

Gravity and complexity

Y. Gaspar^{1,a} and G. Acquaviva²

¹ Università Cattolica del Sacro Cuore, Dipartimento di Matematica e Fisica, Via Musei 41, Brescia, Italy

² Università degli Studi di Trento, Dipartimento di Fisica, Via Sommarive 14, Trento, Italy

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Abstract. We present a heuristic analysis of the dynamics of general solutions to the Einstein Field Equations which highlights the possibility that such systems could possess a degree of unpredictability stronger than that which characterises chaotic systems. Questions regarding features of the complex dynamics of such cosmological models can be *undecidable*. These systems could be qualitatively compared with Turing machines in the sense that even if initial conditions for a dynamical system associated to general solutions to the Einstein Field Equations were known exactly, then the subsequent evolution could still be unpredictable.

1 Introduction

Chaotic behaviour occurs in a wide variety of physical systems. In Newtonian gravity the three-body problem is known to exhibit chaos. Spin orbit couplings between particles, accelerator beams and other similar systems can also display chaotic behaviour. In the context of Einstein's General theory of Gravitation the so-called Bianchi type VIII and IX models without scalar fields can be chaotic [1–4]. The first pioneering work on these general solutions to the Einstein Field Equations has been carried out by Belinski, Khalatnikov and Lifshitz (BKL) [5–7]. Despite the approximate nature of the analysis of the Einstein Field Equations in work of BKL, it seems that several aspects are strongly supported by various rigorous and numerical approaches. The analysis we present in this paper is heuristic and we argue that our approach can guide more rigorous investigations. One might suspect that these anisotropic and homogeneous models are physically irrelevant since our universe is well described on large scales by isotropic and homogeneous Friedmann-Robertson-Walker (FRW) models. However, a cosmological theory ought to explain why our universe is close to an FRW model today, because the FRW models correspond to a set of measure zero in the space of all possible physically admissible solutions to the Einstein Field Equations (EFEs): in this sense the FRW solutions are extraordinarily special and one ought to understand why our universe is described by the most “improbable” and symmetrical models. One of the possible scenarios, which tries to explain this, corresponds to the very important inflationary models. These models claim that, even starting initially with a non-symmetrical and chaotic initial state, through inflation the universe will at late time approach an FRW solution. However, there is no rigorous proof of this mechanism and, in the most general chaotic and inhomogeneous case, it is even possible that inflation will not start (chaotic and eternal inflationary solutions are models which try to tackle this problem). The general Bianchi models offer a leading-order approximation to the most general inhomogeneous solutions to EFEs and, in order to grasp fully the dynamics which leads to an FRW model, these solutions ought to be considered. Furthermore, even if our universe today is described on large scales by an FRW model, in the past very close to the initial singularity space-time can behave like in a chaotic BKL model, which is typical of Bianchi type VIII and IX, but also typical of general inhomogeneous models. Moreover, realistic singularities inside black holes are of BKL type, even if from an outside point of view the solution is highly symmetrical. Furthermore, from a rigorous point of view, it can be shown that a Bianchi type VIII model can be close to an FRW model for an arbitrarily long interval of cosmic clock time: locally, the solution could look like an almost flat FRW model. Also, many general Bianchi models, including type VIII, can be viewed as the result of a superposition of gravitational waves on a simpler more symmetrical background space-time, and, as such, they can provide very interesting insights into the physics of space-times containing strong gravitational waves, as might have occurred close to the initial singularity. These are some of the interesting arguments that show

^a e-mail: yfjmg@cam.ac.uk

why solutions more general than FRW models ought to be considered as part of physical models. Furthermore, by excluding general models, one might miss an important “message” that might be hidden deeply within the gravity theory and the EFEs and therefore the possibility of Gödel undecidability within the physical equations for gravity ought to be considered. It is of some importance to note that profound limits to computability can exist in the context of our modern gravitational theory.

The equations of motion for the chaotic systems that we will consider can not be integrated and no general formula exists which gives their state at all times. Physicists study various approximations and simulations of these systems and various statistical or scaling properties, escape rates and other average quantities can be computed, because the trajectories characterising the evolution of these systems are essentially random. However, as is studied by C. Moore in [8], dynamical systems exist in which these computations of average quantities are impossible: their dynamics is not only random, it exhibits an additional degree of *complexity*. In [8] an example is constructed of a three-dimensional potential in which the motion of a single particle possesses this “new” degree of complexity which is related to the *undecidability* regarding some features of the system. For such complex systems the basins of attraction are not merely recursive. The *undecidability* regarding features of the dynamics causes these sets to be more complex than fractal sets: at every scale of magnification, qualitatively new behaviour shows up. Even if the initial conditions were known exactly, the evolution of the system would still be unpredictable because of this additional degree of complexity which superposes itself onto the chaoticity of the system.

In [8] it is argued that such systems can be understood by comparing them with Turing machines. These machines correspond to idealised computers consisting essentially in a box containing a finite number of states and an infinite “tape” on which sequences of symbols may be written. The machine can read only one symbol for a given position on the tape and as a consequence it can alter its internal state, change the tape symbol and move one space left or right on the tape. Turing machines are capable of universal computation: for any given program, there exists a Turing machine which will perform it using the tape as its registers and memory, such that it will be capable of any finite computation. An interesting problem regarding Turing machines is the so-called *halting problem*: given some initial state, will the machine ever halt? For instance, one could construct a Turing machine which searches for counterexamples to Riemann’s hypothesis and such that it would halt only if it finds one. Alan Turing proved this kind of question to be undecidable: in order to answer this halting problem one would have to prove (or disprove) Riemann’s hypothesis. Thus one needs more information than is actually available to solve the problem. Consider the set H of sequences on which the Turing machine will eventually halt: Rice [9] proved that virtually any question about H is undecidable, such as whether H is finite, dense, non-empty etc. Even a measure of the set H can not be computed. From a dynamical systems viewpoint, these are questions regarding a basin of attraction whose features can be undecidable. Furthermore, this implies that Turing machines are unpredictable even if the initial conditions are known exactly.

In this work we will first overview the particular nature of the chaotic dynamics of the homogeneous Bianchi type VIII cosmological models. These models have been shown to exhibit an interesting type of *time asymmetry* [10]: their behaviour towards the past is chaotic, whereas their behaviour towards the future, away from the initial singularity, is characterised by non-chaotic oscillations. This behaviour will be reconsidered from a new perspective using the orthonormal frame approach and the Hamiltonian formulation of the problem and it will be shown how these particular properties can lead to new features of the dynamical system related to undecidability.

2 Hamiltonian evolution in minisuperspace

It turns out to be useful to discuss the problem of the time evolution of type VIII models in the so-called *Hamiltonian formalism* [8,11]. The features of the type VIII model presented in this section can also be derived from the results which were obtained by H. Ringström and J.T. Horwood, J. Wainwright *et al.* [3,4] using a rigorous analysis of the system, but we discuss these features differently in order to stress particular characteristics of the dynamics in a clear and transparent way: in the subsequent section these properties will be used to derive new results, which can not be found elsewhere in the literature.

In the Hamiltonian picture the evolution of the type VIII model corresponds to the motion of a point particle in a two-dimensional billiard, where the billiard walls correspond to a triangular shaped potential in minisuperspace (β^+, β^-) [11]. The parameters β^+ and β^- are related to the metric in the following way: considering the metric components to be the basic variables for the gravitational field, the general form for a line element for Bianchi class A models can be written as

$$ds^2 = -N(t')dt'^2 + g_{ab}W^aW^b,$$

where W^a are time-independent one-forms dual to suitable frame vectors e_a . Three time-dependent scale factors can be introduced such that

$$g_{ab} = \text{diag}(a^2, b^2, c^2),$$

which can be rewritten as

$$g_{ab} = \text{diag}(e^{2\beta_1}, e^{2\beta_2}, e^{2\beta_3}),$$

with

$$\begin{aligned} \beta_1 &= \beta^0 - 2\beta^+, \\ \beta_2 &= \beta^0 + \beta^+ + \sqrt{3}\beta^-, \\ \beta_3 &= \beta^0 + \beta^+ - \sqrt{3}\beta^-. \end{aligned}$$

The triangular-shaped potential in minisuperspace in which the universe point will evolve has also an infinite open channel along the β^+ axis. The triangular shape of the potential allows the evolution to be *chaotic*: the universe point can bounce off the walls in a chaotic sequence, in principle *ad infinitum*, such that the so-called mixmaster behaviour takes place. Now if the type VIII system is studied towards the past, close to the initial singularity, then the triangular potential expands and it has been shown that an infinity of bounces are possible which lead to a chaotic evolution. If one studies the evolution towards the future, far from the initial singularity, then the triangular potential contracts and one would expect that the universe point could bounce off the walls in an infinite chaotic sequence as well. However, our previous analysis [1] based on a combination of the Hamiltonian formalism and the orthonormal frame approach shows that this can not happen: unlike in a “classical Newtonian” billiard, the universe point is forced to leave the triangular region of the potential and to escape along the infinite open channel along the β^+ axis, such that $\beta^- \rightarrow 0$ and $\beta^+ \rightarrow +\infty$. This late-time evolution is characterised by *non-chaotic* oscillations along the two walls of the infinite channel. Note that the channel becomes increasingly narrow as $\beta^+ \rightarrow +\infty$ and the universe point will exhibit increasingly rapid non-chaotic oscillations about the β^+ axis. As a consequence the so-called shear parameter Σ^2 , given by

$$\Sigma^2 = \frac{\sigma^2}{3H^2},$$

with σ^2 being the shear scalar and H being the Hubble parameter, exhibits increasingly rapid non-chaotic oscillations. In fact, if one defines

$$\Sigma_{\pm} = \frac{\sigma_{\pm}}{H},$$

with

$$\begin{aligned} \sigma_+ &= \frac{1}{2}(\sigma_{22} + \sigma_{33}), \\ \sigma_- &= \frac{1}{2\sqrt{3}}(\sigma_{22} - \sigma_{33}), \end{aligned}$$

then one can show that, as $\tau \rightarrow +\infty$ (note that the expression is slightly different from the one used in [1–4]),

$$\Sigma_-(\tau) \sim e^{-f(\tau)} \text{osc}(be^{a\tau} + \phi), \tag{1}$$

where $\text{osc}(\tau)$ represents some bounded non-chaotic oscillatory function, a and b are suitable constants and ϕ represents a phase term (as we will explain below, the present analysis will focus on this phase term) and where $e^{-f(\tau)}$ can not decrease more quickly than any exponential function. The exact asymptotic form for $\Sigma_-(\tau)$ can be found in the work of J.T. Horwood, J. Wainwright *et al.* [3] but for our purpose only the qualitative expression given by eq. (1) will be sufficient because we will examine the motion of the universe point only as it enters the open infinite channel.

Let us note that unbounded scalars can be derived from the *dimensionless electric and magnetic parts of the Weyl curvature* [11],

$$\begin{aligned} \tilde{E}_{ab} &= E_{ab} \frac{1}{H^2}, \\ \tilde{H}_{ab} &= H_{ab} \frac{1}{H^2}. \end{aligned}$$

For all Class A models one can define

$$\begin{aligned} \tilde{H}_+ &= \frac{1}{2}(\tilde{H}_{22} + \tilde{H}_{33}), \\ \tilde{H}_- &= \frac{1}{2\sqrt{3}}(\tilde{H}_{22} - \tilde{H}_{33}), \end{aligned}$$

and likewise for \tilde{E}_{\pm} .

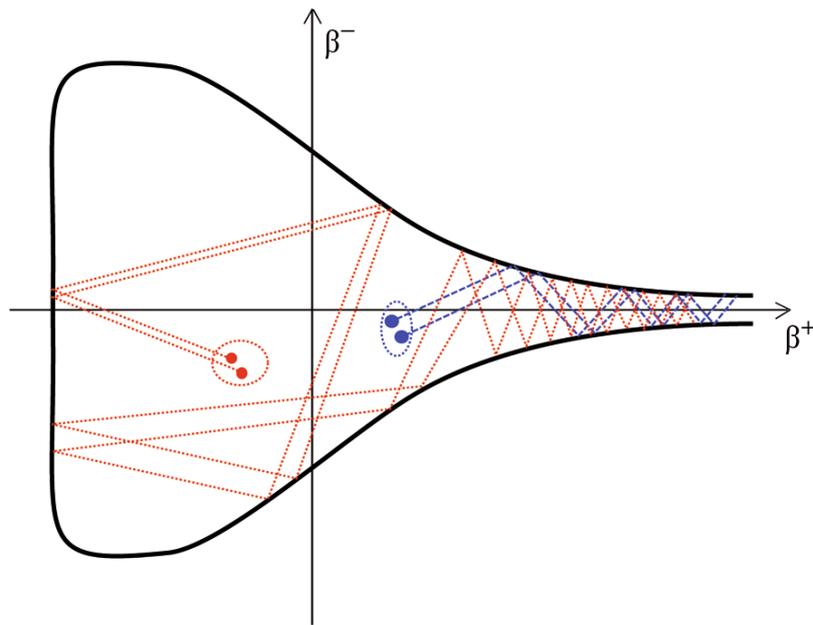


Fig. 1. The evolution of different sets of initial conditions in minisuperspace.

The magnetic part of the Weyl curvature tensor describes *intrinsic general relativistic effects* which have *no Newtonian counterpart*: the study of its properties is thus important in order to understand profound differences between Newton's and Einstein's theory of gravity.

Using the relations between the expansion normalised variables and the Hamiltonian variables, one can show that there exists an n -th order derivative of \tilde{H}_{\pm} which will diverge as $\tau \rightarrow +\infty$. As a consequence, one can show that small phase differences lead to

$$|H_{\pm}^{(n)}(\phi_1) - H_{\pm}^{(n)}(\phi_2)| \rightarrow +\infty,$$

if $|\phi_1 - \phi_2| < \epsilon$ for any $\epsilon > 0$. It has also been shown by J. Wainwright [12] that the expansion normalised Weyl parameter $W = \tilde{H}_+^2 + \tilde{H}_-^2 + \tilde{E}_+^2 + \tilde{E}_-^2$ diverges as $\tau \rightarrow +\infty$, and one can show that slightly different initial conditions can lead to important differences in the evolution of the Weyl curvature parameter.

In what follows, we will use the general properties presented in this section to show how *undecidability* emerges in the dynamics of the type VIII system.

3 The boundary between chaos and order

The analysis of the dynamics of the type VIII cosmological model can lead to undecidability. Recall that a formal system is decidable if, for every statement S , one can prove whether it is true or false.

Consider a set of initial conditions which in the Hamiltonian picture would correspond to points lying in a region at the beginning of the open infinite channel of the type VIII potential in minisuperspace.

Consider a point p_i in such a region R_1 close to the channel: this point will lead to non-chaotic oscillations between the two walls of the channel with an associated phase ϕ_i , see the discussion in the previous section. Around the point p_i , there exists a neighbourhood (of non-zero measure) of points q_i with phase factors ϕ'_i such that $|\phi_i - \phi'_i| < \epsilon$, for any given ϵ : this means that points belonging to this neighbourhood in R_1 will not display sensitivity to initial conditions and the oscillations will display only small phase differences (see points in blue of fig. 1).

However if one considers the region outside the channel in the triangular part of the potential, then there exist points p_i in a region R_2 such that initial conditions belonging to a neighbourhood (of non-zero measure) of p_i will lead to chaotic trajectories characterised by a strong sensitivity to initial conditions and the resulting phase differences between oscillations when the universe point enters the open channel would be essentially random and unpredictable (see the points in red in fig. 1). One could construct a basin of attraction by choosing some suitable value of the phase term ϕ_0 and consider initial conditions which lead to a phase term $\phi_i < \phi_0$ or $\phi_i > \phi_0$. For initial conditions belonging to the non-chaotic region R_1 one would not obtain a fractal structure, while for points belonging to the chaotic R_2

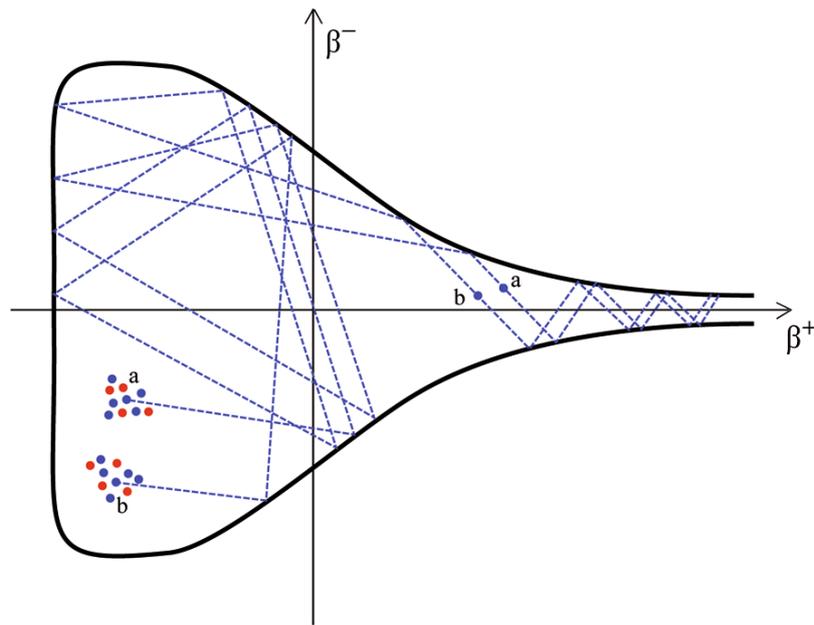


Fig. 2. The mixing of different sets of initial conditions.

region and which would be sufficiently close to the initial singularity, the basin of attraction would approach a fractal set. Thus within the same dynamical system, part of the set of initial conditions leads to non-chaotic behaviour with respect to the phase differences, while the other part is characterised by chaotic behaviour with respect to the phase factor ϕ . In fact, as is represented in fig. 2, the evolution backwards in time towards the initial singularity of two neighbouring points belonging to a region R_1 will lead to two points which are no longer close to each other in minisuperspace: the blue points will merge with red points. The closer one approaches the initial singularity, the stronger the mixing will be between points originating within R_1 and R_2 regions.

This implies that it would be hard to determine whether given initial conditions close to the initial singularity lead to chaotic behaviour with respect to the phase or not. Chaoticity itself *would depend on the initial conditions* in a highly complex way. Therefore, if one studies the type VIII system, using some approximation or map (the type VIII equations are non-integrable in the general case of interest), *even if initial conditions would be known exactly*, then the subsequent evolution could still be unpredictable. For initial conditions close to the initial singularity, the chaoticity itself would be *undecidable*: for any map, approximation or numerical simulation to be meaningful when trying to tell something about the late-time behaviour, even knowing initial conditions exactly, one would have to know whether or not the late-time behaviour is chaotic. But this is what one tries to find out using this approximation of the system. Thus one needs more information than is actually available to solve the problem: in order to determine the future behaviour, one would have to know not only the initial conditions as accurately as possible, but also the dependence of the escape rates on all the initial conditions should be known and this would require knowledge of the exact full solution to the Einstein Field Equations for the type VIII model, which is impossible.

For initial conditions close to the initial singularity, the future behaviour of the unbounded scalars such as the Weyl parameter presented in the previous section would be unpredictable in the same sense as explained above, even if these initial conditions were known exactly, because these scalars at late times are function of a phase factor ϕ which depends in an undecidable way on the considered initial conditions. Questions regarding the future behaviour of the dynamics are analogous to questions about the future behaviour of Turing machines: in a sense the space-time in a neighbourhood of the singularity can be compared with a universal Turing machine, as far as phase differences between late-time oscillations are concerned.

Also, it would have no meaning to try compute any average statistical quantity over all initial conditions, since the full dynamics would not just be random: the basins of attraction would not be purely recursive fractal sets, an additional degree of complexity would superpose itself on the chaoticity of the dynamics. The nature of these fractal basins might correspond to the recently studied notion of superfractals [15]. Within such sets the properties of the fractals possess an additional random variability.

In the next section we shall try to compare the degree of complexity or unpredictability in Newton's and in Einstein's theory of gravitation.

4 Comparing the complexity of Newton's and Einstein's theory of gravity

Surprisingly, an infinity of oscillations in a finite time interval can also occur in the Newtonian many body problem, see the work of J. Xia [16]. One might then suspect that complexity or undecidability could occur as well in Newtonian gravity, because this type of oscillations play a crucial role in the dynamics of general homogeneous solutions to the Einstein Field Equations. However, the configuration which would lead to an infinity of oscillations in a finite interval of time is special: one has to consider four particles of equal mass forming two binary pairs with opposite angular momentum and orbiting on two different parallel planes. Next a fifth lighter particle has to be considered oscillating between the centres of the two binary pairs along a line perpendicular to the two planes: for this configuration it has been shown that the lighter particle will undergo an infinity of oscillations in a finite time. This means that, since the initial configurations leading to an infinity of oscillations are special, knowing the initial conditions *exactly* one could predict the future evolution completely: the Newtonian gravitational system can not thus be compared with the type VIII dynamical system or with Turing machines. Furthermore, the complexity that emerges in the dynamics of solutions to the Einstein Field Equations which we discussed in the previous sections is related to the *time asymmetry* exhibited by those solutions: the Bianchi type VIII model is chaotic towards the past and it can be shown to be non-chaotic towards the future. This duality regarding chaotic behaviour is linked with the complexity of the dynamics of those models. One would not expect such a level of complexity to emerge in Newtonian many body problems because the solutions exhibit *time symmetry* between past and future.

Another reason which might imply that Newtonian gravity does not lead to undecidability regarding features of the dynamics is the following: as explained in sect. 2, although the Weyl curvature parameter and other scalars derived from it are known to be unbounded towards the future, the dependence of the phase term associated to their late-time oscillations on the initial conditions can be undecidable. The magnetic part of the Weyl curvature describes intrinsically general relativistic effects which have *no Newtonian counterpart*: one might thus suspect that the dynamical features exhibited by those scalars possess no Newtonian analogue.

5 Conclusion

Since the work of K. Gödel [17] on incompleteness and undecidability, several important examples of undecidable propositions in pure mathematics have been found. But also in the context of systems which might be relevant for physics [14] interesting results have been obtained: an example is provided by the work of F. Doria and N. da Costa [18, 19]. In the latter it was shown that it is impossible *in general* to demonstrate the stability or instability of equilibrium points of differential equations: the stability is in general undecidable. In order for these results to be of physical relevance, the equilibria have to involve the interplay of a very large number of different forces: however, this situation has not arisen yet in real physical problems. In the present work, our heuristic analysis has shown that undecidability can arise in a physically important problem (although it regards only some details of the dynamics): the study of general initial conditions in cosmology. For a Bianchi type VIII model, the question whether given sets of initial conditions close to the initial singularity lead to chaotic behaviour with respect to phase differences or not can be undecidable: unbounded scalars such as the Weyl parameter at late times are function of a phase term which depends in an undecidable way on the initial conditions. Even knowing the initial conditions exactly, one would not be able to predict the future dynamics fully because uncertainties in the phase differences can lead to a different evolution of the unbounded scalars associated to the type VIII solution. The fact that one can not solve the equations for the type VIII model *exactly* implies not only (usually inevitable) quantitative errors in the study of the future evolution, but even important qualitative errors will emerge: the error can correspond to exchanging chaotic and non-chaotic behaviour.

It would be interesting to try to apply the techniques used in [18, 19] to show rigorously that stability with respect to phase factors is undecidable for the type VIII differential equations, and hence that chaos is undecidable as well. Also, numerical simulations might be able to show the presence of non-recursive fractal sets in the type VIII dynamics.

The type VIII solution might provide a leading-order approximation to part of the general inhomogeneous solution to the Einstein Field Equations and one might suspect that a similar level of complexity could occur as well for inhomogeneous solutions. Note that the presence of the singularity is crucial in determining the undecidability with regard to the features of the type VIII dynamics discussed in sect. 3. One might then argue at a heuristic level, that by eliminating the singularity by an appropriate high-energy physics theory, one could also erase these fundamental problems and obtain a consistent and decidable cosmological model at all levels. However, theories such as quantum gravity or some other high-energy physics theory make use of quantum physics (quantum field theory), which introduces a fundamental unpredictability and could even be itself undecidable, see [20] where undecidability in quantum field theory was discussed, see also [21] where it is shown that calculations of a wave function for a cosmological quantity can turn out to be uncomputable.

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