f(t,s,y,z,u) \leq \overline{f}(t,s,y,z,u), \forall (t,s,y,z,u) \in [0,T]^2 \times \mathbb{R} \times \mathbb{R}^d \times \ell^2, a.s.$$

and

$$\varphi(t) \geq \overline{\varphi}(t), \forall t \in [0,T], a.s. \tag{31}$$

Let $(Y(\cdot), Z(\cdot,\cdot), U(\cdot,\cdot))$ (resp. $(\overline{Y}(\cdot), \overline{Z}(\cdot,\cdot), \overline{U}(\cdot,\cdot))$) be the adapted M-solution to BSVIEL (30) corresponding to (f, φ) (resp. $(\overline{f}, \overline{\varphi})$), then

$$Y(t) \leq \overline{Y}(t), \quad \forall t \in [0,T], a.s. \tag{32}$$

Proof. For $\forall t \in [0,T]$, we have

$$\begin{aligned}
& Y(t)-\overline{Y}(t) \\
= \ & \varphi(t)-\overline{\varphi}(t)+\int_t^T f(t,s,Y(s-),Z(s,t),U(s,t))-\overline{f}(t,s,\overline{Y}(s),\overline{Z}(s,t),\overline{U}(s,t))ds \\
& -\int_t^T [Z(t,s)-\overline{Z}(t,s)]dW_s-\sum_{i=1}^{\infty}\int_t^T [U^{(i)}(t,s)-\overline{U}^{(i)}(t,s)]dH_s^i \\
= \ & \widehat{\varphi}(t)+\int_t^T \{B_0(t,s)[Y(s)-\overline{Y}(s)]+B(t,s)[Z(t,s)-\overline{Z}(t,s)] \\
& +\sum_{i=1}^{\infty}C^{(i)}(t,s)[U^{(i)}(t,s)-\overline{U}^{(i)}(t,s)]\}ds-\int_t^T [Z(t,s)-\overline{Z}(t,s)]dW_s \\
& -\sum_{i=1}^{\infty}\int_t^T [U^{(i)}(t,s)-\overline{U}^{(i)}(t,s)]dH_s^i, \qquad (33)
\end{aligned}$$

where

$$\begin{aligned}
\widehat{\varphi}(t) &= \varphi(t)-\overline{\varphi}(t)+\int_t^T f(t,s,\overline{Y}(s),\overline{Z}(s,t),\overline{U}(s,t)) \\
& \qquad -\overline{f}(t,s,\overline{Y}(s),\overline{Z}(s,t),\overline{U}(s,t))ds \leq 0, \\
B_0(t,s) &= [f(t,s,Y(s-),\overline{Z}(s,t),\overline{U}(s,t))-f(t,s,\overline{Y}(s),\overline{Z}(s,t),\overline{U}(s,t))] \\
& \quad [Y(s)-\overline{Y}(s)]^{-1}\mathbf{1}_{\{Y(s)\neq\overline{Y}(s)\}},
\end{aligned}$$

and $B(t,s)=(B_1(t,s),\cdots,B_d(t,s))^T$, $C(t,s)=(C^{(1)}(t,s),\cdots,C^{(i)}(t,s)\cdots)$. Here, for $j=1,\cdots,d$,

$$\begin{aligned}
B_j(t,s) &= [f(t,s,\overline{Y}(s),\widehat{Z}_{j-1}(s,t),\overline{U}(s,t))-f(t,s,\overline{Y}(s),\widehat{Z}_j(s,t),\overline{U}(s,t))] \\
& \quad [Z_j(s,t)-\overline{Z}_j(s,t)]^{-1}\mathbf{1}_{\{Z_j(s,t)\neq\overline{Z}_j(s,t)\}}, \\
\widehat{Z}_j(s,t) &= (\overline{Z}_1(s,t),\cdots,\overline{Z}_j(s,t),Z_{j+1}(s,t),Z_d(s,t))
\end{aligned}$$

and for $i\geq 1$,

$$\begin{aligned}
C^{(i)}(t,s) &= [f(t,s,\overline{Y}(s),\overline{Z}(s,t),\widehat{U}^{(i-1)}(s,t))-f(t,s,\overline{Y}(s),\overline{Z}(s,t),\widehat{U}^{(i)}(t,s))] \\
& \quad [U^{(i)}(s,t)-\overline{U}^{(i)}(s,t)]^{-1}\mathbf{1}_{\{U^{(i)}(s,t)\neq\overline{U}^{(i)}(s,t)\}}, \\
\widehat{U}^{(i)}(t,s) &= (\overline{U}^{(1)}(s,t),\cdots,\overline{U}^{(i)}(s,t),U^{(i+1)}(s,t),\cdots).
\end{aligned}$$

From Lemma 4.1, we can prove the result. $\square$

5 Applications in Finance

In this Section, we define a class of continuous-time dynamic risk measures by means of BSVIELs.

The following definitions are borrowed from [19].

Definition 5.1 *A map* $\rho : \mathbb{L}^2_{\mathcal{F}_T}(0,T;\mathbb{R}) \to \mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R})$ *is called* a dynamic risk measure *if the following hold :*

(i) For any $\varphi(\cdot), \overline{\varphi}(\cdot) \in \mathbb{L}^2_{\mathcal{F}_T}(0,T;\mathbb{R})$, *if* $\varphi(s) = \overline{\varphi}(s), a.s. \omega \in \Omega, s \in [t,T]$ *for some* $t \in [0,T)$, *then* $\rho(t;\varphi(\cdot)) = \rho(t;\overline{\varphi}(\cdot)), a.s. \omega \in \Omega$.

(ii) For any $\varphi(\cdot), \overline{\varphi}(\cdot) \in \mathbb{L}^2_{\mathcal{F}_T}(0,T;\mathbb{R})$, *if* $\varphi(s) \geq \overline{\varphi}(s), a.s. \omega \in \Omega, s \in [t,T]$ *for some* $t \in [0,T)$, *then* $\rho(s;\varphi(\cdot)) \leq \rho(s;\overline{\varphi}(\cdot)), a.s. \omega \in \Omega, s \in [t,T]$.

Definition 5.2 *A dynamic risk measure* $\rho : \mathbb{L}^2_{\mathcal{F}_T}(0,T;\mathbb{R}) \to \mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R})$ *is called a* coherent risk measure *if the following hold :*

(i) There exists a deterministic integral function $r(\cdot)$ *such that for any* $\varphi(\cdot) \in \mathbb{L}^2_{\mathcal{F}_T}(0,T;\mathbb{R})$,

$$\rho(t;\varphi(\cdot)+c) = \rho(t;\varphi(\cdot)) - ce^{-\int_t^T r(s)ds}, a.s., t \in [0,T].$$

(ii) For any $\varphi(\cdot) \in \mathbb{L}^2_{\mathcal{F}_T}(0,T;\mathbb{R})$ *and* $\lambda > 0$,

$$\rho(t;\lambda\varphi(\cdot)) = \lambda\varphi(t;\cdot), a.s., t \in [0,T].$$

(iii) For any $\varphi(\cdot), \overline{\varphi}(\cdot) \in \mathbb{L}^2_{\mathcal{F}_T}(0,T;\mathbb{R})$,

$$\rho(t;\varphi(\cdot)+\overline{\varphi}(\cdot)) \leq \rho(t;\varphi(\cdot)) + \rho(t;\overline{\varphi}(\cdot)), a.s., t \in [0,T].$$

In what follows, we denote by

$$\rho(t;\varphi(\cdot)) = Y(t), \tag{34}$$

where $(Y(\cdot), Z(\cdot,\cdot), U(\cdot,\cdot)$ is the unique M-solution of the following BSVIEL:

$$\begin{aligned} Y(t) &= -\varphi(t) + \int_t^T f(t,s,Y(s-),Z(s,t),U(s,t))ds \\ &\quad - \int_t^T Z(t,s)dW_s - \sum_{i=1}^{\infty} \int_t^T U^{(i)}(t,s)dH^i_s, \; t \in [0,T]. \end{aligned} \tag{35}$$

Lemma 5.1 *Let $f : [0,T]^2 \times \mathbb{R} \times \mathbb{R}^d \times \ell^2 \to \mathbb{R}$ satisfy (H1) and (H3), suppose f is sub-additive, i.e.,*

$$f(t,s,y_1+y_2,z_1+z_2,u_1+u_2) \le f(t,s,y_1,z_1,u_1) + f(t,s,y_2,z_2,u_2),$$
$$\forall (t,s) \in [0,T]^2, y_1,y_2 \in \mathbb{R}, z_1,z_2 \in \mathbb{R}^d, u_1,u_2 \in \ell^2, a.e.$$

then $\varphi(\cdot) \to \rho(t;\varphi(\cdot))$ is sub-additive, i.e.,

$$\rho(t;\varphi_1(\cdot)+\varphi_2(\cdot)) \le \rho(t;\varphi_1(\cdot)) + \rho(t;\varphi_2(\cdot)), a.e.$$

Proof. We can get the conclusion by Theorem 4.2. $\square$

Lemma 5.2 *(i) If the generator f is of form*

$$f(t,s,y,z,u) = r(s)y + \widetilde{f}(t,s,z,u),$$

with $r(\cdot)$ being a deterministic integral function, then $\varphi(\cdot) \to \rho(t;\varphi(\cdot))$ is transition invariant, i.e.,

$$\rho(t;\varphi(\cdot)+c) = \rho(t;\varphi(\cdot)) - ce^{-\int_t^T r(s)ds}, a.s., t \in [0,T], a.s., \forall c \in \mathbb{R}.$$

In particular, if $r(\cdot) = 0$, then

$$\rho(t;\varphi(\cdot)+c) = \rho(t;\varphi(\cdot)) - c, a.s., t \in [0,T], a.s., \forall c \in \mathbb{R}.$$

(ii) If $f : [0,T]^2 \times \mathbb{R} \times \mathbb{R}^d \times \ell^2 \to \mathbb{R}$ is positively homogeneous, i.e., $f(t,s,\lambda y,\lambda z,\lambda u) = \lambda f(t,s,y,z,u), \forall (t,s) \in [0,T]^2, \lambda \in \mathbb{R}^+, y \in \mathbb{R}, z \in \mathbb{R}^d, u \in \ell^2, a.e.$ So is $\varphi(\cdot) \to \rho(t;\varphi(\cdot))$.

Proof. The proof is obvious. $\square$

By Lemmas 5.1 and 5.2, we are able to construct a class of dynamic coherent risk measures by means of solution of certain BSVIELs. The proof of the following theorem is obvious, so we omit it.

Theorem 5.3 *Suppose f satisfy (H1) and (H3). Moreover,*

$$f(t,s,y,z,u) = r(s)y + \widetilde{f}(t,s,z,u),$$

with $r(\cdot)$ being a bounded and deterministic integral function, then $\rho(\cdot)$ defined by (34) is a dynamic coherent risk measure if $\widetilde{f}(t,s,z,u)$ is positively homogeneous and sub-additive.

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