On the Three Colorability of Planar Graphs

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Abstract

The chromatic number of an planar graph is not greater than four and this is known by the famous four color theorem and is equal to two when the planar graph is bipartite. When the planar graph is even-triangulated or all cycles are greater than three we know by the Heawood and the Grotszch theorems that the chromatic number is three. There are many conjectures and partial results on three colorability of planar graphs when the graph has specific cycles lengths or cycles with three edges (triangles) have special distance distributions. In this paper we have given a new three colorability criteria for planar graphs that can be considered as an generalization of the Heawood and the Grotszch theorems with respect to the triangulation and cycles of length greater than ≥ 4. We have shown that an triangulated planar graph with k disjoint holes is 3-colorable if and only if every hole satisfies the parity symmetric property, where a hole is a cycle (face boundary) of length greater than 3.

1 Introduction

Four coloring of planar graphs is a famous theorem (4CT) and it has been proved twice by the same method by the assistance of an computer and correctness of the proof has been verified by another computer program [1],[2],[3]. The author has given an non-computer proof of the four color theorem by using spiral chains and spiral chain coloring in the maximal planar graphs [2],[3],[17]. However, for an given planar graph the question “When three color suffice?” has not been completely solved. When the planar graph is even-triangulated or all cycles are greater than three we know by the Heawood and the Grotszch theorems that the chromatic number is three [1]. In the literature, there are several proofs of Grotszch theorems [11],[15],[16],[18] and the simplest and efficient algorithmic proof is appear to be given by the author [17]. It has been also considered on the other surfaces [12],[13],[15]. Let $C = \{White, Gray, Black\} = \{W, G, B\}$ be the set of three colors and when we needed we use $Red = \{R\}$ as the fourth color.

In this paper we have given a new three colorability criteria for planar graphs that can be considered as an generalization of the Heawood and the Grotszch theorems with respect to the triangulation and cycles of length greater than ≥ 4.

First we have defined the triangulated ring and gave necessary and sufficient condition for three colorability ring. Next we have given an generaliztion of triangulated rings for an triangulated planar graph with k disjoint holes which is 3-colorable if and only if every hole satisfies the parity symmetric property, where a hole is a cycle (face boundary) of length greater than 3.

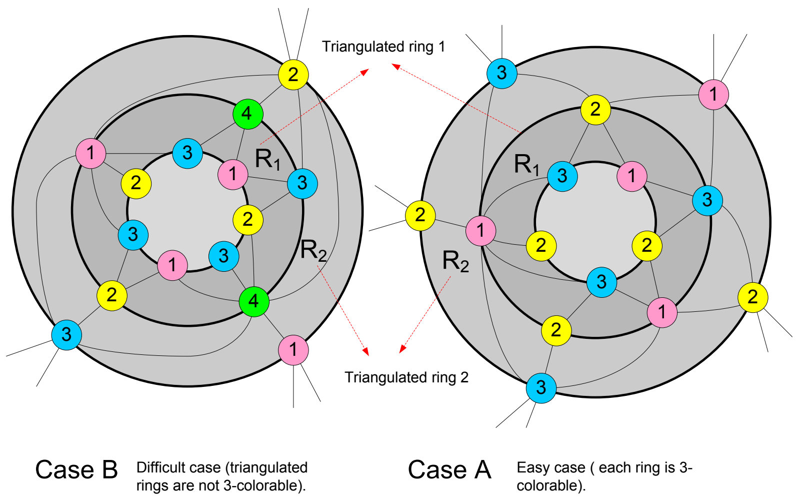

Figure 1: Illustration of the triangulated rings for 3 and 4 colorings.

2 Triangulated ring

In Fig. 1 we have shown two double triangulated rings together with their colorings. The double triangulated ring shown on the right (Case A) is 3-colorable while double triangulated ring shown on the left (Case B) is 4-colorable.

We will make a few formal definitions first. A triangulated ring is a 2-connected planar graph G with minimum degree ≥ 3 with two faces F_i and F_o whose facial walks are the (induced) cycles C_i and C_o, respectively, such that:

(1) $V(C_i)$ and $V(C_o)$ partition $V(G)$. (That is $V(C_i) \cup V(C_o) = V(G)$ and $V(C_i) \cap V(C_o) = \emptyset$), where indices i and o are being used to denote the inner and outer cycles (faces) of the graph.

(2) Every face other than F_i and F_o is a triangle.

Every vertex v on C_i or C_o has a fan, namely the triangles incident with v (other than possibly C_i and C_o). These fans, along with F_i and F_o, partition the faces of G. Let the fan graph of G be the graph $F(G)$ whose vertices f_i represent the fans F_i of G, where an edge is in $F(G)$ if the fans share a common edge. The vertices f_i can (and will be) identified with the of the fan (the vertex of the fan adjacent to all others). The graph $F(G)$ is also the subgraph of G induced by $\{f_0, f_1, ..., f_{2k-1}\}$. The fan graph of G is a even cycle $f_0, f_1, ..., f_{2k-1}$. The *cyclic parity sequence* of $(cps(G))$ is the cyclic sequence $p_0 p_1 p_2 ... p_{2k-1}$ where is the parity (even/odd) of the number of triangles in F_i. Note that every triangulated ring has at most one 3-coloring (up to permutation of colors), because a triangle

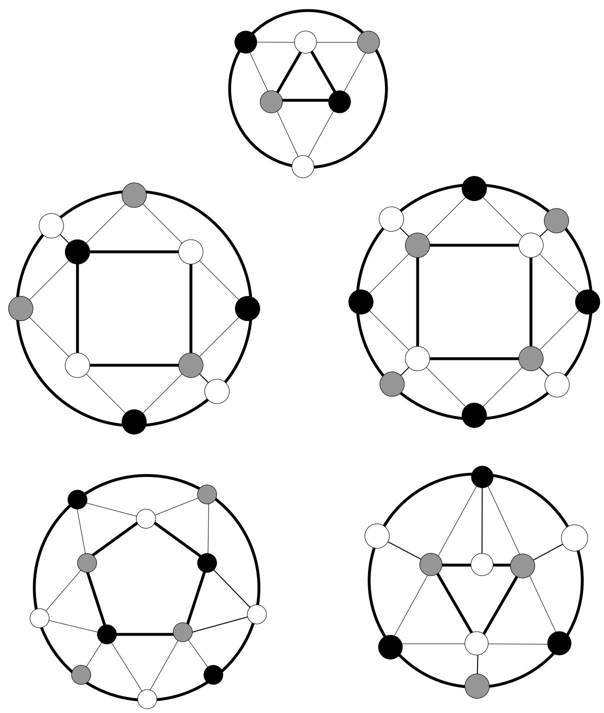

Figure 2: All possible three colorings of the triangulated rings with inner-cycles of length 3,4 and 5.

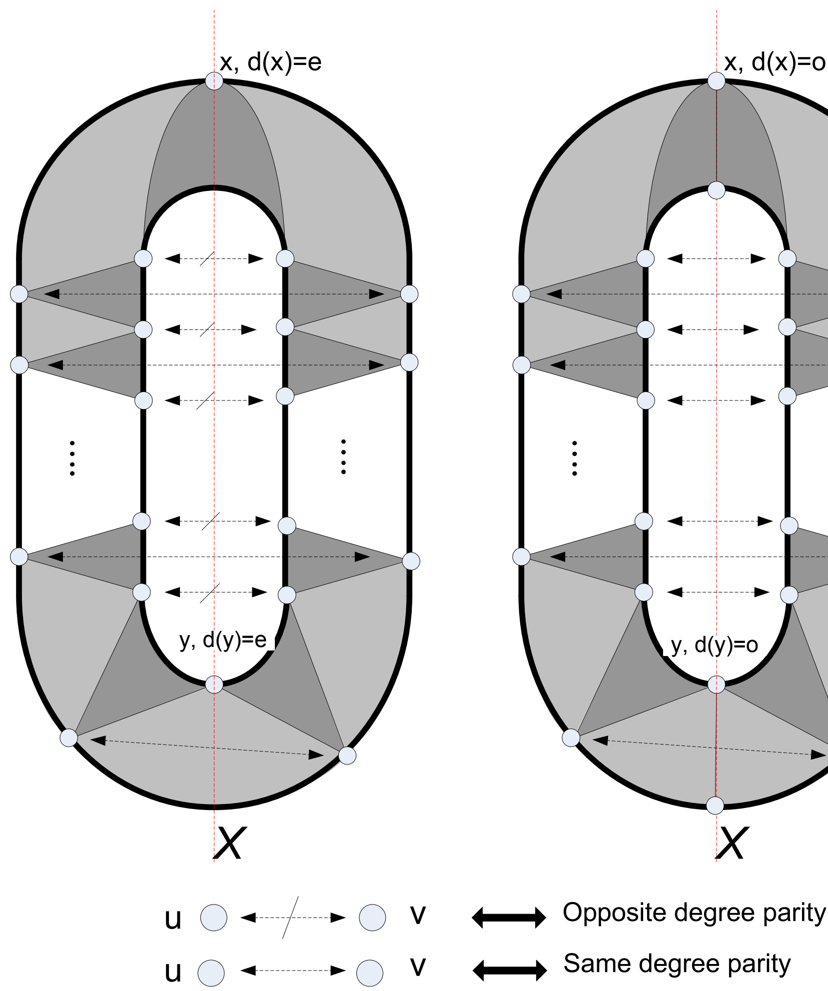

Figure 3: Three colorable triangulated rings that $cps(G_i) \in T, i = 1, 2$, where X is the symmetry axis of the graph.

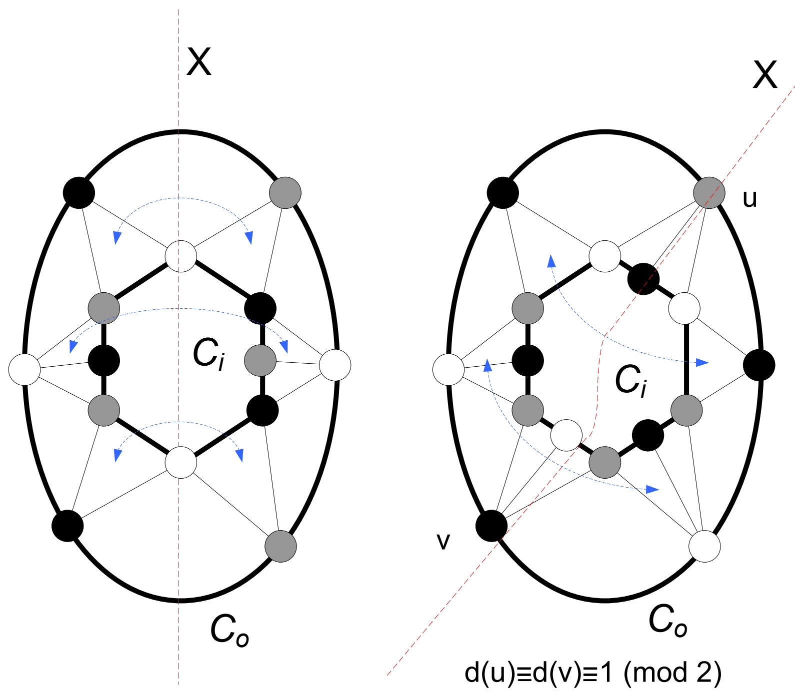

Figure 4: Three colorings of the triangulated rings with $|C_i| = 8, |C_o| = 6$ and $|C_i| = 10, |C_o| = 6$

has exactly one 3-coloring (again, up to permutation of colors), and every edge not in $C_o \cup C_i$ is in a triangle. The next lemma takes this result one step further. For that, we must introduce some terminology related to the *cps* of a graph. If $p_0p_1...p_{2k-1}$ is a *cps* with $k \geq 2$ and $p_j = e$, then define the e-collapse of the *cps* at j to be the *cps* as $p_0p_1...p_{j-2}(p_{j-1} + p_{j+1})p_{j+2}...p_{2k}$, where addition is of parities (modulo 2). That is, the e-collapse of *ooeeoeoo* at 5 is $ooee(o + o)o = ooeeeo$. Finally, let T be the set of all *cps*'s which can be transformed into ee or $(o)^{6m}$ (i.e., $6mo$'s, for some integer m) by a finite number of e-collapses.

Let us give a simple lemma.

Lemma 1. If $G(C_i \cup C_o)$ is 3-colorable triangulated ring with all even fans then $|C_i| \equiv 0(mod3)$ or with all odd fans then $|C_i| = |C_o| \equiv 0(mod2), |C_i| \neq 4$.

Lemma 2. If G is a triangulated ring, then G is 3-colorable iff $cps(G) \in T$.

First, we need some lemmas. The first is easily proven using induction:

*Lemma 2.1. Suppose F is a fan graph, where v is the vertex of F adjacent to all others, and u and w are the two vertices of F
v with degree 1. Then:*

(a) If F is even and $u, v$and w are colored so that u and w receive the same color, then this partial 3-coloring extends to a proper 3-coloring of F;

(b) In a proper 3-coloring of an even fan F, u and w receive the same color;

(c) If F is odd and u, v, and w are colored so that u, v and w all receive different colors, then this partial 3-coloring extends to a proper 3-coloring of F; and

(d) In a proper 3-coloring of an odd fan F, u, v, and w all receive different colors.

If G is a triangulated ring with at least four fans, and F_j an even fan of G, we will call the [even] fan collapse of G at F_j the graph H obtained from G by deleting the vertices of F_j other than f_{j-1}, f_j , and f_{j+1}, and then identifying the vertices f_{j-1} and f_{j+1}.

Lemma 2.2. Let G be a triangulated ring, and H the graph resulting from a fan collapse of G at F_j . Then G is 3-colorable if and only if H is. Furthermore, $cps(H)$ can be obtained from $cps(G)$ by an e-collapse at$j$.

Proof: If c is a proper 3-coloring of G, then since F is a fan, $c(u) = c(w)$, by Lemma 2.1(b). This means that when H is created, some vertices will be deleted (which makes this smaller graph 3-colorable), and two vertices with the same color will be identified. This means H is 3-colorable. Now suppose c is a proper 3-coloring of H. To go from H to G, a vertex v_0 will have to be split into u and w. We will let $c(u) = c(w) = c(v_0)$, and we still have a 3-coloring. To get G, we need to add some vertices, so we need to make sure that we can extend the 3-coloring to these vertices. Since the fan which will be created is even, Lemma 2.1(a) implies that this new fan is 3-colorable. The union of these two proper 3-colorings gives a proper 3-coloring of G.

To prove the second result: The graph H is a triangulated ring with two fewer fans than G. All but one of the fans of H come from fans of G, with the other fan of H being the combination of two fans F_{j-1} and F_{j+1} of G; the parity of this new fan is the sum of the parities of F_{j-1} and F_{j+1}. QED.

And the following result makes life easier:

Lemma 2.3. If G_1 and G_2 are triangulated rings with the same cps, then G_1 is 3-colorable iff G_2 is.

Proof: If G_1 and G_2 are as stated above, then G_1 can be transformed into G_2 by repeatedly adding two vertices to a fan, or by removing two vertices from a fan. If this is done to fan Fi, it doesn't affect the coloring of f_{i-1}, f_i, or f_{i+1}. QED.

Proof of Lemma 2: Let $F(G)$ be the fan graph of G. Note that fan F_i contains the vertices f_{i_1}, f_i, and f_{i+1} (where all indices are modulo $2k$), which correspond to the vertices u, v, and w in the statement of Lemma 2.1.

We will begin by proving that if $cps(G) \in T$, then G is 3-colorable. The proof will be by induction on the number of e-collapses. If there are no e-collapses, then we only have to settle the cases ee and $(o)^{6m}$, since $eo, oo \notin T$. If $cps(G) = ee$, then $F(G)$ contains two vertices v_1 and v_2, and (technically) 2 parallel edges. (Note that G itself can itself be a simple graph, because the e in the cps means there's an even number of triangles in each fan, not zero.) If we color f_0 with 1 and f_1 with 2, then note that for all $i, p_i = e$, and f_{i-1} and f_{i+1} are (trivially) colored with the same color. Lemma 2.1(a) implies that G is 3-colorable. Similarly, if $cps(G)$ consists of $6mo$'s, we color the fan graph as follows. (Note that in this case, $2k = 6m$ is a multiple of 6, so it's also a multiple of 3, and the coloring is well-defined:

$$c(f_i) = \begin{cases} 1 & \text{if } i \equiv 0(mod 3) \\ 2 & \text{if } i \equiv 1(mod 3) \\ 3 & \text{if } i \equiv 2(mod 3) \end{cases}$$

Now, since every p_i is odd, all we need to do is to verify that f_{i-1}, f_i, and f_{i+1} all receive different colors, for all i. This follows immediately from the definition, and Lemma 2.1(c) then implies that G is 3-colorable. Now suppose the result is true for $N-1$ e-collapses, with $N \geq 1$. Let G be a triangulated ring whose $cps(G)$ can be transformed into ee or $(o)^{6m}$ by N e-collapses. Consider the first e-collapse, which we will assume occurs at j and results in the cps sequence S. If H is the fan collapse of G at F_j , then $cps(H) = S$, by Lemma 2.2.

But since S can be transformed into ee or $(o)^{6m}$ with $N-1$ e-collapses, H is 3-colorable by the induction hypothesis. Lemma 2.2 states that H is 3-colorable iff G is, so G is 3-colorable.

Now we have to show that if a triangulated ring G is 3-colorable, then $cps(G) \in T$. Assume that G is 3-colorable. First, we will settle the case where $cps(G)$ has no e's in it. In this case, every p_i is o, so the3-coloring must satisfy $c(f_{i+3}) = c(f_i)$ for all i, where indices are taken modulo $2k$; this follows because $c(f_i), c(f_{i+1})$, and $c(f_{i+2})$ must all be distinct, since $p_{i+1} = o$, by Lemma 4.1; and $c(f_{i+1}), c(f_{i+2})$, and $c(f_{i+3})$ must all be distinct, since $p_{i+2} = o$. This forces $c(f_i) = c(f_{i+3})$. However, if the number of o's is not a multiple of 6, then $2k$ (the length of the cps) is not a multiple of 3. That means that the condition $c(f_{i+3}) = c(f_i)$ forces all vertices f_i to be colored the same, which is not allowed by Lemma 2.1(c). Hence there is no proper 3-coloring of G, contrary to assumption. Thus the number of o's is a multiple of 6, so $cps(G) = (o)^{6m} \in T$, as claimed. Now suppose that $p_j = e$. If $k = 1$, then $cps(G) = oe, eo$, oree. It is easily seen that if $cps(G) = oe$ (or eo) then G is not 3-colorable, so $cps(G) = ee \in T$.

So now we may assume that $k \geq 2$. Define a sequence of graphs G_i in the following way: Let G_0 be G, and for all $i \geq 0$, if $p_j = e$ in $cps(G_i)$ and $cps(G_i)$ has length at least two, then let G_{i+1} be the fan collapse of G_i at F_j . Suppose we cannot continue from G_N. Note that, for all applicable i, $cps(G_{i+1})$ can be obtained from $cps(G_i)$ by e-collapse. Then $cps(G_N)$ either has no e's, or has length two. Furthermore, G_N is 3-colorable, by repeated application of Lemma 2.2. But we have seen G_N can only be 3-colorable if $cps(G_N) = ee$ or $cps(G_N) = (o)^{6m}$. In either case, we can obtain ee or $(o)^{6m}$ from $cps(G)$ by repeated e-collapsings. Thus $cps(G) \in T$, which proves the lemma.

The following lemma is given without proof and useful for 3-colorable triangulated rings.

Lemma 3. Triangulated ring G is 3-colorable if $cps(G)$ is symmetric and $|C_o| \equiv 0(mod3)$ or $|C_o \cup C_i| \equiv 0(mod3)$.

For illustrations see the 3-colorings of the triangulated rings given in Fig. 2 and 4. In Fig. 4 the symmetry axis is denoted by X.

In Fig. 3 a more general three colorable triangulated rings have been shown. The $csp(G)$'s respectively are:

$cps(G_1) = \{oab\text{'}...ab\text{'}ab\text{'}obaba...ba\}$ (Fig.2 (left))

$cps(G_2) = \{eab...babaebaba...ba\}$ (Fig.2 (right)),

where if $b = e$ then $b\text{'} = 0$ and if $b\text{'} = e$ then $b = o$, and o and e denote odd and even parities.

3 Planar graphs with holes

In this section we will extend the result obtained for the three-colorability of the triangulated rings to triangulated planar graphs with vertex disjoint k cycles (holes) h_i of lengths greater than three. Let $G(h_i)$, $i = 1, 2, ..., k$ be the set of triangulated rings arround the holes h_i, where $V(h_1) \cap V(h_2) \cap ... \cap V(h_k) = \emptyset$. Let $V(C_o)$ be the outer-cycle of G. Clearly $V(C_o) \cap \{V(h_1) \cup V(h_2) \cup ... \cup V(h_k)\} = \emptyset$.

Theorem 3. Let G be an triangulated planar graph with k disjoint holes h_i, $i = 1, 2, ..., k$. Then G is 3-colorable iff for every triangulated ring $G(h_i)$ we have $csp(G(h_i)) \in T$.

Proof. Necessity of the theorem can be easily seen by the cyclic parity sequence of $(cpsG(h_i)$ which we have assumed that can be transformed into ee or $(o)^{\{6m\}}$ (see Lemma 2).

Now define the graph $H(V, E)$ where the vertex set $V(H) = \{h_1, h_2, ..., h_k\}$ are the holes of G and $e_{i,j} = (h_i h_j) \in E(H)$ if $E(G((h_i)) \cap E(G(h_j)) \neq \emptyset$. Let T_G be an spanning tree of $H(V, E)$. Then start coloring of the vertices $G(h_1)$ first and next select an vertex $h_i, i \neq 1$such that $(h_i h_1) \in T_G$. Color the vertices of $G(h_\imath)$. Repeat this step for the other vertices of T_G. It is clear that since

$cspG(h_i) \in T$ and $(G(h_i)$is an triangulated ring at the end G would be colored properly with three colors.

Corollary. The planar triangulated graph $G(h_i)$ with k holes $h_i = 1, 2, ..., k$ can be made 3-colorable triangulated ring with $C_i = h_1 \cup h_2 \cup ... \cup h_k$.

Proof. Delete suitable edges of the spanning tree T_G in merging two adjacent holes h_i and h_j. At the end the inner cycle C_i will be the union of the holes.

Next define the semi-triangulated graph G^*as an triangulated ring in which there exists at least one cycle C_r of length greater than three such that $C_i \cap C_r \neq \emptyset$ and $C_o \cap C_r \neq \emptyset$. Note that non-triangulated ring has no triangle; hence is 3-colorable by Grotzsch theorem. The following simple theorem gives useful information when $G^*$is 3-colorable.

Theorem 4. Let G be an triangulated ring. If $csp(G) \notin T$ and G^*be any semi-triangulated ring obtained by the addition of an single cycle C_r of length $k \geq 4$ then $G^*$is 3-colorable. If $csp(G) \in T$ and G^*be any semi-triangulated ring obtained by the addition of an single cycle C_r such that $|E(C_o) \cap E(C_r)| \geq 2$ then $G^{^*}$ is 3-colorable.

Proof. If $csp(G) \notin G$ then for any 3-coloring c there must be an vertex v such that $c(v) = i = j$, where $i, j \in \{1, 2, 3\}, i \neq j$. Note that such a vertex v can be selected freely beforehand since G is an triangulated ring. Then the vertex v is splited into two $v^{'}$and $v^{''}$vertices in G^*such that $(v^{'}v^{''}) \in C_r$. Hence the 3-coloring c of G^*can be made proper by $c(v^{'}) = i$ and $c(v^{''}) = j$ and the other vertices of $C_r$can be colored alternatingly by two colors $1, 2$ or $3, 2$. Second part of the theorem is similar to the first part but this time since $csp(G) \in G$ the coloring c of G is an proper 3-coloring and the vertex v with $c(v) = i$ must be splited into three vertices i.e., $|E(C_o) \cap E(C_r)| \geq 2$. Therefore coloring c will be proper three coloring of $G^* = G \cup C_r$.

In fact the above theorem can be generalized to semi-triangulated rings with k cycles of length ≥ 4. That is think of G^*as disjoint triangular ladders (all with triangles) separated by cycles of length ≥ 4. Let us denote the semi-triangulated graph $G^* = G^*(C_1 \cup C_2 \cup ... \cup C_k)$. This suggest that as we are inserting large cycles into 3-colorable planar graph three colorability maintained.

4 Another wave of conjectures on 3-colorability

As early as 1959, Grötzsch proved that every planar graph without 3-cycles is 3-colorable. This result was later improved by Aksionov in 1974. He proved that every planar graph with at most three 3-cycles is 3-colorable. In 1976, Steinberg conjectured the following :

Steinberg's Conjecture (1976) [5] : Every planar graph without 4- and 5-cycles is 3-colorable.

An algorithmic proof to Steinberg's conjecture has been proposed by the author in 2006 [4]. We note that the statement of the Steinberg's conjecture is not sharp since there are 3-colorable planar graphs with four and five cycles (see Fig. 7).

In 1969, Havel posed the following problem:

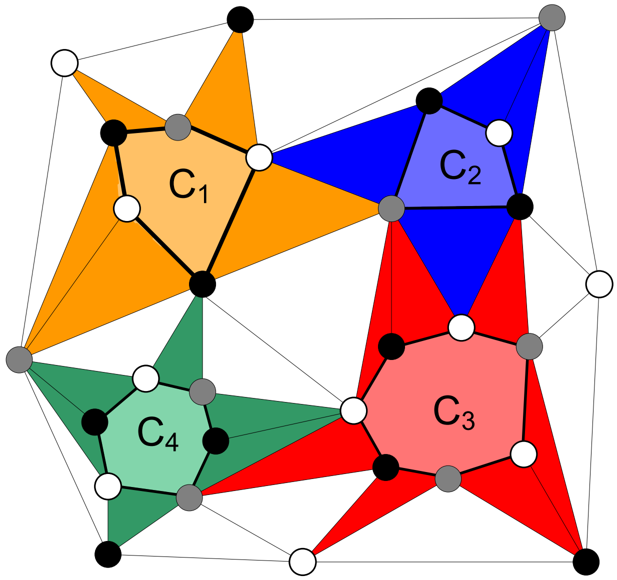

Figure 5: Three coloring of an planar graph with 4 holes.

Havel's Problem (1969) [20] : Does there exist a constant C such that every planar graph with the minimal distance between triangles at least C is 3-colorable?

Aksionov and Mel'nikov proved that if C exists, then $C \geq 4$, and conjectured that $C = 5$ [6]. These two problems remain widely open. In 1991, Erdös suggested the following relaxation of Steinberg's Conjecture: Determine the smallest value of k, if it exists, such that every planar graph without any cycles of length 4 to k is 3-colorable. The best known bound for such a k is 7 [19]. Many other sufficient conditions of 3-colorability considering planar graphs without cycles of specific lengths were proposed.

At the crossroad of Havel's and Steinberg's problems (since the authors consider planar graphs without cycles of specific length and without close triangles), Borodin and Raspaud proved that every planar graph without 3-cycles at distance less than four and without 5-cycles is 3-colorable (the distance was later decreased to three by Xu, and to two by Borodin and Glebov). As well, they proposed the following conjecture:

Strong Bordeaux Conjecture (2003) [21]: Every planar graph without 5-cycle and without adjacent triangle is 3 colorable.

By adjacent cycles, we mean those with an edge in common. This conjecture implies Steinberg's Conjecture. Finally, Borodin et al. considered the adjacency between cycles in planar graphs where all lengths of cycles are authorized, which seems to be the closest from Havel's problem ; they proved that every planar graphs without triangles adjacent to cycles of length from 3 to 9 is 3-colorable . Moreover they proposed the following conjecture:

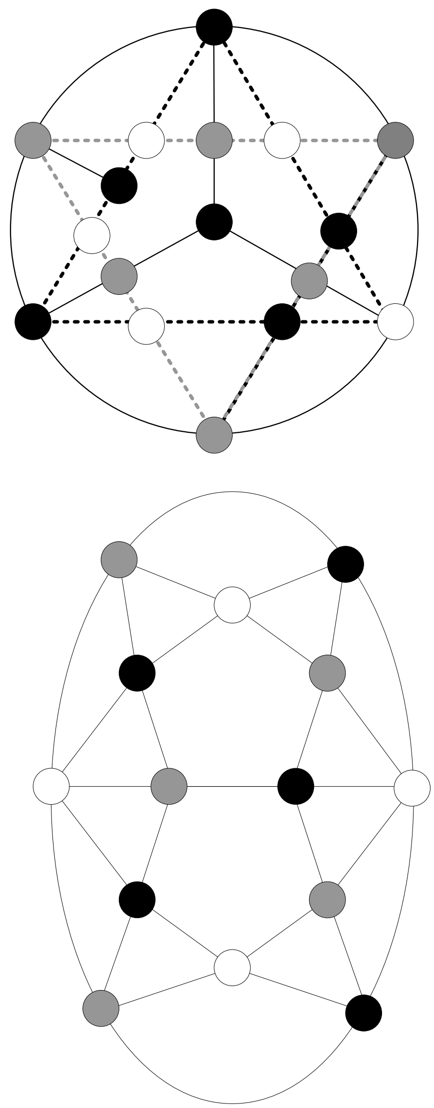

Figure 6: Three colorable planar graphs with triangulated ring (*upper*): with three cycles of length 5 and 6, (*lower*): with two cycles of length 5.

Novosibirsk 3-Color Conjecture (2006) [22]: Every planar graph without 3-cycles adjacent to cycles of length 3 or 5 is 3-colorable. This implies Strong Bordeaux Conjecture and Steinberg's Conjecture.

We claim that the above two conjectures can be settled by the use of spiral chains and this will be given in the updated version of [4].

In Fig. 9 we have shown triangulated rings with 3- and 4-colorings. Note that the Kempe chains $K(W,B), K(G,B)$ in Fig. 9(b) and $K(W,G)$ in Fig. 9(c) prevent the three colorability of the triangulated rings.

4 Concluding remarks

In this paper we have given a new three coloring criteria, that is three colorability of triangulated rings. We have generalized three colorable rings to some extend to the triangulated planar graphs with disjoint large cycles (cycles of length ≥ 4). Although we know that planar 3-colorability is *NP*-complete [10] but the result obtained here may rise some hopes to devise an efficient algorithm for three colorable planar graphs.

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References

[1] H. Groetzsch, Ein Dreifarbensatz fur Dreikreisfrei Betz auf der Kugel, Wiss, Z. Martin Luther Univ., Halle-Wittenburg Math. Naturwise. Reihe, 8 (1958).

[2] I. Cahit, Spiral Chains: A New Proof of the Four Color Theorem,arXiv:math/0408247v1 [math.CO], 2004.

[3] I. Cahit, Spiral Chains: The Proofs of Tait's and Tutte's Three-Edge-Coloring Conjectures,arXiv:math/0507127v1 [math.CO],2005.

[4] I. Cahit, The proof of Steinberg's three coloring conjecture, arXiv:math/0607497v1 [math.CO], 2006.

[5] R. Steinberg, The state of the three color problem, Quo Vadis, Graph Theory? J. Gimbel, J.W. Kennedy and L.V. Quintas (eds), Ann. Disc. Math. 55(1993) 211-248.

[6] L. S. Meinikov, and V. A. Aksionov, Some counter-examples associated with the 3-colour problem, Journal of Combinatorial Theory, B (28), 1-9, 1980.

[7] B. Grünbaum, Grötzsch's theorem on 3-colorings, Michigan Math. J., 10, 1963, 303-310.

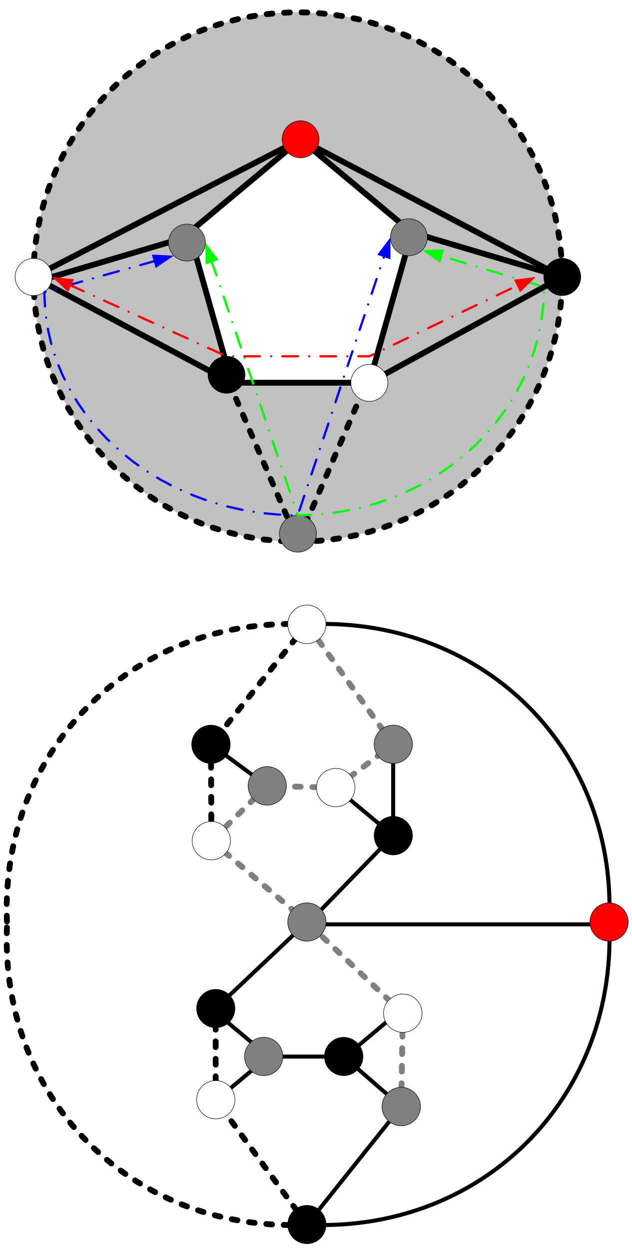

Figure 7: (a) Four colorable triangulated ring with only an single C_5 (subgraph in bold lines is not 3-colorable).(b) Four colorable graph without $C4$.

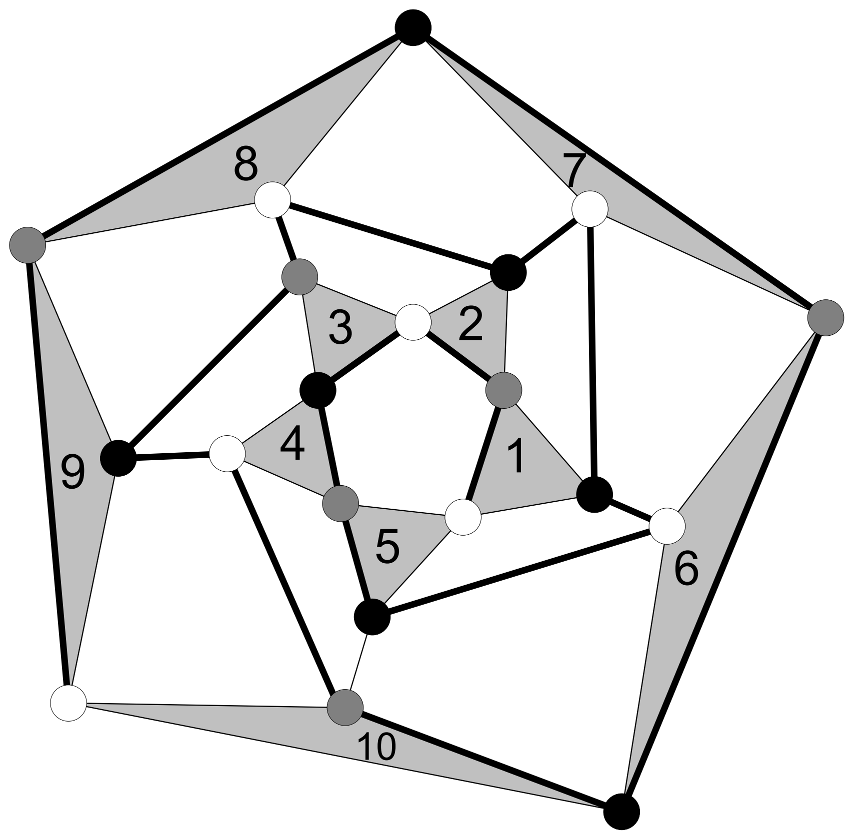

Figure 8: Three colorable semi-triangulated rings.

[8] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, Vierteljahrschriften der Naturforschungsgellschaft Zürich, 88 (1943) 133-142.

[9] K. Appel, W. Haken, Every planar map is four-colorable, Bull. Amer. Math. Soc. 82 (1976), 711-712.

[10] L. J. Stockmeyer, Planar 3-colorability is NP-complete, SIGACT News, 5(3), 1973, 19-25.

[11] L. Kowalik, Fast 3-coloring triangle-free planar graphs, Algorithms-ESA 2004, 436-447, Lec. Notes in Comput. Sci., 3221, Spinger Berlin, 2004.

[12] J. Hutchinson, Three coloring graphs embeded on surfaces with all even-sided, J. Combin. Th. Ser B, 65 (1995) 139-155.

[13] Z. Dvŏrák, D. Král, R. Thomas, Coloring triangle-free graphs on surfaces, preprint, 2007.

[14] D. Surendran, The three colour problem, Resonance, February 1999, 76-81.

[15] C. Thomassen, Grötzsch's 3-color theorem and its counterparts for the torus and projective plane, J. of Combin. Theory, Series B, 62, 268-279, 1994.

[16] C. Thomassen, A short list color proof Grötzsch's theorem, J. of Combin. Theory, Series B, 88, 189-192, 2003.

[17] I. Cahit, A unified spiral chain coloring algorithm for planar graphs, arXiv:0710.2066v1 [math.CO], October 2007

[18] R. Thomas, Z. Dvorak and K. Kawarabayashi, Three-coloring triangle-free planar graphs in linear time, SODA 09 to appear 2009.

[19] B. Xu, On 3-colorable plane graphs without 5- and 7-cycles. Journal of Combinatorial Theory, Series B, 96:958–963, 2006.

[20] I. Havel, On a conjecture of Grünbaum. Journal of Combinatorial Theory, Serie B, 7:184–186, 1969.

[21] O.V. Borodin and A. Raspaud, A sufficient condition for a planar graph to be 3-colorable. Journal of Combinatorial Theory, Serie B, 88:17–27, 2003.

[22] O.V. Borodin, A.N. Glebov, T.R. Jensen, and A. Raspaud, Planar graphs without triangles adjacent to cycles of length from 3 to 9 are 3-colorable. Sib. Elec. Math. Rep. 3:428–440, 2006.

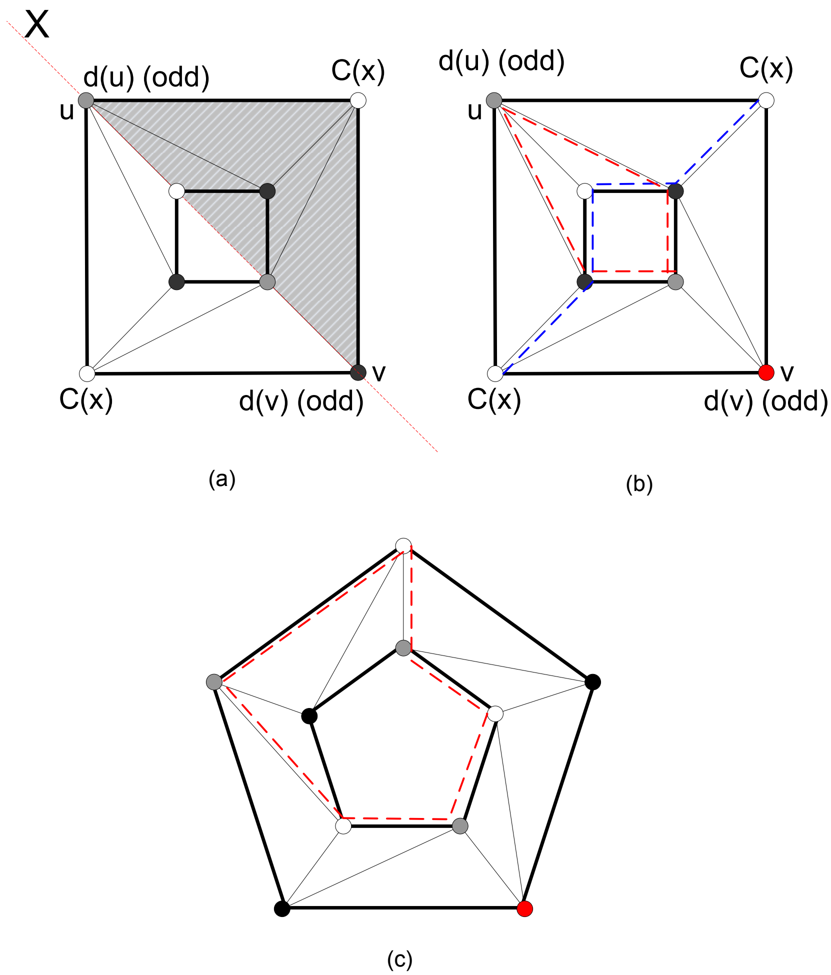

Figure 9: Some triangulated rings with and without three coloring (dashed lines are the Kempe-chains that prevent to reduce the chromatic number.