ON A GENERALIZATION OF BERNOULLI AND EULER NUMBERS

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10th International Seminar "Discrete Mathematics and its Applications" , 2010

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We shall consider permutations on the set $\{0, \ldots, N\}$ (i.e. one-to-one mappings from this set onto itself), where $N \in \mathbb{Z}_+$. We say that such permutation τ *increases* (*decreases*) on the set $\{p, \ldots, q\} \subseteq \{0, \ldots, N\}$ if for any $j = p, \ldots, q-1$ we have $\tau(j) < \tau(j+1)$ $(\tau(j) > \tau(j+1))$.

Take $n \in \mathbb{N}$, $i_1, j_1, \ldots, i_n, j_n \in \mathbb{Z}_+$, $N := i_1 + j_1 + \ldots + i_n + j_n$. The number of permutations on $\{0, \ldots, N\}$ which increase on $\{s''_{k-1}, \ldots, s'_k\}$ and decrease on $\{s'_k, \ldots, s''_k\}$ for each $k = 1, \ldots, n$ will be denoted as $\Omega(i_1, j_1, \ldots, i_n, j_n)$. From now on $s'_k := i_1 + j_1 + \ldots + i_k$, $s''_k := i_1 + j_1 + \ldots + i_k + j_k$.

The last argument j_n is omitted if $j_n = 0$. It is convenient to let Ω with no arguments be equal to 1.

Lemma 1. $\Omega(p) = 1$, $\Omega(p, q) = (p+q)!/(p!q!)$ *for* $p, q \in \mathbb{Z}_+$.

This is a generalization of *Andre problem*: to find $b_n := \Omega(1, \ldots, 1)$ ($n-1$ unit arguments). Define *Bernoulli numbers* B_n and *Euler numbers* E_n as the coefficients in these series:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{+\infty} \frac{B_n}{n!} x^n, \quad \frac{1}{\operatorname{ch} x} = \sum_{n=0}^{+\infty} \frac{E_n}{n!} x^n.$$

Then (see [1], Chapter 3, \$1)

$$b_{2n-1} = \frac{(-1)^{n-1}}{n} 2^{2n-1}(2^{2n} - 1) B_{2n}, \quad b_{2n} = (-1)^n E_{2n}.$$

Hence the Ω numbers are a generalization of Bernoulli and Euler numbers and binomial coefficients (recall Lemma 1).

Theorem 1. *For any* $n, i_1, \ldots, i_n \in \mathbb{N}$ *we have* $\Omega(i_1, \ldots, i_n) = \Omega(i_1 - 1, \ldots, i_n) + \Omega(i_1, i_2 - 1, i_3, \ldots, i_n) + \ldots + \Omega(i_1, \ldots, i_n - 1)$.

Consider a generating function

$$F_n(x_1, \ldots, x_n) := \sum_{i_1=0}^{+\infty} \ldots \sum_{i_n=0}^{+\infty} \Omega(i_1, \ldots, i_n) x_1^{i_1} \ldots x_n^{i_n}.$$

It is not difficult to deduce $\Omega(i_1, \ldots, i_n) \le (i_1 + \ldots + i_n)!/(i_1! \ldots i_n!)$ from Theorem 1. Therefore, this series absolutely converges if $|x_1| + \ldots + |x_n| < 1$. For any n the function F_n is rational. One can calculate it for any given n. For example,

$$F_3(x_1, x_2, x_3) = \frac{1 - x_1 - x_3}{(1 - x_1)(1 - x_3)(1 - x_1 - x_2 - x_3)},$$

$$F_4(x_1, x_2, x_3, x_4) = \frac{1}{1 - x_1 - x_2 - x_3 - x_4} \times$$
$$\times \left(\frac{(1 - x_1)(1 - x_1 - x_2 - x_4)}{(1 - x_1 - x_2)(1 - x_1 - x_4)} + \frac{(1 - x_4)(1 - x_1 - x_3 - x_4)}{(1 - x_1 - x_4)(1 - x_3 - x_4)} - 1 \right).$$

The formulae for Ω with $3, 4$ or more arguments can be obtained form these expressions.

Let us introduce *exponential generating functions*: let $\Omega'(i_1, \ldots, i_n) := \Omega(i_1, \ldots, i_n)/(i_1 + \ldots + i_n + 1)!$ and

$$G_n(x_1, \ldots, x_n) := \sum_{i_1=0}^{+\infty} \ldots \sum_{i_n=0}^{+\infty} \Omega'(i_1, \ldots, i_n) x_1^{i_1} \ldots x_n^{i_n}.$$

This term is commonly used for a generating function of $\Omega(i_1,\ldots,i_n)/(i_1!\ldots i_n!)$. But this is not convenient for us. - но это нам будет неудобно. Note that $\Omega'(i_1,\ldots,i_n)$ is a probability that an arbitrarily chosen permutation on $\{0,\ldots,i_1+\ldots+i_n\}$ has the required monotonicity intervals. The series for G_n converges for any values of its arguments.

Theorem 2. *For any $n, i_1, j_1, \ldots, i_n, j_n \in \mathbb{N}$, $N := i_1+\ldots+j_n$ we have*

$$\Omega'(i_1,\ldots,j_n) = \frac{1}{N+1}\sum_{k=1}^{n}\Omega'(i_1,j_1,\ldots,i_k-1)\Omega'(j_k-1,\ldots,i_n,j_n),$$

$$\Omega'(i_1,\ldots,j_n) = \frac{1}{N+1}\sum_{k=0}^{n}\Omega'(i_1,j_1,\ldots,j_k-1)\Omega'(i_{k+1}-1,\ldots,i_n,j_n).$$

It is straightforward to state and prove the similar theorems for odd number of arguments of Ω'. The following theorem can be easily deduced from these four ones:

Theorem 3. *For any $n, i_1, \ldots, i_n \in \mathbb{N}$, $N := i_1+\ldots+i_n$ we have*

$$\Omega'(i_1,\ldots,i_n) = \frac{1}{2(N+1)}\sum_{k=0}^{n}\Omega'(i_1,\ldots,i_k-1)\Omega'(i_{k+1}+1,\ldots,i_n).$$

Let us apply this theory to probabilistic problems. Suppose $X_t,\ t\in\mathbb{Z}$ are independent identically distributed random variables with continuous distribution function.

Theorem 4. *Let $n\in\mathbb{N}$, $i_1, j_1, \ldots, i_n, j_n \in \mathbb{Z}_+$, $N := i_1+\ldots+j_n$. Then the following event: for all $k=1,\ldots,n$*

$$X_{s''_{k-1}} < X_{s''_{k-1}+1} < \ldots < X_{s'_k} > X_{s'_k+1} > \ldots > X_{s''_k},$$

has the probability $\Omega'(i_1,j_1,\ldots,i_n,j_n)$.

Let us call any $n\in\mathbb{Z}$ *the maximum point* (*the minimum point*) if $X_n > X_{n-1},\ X_{n+1}$ ($X_n < X_{n-1}, X_{n+1}$). The maximum points alternate with the minimum points. From now on we shall suppose that 0 is a maximum point. Let μ_0 be the distance from 0 to the next minimum point, let μ_1 be the distance from this point to the next maximum point, etc. Then the sequence (μ_t) is strictly stationary. It is easy to prove that for any $k_0,\ldots,k_n\in\mathbb{N}$ $\mathbf{P}\{\mu_0=k_0,\ldots,\mu_n=k_n\} = 3\Omega'(1,k_0,\ldots,k_n,1)$.

After some calculations, we obtain $\mathbf{P}\{\mu_0=k\} = 3(k^2+3k+1)/(k+3)!$, $\mathbf{E}\mu_0 = 3/2$, $Var\mu_0 = 6e-63/4 \approx 0.560$, and $corr(\mu_0,\mu_1) = (2e^2-8e+7)/(8e-21)\approx 0.0427$.

Conjecture. The sequence (μ_t) satisfies the Law of Large Numbers:

$$\frac{1}{n}\sum_{k=0}^{n-1}\mu_k \to \mathbf{E}\mu_0 = \frac{3}{2},\ n\to+\infty.$$

It is sufficient to prove that (μ_t) is ergodical. If this conjecture is true, one may try to construct a statistical test which checks whether any given observations $X_1,\ldots,X_n$ are independent and identically distributed.

References

1. V. N. Sachkov, Introduction to Combinatorical Methods in Discrete Mathematics. Moscow, MCCME Publishing House, 2004.