The quantum fields of particles with the half-integer spins

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Abstract

It is shown that all spin possibilities can be constructed by combining the fundamental spin 1/2 system a sufficient number of times. In order to express all spin possibilities mathematically we introduce the multispinor source. All component spins are on the same footing and additional symmetry requirements can be imposed on the multispinor. The multispinor description provides a unified approach to all spin values, with the first-order differential equation of spin 1/2 setting the pattern of field equations.

1 Introduction

Spin was originally introduced as the rotation with an angular momentum of a particle around some axis. On the other hand, spin has some peculiar properties that distinguish it from orbital angular momenta: a) Spin quantum numbers may take half-integer values. b) Although the direction of its spin can be changed, an elementary particle cannot be made to spin faster or slower. c) The spin of a charged particle is associated with a magnetic dipole moment with a g-factor differing from 1. This could only occur classically if the internal charge of the particle were distributed differently from its mass.

The conventional definition of the spin quantum number, s, is s = n/2, where n can be any non-negative integer. So, the allowed values of s are 0, 1/2, 1, 3/2, 2, etc. The value of s for an elementary particle depends only on the type of particle, and cannot be altered in any known way (in contrast to the spin direction).

We approach here the way from the spin 1/2 fields to the spin 3/2 fields and spin 5/2 fields for massive and massless particles and we derive the fields. Then, we approach the half-integer spin.

2 Spin 1/2 fields

The mathematical structure of the action for the spin 1/2 particles is introduced as the analogue with the scalar and vector sources with the substantial difference that the sources are elements of the Grassmann algebra. The historical discussion concerning the spin of electron was presented by Tomonaga (Tomonaga, 1997).

It is well known that fields with spin 1/2 are described by the Dirac equation which can be written for the free particle in the form

$$\left(\gamma^\mu \frac{1}{i} + m\right)\psi(x) = \eta(x), \tag{1}$$

where $\psi(x)$ is the four-component spinor and $\eta(x)$ is the four component source which it is necessary to define rigorously. It will be done later in the text.

The action corresponding to eq. (1) is of the form

$$W(\eta) = \frac{1}{2}\int (dx)(dx')\eta(x)\gamma^0 G_+(x-x')\eta(x'), \tag{2}$$

where $G_+(x-x')$ is the Green function of the Dirac equation and it may be easy to show that

$$G_+(x-x') = \left(m - \frac{1}{i}\gamma^\mu\partial_\mu\right)\Delta_+(x-x'), \tag{3}$$

where $\Delta_+(x-x')$ is the Green function of the scalar particle.

The mathematical structure of $W(\eta)$ is similar to the mathematical structure of the action for the scalar or vector sources, however, the fundamental difference is that function $\eta(x)$ are anticommuting numbers, or

$$\eta_\alpha(x)\eta_\beta(x') = -\eta_\beta(x')\eta_\alpha(x). \tag{4}$$

In other words they are elements of the Grassmann algebra. The necessity of the definition of $\eta(x)$ as the elements of the Grassmann algebra is given by the situation where action (2), or,

$$W(\eta) = \frac{1}{2}\int (dx)(dx')\eta_\alpha(x)K_{\alpha\alpha'}(x,x')\eta_{\alpha'}(x') \tag{5}$$

is nonzero only if η-s are the elements of the Grasmann algebra (Schwinger, 1970; 2018).

The quantities γ are so called Dirac 4×4 matrices which have the following structure in the Majorana representation:

$$\gamma^0 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\ \gamma^k \equiv \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}; \quad k = 1,2,3..., \tag{6}$$

where $\boldsymbol{\sigma}$ are the Pauli 2 x 2 matrices defined as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

In the framework of the gamma-matrices it is still defined

$$\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3 \tag{8}$$

and some ingredients of the algebra of the gamma-matrices can be expressed as follows:

$$\gamma_4 = i\gamma^0, \quad \left(\gamma^0\right)^2 = 1, \quad (\gamma_k)^2 = -1, \quad \gamma_5^T = -\gamma_5 \tag{9}$$

$$\{\gamma_0, \gamma_k\} = 0 \tag{10}$$

$$\frac{1}{2}\{\gamma_k, \gamma_l\} = -\delta_{kl} \tag{11}$$

$$\frac{1}{2}\{\gamma_\mu, \gamma_\nu\} = -g_{\mu\nu} \tag{12}$$

$$\{\gamma_\mu, \gamma_5\} = 0 \tag{13}$$

$$\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda = \varepsilon^{\mu\nu\kappa\lambda}\gamma_5 \tag{14}$$

$$\frac{1}{4}\mathrm{tr}\ \gamma_\mu\gamma_\nu = \frac{1}{4}\mathrm{tr}\ \gamma_\mu\gamma_5\gamma_\nu\gamma_5 = -g_{\mu\nu} \tag{15}$$

$$\mathrm{tr}\ \gamma_\mu = -\frac{1}{2}\mathrm{tr}\ \{\gamma_\mu, \gamma_5\} = 0 \tag{16}$$

$$\gamma^\mu\gamma_\mu = -4; \quad \frac{1}{4}\mathrm{tr}\,[\gamma_\mu, \gamma_\nu] = -g_{\mu\nu} \tag{17}$$

$$\gamma^\mu(\gamma p)\gamma_\mu = 2\gamma p. \tag{18}$$

Other relations will be mentioned during the calculations in the text.

Analogically to the situation with the integer spins we write for the spin 1/2 fields for the vacuum-vacuum amplitude

$$\langle 0_+|0_-\rangle = \exp\{iW(\eta)\}, \tag{19}$$

from which follows, as it is known,

$$|\langle 0_+|0_-\rangle|^2 = \exp\{-\mathrm{Im}W\} \tag{20}$$

and in ref. (Schwinger, 1970; 2018) is proven that $|\langle 0_+|0_-\rangle|^2 \leq 1$.

From eq. (3) follows immediately for the Green function

$$G_+(x - x') = \int \frac{(dp)}{(2\pi)^4}\frac{m - \gamma p}{p^2 - m^2 - i\varepsilon}\ e^{ip(x-x')} \quad =$$

$$\int \frac{(dp)}{(2\pi)^4}\frac{e^{ip(x-x')}}{\gamma p + m - i\varepsilon}, \tag{21}$$

from which, for instance, follows that

$$G_+(x - x') = i\int d\omega_p(m - \gamma p)\ e^{ip(x-x')}; \qquad x^0 > x'^0. \tag{22}$$

The fields of the spin 1/2 particles is defined by analogy with the integer spin situation as follows:

$$\delta W(\eta) = \int (dx)\delta\eta(x)\gamma^0\psi(x) = \int (dx)\psi(x)\gamma^0\delta\eta(x). \tag{23}$$

The presence of the antisymmetrical matrix γ^0 compensates the anticommutativity of the probe source $\delta\eta(x)$ with $\psi(x)$, which is formed from and it is of the same nature as $\eta(x)$. From eq. (23) but also from the Dirac equation we get

$$\psi(x) = \int (dx')G_+(x-x')\eta(x'), \tag{24}$$

from which follows

$$\psi(x)\gamma^0 = \int (dx')\eta(x')\gamma^0 G_+(x'-x) \tag{25}$$

as a consequence of

$$\left[\gamma^0 G_+(x'-x)\right]^T = -\gamma^0 G_+(x-x'). \tag{26}$$

The situation with the real source $\eta(x)$ can be generalized to the situation with the complex sources with components $\eta_{(1)}(x)$ and $\eta_{(2)}(x)$, or,

$$\eta(x) = \frac{1}{2}\left(\eta_{(1)}(x) - i\eta_{(2)}(x)\right) \tag{27}$$

$$\eta^*(x) = \frac{1}{2}\left(\eta_{(1)}(x) + i\eta_{(2)}(x)\right). \tag{28}$$

Then, the corresponding action is of the form

$$W(\eta) = \int (dx)(dx')\bar{\eta}(x)G_+(x-x')\eta(x'), \tag{29}$$

where

$$\bar{\eta}(x) = \eta^*(x)\gamma^0 \tag{30}$$

and fields are defined through relation

$$\delta W(\eta) = \int (dx)\left[\delta\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\delta\eta(x)\right], \tag{31}$$

which leads to

$$\psi(x) = \int (dx')G_+(x-x')\eta(x') \tag{32}$$

and

$$\bar{\psi}(x) = \int (dx')\bar{\eta}(x')G_+(x-x') \tag{33}$$

with the corresponding differential equations for ψ and $\bar{\psi}$:

$$\left(\gamma^\mu \frac{1}{i} + m\right)\psi(x) = \eta(x) \tag{34}$$

and

$$\bar{\psi}(x)\left(\gamma^{\mu}\frac{1}{i}+m\right)=\bar{\eta}(x), \tag{35}$$

where

$$\bar{\psi}(x)\partial_{\mu}^{T}=-\partial_{\mu}\bar{\psi}(x). \tag{36}$$

3 Spin 3/2 fields

From the viewpoint of history, the starting point is the so called Rarita-Schwinger equation for spin 3/2 fields (RaritaSchwinger, 1941), where the Weinberg comment is necessary to accept (Weinberg, 2000; p., 333).

The RaritaSchwinger equation is at present time the standard relativistic field equation for the gravitino, the spin-3/2 gauge field of supergravity. In four-dimensional supergravity the gravitino may be written as a Weyl vector-spinor, or equivalently as a Majorana Rarita-Schwinger field. When supersymmetry is spontaneously broken, the super-Higgs mechanism gives the gravitino a Majorana mass. In extended supergravity there are several gravitini. In broken supergravity, there are two Majorana gravitini in the form of the Dirac gravitino if an appropriate residual symmetry is present.

Description of fields with spin 3/2 can be realized by combining the four-vector treatment of unit spin with the four-component spinor of fields with spin 1/2. We get the hybrid description in the form of the so called vector-spinor wave function $\psi_{\mu}(x)$ and the corresponding vector-spinor source $\eta_{\mu}(x)$. The source $\eta_{\alpha}^{\mu}(x)$ has 16 components, from additional charge multiplicity. The reduction of this system to the system which corresponds to the spin 3/2 multiplicity can be achieved by the projection matrices appropriate to the constituents which are $\bar{g}_{\mu\nu}(p)$ for spin 1 and $m-\gamma p$ for spin 1/2. On considering the source coupling in the rest frame of the particle, this procedure supplies the effective source $(1+\gamma^{0})\eta_{k}/2$ which has six components. The final reduction to the four components corresponding to spin 3/2 is accomplished by projection matrix

$$M_{ij}=\frac{l+1/2\pm(1/2+\mathbf{L}\cdot\boldsymbol{\sigma})}{2l+1}=\frac{j+1/2\pm\mathbf{L}\cdot\boldsymbol{\sigma}}{2j+1\pm1} \tag{37}$$

with $l=1$ and $j=3/2$. In the context of the three-component vectors and two-component spinors it is represented by spinors it is represented by

$$(M_{3/2})_{kl}=\delta_{kl}-\frac{1}{3}\sigma_{k}\sigma_{l}. \tag{38}$$

The projector character of this matrix is equivalent to the property

$$\sigma_{k}(M_{3/2})_{kl}=(M_{3/2})_{kl}\sigma_{l}=0 \tag{39}$$

and its specific identification is confirmed by evaluating the trace over the six-dimensional space. Let us remark that the projection matrix obeys relations:

$$M_{lj}M_{lj'}=\delta_{jj'}M_{lj};\quad \sum_{j}M_{lj}=1 \tag{40}$$

and has a trace appropriate to the multiplicity $2j+1$

$$\mathrm{tr}\ M_{lj} = 2j+1. \tag{41}$$

The resulting form of the source coupling in the rest frame is

$$\eta_k^* \frac{1}{2}(1+\gamma^0)(\delta_{kl} - \frac{1}{3}\sigma_k\sigma_l) = \bar{\eta}_k^* \frac{1}{2}(1+\gamma^0)\bar{\eta}_k, \tag{42}$$

where

$$\bar{\eta}_k = \eta_k - \frac{1}{3}\sigma_k\sigma_l \tag{43}$$

with property

$$\sigma_k\bar{\eta}_k = 0, \tag{44}$$

which makes explicit the rejection of the spin 1/2 composite system.

The transition to the moving system is facilitated by the relation

$$\sigma_k\sigma_l = \gamma_k i\gamma_5\gamma_l i\gamma_5, \tag{45}$$

which gives the straight forward generalization of eq. (42) (apart from a factor $2m$)

$$\eta^{*\mu}(p)\gamma^0(m-\gamma p)\left[\bar{g}_{\mu\nu}(p) - \frac{1}{3}\bar{g}_{\mu\kappa}\gamma^\kappa i\gamma_5(p)\gamma^\lambda i\gamma_5\right]\eta^\nu(p), \tag{46}$$

which can be simplified using relation for the second term:

$$\begin{aligned}
(m-\gamma p)\bar{g}_{\mu\kappa}(p)\gamma^k i\gamma_5\bar{g}_{\nu\lambda}(p)\gamma^\lambda i\gamma_5 \quad &= \\
\bar{g}_{\mu\kappa}(p)\gamma^\kappa i\gamma_5(m-\gamma p)\bar{g}_{\nu\lambda}(p)\gamma^\lambda i\gamma_5 \quad &= \\
-\bar{g}_{\mu\kappa}(p)(m+\gamma p)\bar{g}_{\nu\lambda}(p)\gamma^\lambda \quad &= \\
-(\gamma_\mu - \frac{1}{m}p_\mu)(m+\gamma p)(\gamma_\nu + \frac{1}{m}p_\nu), &
\end{aligned} \tag{47}$$

where we used the commutativity of the term $(1+\gamma^0)$ with σ_k in the first rearrangement, then properties of $i\gamma_5$ and the last rearrangement uses the presence of the factor $m+\gamma p$ to substitute m for γp. An alternative form replaces $\gamma_\mu + (1/m)p_\mu$ with $-(i/m)\sigma_{\mu\lambda}p^\lambda$.

The resulting action of spin 3/2 fields in the vacuum-vacuum amplitude

$$\langle 0_+|0_-\rangle = \exp\left\{iW_{3/2}(\eta)\right\} \tag{48}$$

is

$$\begin{aligned}
W_{3/2}(\eta) = \frac{1}{2}\int \frac{(dp)}{(2\pi)^4}\eta^\mu(-p)\gamma^0\Big[(m-\gamma p)(g_{\mu\nu} + \frac{1}{m^2}p_\mu p_\nu) \quad + \\
\frac{1}{3}(\gamma_\mu + \frac{1}{m}p_\mu)(m+\gamma p)(\gamma_\nu + \frac{1}{m}p_\nu)\Big]\frac{1}{p^2+m^2-i\varepsilon}\eta^\nu(p).
\end{aligned} \tag{49}$$

The kernel of this quadratic form is antisymmetrucal with regard to simultaneous transformation $p_\mu \to -p_\mu$ and transformation of the matrix and vector indices. In other words it is in agreement with the anticommutating of sources, or with Fermi-Dirac statistics.

The field of the spin 3/2 particles is related to action $W_{3/2}$ in analogy with the spin 1/2, or,

$$\delta W(\eta) = \int \frac{(dp)}{(2\pi)^4}\delta\eta^\mu(-p)\gamma^0\psi_\mu(p), \tag{50}$$

which gives

$$\psi_\mu(p) = \frac{1}{p^2+m^2-i\varepsilon} \quad \times$$

$$\left[(m-\gamma p)\bar{g}_{\mu\nu}(p) + \frac{1}{3}(\gamma_\mu + \frac{1}{m}p_\mu)(m+\gamma p)(\gamma_\nu + \frac{1}{m}p_\nu)\right]\eta^\nu(p). \tag{51}$$

It enables to derive equations

$$p^\mu\psi_\mu = \frac{1}{m^2}\left[(m-\gamma p)p_\nu + \frac{1}{3}\gamma p(m\gamma_\nu + p_\nu)\right]\eta^\nu \tag{52}$$

and

$$\gamma^\mu\psi_\mu = \frac{1}{m^2}\left[p_\nu - \frac{1}{3}(m\gamma_\nu + p_\nu)\right]\eta^\nu, \tag{53}$$

which supply the linear combination

$$p^\mu\psi_\mu + \gamma p\psi_\mu(p) = \frac{1}{m}p_\nu\eta^\nu \tag{54}$$

and

$$-p^\mu\psi_\mu + (m-\gamma p)\gamma^\mu\psi_\mu = -\frac{1}{3m}(m\gamma_\nu + p_\nu)\eta^\nu. \tag{55}$$

The use of identities

$$(m+\gamma p)(m\gamma_\mu + p_\mu) = (m\gamma_\mu - p_\mu)(m-\gamma p), \tag{56}$$

$$(m\gamma_\mu + p_\mu)(m+\gamma p) = (m-\gamma p)(m\gamma_\mu - p_\mu) \tag{57}$$

produces the following equation

$$(\gamma p + m)\psi_\mu = \eta_\mu + \frac{1}{3}\gamma_\mu(\gamma_\nu + \frac{1}{m})p_\nu\eta^\nu \quad +$$

$$\frac{1}{m^2}p_\mu\left[p_\nu - \frac{1}{3}(m\gamma_n + p_\nu)\right], \tag{58}$$

from which immediately follows the final equation for fields of particles with spin 3/2

$$(\gamma p + m)\psi_\mu - p_\mu\gamma^\nu\psi_\nu - \gamma_\mu p^\nu\psi_\nu + \gamma_\mu(m-\gamma p)\gamma\psi_\nu = \eta_\nu, \tag{59}$$

which is the vector spinor equation and it can be transformed into the coordinate form as the following differential equation:

$$(\gamma^\lambda \frac{1}{i}\partial_\lambda + m)\psi_\mu - \frac{1}{i}\partial_\mu \gamma^\nu \psi_\nu - \gamma_\mu \frac{1}{i}\partial^\nu \psi_\nu + \gamma_\nu \left[m - \gamma^\lambda \frac{1}{i}\partial_\lambda \right] \gamma^\nu \psi_\mu = \eta_\mu. \tag{60}$$

The last equation is embodied in the action

$$W = \int (dx) \left[\eta^\mu \gamma^0 \psi_\mu + \mathcal{L} \right], \tag{61}$$

where $\mathcal{L}$ is the Lagrange function of the form

$$\mathcal{L} = -\frac{1}{2} \left\{ \psi^\mu \gamma^0 (\gamma^\lambda \frac{1}{i}\partial_\lambda + m)\psi_\mu - \psi^\mu \gamma^0 \frac{1}{i}\partial_\mu \gamma^\nu \psi_\nu \quad - \right.$$

$$\left. \psi^\nu \gamma^0 \gamma_\nu \frac{1}{i}\partial^\mu \psi_\mu + \psi^\mu \gamma^0 \gamma_\mu (m - \frac{1}{i}\partial_\lambda)\gamma^\lambda \psi_\lambda \right\}, \tag{62}$$

where the second and third term effect the symmetrization between left and right derivatives

$$-\partial_\mu \psi^\mu \gamma^0 \gamma^\nu \psi_\nu = \psi^\nu \gamma^0 \gamma_\nu \partial_\mu \psi_\nu. \tag{63}$$

There exist the massless limit of the spin 3/2 theory where the fundamental ingredient is the action for the massless fields of spin 3/2 and it is of the form:

$$W(\eta, m = 0) = \frac{1}{2} \int (dx)(dx')\eta(x)\gamma^0 \quad \times$$

$$\left[g_{\mu\nu}(-\gamma^\lambda \frac{1}{i}\partial_\lambda) D_+(x - x') - \frac{1}{2}\gamma_\mu(-\gamma^\lambda \frac{1}{i}\partial_\lambda) D_+(x - x')\gamma_\nu \right] \eta^\nu(x') \tag{64}$$

with the simultaneous restriction

$$\partial_\mu \eta^\mu(x) = 0. \tag{65}$$

4 Spin 5/2 fields

The spin 5/2 situation is described by means of so called tensor-spinor source and the tensor-spinor wave function.

Our starting point is to write the action for the maless fields with spin 5/2 in order to generalize it for the nonzero mass situation. While in the spin 3/2 situation we have introduced the vector-spinor source and field, in case of spin 5/2 situation it is necessary to introduce so called tensor-spinor source $\eta^{\mu\nu}(x)$ and tensor-spinor wave function $\psi^{\mu\nu}(x)$. In analogy with the spin 3/2 we write the action for the massless spin 5/2 fields in the following form:

$$W(\eta) = \frac{1}{2} \int \frac{(dp)}{(2\pi)^4} \frac{1}{p^2 - i\varepsilon} \left\{ \eta^{\mu\nu}(-p)\gamma^0(-\gamma p)\eta_{\mu\nu}(p) \quad - \right.$$

$$-\frac{1}{2}\eta^{\mu\nu}(-p)\gamma^0\gamma_\mu(-\gamma p)\gamma^\lambda\eta_{\lambda\nu}(p) - \frac{1}{4}\eta(-p)\gamma^0(-\gamma p)\eta(p)\Big\}, \tag{66}$$

where

$$\eta(p) = g_{\mu\nu}\eta^{\mu\nu}(p) \tag{67}$$

and

$$p_\mu\eta^{\mu\nu}(p) = 0. \tag{68}$$

The tensor-spinor field $\psi_{\mu\nu}$ is defined by the obligate way as follows:

$$\delta W(\eta) = \int \frac{(dp)}{(2\pi)^4}\delta\eta^{\mu\nu}(-p)\gamma^0\psi_{\mu\nu}(p). \tag{69}$$

However, from this definition follows that $\psi_{\mu\nu}$ is not unique, since the source is restricted by eq. (68). Nevertheless, its general form is

$$\psi_{\mu\nu} = \frac{1}{p^2 - i\varepsilon}\Big\{-\gamma p\eta_{\mu\nu} - \frac{1}{4}\gamma_\mu(-\gamma p)\gamma^\lambda\eta_{\lambda\nu} - \frac{1}{4}\gamma_\nu(-\gamma p)\gamma^\lambda\eta_{\mu\lambda}$$

$$-\frac{1}{4}g_{\mu\nu}(-\gamma p)\eta(p)\Big\} + p_\mu\psi_\nu + p_\nu\psi_\mu, \tag{70}$$

where the vector-spinor field ψ_μ is to be determined through the source restriction.

The final form of the field equations are as follows (Schwinger, 1970; 2018):

$$\gamma p\psi_{\mu\nu} - (p_\mu\gamma^\alpha\psi_{\alpha\nu} + \gamma_\mu p^\alpha\psi_{\alpha\nu} + p_\nu\gamma^\alpha\psi_{\mu\alpha} + \gamma_\nu p^\alpha\psi_{\mu\alpha}) + g_{\mu\nu}\gamma^\alpha p^\beta\psi_{\alpha\beta} \quad -$$

$$(\gamma_\mu\gamma p\gamma^\alpha\psi_{\alpha\nu} + \gamma_\nu\gamma p\gamma^\alpha\psi_{\mu\alpha}) + \frac{1}{2}(\gamma_\mu p_\nu + \gamma_\nu p_\mu - g_{\mu\nu}\gamma p) = \eta_{\mu\nu}. \tag{71}$$

Exploring the spin 3/2 situation (9.13), we can derive the action for the nonzero mass fields with spin 5/2 in the following form

$$W(\eta) = \frac{1}{2}\int \frac{(dp)}{(2\pi)^4}\eta^{\mu\nu}(-p)\gamma^0\frac{1}{p^2+m^2-i\varepsilon} \quad \times$$

$$\Big[(m-\gamma p)\left(\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta} - \frac{1}{5}\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta}\right) \quad +$$

$$\frac{2}{5}\left(\gamma_\mu + \frac{1}{m}p_\mu\right)(m+\gamma p)\left(\gamma_\alpha + \frac{1}{m}p_\alpha\right)\Big]\eta^{\alpha\beta}. \tag{72}$$

In order to bring the field equations into first-order form without source derivatives we must introduce the additional contact terms of the following form:

$$W_{cont.}(\eta) = \frac{1}{10m^4}\int \frac{(dp)}{(2\pi)^4}\eta^{\mu\nu}(-p)\gamma^0 p_\mu p_\nu\left[\gamma_\alpha p_\beta\eta^{\alpha\beta}(p) + \frac{m}{4}\eta(p)\right] \quad +$$

$$\frac{1}{20m^4}\int \frac{(dp)}{(2\pi)^4}\eta^{\mu\nu}(-p)\gamma^0\gamma_\mu p_\nu\frac{1}{4}(\gamma p - 5m)\gamma_\alpha p_\beta\eta^{\alpha\beta}(p)$$

$$\frac{1}{10m^3}\int\frac{(dp)}{(2\pi)^4}\eta(-p)\frac{1}{16}(\gamma p-9m)\gamma_\mu p_\nu\eta^{\mu\nu}(p)\quad +$$

$$\frac{1}{20m^2}\int\frac{(dp)}{(2\pi)^4}\eta(-p)\frac{1}{64}(\gamma p-13m)\eta(p). \tag{73}$$

The corresponding field equations then are

$$(\gamma p+m)\psi_{\mu\nu}-(p\gamma^\alpha\psi_{\alpha\nu}+\gamma_\mu p^\alpha\psi_{\alpha\nu}+p_\nu\gamma^\nu\psi_{\mu\alpha}+\gamma_\nu p^\alpha\psi_{\mu\alpha})\quad +$$

$$\gamma_\mu(m-\gamma p)\gamma^\alpha\psi_{\alpha\nu}+\gamma_\nu(m-\gamma p)\gamma^\alpha\psi_{\mu\alpha}+g_{\mu\nu}\gamma^\alpha p^\beta\psi_{\alpha\beta}\quad +$$

$$\frac{1}{2}\left[\gamma_\mu p_\nu+\gamma_\nu p_\mu-g_{\mu\nu}(\gamma p+m)\psi-g_{\mu\nu}m\psi\right]=\eta_{\mu\nu} \tag{74}$$

with

$$(\gamma p-3m)\psi+\frac{5}{12}m\psi=0, \tag{75}$$

where ψ is the auxiliary field.

It has been derived that the function ψ in the momentum representation is of the form:

$$\psi(p)=\frac{1}{8m^3}p_\mu p_\nu\eta^{\mu\nu}(p)+\frac{1}{4}(\gamma p-m)\gamma_\mu p_\nu\eta^{\mu\nu}(p)+\frac{1}{16}(\gamma p-5m)\eta(p). \tag{76}$$

The action corresponding to the field with spin 5/2 involving the Lagrange function can be derived in the x-representation in the following form:

$$W(\eta)=\int(dx)\left[\eta^{\mu\nu}\gamma^0\psi_{\mu\nu}+\mathcal{L}\right], \tag{77}$$

where

$$\mathcal{L}=-\frac{1}{2}\Big\{\psi^{\mu\nu}\gamma^0\left[\gamma^\lambda\frac{1}{i}\partial_\lambda+m\right]\psi_{\mu\nu}-2\psi^{\mu\nu}\gamma^0\frac{1}{i}\partial_\mu\gamma^\lambda\psi_{\lambda\nu}\quad -$$

$$2\psi^{\mu\nu}\gamma^0\gamma_\mu\frac{1}{i}\partial^\lambda\psi_{\lambda\nu}+2\psi^{\mu\nu}\gamma^0\gamma_\mu\left[m-\gamma^\kappa\frac{1}{i}\partial_\kappa\right]\gamma^\lambda\psi_{\mu\nu}\quad +$$

$$\psi\gamma^0\gamma^\mu\frac{1}{i}\partial^\nu\psi_{\mu\nu}+\psi^{\mu\nu}\gamma^0\gamma^\mu\frac{1}{i}\partial_\nu\psi-\frac{1}{2}\psi\gamma^0\left[\gamma^\lambda\frac{1}{i}\partial_\lambda+m\right]\psi\Big\}\quad +$$

$$\frac{1}{2}m\left(\psi\gamma^0\psi-\psi\gamma^0\psi\right)-\frac{5}{6}\psi\gamma^0\left[\gamma^\lambda\frac{1}{i}\partial_\lambda-3m\right]\psi. \tag{78}$$

and the auxiliary spinor ψ is to be varied independently in applying the principle of the stationary action.

5 Multispinor fields

It is shown that all spin possibilities can be constructed by combining the fundamental spin 1/2 system a sufficient number of times. In order to express all spin possibilities mathematically we introduce the multispinor source. All component spins are on the same footing and additional symmetry requirements can be imposed on the multispinor. The multispinor description provides a unified approach to all spin values, with the first-order differential equation of spin 1/2 setting the pattern of field equations.

In the preceding chapters we have discussed fields with spins separately for every spin situation and derived the corresponding actions or field equations. The question arises, if it is possible the unified approach for all spin values. Such program can be realized by the multipsinor formalism.

While the procedures we have followed in the preceding chapters for description the various spin possibilities exploit elementary angular momentum properties, where the integer + 1/2 spin is possible to construct by adding spin 1/2 to the integer spin, it is obvious that all spin possibilities can be constructed by combining the fundamental spin 1/2 system a sufficient number of times. Even for integer spin and odd for n/2 spin.

In order to express all spin possibilities mathematically we introduce the multispinor source $S_{\zeta_1\zeta_2...\zeta_n}(x)$. All component of spins are on the same footing and additional symmetry requirements can be imposed on the multispinor. The most important of these is the requirement of total symmetry:

$$S_{\zeta_1\zeta_2...\zeta_n}(x) = S_{\zeta_{\alpha_1}\zeta_{\alpha_2}...\zeta_{\alpha_n}}(x), \tag{79}$$

where $\alpha_1\alpha_2...\alpha_n$ is any permutation of 1, 2, ... n.

The multispinor description provides a unified approach to all spin values, with the first-order differential equation of spin 1/2 setting the pattern of field equations. In order to realize such program, it is necessary to introduce contact terms in the all situations. First, let us illustrate the nultispinor formalism by considering the unit spin situation.

The situation is provided by the symmetric spinor of the second rank as contained in the action

$$W(\eta) = \frac{1}{2}\int \frac{(dp)}{(2\pi)^4}\eta(-p)\gamma_1^0\gamma_2^0\frac{1}{2m}\left[\frac{(m-\gamma_1 p)(m-\gamma_2 p)}{p^2+m^2-i\varepsilon}+1\right]\eta(p). \tag{80}$$

where the appropriate contact term has already been introduced and the source is of the form involving some normalization factor:

$$\eta(p) = (2m)^{\frac{(n-1)}{2}} S(p). \tag{81}$$

From the field definition

$$\delta W(\eta) = \int \frac{(dp)}{(2\pi)^4}\delta\eta(-p)\gamma_1^0\gamma_2^0\psi(p) \tag{82}$$

it follows

$$\psi(p) = \frac{1}{2m}\left[\frac{(m-\gamma_1 p)(m-\gamma_2 p)}{p^2+m^2-i\varepsilon}+1\right]\eta(p) \tag{83}$$

and we observe that the following equations follow from the last equation

$$(\gamma_1 p + m)\psi(p) = \frac{1}{2m}\left[m-\gamma_2 p + m + \gamma_1 p\right]\eta(p) \tag{84a}$$

$$(\gamma_2 p + m)\psi(p) = \frac{1}{2m}\left[m-\gamma_1 p + m + \gamma_2 p\right]\eta(p), \tag{84b}$$

which gives by their addition

$$\left[\frac{1}{2}(\gamma_1+\gamma_2)p + m\right]\psi(p) = \eta(p). \tag{85}$$

Equation (85) can be transcribed in the coordinate representation as follows:

$$\left[\frac{1}{2}(\gamma_1^\mu+\gamma_2^\mu)\frac{1}{i}\partial_\mu + m\right]\psi(x) = \eta(x) \tag{86}$$

Another consequence of the pair equations (84a,b) is

$$(\gamma_1-\gamma_2)p\psi(p) = \frac{1}{m}(\gamma_1-\gamma_2)p\eta(p), \tag{87}$$

which follows from eq. (85) after multiplication of it by $\gamma_1 p - \gamma_2 p$ and using relation

$$(\gamma_1 p)^2 = (\gamma_2 p)^2. \tag{88}$$

From eq. (88) it follows

$$\gamma_1 p = \pm\gamma_2 p, \tag{89}$$

or, in the rest frame

$$\gamma_1^0 = \pm\gamma_2^0. \tag{90}$$

Setting

$$\gamma_2 p = \gamma_1 p \tag{91}$$

in eq. (85), this equation reduces to the form of the spin 1/2 Dirac equation while the choice

$$\gamma_2 p = -\gamma_1 p \tag{92}$$

removes the coordinate derivatives and supplies a field that vanishes outside the source.

The structure of action $W(\eta)$ involves for $\gamma_2 p = \gamma_1 p$ the form

$$\frac{(m-\gamma_1 p)^2}{p^2+m^2-i\varepsilon} = \frac{(m-\gamma_1 p)}{\gamma_1 p + m - i\varepsilon} = \frac{2m}{\gamma_1 p + m - i\varepsilon} - 1, \tag{93}$$

which puts into evidence the contact term and normalization that are needed to obtain eq. (85).

Introducing the Lagrange function

$$\mathcal{L} = -\frac{1}{2}\psi\gamma_1^0\gamma_2^0\left[\frac{1}{2}(\gamma_1^\mu + \gamma_2^\mu)\frac{1}{i}\partial_\mu + m\right]\psi, \tag{94}$$

we write action of spin 1 and 0 fields as

$$W(\eta) = \int (dx)\left[\eta\gamma_1^0\gamma_2^0 + \mathcal{L}\right]. \tag{95}$$

It should be not forgotten that these even rank spinors are commutative quantities (B.E. statistics) because of the symmetry $\gamma_1^0\gamma_2^0$ and the antisymmetry $\gamma_1^0\gamma_2^0(\gamma_1+\gamma_2)$.

The other possible reformulation of the spin 1 and 0 theory of multispinors is discussed in ref. (Schwinger, 1970; 2018).

Now, let us approach the spin 3/2 fields. They are described by the multispinor of rank 3. The choice

$$\gamma_1 p = \gamma_2 p = \gamma_3 p \tag{96}$$

leads to examination of

$$\frac{(m-\gamma_1 p)^3}{p^2+m^2-i\varepsilon} = \frac{(m-\gamma_1 p)^2}{\gamma_1 p + m - i\varepsilon} = \frac{(2m)^2}{\gamma_1 p + m - i\varepsilon} - 3m + \gamma_1 p, \tag{97}$$

which corresponds to action

$$W(\eta) = \frac{1}{2}\int \frac{(dp)}{(2\pi)^4}\eta(-p)\gamma_1^0\gamma_2^0\gamma_3^0\frac{1}{(2m)^2} \quad \times$$

$$\left[\frac{(m-\gamma_1 p)(m-\gamma_2 p)(m-\gamma_3 p)}{p^2+m^2-i\varepsilon} + 3m - \frac{1}{3}(\gamma_1 p + \gamma_2 p + \gamma_3 p)\right]\eta(p), \tag{98}$$

which displays the field

$$\psi(p) = \frac{1}{(2m)^2}\left[\frac{\prod_\alpha (m-\gamma p)_\alpha}{p^2+m^2-i\varepsilon} + 3m - \frac{1}{3}\sum_\alpha \gamma_\alpha p\right]\eta(p) \tag{99}$$

and which follows also from the definition of field

$$\delta W(\eta) = \int \frac{(dp)}{(2\pi)^4}\delta\eta(-p)\prod_\alpha \gamma_\alpha^0\psi(p). \tag{100}$$

There are only two general alternatives open to $\gamma_\alpha p$:

$$I. \quad \gamma_1 p = \gamma_2 p = \gamma_3 p \tag{101}$$

$$II. \quad \gamma_2 p = -\gamma_3 p \tag{102}$$

For the case I. we have:

$$\psi = \frac{1}{\gamma_1 p + m - i\varepsilon}\eta, \quad \left[\frac{1}{3}\sum_\alpha \gamma_\alpha p + m\right]\psi = \eta \tag{103}$$

and for the case II. it is:

$$\psi = \frac{1}{m^2}\left(m - \frac{1}{3}\gamma_{1p}\right), \quad \left[\frac{1}{3}\sum_\alpha \gamma_\alpha p + m\right]\psi = \left(1 + \frac{1}{(3m)^2}p^2\right)\eta, \tag{104}$$

which can be also presented as the unrestricted equation

$$\frac{1}{2}(\gamma_2 p - \gamma_1 p) = \frac{1}{m^2}\left(m - \frac{1}{3}\gamma_1 p\right)\frac{1}{2}\left(\gamma_2 p - \gamma_3 p\right). \tag{105}$$

An expression that covers all contingencies is

$$\left[\frac{1}{3}\sum_\alpha \gamma_\alpha p + m\right]\psi = \left[1 - \frac{1}{(3m^)2}\frac{1}{2}\sum_\alpha < \beta\left(\frac{1}{2}(\gamma_\alpha p) - \gamma_\beta p\right)^2\right]\eta. \tag{106}$$

The action corresponding to spin 1/2 and/or spin 3/2 fields constructed in order to involve the Lagrange function can be written in the multispinor formalism as follows:

$$W = \int (dx)\left[\eta\gamma\psi + \mathcal{L}\right], \quad \gamma = \prod_\alpha \gamma_\alpha^0 = \gamma_1^0\gamma_2^0\gamma_3^0, \tag{107}$$

where

$$\mathcal{L} = -\frac{1}{2}\psi\gamma\left[\frac{1}{3}\sum_\alpha \gamma_\alpha^\mu \frac{1}{i}\partial_\mu + m\right]\psi \quad -$$

$$\frac{1}{2}\sum_{\alpha<\beta}\left[\psi\gamma\frac{1}{2}(\gamma_\alpha^\mu - \gamma_\beta^\mu)\frac{1}{i}\partial_\mu\psi_{[\alpha\beta]} + \psi_{[\alpha\beta]}\gamma\frac{1}{2}(\gamma_\mu^\alpha - \gamma_\beta^\mu)\frac{1}{i}\partial_\mu\psi\right] \quad -$$

$$3\psi_{[23]}\gamma\left[\gamma_1^\mu\frac{1}{i}\partial_\mu - 3m\right]\psi_{[23]} - ..., \tag{108}$$

where

$$\psi_{[\alpha\beta]} = \frac{1}{(6m)^2}(\gamma_\alpha p - \gamma_\beta p)\eta. \tag{109}$$

The second order formulation of the spin 1/2 and/or spin 3/2 situation is provided by action

$$W(\zeta) = \frac{1}{2m}\int (dx)\left[\zeta\gamma\psi + \mathcal{L}\right], \tag{110}$$

where

$$\zeta = \left[m - \frac{1}{3}\sum_\alpha \gamma_\alpha^\mu \frac{1}{i}\partial_\mu\right]\eta \tag{111}$$

and

$$\mathcal{L} = -\frac{1}{2}\left[\frac{1}{8}\partial^\mu\psi\gamma\partial_\mu\psi - \frac{1}{8}\partial_\mu\psi\gamma\sum_{\alpha,\beta}\gamma_\alpha^\mu\gamma_\beta^\nu\partial_\nu\psi + m^2\psi\gamma\psi\right]. \tag{112}$$

Action $W(\eta)$ can be modified also as

$$W(\zeta) = \frac{1}{2}\left(\frac{1}{2m}\right)\int (dx)\zeta(x)\gamma\psi(x) \quad =$$

$$\frac{1}{2}\left(\frac{1}{2m}\right)\int(dx)\eta(x)\gamma\left[m-\frac{1}{3}\sum_{\alpha}\gamma^{\mu}_{\alpha}\frac{1}{i}\partial_{\mu}\right]\psi \quad =$$

$$\frac{1}{2}\int(dx)\zeta\gamma\psi-\frac{1}{4m}\int(dx)\zeta\gamma\psi \quad +$$

$$\frac{1}{4m}\int(dx)\eta\gamma\sum_{\alpha<\beta}\frac{1}{2}(\gamma^{\mu}_{\alpha}-\gamma^{\mu}_{\beta})\frac{1}{i}\partial_{\mu}\psi_{[\alpha\beta]}, \tag{113}$$

where the last term can be written as

$$9m\int(dx)\sum_{\alpha<\beta}\psi_{[\alpha\beta]}\gamma\psi_{[\alpha\beta]}. \tag{114}$$

The analysis of other other problems is discussed in the Schwinger book (Schwinger, 1970; 2018).

Discussion

All spin possibilities can be constructed by combining the fundamental spin 1/2 system a sufficient number of times. In order to express all spin possibilities mathematically we introduce the multispinor source. All component spins are on the same footing and additional symmetry requirements can be imposed on the multispinor. The multispinor description provides a unified approach to all spin values, with the first-order differential equation of spin 1/2 setting the pattern of field equations. The very interesting treatise of the spin motion including elementary particles is presented by Rivas (2002).

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