Five-Dimensional Tangent Vectors in Space-Time
II. Differential-Geometric Approach

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Abstract

In this part of the series five-dimensional tangent vectors are introduced first as equivalence classes of parametrized curves and then as differential-algebraic operators that act on scalar functions. I then examine their basic algebraic properties and their parallel transport in the particular case where space-time possesses a special local symmetry. After that I give definition to five-dimensional tangent vectors associated with dimensional curve parameters and show that they can be identified with the five-vectors introduced formally in part I. In conclusion I speak about differential forms associated with five-vectors.

1. Five-vectors as equivalence classes of parametrized curves

A. *Definition*

Consider a set $\Re$ of all smooth parametrized curves going through a fixed space-time point Q. I will lable these curves with calligraphic capital Roman letters: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, etc. The parameter of curve $\mathcal{A}$ will be denoted as $\lambda_{\mathcal{A}}$.

If f is a real scalar function defined in the vicinity of Q, one can evaluate its derivative at Q along a given curve $\mathcal{A}$:

$$\left.\frac{df(P(\lambda_{\mathcal{A}}))}{d\lambda_{\mathcal{A}}}\right|_{\lambda_{\mathcal{A}}=\lambda_{\mathcal{A}}(Q)},$$

and I will denote this derivative as $\partial_{\mathcal{A}} f|_Q$.

Let us focus our attention on the behaviour of curves in the infinitesimal vicinity of Q. From that point of view, $\Re$ can be divided into classes of equivalent curves that coincide in direction or in direction and parametrization. One can consider three degrees to which two given curves, $\mathcal{A}$ and $\mathcal{B}$, may coincide:

1. The two curves come out of Q in the same direction. A more precise formulation is the following: there exists a real positive number a such that for any scalar function f
$$\partial_{\mathcal{A}} f|_Q = a \cdot \partial_{\mathcal{B}} f|_Q. \tag{1}$$
2. The two curves come out of Q in the same direction and in the vicinity of Q their parameters change with equal rates. More precisely: for any scalar function f
$$\partial_{\mathcal{A}} f|_Q = \partial_{\mathcal{B}} f|_Q. \tag{2}$$
3. The two curves come out of Q in the same direction; their parameters change with equal rates in the vicinity of Q; and the values of these parameters at Q are the same. This means that
$$\lambda_{\mathcal{A}}(Q) = \lambda_{\mathcal{B}}(Q) \tag{3a}$$
and for any scalar function f
$$\partial_{\mathcal{A}} f|_Q = \partial_{\mathcal{B}} f|_Q. \tag{3b}$$

It is a simple matter to check that relations (1), (2) and (3) are all equivalence relations on $\Re$, and for each of them one can consider the corresponding quotient set—the set whose elements are classes of equivalent curves.

Relation (1) is of no interest to us and I will not consider it any further.

The elements of the quotient set corresponding to relation (2) will be denoted with capital boldface Roman letters: $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, etc. According to relation (2), the derivative of any scalar function f at Q is the same for all curves belonging to a given class $\mathbf{A}$, so it makes sense to introduce the notation $\partial_{\mathbf{A}} f|_Q$.

In a natural way, one can define the addition of two equivalence classes $\mathbf{A}$ and $\mathbf{B}$: $\mathbf{A}+\mathbf{B}$ is such an equivalence class that for any scalar function f one has

$$\partial_{\mathbf{A}+\mathbf{B}} f|_Q = \partial_{\mathbf{A}} f|_Q + \partial_{\mathbf{B}} f|_Q.$$

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It is easy to prove that such a sum exists for any pair of equivalence classes.

In a similar manner one can give definition to the product of an equivalence class $\mathbf{A}$ and a real number k: $k\mathbf{A}$ is such an equivalnce class that

$$\partial_{k\mathbf{A}} f|_Q = k \cdot \partial_{\mathbf{A}} f|_Q$$

for any scalar function f. Again, one can verify that $k\mathbf{A}$ exists for any $\mathbf{A}$ and any k.

With thus defined addition and multiplication by a real number, the set of all equivalence classes corresponding to relation (2) becomes a vector space. This space is four-dimensional, and I will denote it as V_4. As it will be discussed in section 2, the elements of V_4 can be identified with four-dimensional tangent vectors, so in the following I will refer to them as to *four-vectors.*

Let us now turn to the quotient set associated with relation (3). Its elements will be denoted with lowercase boldface Roman letters: $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, etc. As in the case of four-vectors, one can introduce the notation $\partial_{\mathbf{a}} f|_Q$ for the common value of the derivatives of any scalar function f along all the curves belonging to a given equivalence class $\mathbf{a}$. Similarly, the common value of the parameters of all these curves at Q will be denoted as $\lambda_{\mathbf{a}}(Q)$.

One can now give the following definitions to the sum of two equivalence classes $\mathbf{a}$ and $\mathbf{b}$ and to the product of an equivalence class $\mathbf{a}$ and a real number k: $\mathbf{a}+\mathbf{b}$ and $k\mathbf{a}$ are such equivalence classes that

$$\lambda_{\mathbf{a}+\mathbf{b}}(Q) = \lambda_{\mathbf{a}}(Q) + \lambda_{\mathbf{b}}(Q),$$
$$\lambda_{k\mathbf{a}}(Q) = k \cdot \lambda_{\mathbf{a}}(Q)$$

and for any scalar function f

$$\partial_{\mathbf{a}+\mathbf{b}} f|_Q = \partial_{\mathbf{a}} f|_Q + \partial_{\mathbf{b}} f|_Q,$$
$$\partial_{k\mathbf{a}} f|_Q = k \cdot \partial_{\mathbf{a}} f|_Q.$$

One can easily check that such a sum and such a product exist respectively for any two equivalence classes and for any equivalence class and any real number. These two operations turn the quotient set associated with relation (3) into a vector space, whose dimension is evidently five. Let us denote this space as V_5 and call its elements *five-dimensional tangent vectors* or simply *five-vectors.* In section 2 I will consider another, equivalent representation for these vectors and later on will show that they have all the formal properties of those five-vectors that have been introduced in part I.

B. *Structure of the five-vector space*

As any other vector space, V_5 is completely isotropic with respect to its two composition laws and has no distinguished direction nor any other distinguished subspace of nonzero dimension. However, one *can* distinguish two subspaces in V_5 by associating them with certain classes of parametrized curves.

Let us consider all those curves from $\Re$ for which $\partial f|_Q = 0$ for any scalar function f. It is evident that all these curves belong to the same equivalence class with respect to relation (2) and that this class is the zero vector in V_4. With respect to relation (3), the considered curves belong to equivalence classes that make up a one-dimensional subspace in V_5, which I will denote as $\mathcal{E}$. One can say that $\mathcal{E}$ is made up by all those five-vectors that do not correspond to any direction in the manifold.

Another distinguished subspace in V_5 can be obtained by considering all those curves from $\Re$ for which $\lambda(Q) = 0$. The four-vectors corresponding to these curves are all the vectors of V_4. The corresponding five-vectors make up a four-dimensional subspace in V_5, which I will denote as $\mathcal{Z}$. It is easy to see that $\mathcal{E}$ and $\mathcal{Z}$ have only one common element—the zero vector, and that V_5 is the direct sum of $\mathcal{E}$ and $\mathcal{Z}$. The components of an arbitrary five-vector $\mathbf{u}$ in these two subspaces will be denoted as $\mathbf{u}^{\mathcal{E}}$ and $\mathbf{u}^{\mathcal{Z}}$, respectively.

Other properties of $\mathcal{E}$ and $\mathcal{Z}$ will be discussed below.

C. *Relation between four- and five-vectors*

As it follows from the definition of four- and five-vectors given above, there exists a set-theoretic relation between V_4 and V_5: the former is the quotient set corresponding to the following equivalence relation on V_5:

$$\mathbf{a} \equiv \mathbf{b} \Leftrightarrow \partial_{\mathbf{a}} f|_Q = \partial_{\mathbf{b}} f|_Q \text{ for any scalar function } f.$$

Denoting this relation as R, one has $V_4 = V_5/R$. The fact that $\mathbf{A}$ is the equivalence class of $\mathbf{a}$ will be denoted as $\mathbf{a} \in \mathbf{A}$. From the definition of symbols $\partial_{\mathbf{a}}$ and $\partial_{\mathbf{A}}$ it follows that $\mathbf{a} \in \mathbf{A}$ if and only if $\partial_{\mathbf{a}} = \partial_{\mathbf{A}}$. It is a simple matter to see that R has the following linearity properties: if $\mathbf{a} \equiv \mathbf{b} \,(\mathrm{mod} R)$ and $\mathbf{c} \equiv \mathbf{d} \,(\mathrm{mod} R)$, then $\mathbf{a}+\mathbf{c} \equiv \mathbf{b}+\mathbf{d} \,(\mathrm{mod} R)$ and $k\mathbf{a} \equiv k\mathbf{b} \,(\mathrm{mod} R)$, where k is an arbitrary real number. Thus, as any other equivalence relation with such properties, R can be presented in the following form:

$$\mathbf{a} \equiv \mathbf{b} \,(\mathrm{mod} R) \Leftrightarrow \mathbf{a} - \mathbf{b} \in W,$$

where W is the subspace in V_5 that contains all the five-vectors equivalent to the zero vector. It is easy to see that W coincides with the one-dimensional subspace $\mathcal{E}$ introduced in the previous subsection, so R

can be reformulated as:

$$\mathbf{a} \equiv \mathbf{b} \,(\mathrm{mod} R) \Leftrightarrow \mathbf{a} = \mathbf{b} + \mathbf{e}, \text{ where } \mathbf{e} \in \mathcal{E}.$$

The latter condition is equivalent to **a** and **b** having equal components in the four-dimensional subspace $\mathcal{Z}$ or, for that matter, in any subspace complementary to $\mathcal{E}$. This means that there exists a one-to-one correspondence between the five-vectors from $\mathcal{Z}$ and four-vectors, and this correspondence is evidently a homomorphism.

Let me say a few words about the selection of bases in V_4 and V_5 and their transformation.

A typical five-vector basis will be denoted as $\mathbf{e}_A$, where A (as all capital latin indices) runs 0, 1, 2, 3, and 5. One can choose a basis in V_5 arbitrarily, but it is more convenient to select the fifth basis vector belonging to $\mathcal{E}$. Such bases will be called *standard* and will be used in all calculations.

The basis in V_4 can be chosen arbitrarily and independently of the basis in V_5. It is more convenient though to associate it with the five-vector basis. A natural choice is to take $\mathbf{E}_\alpha$ to be the equivalence classes of the basis five-vectors $\mathbf{e}_\alpha$ (the equivalence class of $\mathbf{e}_5$ is the zero four-vector). I will refer to this basis as to the one *associated* with the basis $\mathbf{e}_A$ in V_5.

If $\mathbf{e}_A$ and $\mathbf{e}'_A$ are two standard bases in V_5 and $\mathbf{e}'_A = \mathbf{e}_B L^B_{\ A}$, then $L^B_{\ A}$ can be shown to satisfy the condition

$$L^\alpha_{\ 5} = 0 \text{ for all } \alpha.$$

The corresponding equivalence classes are related as $\mathbf{E}'_\alpha = \mathbf{E}_\beta L^\beta_{\ \alpha}$.

D. *Reminder on the inner product of four-vectors*

Four-vectors inherit their inner product from the Riemannian metric of space-time. The latter is a rule that assigns a certain number, called interval, to each finite continuous line. This number is additive, and for an infinitesimal line connecting two points with coordinates x^α and $x^\alpha + dx^\alpha$ it equals

$$\sqrt{g_{\alpha\beta}(x)dx^\alpha dx^\beta} + \text{ terms of higher order in } dx, \quad (4)$$

where $g_{\alpha\beta}$ is a real nondegenerate 4×4 matrix with the signature $(+, -, -, -)$.

Consider now a parametrized curve coming out of a point Q. According to formula (4), the interval assigned to the part of the curve between Q and a nearby point corresponding to the parameter value $\lambda(Q) + d\lambda$ is

$$\sqrt{g_{\alpha\beta}(Q)(\partial x^\alpha/\partial\lambda)_Q(\partial x^\beta/\partial\lambda)_Q} \cdot d\lambda \\ + \text{ terms of higher order in } d\lambda.$$

Since $(\partial x^\alpha/\partial\lambda)_Q$ is the same for all curves from a given equivalence class associated with relation (2), the expression under the radical sign is a function of the four-vector corresponding to the curve rather than of the curve itself. This enables one to assign a number to each four-vector, which is interpreted as its length squared. More precisely, the inner product g is defined as a real bilinear symmetric function of two four-vectors such that for any four-vector **U**

$$g(\mathbf{U}, \mathbf{U}) = g_{\alpha\beta}(Q)(\partial_{\mathbf{U}} x^\alpha)_Q(\partial_{\mathbf{U}} x^\beta)_Q.$$

The interval is a dimensional quantity. It is measured in centimeters or seconds or in any other units of length or time. Accordingly, the quantity under the radical sign in formula (4) is measured in cm^2 or sec^2 or in some other squared units. Throughout sections 1 and 2 of this paper I will consider only dimensionless coordinates and curve parameters. Then, if the interval is measured, say, in centimeters, the elements of the matrix $g_{\alpha\beta}$ will be measured in cm^2, $g^{\alpha\beta}$ will be measured in cm^{-2}, and the connection coefficients for four- and five-vector fields will be dimensionless.

E. *Symmetries*

The set $\Re$ of all parametrized curves going through an arbitrary point Q has a certain symmetry with respect to the behaviour of curves in the infinitesimal vicinity of Q. Namely, there exist certain maps of $\Re$ onto itself that have the following properties:

1. If $\mathcal{A} \mapsto \mathcal{A}'$, then $\lambda_{\mathcal{A}}(Q) = \lambda_{\mathcal{A}'}(Q)$.
2. If $\mathcal{A} \mapsto \mathcal{A}'$ and $\mathcal{B} \mapsto \mathcal{B}'$, and for any scalar function f one has $\partial_{\mathcal{A}} f|_Q = k \cdot \partial_{\mathcal{B}} f|_Q$, where k is some constant factor, then for any scalar function f one has $\partial_{\mathcal{A}'} f|_Q = k \cdot \partial_{\mathcal{B}'} f|_Q$.
3. If $\mathcal{A} \mapsto \mathcal{A}'$, $\mathcal{B} \mapsto \mathcal{B}'$, and $\mathcal{C} \mapsto \mathcal{C}'$, and for any scalar function f one has $\partial_{\mathcal{A}} f|_Q + \partial_{\mathcal{B}} f|_Q = \partial_{\mathcal{C}} f|_Q$, then for any scalar function f one has $\partial_{\mathcal{A}'} f|_Q + \partial_{\mathcal{B}'} f|_Q = \partial_{\mathcal{C}'} f|_Q$.
4. If $\mathcal{A} \mapsto \mathcal{A}'$, then
$$g_{\alpha\beta}(Q)(\partial_{\mathcal{A}} x^\alpha)_Q(\partial_{\mathcal{A}} x^\beta)_Q \\ = g_{\alpha\beta}(Q)(\partial_{\mathcal{A}'} x^\alpha)_Q(\partial_{\mathcal{A}'} x^\beta)_Q.$$

Property 2 at $k = 1$ means that such transformations of $\Re$ induce maps of V_4 onto itself. Properties 2 and 3 mean that these transformations of V_4 are linear, and property 4 means that they conserve the inner product of two four-vectors.

Property 1 and property 2 at $k = 1$ mean that the considered transformations of $\Re$ also induce maps of V_5 onto itself. Properties 1, 2, and 3 mean that these maps are linear. Property 1 means that a vector from $\mathcal{Z}$ is transformed into a vector from $\mathcal{Z}$. And properties 1 and 2 mean that vectors from $\mathcal{E}$ are not changed at all.

Let us now find the corresponding transformation matrices for V_4 and V_5.

Let $\mathbf{E}_\alpha$ be an arbitrary orthonormal basis in V_4 and let us take that under the considered transformation these basis vectors are transformed into $\mathbf{E}'_\alpha = \mathbf{E}_\beta \Lambda^\beta_{\ \alpha}$. Since the transformation should conserve the inner product, and the basis $\mathbf{E}_\alpha$ is orthonormal, $\Lambda^\beta_{\ \alpha}$ should be a matrix from O(3,1). As a basis in V_5 let us take a standard basis where $\mathbf{e}_\alpha \in \mathcal{Z}$ and $\mathbf{e}_\alpha \in \mathbf{E}_\alpha$. Let us suppose that $\mathbf{e}_A$ are transformed into $\mathbf{e}'_A = \mathbf{e}_B L^B_{\ A}$, where $L^B_{\ A}$ is some real nondegenerate 5×5 matrix. Since vectors from $\mathcal{E}$ do not change under the considered transformation, one should have $L^5_{\ 5} = 1$ and $L^\alpha_{\ 5} = 0$ for all α. Since vectors from $\mathcal{Z}$ are transformed into vectors from $\mathcal{Z}$, one should have $L^5_{\ \alpha} = 0$ for all α. Finally, owing to the one-to-one correspondence between $\mathcal{Z}$ and V_4, one should have $L^\alpha_{\ \beta} = \Lambda^\alpha_{\ \beta} \in$ O(3,1).

F. *Inner product of five-vectors*

The method used in subsection D to define the inner product g for four-vectors is also applicable in the case of five-vectors. The resulting inner product on V_5, which for the time being I will denote as h', is a real bilinear symmetric function of two five-vectors such that for any five-vector $\mathbf{u}$

$$h'(\mathbf{u},\mathbf{u}) = g_{\alpha\beta}(Q)(\partial_\mathbf{u} x^\alpha)_Q (\partial_\mathbf{u} x^\beta)_Q$$

(Q is the space-time point where one considers the tangent space of five-vectors). Since the value of the derivative $\partial_\mathbf{u}$ is the same for all five-vectors corresponding to the same four-vector, h' will be a degenerate inner product. It is not difficult to see that the subspace of all degenerate five-vectors for h' (of all such five-vectors $\mathbf{u}$ that $h'(\mathbf{u},\mathbf{v}) = 0$ for any $\mathbf{v}$) coincides with $\mathcal{E}$ and that h' is nondegenerate within any subspace complementary to $\mathcal{E}$. It is also apparent that for any $\mathbf{u}$ and $\mathbf{v}$ one has

$$h'(\mathbf{u},\mathbf{v}) = g(\mathbf{U},\mathbf{V}), \tag{5}$$

where $\mathbf{u} \in \mathbf{U}$ and $\mathbf{v} \in \mathbf{V}$.

It is not difficult to construct from h' a nondegenerate inner product on V_5. For that one should consider another natural measure that exists for five-vectors: to each five-vector $\mathbf{u}$ one can put into correspondence the value of the relevant curve parameter, $\lambda_\mathbf{u}$. If one then interprets this latter number as the length of vector $\mathbf{u}$, one will obtain another inner product—let us denote it as h''—which will also be degenerate. It is easy to see that $h''(\mathbf{u},\mathbf{v}) = \lambda_\mathbf{u} \cdot \lambda_\mathbf{v}$. Consequently, the subspace of all degenerate vectors for h'' coincides with $\mathcal{Z}$ and h'' is nondegenerate within any (one-dimensional) subspace complementary to $\mathcal{Z}$.

One should now notice that the subspaces of degenerate vectors for h' and h'' are complementary to each other, which means that the sum of h' and h'' will be a nondegenerate inner product on V_5. The only problem in constructing such a sum is that h' is a dimensional quantity and is measured in the same units as g, whereas h'', being the product of curve parameters, does not have a dimension. We thus see that to construct a nondegenerate inner product on V_5 from h' and h'', one needs a dimensional constant, ξ, which would play a role similar to that of the speed of light: it would establish a relation between different units used to measure the same quantity. The resulting inner product measured in the same units as g will be

$$h(\mathbf{u},\mathbf{v}) = h'(\mathbf{u},\mathbf{v}) + \xi \cdot h''(\mathbf{u},\mathbf{v}). \tag{6}$$

The same result can be obtained from considerations of another kind. For that one should adopt the view-point that four-vectors and five-vectors are subordinate objects, whose algebraic properties are determined by the properties of the manifold with which they are associated. In particular, this means that the structure of V_5 should have a symmetry no less than the symmetry of $\Re$. This, in its turn, means that *any* inner product of five-vectors should be invariant under the transformations discussed in the previous subsection.

Let us consider the same five-vector basis $\mathbf{e}_A$ that has been used in subsection E. It is a simple matter to show that the matrix $h_{AB} \equiv h(\mathbf{e}_A,\mathbf{e}_B)$ of any nondegenerate inner product h satisfying the above symmetry requirement has to be of the form

$$h_{\alpha\beta} = a \cdot \eta_{\alpha\beta},\ h_{\alpha 5} = h_{5\alpha} = 0,\ h_{55} = b, \tag{7}$$

where a and b are some nonzero constants. A direct consequence of these formulae is that any five-vector from $\mathcal{Z}$ is orthogonal to any five-vector from $\mathcal{E}$, so for any $\mathbf{u}$ and $\mathbf{v}$

$$h(\mathbf{u},\mathbf{v}) = h(\mathbf{u}^\mathcal{Z},\mathbf{v}^\mathcal{Z}) + h(\mathbf{u}^\mathcal{E},\mathbf{v}^\mathcal{E}). \tag{8}$$

Another consequence of formulae (7) is that the inner product of any two five-vectors from $\mathcal{Z}$ is proportional to the inner product of the corresponding four-vectors. Thus, if the overall normalization of h is selected in such a way that the proportionality factor between h and g be unity, one will have

$$h(\mathbf{u}^{\mathcal{Z}}, \mathbf{v}^{\mathcal{Z}}) = g(\mathbf{U}, \mathbf{V}) = h'(\mathbf{u}, \mathbf{v}).$$

Finally, one should observe that the $\mathcal{E}$-component of any five-vector $\mathbf{u}$ equals $\lambda_{\mathbf{u}} \cdot \mathbf{i}$, where $\mathbf{i}$ is the vector from $\mathcal{E}$ that corresponds to the unity value of the parameter: $\lambda_{\mathbf{i}} = 1$. Consequently,

$$h(\mathbf{u}^{\mathcal{E}}, \mathbf{v}^{\mathcal{E}}) = \lambda_{\mathbf{u}}\lambda_{\mathbf{v}}\, h(\mathbf{i}, \mathbf{i}) = h(\mathbf{i}, \mathbf{i})\, h''(\mathbf{u}, \mathbf{v}),$$

and formula (8) acquires the form of formula (6) with $\xi = h(\mathbf{i}, \mathbf{i})$. Thus, at an appropriate choice of its overall normalization factor, any nondegenerate inner product on V_5 satisfying the above, quite natural symmetry requirement has the form indicated in formula (6).

It is obvious that constant ξ is not determined by the Riemannian metric of space-time nor by symmetry considerations, and consequently the same is true of the nondegenerate inner product of five-vectors. This is a distinctive feature of five-dimensional tangent vectors (and of similar objects in other manifolds) and is a consequence of that specific way in which five-vectors are associated with space-time.

In the previous subsection I have introduced a five-vector basis where $\mathbf{e}_\alpha \in \mathcal{Z}$. As we have seen above, in terms of the five-vector inner product this means that all $\mathbf{e}_\alpha$ are orthogonal to $\mathbf{e}_5$. This is one of the two conditions satisfied by a *regular* five-vector basis defined in section 3 of part I within the formal theory, the other condition being that $h(\mathbf{e}_5, \mathbf{e}_5) = 1$. When five-vectors are introduced as equivalence classes of parametrized curves, it is more convenient to define the regular basis in a slightly different way, equating to unity not the value of $h(\mathbf{e}_5, \mathbf{e}_5)$ (which depends on the choice of ξ) but the value of $\lambda_{\mathbf{e}_5}$. A regular basis will thus be a standard five-vector basis where all $\mathbf{e}_\alpha \in \mathcal{Z}$ and $\mathbf{e}_5 = \mathbf{i}$.

2. Five-vectors as operators

A. *Another representation for five-vectors*

In modern textbooks on differential geometry, ordinary tangent vectors are usually introduced by identifying their fields with linear differential operators (derivations) that act upon scalar functions from a certain set $\Im$ which determines the topological and differential properties of the manifold. Each derivation is a map

$$\mathbf{U} : \Im \to \Im$$

that satisfies the following requirements:

$$\begin{aligned} &\mathbf{U}[k] = 0 \text{ for any constant function } k \in \Im, \\ &\mathbf{U}[f + g] = \mathbf{U}[f] + \mathbf{U}[g] \text{ for any } f, g \in \Im, \\ &\mathbf{U}[fg] = \mathbf{U}[f] \cdot g + f \cdot \mathbf{U}[g] \text{ for any } f, g \in \Im. \end{aligned} \tag{9}$$

One can then prove a theorem that in a local coordinate system each derivation can be presented as the following differential operator:

$$\mathbf{U} = U^\alpha(\partial/\partial x^\alpha), \tag{10}$$

where $\partial/\partial x^\alpha$ are derivatives along coordinate lines and U^α are scalar functions from $\Im$. It is evident that at each point in space-time there exists a natural isomorphism between the equivalence classes of parametrized curves corresponding to relation (2) and operators of the form (10):

$$\mathbf{A} \mapsto \partial_{\mathbf{A}},$$

and basing on this isomorphism one can identify the elements of V_4 with four-dimensional tangent vectors.

Let us now find a similar operator representation for five-vectors. First, one should notice that the two conditions that determine the equivalence relation (3) can be replaced with a single requirement: that for any scalar function f

$$\partial_{\mathcal{A}} f|_Q + \lambda_{\mathcal{A}}(Q) f(Q) = \partial_{\mathcal{B}} f|_Q + \lambda_{\mathcal{B}}(Q) f(Q).$$

This enables one to establish a one-to-one correspondence between the equivalence classes of parametrized curves associated with relation (3) and differential-algebraic operators of the form

$$\mathbf{u} = u^\alpha(\partial/\partial x^\alpha) + u^5 \cdot \mathbf{1}, \tag{11}$$

where $\mathbf{1}$ is the identity operator. The simplest variant of such a correspondence is evidently

$$\mathbf{a} \mapsto \partial_{\mathbf{a}} + \lambda_{\mathbf{a}} \cdot \mathbf{1}. \tag{12}$$

One can then consider five-vector fields and basing on the above correspondence, relate them to such maps $\mathbf{u} : \Im \to \Im$ which in any local coordinate system can be presented in the form (11), where u^A are now scalar functions.

Finally, one can find a set of formal requirements, similar to conditions (9) for derivations, that enable one to introduce the above maps without referring to any coordinates. One possible set of such requirements is the following:

$$\begin{aligned} &\mathbf{u}[k] = \upsilon \cdot k \text{ for any constant } k \in \Im, \\ &\quad \text{where } \upsilon \in \Im \text{ is characteristic of } \mathbf{u}, \\ &\mathbf{u}[f + g] = \mathbf{u}[f] + \mathbf{u}[g] \text{ for any } f, g \in \Im, \\ &\mathbf{u}[fg] = \mathbf{u}[f] \cdot g + f \cdot \mathbf{u}[g] - \mathbf{u}[\mathit{1}] fg \text{ for} \\ &\quad \text{any } f, g \in \Im, \text{ where } \mathit{1} \text{ is the constant} \\ &\quad \text{unity function.} \end{aligned} \tag{13}$$

It is evident that any operator of the form (11) satisfies these three requirements. Let us now prove the reverse statement:

In any local coordinate system each map $\mathbf{u} : \Im \to \Im$ satisfying requirements (13) can be presented in the form (11), where u^A are scalar functions from $\Im$.

Proof : Let us consider the operator

$$\mathbf{w} \equiv \mathbf{u} - \upsilon \cdot \mathbf{1},$$

where υ is the scalar function from $\Im$ defined by the first of the requirements (13). It is a simple matter to check that $\mathbf{w}$ satisfies conditions (9) for derivations and therefore can be presented in any local coordinate system as

$$\mathbf{w} = w^\alpha(\partial/\partial x^\alpha),$$

where $w^\alpha \in \Im$. Consequently, in any such system $\mathbf{u}$ can be presented in the form (11) with $u^\alpha = w^\alpha$ and $u^5 = \upsilon$. ▮

One may observe that the operator corresponding to a given four-vector $\mathbf{U}$ is exactly the differential part of the operator that corresponds to any five-vector belonging to $\mathbf{U}$. This coincidence is a manifestation of the fact that V_4 is isomorphic to $\mathcal{Z}$. This does not mean, however, that one can identify four-vectors with $\mathcal{Z}$-components of five-vectors, for as one will see in section 3, the isomorphism between V_4 and $\mathcal{Z}$ is not preserved by parallel transport.

The representation of five-vectors with operators enables one to introduce the former in another way: as maps $\Im \to \Im$ that satisfy requirements (13). In its mathematical qualities, such a definition of five-vectors is superior to the one given in section 1 and enables one to introduce in a natural way the commutator of two five-vector fields. On the other hand, in this case one cannot see as clearly the correspondence between five-vectors and parametrized curves, and this is why in this paper I have first considered the representation of five-vectors in the form of equivalence classes associated with relation (3). It turns out, however, that one should make a distinction between a given equivalence class and the five-vector corresponding to it. In view of this, in the following five-vector fields will always be identified with operators satisfying requirements (13), the set of which will be denoted as $\boldsymbol{\mathcal{F}}$.

As in the case of four-vectors, tangent five-vectors at a given point Q can be defined as equivalence classes of maps from $\boldsymbol{\mathcal{F}}$ with respect to the equivalence relation

$$\mathbf{u} \equiv \mathbf{v} \Leftrightarrow \mathbf{u}[\,f\,](Q) = \mathbf{v}[\,f\,](Q) \text{ for any } f \in \Im.$$

The algebraic properties of the five-vectors defined this way are the same as of those defined as classes of equivalent curves, and their analysis would have been almost an exact repetition of the one made in section 1, except for a few obvious changes in the definitions. Let me only mention that a *regular* five-vector basis can now be defined as a basis where all $\mathbf{e}_\alpha$ are purely differential operators and $\mathbf{e}_5 = \mathbf{1}$.

One should also note that the correspondence between equivalence classes of parametrized curves and operators from $\boldsymbol{\mathcal{F}}$ given by formula (12) is not the only one possible. A more general form of such a correspondence is

$$\mathbf{a} \mapsto a \cdot \partial_{\mathbf{a}} + b \cdot \lambda_{\mathbf{a}} \cdot \mathbf{1}, \qquad (14)$$

where a and b are some nonzero coefficients independent of $\mathbf{a}$. Since the overall normalization of the operators representing five-vectors is of no importance, one can always choose it so that $a = 1$. In formula (12) the second coefficient has been selected in the simplest way: $b = 1$. However, as one will see in section 3, to give a consistent definition to the five-vectors associated with curves parametrized by *dimensional* parameters, one has to assign to b a certain dimension, so it will equal unity only at some particular choice of the corresponding measurement units.

B. *Commutator of five-vector fields*

The representation of five-vectors with operators enables one to introduce the commutator of five-vector fields. Namely, if $\mathbf{u} = u^\alpha(\partial/\partial x^\alpha) + u^5 \cdot \mathbf{1}$ and $\mathbf{v} = v^\alpha\ (\partial/\partial x^\alpha) + v^5 \cdot \mathbf{1}$, then by definition,

$$[\mathbf{u}, \mathbf{v}](f) = \mathbf{u}(\mathbf{v}(f)) - \mathbf{v}(\mathbf{u}(f))$$

for any scalar function f, and one can show that $\mathbf{w} \equiv [\mathbf{u}, \mathbf{v}]$ is an operator of the form (11) with components

$$w^A = u^\beta(\partial v^A/\partial x^\beta) - v^\beta(\partial u^A/\partial x^\beta). \qquad (15)$$

For an arbitrary five-vector basis $\mathbf{e}_A$ one can define the commutation constants, $C_{AB}{}^D$, as

$$[\mathbf{e}_A, \mathbf{e}_B] = C_{AB}{}^D \mathbf{e}_D,$$

and show that the components of $[\mathbf{u}, \mathbf{v}]$ in this basis are

$$\partial_{\mathbf{u}} v^A - \partial_{\mathbf{v}} u^A + u^B v^D C_{BD}{}^A.$$

This is the analog of the well-known formula for components of the commutator of two four-vector fields $\mathbf{U}$ and $\mathbf{V}$ in an arbitrary basis $\mathbf{E}_\alpha$:

$$\partial_{\mathbf{U}} V^\mu - \partial_{\mathbf{V}} U^\mu + U^\alpha V^\beta C_{\alpha\beta}{}^\mu,$$

where $[\mathbf{E}_\alpha, \mathbf{E}_\beta] = C_{\alpha\beta}{}^\mu \mathbf{E}_\mu$.

If $\mathbf{e}_A$ is a standard basis, one has $C_{\mu 5}{}^{\alpha} = 0$. It is a simple matter to show that if $\mathbf{u} \in \mathbf{U}$ and $\mathbf{v} \in \mathbf{V}$, then $[\mathbf{u}, \mathbf{v}] \in [\mathbf{U}, \mathbf{V}]$. Thus, if $\mathbf{e}_\alpha \in \mathbf{E}_\alpha$, then $C_{\alpha\beta}{}^{\mu}|_{\text{for five-vectors}} = C_{\alpha\beta}{}^{\mu}|_{\text{for four-vectors}}$.

Let us now consider two subsets of five-vector fields from $\boldsymbol{\mathcal{F}}$: (*i*) the subset $\boldsymbol{\mathcal{F}}_{\mathcal{Z}}$ of all purely differential operators, and (*ii*) the subset $\boldsymbol{\mathcal{F}}_{\mathcal{E}}$ of all purely algebraic operators. It is evident that any element of $\boldsymbol{\mathcal{F}}$ can be uniquely presented as a sum of an operator from $\boldsymbol{\mathcal{F}}_{\mathcal{Z}}$ and an operator from $\boldsymbol{\mathcal{F}}_{\mathcal{E}}$, so $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_{\mathcal{Z}} \oplus \boldsymbol{\mathcal{F}}_{\mathcal{E}}$. The components of an arbitrary five-vector field $\mathbf{u}$ in these two subspaces will be denoted as $\mathbf{u}^{\mathcal{Z}}$ and $\mathbf{u}^{\mathcal{E}}$. It is evident that they correspond to the operators $u^\alpha(\partial/\partial x^\alpha)$ and $u^5 \cdot \mathbf{1}$, respectively.

One can easily see that the commutator of two five-vector fields from $\boldsymbol{\mathcal{F}}_{\mathcal{Z}}$ is, again, a field from $\boldsymbol{\mathcal{F}}_{\mathcal{Z}}$, so $\boldsymbol{\mathcal{F}}_{\mathcal{Z}}$ is a subalgebra: $[\boldsymbol{\mathcal{F}}_{\mathcal{Z}}, \boldsymbol{\mathcal{F}}_{\mathcal{Z}}] \subset \boldsymbol{\mathcal{F}}_{\mathcal{Z}}$. Furthermore, the commutator of a field from $\boldsymbol{\mathcal{F}}_{\mathcal{E}}$ with any other field from $\boldsymbol{\mathcal{F}}$ is an element of $\boldsymbol{\mathcal{F}}_{\mathcal{E}}$, so $\boldsymbol{\mathcal{F}}_{\mathcal{E}}$ is an ideal: $[\boldsymbol{\mathcal{F}}_{\mathcal{E}}, \boldsymbol{\mathcal{F}}] \subset \boldsymbol{\mathcal{F}}_{\mathcal{E}}$.

Commutators of four-vector fields enable one to tell whether or not a given four-vector basis is holonomic. Namely, for a given set of basis fields $\mathbf{E}_\alpha$ there exists a system of local coordinates x^α such that $\mathbf{E}_\alpha$ are tangent vectors to coordinate lines ($\mathbf{E}_\alpha = \partial/\partial x^\alpha$) iff $[\mathbf{E}_\alpha, \mathbf{E}_\beta] = 0$. A similar statement for five-vectors is the following:

> For a given set of standard five-vector basis fields $\mathbf{e}_A$ there exists a system of local coordinates x^α such that $\mathbf{e}_\alpha$ are tangent five-vectors to coordinate lines iff
>
> $$[\mathbf{e}^{\mathcal{Z}}_\alpha, \mathbf{e}^{\mathcal{Z}}_\beta] = 0, \qquad (16a)$$
> $$[\mathbf{e}^{\mathcal{Z}}_\alpha, \mathbf{e}^{\mathcal{E}}_\beta] = \delta_{\alpha\beta} \cdot \mathbf{1}. \qquad (16b)$$
>
> where $\delta_{\alpha\beta}$ is the Kronecker symbol.[1]

Proof: If $\mathbf{e}_\alpha$ are tangent vectors to coordinate lines x^α, then $\mathbf{e}_\alpha = \partial/\partial x^\alpha + x^\alpha \cdot \mathbf{1}$, and equations (16) are evidently obeyed.

If $\mathbf{e}_\alpha$ satisfy equations (16) and $\mathbf{E}_\alpha$ are such that $\mathbf{e}_\alpha \in \mathbf{E}_\alpha$, then

$$\begin{aligned} 0 = [\mathbf{e}^{\mathcal{Z}}_\alpha, \mathbf{e}^{\mathcal{Z}}_\beta] &= \partial_{\mathbf{e}_\alpha}\partial_{\mathbf{e}_\beta} - \partial_{\mathbf{e}_\beta}\partial_{\mathbf{e}_\alpha} \\ &= \partial_{\mathbf{E}_\alpha}\partial_{\mathbf{E}_\beta} - \partial_{\mathbf{E}_\beta}\partial_{\mathbf{E}_\alpha} = [\mathbf{E}_\alpha, \mathbf{E}_\beta], \end{aligned}$$

and by virtue of the corresponding theorem for four-vectors, there exists a system of local coordinates x^α such that $\partial/\partial x^\alpha = \partial_{\mathbf{E}_\alpha} = \partial_{\mathbf{e}_\alpha}$. In these coordinates each $\lambda_{\mathbf{e}_\alpha}$ is a certain real function, which according to (16b) satisfies the equation

$$\partial\lambda_{\mathbf{e}_\beta}(x)/\partial x^\alpha = \delta_{\alpha\beta}.$$

This is only possible if $\lambda_{\mathbf{e}_\alpha}(x) = x^\alpha + c^\alpha$, where c^α are integration constants. Consequently, one has $\mathbf{e}_\alpha = \partial/\partial y^\alpha + y^\alpha \cdot \mathbf{1}$, where $y^\alpha = x^\alpha + c^\alpha$. ∎

By analogy with four-vectors, a standard five-vector basis satisfying requirements (16) can be called a *coordinate* basis. In certain cases, however, it proves to be more convenient to select the $\mathcal{E}$-components of the first four basis five-vectors in a different way, for example, equal to zero. Since such five-vector bases still correspond to a coordinate four-vector basis, it makes sense to call them coordinate, too.

C. *Five-vector Lie derivative*[2]

The formal definition of the Lie derivative with respect to a four-vector field $\mathbf{U}$ is the following:

- the Lie derivative of a four-vector field $\mathbf{V}$ is

$$\boldsymbol{\pounds}_{\mathbf{U}}\mathbf{V} \equiv [\mathbf{U}, \mathbf{V}]; \qquad (17)$$

- the Lie derivative of a scalar function f is

$$\boldsymbol{\pounds}_{\mathbf{U}} f \equiv \mathbf{U} f; \qquad (18)$$

- the Lie derivatives of all other four-tensor fields can be found from formulae (17) and (18) by using the Leibniz rule, which in schematic form can be presented as

$$\boldsymbol{\pounds}_{\mathbf{U}}(\mathcal{A} * \mathcal{B}) = \boldsymbol{\pounds}_{\mathbf{U}}\mathcal{A} * \mathcal{B} + \mathcal{A} * \boldsymbol{\pounds}_{\mathbf{U}}\mathcal{B}, \qquad (19)$$

where $\mathcal{A}$ and $\mathcal{B}$ are any two four-tensor fields and $*$ denotes contraction or tensor product.

In a similar manner one can give a formal definition to the Lie derivative with respect to a *five*-vector field $\mathbf{u}$. I will denote this latter derivative as $\boldsymbol{\pounds}_{\mathbf{u}}$ and will call it the *five-vector Lie derivative.* The analog of rule (17) is quite apparent:

- the five-vector Lie derivative of a five-vector field $\mathbf{v}$ is

$$\boldsymbol{\pounds}_{\mathbf{u}}\mathbf{v} \equiv [\mathbf{u}, \mathbf{v}]. \qquad (20)$$

As the analog of rule (18) it seems reasonable to take the following one:

- the five-vector Lie derivative of a scalar function f is

$$\boldsymbol{\pounds}_{\mathbf{u}} f \equiv \mathbf{u} f. \qquad (21)$$

[1] For simplicity, this theorem is formulated and proved for $a = b = 1$ in formula (14).

[2] The contents of this subsection is not necessary for understanding the rest of the material and can be skipped by a reader not familiar enough with or not interested in this particular subject.

It is easy to check that the five-vector Lie derivative of the product of two scalar functions and the five-vector Lie derivative of the product of a scalar function and a five-vector field are expressed in terms of the five-vector Lie derivatives of the factors not according to the Leibniz rule but according to the rule

$$\boldsymbol{\pounds}_{\mathbf{u}}(\mathcal{A}*\mathcal{B}) = \boldsymbol{\pounds}_{\mathbf{u}}\mathcal{A}*\mathcal{B}+\mathcal{A}*\boldsymbol{\pounds}_{\mathbf{u}}\mathcal{B}-\boldsymbol{\pounds}_{\mathbf{u}}\mathit{1}\cdot(\mathcal{A}*\mathcal{B}), \quad (22)$$

where, as before, *1* is the constant unity scalar function. In view of this, it is not clear which of the rules — (19), (22) or some other — should hold for the contraction and tensor product. To answer this question and to gain a better understanding of the five-vector Lie derivative, let us find for the latter an interpretation similar to the one that can be given to the ordinary Lie derivative in terms of the one-parameter local group of diffeomorphisms generated by a four-vector field.

Let us recall that any sufficiently smooth four-vector field $\mathbf{U}$ defines in the neighbourhood of any point Q of the space-time manifold $\mathcal{M}$ a congruence of integral curves, and that there always exist such an open neighbourhood $\mathcal{U}$ of Q and such a real number $\varepsilon > 0$ that the map ϕ_t obtained by taking each point of $\mathcal{U}$ a parametric distance t along the corresponding integral curve, at $|t| < \varepsilon$ is a diffeomorphism of $\mathcal{U}$ into $\mathcal{M}$. At sufficiently small s and t one has $\phi_s \circ \phi_t = \phi_{s+t}$ and $(\phi_t)^{-1} = \phi_{-t}$, so these diffeomorphisms form a one-parameter local group.

At each t map ϕ_t defines a certain transformation, Φ_t, of scalar functions: the image $\Phi_t\{f\}$ of a scalar function f is such that

$$\Phi_t\{f\}|_{\phi_t(P)} = f|_P. \quad (23)$$

This transformation, in its turn, generates a certain transformation of four-vector and other four-tensor fields, which is determined by the following rules:

• the image $\Phi_t\{\mathbf{V}\}$ of a four-vector field $\mathbf{V}$ is such that for any scalar function f

$$\Phi_t\{\mathbf{V}\}\Phi_t\{f\} = \Phi_t\{\mathbf{V}f\}; \quad (24)$$

• the image $\Phi_t\{\widetilde{\mathbf{W}}\}$ of a four-vector 1-form field $\widetilde{\mathbf{W}}$ is such that for any four-vector field $\mathbf{V}$

$$< \Phi_t\{\widetilde{\mathbf{W}}\}, \Phi_t\{\mathbf{V}\} > = \Phi_t\{< \widetilde{\mathbf{W}}, \mathbf{V} >\}; \quad (25)$$

• the image $\Phi_t\{\mathcal{A}\otimes\mathcal{B}\}$ of the tensor product of two four-tensor fields $\mathcal{A}$ and $\mathcal{B}$ is such that

$$\Phi_t\{\mathcal{A}\otimes\mathcal{B}\} = \Phi_t\{\mathcal{A}\}\otimes\Phi_t\{\mathcal{B}\}. \quad (26)$$

Within this approach, the Lie derivative of an arbitrary four-tensor field $\mathcal{S}$ is defined as

$$\boldsymbol{\pounds}_{\mathbf{U}}\mathcal{S} \equiv -\,(d/dt)\,\mathit{\Phi}_t\{\mathcal{S}\}|_{t=0}. \quad (27)$$

It is easy to see that at small t

$$\Phi_t\{f\} = f - t\cdot\mathbf{U}f + O(t^2), \quad (28)$$

from which, using definition (27), one obtains rule (18). In a similar manner, after rewriting equation (24) as

$$\Phi_t\{\mathbf{V}\}f = \Phi_t\{\mathbf{V}\Phi_t^{-1}\{f\}\}$$

and using definition (27), one obtains rule (17). From equation (25) it follows that the Leibniz rule holds for the contraction of a four-vector field and a four-vector 1-form field and from equation (26) it follows that it also holds for the tensor product of any two four-tensor fields. Thus, the definition of the Lie derivative by means of equations (23)–(27) is equivalent to its formal definition according to equations (17)–(19).

It is now apparent that to obtain the desired interpretation of the five-vector Lie derivative, one should associate with every sufficiently smooth five-vector field a certain one-parameter group of transformations of scalar functions and five-tensor fields. Let us denote the transformations from this group as Ψ_t and define the five-vector Lie derivative of an arbitrary five-tensor field $\mathcal{S}$ as

$$\boldsymbol{\pounds}_{\mathbf{u}}\mathcal{S} \equiv -\,(d/dt)\,\mathit{\Psi}_t\{\mathcal{S}\}|_{t=0}. \quad (29)$$

Considering what has been said above, it seems reasonable to take that at small t

$$\Psi_t\{f\} = f - t\cdot\mathbf{u}f + O(t^2) \quad (30)$$

for any scalar function f, which together with definition (29) gives us rule (21). If, by analogy with rule (24), one then takes that

$$\Psi_t\{\mathbf{v}\}\Psi_t\{f\} = \Psi_t\{\mathbf{v}f\} \quad (31)$$

for any $\mathbf{v}$ and f, from formulae (29) and (30) one will obtain rule (20). Thus, the infinitesimal transformation (30) produces the desired result. Let us now find the corresponding finite transformation.

It is evident that for any sufficiently smooth five-vector field $\mathbf{u}$, in the vicinity of any point Q one can construct a congruence of integral curves of the corresponding four-vector field $\mathbf{U}$. In this case these curves will be called the integral curves of field $\mathbf{u}$. It is not difficult to prove that at finite t the image $\Psi_t\{f\}$ of any scalar function f of class C^∞ equals

$$\Psi_t\{f\}(\lambda) = \exp\{-\int_{\lambda-t}^{\lambda} u^5(\lambda')\,d\lambda'\,\}\,f(\lambda-t), \quad (32)$$

where λ is the parameter of the integral curve of field $\mathbf{u}$ and u^5 is the fifth component of the latter in a regular basis. We thus see that transformation Ψ_t consists in "shifting" every value of the function a parametric distance t along the corresponding integral curve and then multiplying it by a certain exponential factor. It is easy to see that this latter factor equals the corresponding value of $\Psi_t\{\mathit{1}\}$, so for an arbitrary scalar function f one has

$$\Psi_t\{f\} = \Psi_t\{\mathit{1}\}\Phi_t\{f\}. \tag{33}$$

From the latter formula it follows that transformations Φ_t induced by four-vector fields are a particular case of transformations Ψ_t — a case that corresponds to the five-vector fields from $\mathcal{F}_{\mathcal{Z}}$. Another particular case are the transformations Ψ_t induced by five-vector fields from $\mathcal{F}_{\mathcal{E}}$. In this case

$$\Psi_t\{f\}(P) = \exp\{-t \cdot u^5(P)\}f(P).$$

It is evident that to each transformation Ψ_t one can put into correspondence a certain map of $\mathcal{U}$ into $\mathcal{M}$, namely, the map ϕ_t induced by the four-vector field corresponding to $\mathbf{u}$. Thus, both in the case of four-vector fields and in the case of five-vector fields one is actually dealing with *two* maps: (*i*) a map from $\mathcal{U}$ to $\mathcal{M}$ and (*ii*) a map from the set of restrictions to $\mathcal{U}$ of all the functions from $\Im$ to the set of restrictions of all these functions to $\phi_t(\mathcal{U})$. In the case of four-vector fields there exists a one-to-one correspondence bewteen these two maps, which enables one to think that the second map is induced by the first one. This is not so in the case of five-vector fields: for example, the identity map from $\mathcal{U}$ to $\mathcal{M}$ may correspond to different *non*identical transformations of scalar functions.

From equation (32) it is not difficult to derive that for any two scalar functions f and g

$$\Psi_t\{fg\} = \Phi_t\{f\}\Psi_t\{g\} = \Psi_t\{f\}\Phi_t\{g\}, \tag{34}$$

so in the general case the image of the product of two scalar functions with respect to Ψ_t is not the product of their images. By substituting $\Psi_t^{-1}\{\mathit{1}\}\Psi_t\{f\}$ for $\Phi_t\{f\}$ in formula (34) and differentiating both sides of the latter with respect to t, one can verify that in this case rule (22) is indeed obeyed.

It is natural to define the action of Ψ_t on a tensor product in the following way:

$$\Psi_t\{\mathcal{A} \otimes \mathcal{B}\} = \Psi_t\{\mathcal{A}\} \otimes \Psi_t\{\mathcal{B}\}, \tag{35}$$

where $\mathcal{A}$ and $\mathcal{B}$ are any two five-tensor fields of nonzero rank. This formula does not work, however, if one of the fields or both of them are of rank zero. In the second case this can be seen from formula (34), if one considers that for scalar functions $f \otimes g = fg$. In the first case, if, for example, $\mathcal{A} = f$ and $\mathcal{B} = \mathbf{v}$, from formula (34) and definition (31) one can easily obtain that

$$\Psi_t\{f \otimes \mathbf{v}\} = \Psi_t\{f\mathbf{v}\} = \Phi_t\{f\}\Psi_t\{\mathbf{v}\}. \tag{36}$$

Difficulties also occur with the definition of the action of Ψ_t on five-vector 1-forms. The direct analog of rule (25) is

$$< \Psi_t\{\widetilde{\mathbf{w}}\}, \Psi_t\{\mathbf{v}\} > = \Psi_t\{< \widetilde{\mathbf{w}}, \mathbf{v} >\}, \tag{37}$$

which means that the operation of contraction is "correlated" with transformation Ψ_t in the sense that the contraction of the image of a five-vector field $\mathbf{v}$ with the image of a five-vector 1-form field $\widetilde{\mathbf{w}}$ equals the image of the scalar function equal to the contraction of $\mathbf{v}$ with $\widetilde{\mathbf{w}}$. The quantity $< \widetilde{\mathbf{w}}, \mathbf{v} >$ can also be regarded as a five-tensor field of rank zero obtained by contracting the field $\widetilde{\mathbf{w}} \otimes \mathbf{v}$ of rank $(1, 1)$. A similar operation can be performed on other five-tensor fields, for example, on the field $\widetilde{\mathbf{w}} \otimes \mathbf{v} \otimes \mathbf{s}$. For the contraction of this latter field to be correlated with Ψ_t it is necessary that there would hold not rule (37) but the rule

$$< \Psi_t\{\widetilde{\mathbf{w}}\}, \Psi_t\{\mathbf{v}\} > = \Phi_t\{< \widetilde{\mathbf{w}}, \mathbf{v} >\}. \tag{38}$$

Therefore, in those cases where Ψ_t does not coincide with Φ_t, the requirements of correlation between the contraction and transformation Ψ_t for five-tensor fields of rank $(1, 1)$ and for five-tensor fields of other ranks are conflicting.

It is also useful to look at the components of the five-vector Lie derivatives of five-tensor fields of different ranks, in a regular coordinate basis. Let us write out these components for the case where the rule that determines the action of Ψ_t on 1-form fields is

$$< \Psi_t\{\widetilde{\mathbf{w}}\}, \Psi_t\{\mathbf{v}\} >= (\Psi_t\{\mathit{1}\})^k \Psi_t\{< \widetilde{\mathbf{w}}, \mathbf{v} >\}. \tag{39}$$

According to equation (21), the five-vector Lie derivative of function f is

$$\pounds_{\mathbf{u}} f = u^\alpha \partial_\alpha f + u^5 f. \tag{40}$$

From equations (20) and (15) one finds that the components of the five-vector Lie derivative of a five-vector field $\mathbf{v}$ are

$$(\pounds_{\mathbf{u}}\mathbf{v})^A = u^B(\partial_B v^A) - v^B(\partial_B u^A), \tag{41}$$

where, for convenience, I have introduced the notation $\partial_A \equiv \partial_{\mathbf{e}_A}$, so $\partial_\alpha = \partial/\partial x^\alpha$ and $\partial_5 = 0$. From equation (39) one can easily derive that in the dual basis of five-vector 1-forms $\widetilde{\mathbf{o}}^A$,

$$\begin{aligned}(\pounds_{\mathbf{u}}\widetilde{\mathbf{w}})_A = u^B(\partial_B w_A) + w_B(\partial_A u^B) \\ + (1 + k)\, u^5 w_A\end{aligned} \tag{42}$$

Finally, in the general case of an arbitrary five-tensor field

$$\mathbf{T} = T^{A_1\ldots A_m}_{B_1\ldots B_n}\,\mathbf{e}_{A_1}\otimes\ldots\otimes\mathbf{e}_{A_m}\otimes\widetilde{\mathbf{o}}^{B_1}\otimes\ldots\otimes\widetilde{\mathbf{o}}^{B_n}$$

one has

$$\begin{aligned}(\boldsymbol{\pounds}_{\mathbf{u}}\mathbf{T})^{A_1\ldots A_m}_{B_1\ldots B_n} &= u^H(\partial_H T^{A_1\ldots A_m}_{B_1\ldots B_n})\\ &+ n(1+k)\cdot u^5\,T^{A_1\ldots A_m}_{B_1\ldots B_n}\\ &- T^{H\ldots A_m}_{B_1\ldots B_n}(\partial_H u^{A_1})\\ -\ldots &- T^{A_1\ldots H}_{B_1\ldots B_n}(\partial_H u^{A_m})\\ &+ T^{A_1\ldots A_m}_{H\ldots B_n}(\partial_{B_1}u^H)\\ +\ldots &+ T^{A_1\ldots A_m}_{B_1\ldots H}(\partial_{B_n}u^H).\end{aligned} \tag{43}$$

As one can see from the formulae obtained, there exists a distinguished value of parameter k: $k=-1$, at which the terms proportional to u^5 in equations (42) and (43) vanish, and the five-vector Lie derivative of any five-tensor field that has at least one lower index depends only on the *derivative* of u^5, as does the five-vector Lie derivative of a five-vector field. One can also see that in the case of a five-tensor field of rank zero (at $m=n=0$) formula (43) disagrees with formula (40) for the five-vector Lie derivative of a scalar function.

All these observations suggest that in the case of transformations Ψ_t induced by five-vector fields, one should make a distinction between scalar functions which are elements of $\Im$ and scalar functions which are five-tensor fields of rank zero. Formally, these two types of objects are of different nature: the former are the functions upon which act the operators of five-vector fields; the latter are elements of a commutative ring, by which one can multiply five-vector fields, obtaining five-vector fields again. To establish order in the theory, one should suppose that these two types of functions are transformed by Ψ_t *differently*: the elements of $\Im$ are transformed according to formula (32), whereas the five-tensor fields of rank zero are transformed according to the formula

$$\Psi_t\{\mathrm{f}\}(\lambda) = \mathrm{f}(\lambda - t), \tag{44}$$

which means that for them transformation Ψ_t coincides with Φ_t. Under this assumption formula (35) for the tensor product will be valid for five-tensor fields of zero rank as well. Moreover, since the contraction of a vector and a 1-form is a tensor of rank zero, formula (37) will coincide with formula (38), and consequently the contraction will be correlated with transformation Ψ_t for tensor fields of any rank for which it makes sense. Among other things, the latter two facts mean that the five-vector Lie derivative of a contraction and of a tensor product is expressed in terms of the five-vector Lie derivatives of the factors according to the Leibniz rule. In formulae (42) and (43) one should now put $k=-1$, and so the derivatives $\boldsymbol{\pounds}_{\mathbf{u}}$ of the corresponding five-tensor fields will depend only on the derivative of u^5. Finally, the five-vector Lie derivative of an arbitrary five-tensor field f of rank zero will be

$$\boldsymbol{\pounds}_{\mathbf{u}}\mathrm{f} = \partial_{\mathbf{u}}\mathrm{f} = u^\alpha\partial_\alpha\mathrm{f}, \tag{45}$$

which agrees with formula (43). Let me emphasize once more that in the case of scalar fields from $\Im$, the image of the product of two such functions with respect to Ψ_t will not equal the product of their images, which is inevitable and has no relation to the definition of Ψ_t for five-tensor fields.

3. Some other properties of five-vectors

A. *Parallel transport of five-vectors*

As for any other type of vector-like objects considered in space-time, one can speak of parallel transport of five-vectors from one space-time point to another. One can then define the covariant derivative of five-vector fields; introduce the connection coefficients corresponding to a given five-vector basis; construct the corresponding curvature tensor; etc. In doing all this one does not have to use in any way the fact that five-vectors are associated with space-time by their definition.

One should expect that the origin of five-vectors manifests itself in that the rules of their parallel transport are related in some way to similar rules for four-vectors and to the Riemannian geometry of space-time. It is obvious that this relation cannot be derived from the algebraic properties of five-vectors, and to obtain it one has to make some new assumptions about five-vectors, which ought to be regarded as part of their definition.

Let us first consider the relation between the rules of parallel transport for four- and five-vectors. The simplest and the most natural form of this relation is obtained by postulating that parallel transport preserves the algebraic relation between four- and five-vectors discussed in subsection 1.C. A more precise formulation of this statement is the following:

> *If four-vector* **U** *is the equivalence class of five-vector* **u**, *then the transported* **U** *is the equivalence class of the transported* **u**. (46)

This assumption is quite natural considering that $\mathbf{u}\in\mathbf{U}$ means that **u** and **U** correspond to the same direction in the manifold. It has two consequences, which can be conveniently expressed in terms of connection coefficients (the latter are defined in section 4 of part I).

Let us consider the parallel transport of vectors from an arbitrary point Q to a nearby point Q'. If two five-vectors at Q belong to the same equivalence class, then according to our assumption, the transported five-vectors should also be equivalent. Since parallel transport is a linear operation, this means that vectors from $\mathcal{E}_{\text{at } Q}$ are transported into vectors from $\mathcal{E}_{\text{at } Q'}$. Consequently, in *any* standard five-vector basis,

$$G^{\alpha}_{\ 5\mu} = 0. \tag{47}$$

Let $\mathbf{e}_A$ be an arbitrary standard five-vector basis and let $\mathbf{E}_\alpha$ be the associated basis of four-vectors. If $\mathbf{E}_\alpha(Q)$ are transported into vectors $\mathbf{E}_\beta(Q')C^{\beta}_{\ \alpha}$, then according to our assumption, $\mathbf{e}_\alpha(Q)$ should be transported into vectors $\mathbf{e}_\beta(Q')C^{\beta}_{\ \alpha} + \mathbf{e}_5(Q')C^{5}_{\ \alpha}$, where the coefficients $C^{\beta}_{\ \alpha}$ are the same in both cases. This means that in the selected bases,

$$G^{\alpha}_{\ \beta\mu} = \Gamma^{\alpha}_{\ \beta\mu}. \tag{48}$$

It is evident that assumption (46) tells one nothing about $G^{5}_{\ \alpha\mu}$ and $G^{5}_{\ 5\mu}$. To get an idea of what these coefficients can be like, let us now consider a particular case where the connection for five-vector fields is such that there exists a certain local symmetry which can be formulated as the following principle:

> For any set of scalar, five-vector and five-tensor fields defined in the vicinity of any point Q in space-time, by means of a certain procedure one can construct a set of fields in the vicinity of any other point Q', such that at Q' these new fields (which will be called *equivalent*) satisfy the same algebraic and first-order differential relations that the original fields satisfy at Q. (49)

The procedure by means of which the equivalent fields are constructed can be formulated as follows:

1. Introduce at Q a system of local Lorentz coordinates x^α.
   Introduce the corresponding *regular coordinate* five-vector basis $\mathbf{e}_A$.
   Introduce the corresponding bases for all other five-tensors.
2. Each scalar field f in the vicinity of Q will then determine and be determined by one real coordinate function $f(x)$.
   Each five-vector field $\mathbf{u}$ in the vicinity of Q will determine and be determined by five real coordinate functions $u^A(x)$ (= components of $\mathbf{u}$ in the basis $\mathbf{e}_A$).
   Each five-tensor field $\mathbf{T}$ in the vicinity of Q will determine and be determined by an appropriate number of real coordinate functions $T^{AB\ldots C}_{DE\ldots F}(x)$ (= components of $\mathbf{T}$ in the relevant tensor basis corresponding to $\mathbf{e}_A$).
3. Introduce at Q' a system of local Lorentz coordinates x'^α such that $x'^\alpha(Q') = x^\alpha(Q)$.
   Introduce the corresponding regular coordinate five-vector basis $\mathbf{e}'_A$.
   Introduce the corresponding bases for all other five-tensors.
4. Then the equivalent scalar, five-vector and five-tensor fields in the vicinity of Q' will be determined in coordinates x'^α and in the corresponding bases by the *same functions* $f(\cdot)$, $u^A(\cdot)$, ..., $T^{AB\ldots C}_{DE\ldots F}(\cdot)$ that determine the original fields in the vicinity of Q in coordinates x^α and in the corresponding bases.

At $Q' = Q$ the two mentioned systems of local Lorentz coordinates, x^α and x'^α, are related as follows:

$$x'^\alpha(P) = x^\alpha(Q) + \Lambda^{\alpha}_{\ \beta}[x^\beta(P) - x^\beta(Q)] \\ + \text{ terms of order } [x^\alpha(P) - x^\alpha(Q)]^3,$$

where P is an arbitrary point in the vicinity of Q and $\Lambda^{\alpha}_{\ \beta}$ is a matrix from O(3,1). Reasoning as in section 4 of part I, one can show that in the regular basis associated with either of these coordinate systems one should have $G^{\alpha}_{\ \beta\mu}(Q) = G^{5}_{\ 5\mu}(Q) = 0$ and $G^{5}_{\ \alpha\mu}(Q) \propto \eta_{\alpha\mu}$. Since in these coordinates $g_{\alpha\beta}(Q) \propto \eta_{\alpha\beta}$, too, this means that the connection coefficients $G^{5}_{\ \alpha\mu}(Q)$ are proportional to the components of the metric tensor. Denoting the proportionality factor as $-\varsigma$ and using the obvious transformation formulae for five-vector connection coefficients, one can show that in *any* regular five-vector basis

$$G^{5}_{\ 5\mu} = 0 \tag{50}$$

and

$$G^{5}_{\ \alpha\mu} = -\varsigma g_{\alpha\mu}. \tag{51}$$

From requirement (49) it also follows that five-vector connection coefficients should have the same form at any two points in space-time in similar five-vector bases. In the case of four-vector connection coefficients a similar condition is satisfied automatically, and therefore is not necessary. For five-vectors this is a nontrivial requirement, which means that ς in equation (51) should be a constant.

It is evident that the value of ς is not fixed by the symmetry principle. Since for dimensionless coordinates and curve parameters the connection coefficients are dimensionless and $g_{\alpha\beta}$ are measured in the units of interval squared, ς should have the dimension $(interval)^{-2}$. There is no sense in talking about five-vectors if $\varsigma = 0$, for it is impossible to distinguish a five-vector with such rules of parallel transport from a pair consisting of a four-vector and a scalar. Indeed, V_5 is isomorphic to the direct sum of V_4 and the space of scalars (regarded as one-dimensional vectors), and it is apparent that at $\varsigma = 0$ this isomorphism is preserved by parallel transport. Considering this, I will always assume that $\varsigma \neq 0$.

B. *Five-vectors associated with dimensional curve parameters*

So far we have been dealing with dimensionless curve parameters and coordinates. In practice, the latter are usually selected in such a way so that their values would be associated in some particular way with certain lengths, time intervals or angles determined by the space-time metric. For example, any system of dimensionless Lorentz coordinates in flat space-time is such that the square of the interval between any two events A and B, measured in certain units ℓ, equals

$$[x^0(A) - x^0(B)]^2 - [x^1(A) - x^1(B)]^2 - [x^2(A) - x^2(B)]^2 - [x^3(A) - x^3(B)]^2.$$

It is evident that if one changes the unit for measuring the interval as

$$\ell \to k\ell \ \ (k > 0), \tag{52}$$

the dimensionless Lorentz coordinates will change in the inverse proportion. This enables one to consider the latter as numerical values of certain dimensional quantities, $\bar{x}^\alpha$, measured in the units of interval, and it is these latter quantities one usually has in mind when using the term "Lorentz coordinates".

The situation is similar in all other cases and as in the above example, enables one to introduce the corresponding dimensional coordinates. For simplicity, in the following I will suppose that all four coordinates are measured in the units of interval. A convenient property of such dimensional coordinates is that the corresponding metric coefficients, defined by the equation

$$ds^2 = \bar{g}_{\alpha\beta}\, d\bar{x}^\alpha d\bar{x}^\beta,$$

are all dimensionless quantities. It is easy to see that $\bar{g}_{\alpha\beta}$ are the values of the dimensional metric coefficients $g_{\alpha\beta}$ that correspond to the dimensionless coordinates x^α which are the values of $\bar{x}^\alpha$ at the given ℓ.

The same idea can be used to define dimensional curve parameters (for simplicity, let us consider only those of them which are measured in the units of interval). One can then introduce the notion of a tangent four-vector corresponding to a curve parameterized by a given dimensional parameter $\bar{\lambda}$. Such four-vectors behave as dimensional quantities in the sense that at each ℓ they have a certain "value", which, by definition, is the four-vector that corresponds to the dimensionless parameter λ which is the value of $\bar{\lambda}$ for the given ℓ. The algebraic operations and parallel transport for such dimensional four-vectors are defined on the basis of the corresponding operations for four-vectors associated with dimensionless parameters. For example, a sum of two dimensional four-vectors $\mathbf{U}$ and $\mathbf{V}$ is a dimensional four-vector whose value at any ℓ equals the sum of the corresponding values of $\mathbf{U}$ and $\mathbf{V}$. It is evident that when one changes ℓ according to formula (52), the value of each dimensional four-vector changes in the same proportion, owing to which the inner product of any two such four-vectors is a dimensionless quantity. This and other properties of four-vectors associated with dimensional curve parameters are well known, and I will not discuss them any further.

Let us now see how one can define a tangent five-vector corresponding to a curve parametrized by some dimensional parameter $\bar{\lambda}$. Following the same idea that has been used for tangent four-vectors, one should consider such a five-vector as a quantity that has a certain "value" at every choice of ℓ. This "value" is the tangent five-vector that corresponds to the dimensionless parameter λ which is the value of $\bar{\lambda}$ for the given ℓ. Let us now find the operator that corresponds to this latter five-vector.

According to section 2, the general form of the operator representing the five-vector tangent to a curve parametrized by a given dimensionless parameter λ is

$$a \cdot d/d\lambda + b \cdot \lambda \cdot \mathbf{1}, \tag{53}$$

where a and b are some arbitrary nonzero constants. As it has been said above, the overall normalization of the operators representing five-vectors can always be chosen in such a way that a be unity. When dimensionless curve parameters are considered by themselves—not as values of some dimensional parameters, one can take $b = 1$, too, as it has been done in formula (12). However, if operator (53) represents the value of a five-vector associated with a dimensional parameter $\bar{\lambda}$, the value of b has to depend on the choice of ℓ. Indeed, let us suppose that one has a dimensional five-vector, $\mathbf{u}$, represented by a purely differential operator and one parallel transports it from a given space-time point Q to some other point Q'. By definition, $\mathbf{u}^{\text{transported}}$ is the five-vector at Q' whose value at any ℓ equals the value of $\mathbf{u}$ at Q transported from Q to Q' along the selected path. It is evident that if one changes ℓ according to formula (52), the value of $\mathbf{u}$ will change in the same proportion, and since parallel transport is a linear operation, so will the value of $\mathbf{u}^{\text{transported}}$. Consequently, the algebraic part of the operator representing $\mathbf{u}^{\text{transported}}$, which in the general case will not be zero, should change in the same proportion as ℓ, which is only possible if b changes as $b \to k^2 b$.

We thus see that in the case of five-vectors associated with dimensional curve parameters, the coefficient b in formula (53) has to be the value of some nonzero constant with dimension $(interval)^{-2}$. Apart from being nonzero, this constant is absolutely arbitrary, and it is convenient to choose it equal to the constant ς introduced in the previous subsection. The operator representing a five-vector associated with a dimensional parameter $\bar{\lambda}$ can then be presented in the following form:

$$d/d\bar{\lambda} + \bar{\lambda} \cdot \varsigma \cdot \mathbf{1}. \tag{54}$$

In a similar manner one can introduce five-vectors corresponding to parameters with dimension other than that of the interval. The algebraic and differential properties of all such five-vectors will be practically the same as those of the five-vectors associated with dimensionless parameters, and only the dimension of certain relevant quantities will be different. For example, in the particular case considered above, both the inner product h' induced by the metric and the nondegenerate inner product h are dimensionless. The relation between the two is still given by formula (6), only now ξ has the dimension $(interval)^{-2}$.

In the case of dimensional five-vectors, there exist *three* convenient ways to normalize the fifth basis vector in a standard five-vector basis and, accordingly, there are three ways to define a regular basis.

In those cases where the emphasis is made on parallel transport of five-vectors, it is convenient to choose $\mathbf{e}_5 = \varsigma \cdot \mathbf{1}$. Then, in the corresponding regular basis (in the one where the other four basis five-vectors belong to $\mathcal{Z}$) one will have $G^5{}_{\alpha\mu} = -\,g_{\alpha\mu}$, and the fifth component of any five-vector $\mathbf{u}$ will equal $\lambda_{\mathbf{u}}$. In the following, such a basis will be referred to as an *active* regular basis.

In those cases where the emphasis is made on the action of five-vectors on scalar functions, it is convenient to take $\mathbf{e}_5 = \mathbf{1}$. In the corresponding regular basis one will then have $G^5{}_{\alpha\mu} = -\,\varsigma g_{\alpha\mu}$ and the fifth component of any five-vector $\mathbf{u}$ will equal $\varsigma\lambda_{\mathbf{u}}$. In the following, such a basis will be referred to as a *passive* regular basis.

Finally, in those cases where the emphasis is made on the inner product of five-vectors (at some particular choice of ξ), it is convenient to normalize $\mathbf{e}_5$ by the requirement $h(\mathbf{e}_5, \mathbf{e}_5) = \operatorname{sign}\xi$. It is evident that this equation has two solutions: $\mathbf{e}_5 = +\,|\xi|^{-1/2}\,\varsigma \cdot \mathbf{1}$ and $\mathbf{e}_5 = -\,|\xi|^{-1/2}\,\varsigma \cdot \mathbf{1}$, and to be definite, I will choose the first one. In the corresponding regular basis one will then have $G^5{}_{\alpha\mu} = -\,|\xi|^{1/2} g_{\alpha\mu}$, and the fifth component of any five-vector $\mathbf{u}$ will equal $|\xi|^{1/2}\lambda_{\mathbf{u}}$. In the following, such a basis will be referred to as a *normalized* regular basis and the operator $|\xi|^{-1/2}\,\varsigma \cdot \mathbf{1}$ will be denoted as $\mathbf{n}$.

From now on, unless it is stated otherwise, I will talk only about five-vectors associated with dimensional curve parameters and coordinates, and will omit the bar over the dimensional x^α and λ. It is evident that any result obtained for such five-vectors can readily be reformulated for five-vectors corresponding to dimensionless parameters.

C. *Four-vectors as simple bivectors over V_5*

We are now ready to demonstrate that the five-vectors introduced formally in part I can be identified with the five-dimensional tangent vectors introduced in this paper. More precisely, it will be shown that there can be established a natural isomorphism between the space of four-vectors and one of the maximal vector spaces of simple bivectors over V_5 and that in those cases where the connection for five-vectors possesses the local symmetry considered in subsection A, this isomorphism is preserved by parallel transport. This will mean that the five-vectors considered in this paper have all the formal properties postulated for five-vectors in part I.

Let us fix a nonzero five-vector $\mathbf{e} \in \mathcal{E}$ and consider all simple bivectors of the form $\mathbf{u} \wedge \mathbf{e}$, where $\mathbf{u} \in V_5$. It is evident that $\mathbf{u} \wedge \mathbf{e} = \mathbf{v} \wedge \mathbf{e}$ if and only if $\mathbf{u} - \mathbf{v} \in \mathcal{E}$, which is exactly the equivalence relation R of subsection 1.C. Thus, one is able to establish a one-to-one correspondence between four-vectors and elements of the maximal vector space of simple bivectors over V_5 with the directional vector belonging to $\mathcal{E}$. It is evident that this correspondence is a homomorphism and that it depends on the choice of the arbitrary nonzero vector $\mathbf{e}$. Let us fix the latter by requiring that the considered correspondence be an isomorphism.

Let us consider some particular nondegenerate inner product on V_5, where the constant ξ has been chosen positive, so that h would have the signature $(+ - - - +)$. It is not difficult to check that if $\mathbf{u} \in \mathbf{U}$

and $\mathbf{v} \in \mathbf{V}$, then

$$g(\mathbf{U}, \mathbf{V}) = h(\mathbf{u}, \mathbf{v}) - \frac{h(\mathbf{e}, \mathbf{u}) h(\mathbf{e}, \mathbf{v})}{h(\mathbf{e}, \mathbf{e})}. \qquad (55)$$

On the other hand, the inner product of $\mathbf{u} \wedge \mathbf{e}$ and $\mathbf{v} \wedge \mathbf{e}$ induced by h is

$$h(\mathbf{u} \wedge \mathbf{e}, \mathbf{v} \wedge \mathbf{e}) = h(\mathbf{u}, \mathbf{v}) h(\mathbf{e}, \mathbf{e}) - h(\mathbf{u}, \mathbf{e}) h(\mathbf{v}, \mathbf{e}).$$

For the correspondence $\mathbf{U} \mapsto \mathbf{u} \wedge \mathbf{e}$ to be an isomorphism $g(\mathbf{U}, \mathbf{V})$ should equal $h(\mathbf{u} \wedge \mathbf{e}, \mathbf{v} \wedge \mathbf{e})$ for all $\mathbf{u}$ and $\mathbf{v}$, which is only possible if $h(\mathbf{e}, \mathbf{e}) = 1$. This means that $\mathbf{e}$ is either $+\mathbf{n}$ or $-\mathbf{n}$. We thus see that (for the given $\xi > 0$) there exist *two* isomorphisms of V_4 onto the considered maximal vectors space of simple bivectors, and unless additional requirements are imposed, the choice between the two is a matter of convention. To be definite, I will take $\mathbf{e} = \mathbf{n}$.

The fact that the above isomorphism (actually, both of them) is preserved by parallel transport becomes evident if one considers that the relation $\mathbf{u} \in \mathbf{U}$ is invariant under parallel transport and that $\mathbf{n}$ is transported into $\mathbf{n}$.

One can now use all the results obtained within the formal theory of five-vectors. Most of the definitions made in the present paper correspond to those made in part I. The only essential difference concerns the associated four-vector basis.

When introducing five-vectors formally, one has no means of associating them with four-dimensional tangent vectors other than saying that a five-vector $\mathbf{u}$ corresponds to the four-vector identified with the bivector $\mathbf{u} \wedge \mathbf{e}$, where $\mathbf{e}$ is some directional vector. The only way one can fix $\mathbf{e}$ within the formal theory is to require that it be of certain length. However, since the inner product of five-vectors is an object of study itself, one prefers to have a purely "kinematic" relation between the four- and five-vector bases, and the only sensible choice is to take $\mathbf{E}_\alpha = \mathbf{e}_\alpha \wedge \mathbf{e}_5$. This means that

$$\mathbf{E}_\alpha = \xi^{1/2} \lambda_{\mathbf{e}_5} \times (\text{the equivalence class of } \mathbf{e}_\alpha). \quad (56)$$

Considering that $\xi\,(\lambda_{\mathbf{e}_5})^2 = h(\mathbf{e}_5, \mathbf{e}_5)$, from formula (55) one obtains the relation between the components of g and h derived in part I:

$$g_{\alpha\beta} = h_{55} h_{\alpha\beta} - h_{\alpha 5} h_{\beta 5}.$$

Furthermore, if $\nabla_\mu \mathbf{n} = 0$, then $G^5_{5\mu} = (\lambda_{\mathbf{e}_5})^{-1} \partial_\mu \lambda_{\mathbf{e}_5}$, and for the four-vector connection coefficients corresponding to basis (56) one has

$$\Gamma^\alpha_{\beta\mu} = G^\alpha_{\beta\mu} + \delta^\alpha_\beta G^5_{5\mu},$$

which is exactly the relation obtained in part I. Finally, if one assumes that flat space-time possesses the symmetry considered in subsection A, then in any orthonormal standard five-vector basis one will have

$$G^5_{5\mu} = 0 \quad \text{and} \quad G^5_{\alpha\mu} = -\,\kappa \eta_{\alpha\mu},$$

where $\kappa = \xi^{1/2}$ (if we had taken $\mathbf{e} = -\mathbf{n}$, we would have had $\kappa = -\,\xi^{1/2}$).

Let me also say a few words about the equation for the first covariant derivative of h. Straightforward calculations similar to those made in part I give the following result:

$$\begin{aligned} \{\nabla_{\mathbf{U}} h\}(\mathbf{v}, \mathbf{w}) &\\ = \kappa g(\mathbf{U}, \mathbf{V}) h(\mathbf{w}, \mathbf{n}) &\\ + \kappa g(\mathbf{U}, \mathbf{W}) h(\mathbf{v}, \mathbf{n}), & \end{aligned} \qquad (57)$$

where it is assumed that $\mathbf{v} \in \mathbf{V}$ and $\mathbf{w} \in \mathbf{W}$. Since for any $\mathbf{v}$ one has $\kappa h(\mathbf{v}, \mathbf{n}) = \xi \cdot \lambda_{\mathbf{v}}$, this equation can also be presented as

$$\{\nabla_{\mathbf{U}} h\}(\mathbf{v}, \mathbf{w}) = \xi\, g(\mathbf{U}, \mathbf{V}) \lambda_{\mathbf{w}} + \xi\, g(\mathbf{U}, \mathbf{W}) \lambda_{\mathbf{v}}. \quad (58)$$

It is easy to see that for an arbitrary nonzero $\mathbf{e} \in \mathcal{E}$ the bivectors $\mathbf{v} \wedge \mathbf{e}$ and $\mathbf{w} \wedge \mathbf{e}$ correspond to the four-vectors $\xi^{1/2} \lambda_{\mathbf{e}} \mathbf{V}$ and $\xi^{1/2} \lambda_{\mathbf{e}} \mathbf{W}$, respectively. Thus, by multiplying both sides of equation (57) by $\xi\,(\lambda_{\mathbf{e}})^2 = h(\mathbf{e}, \mathbf{e})$ one obtains

$$\begin{aligned} h(\mathbf{e}, \mathbf{e}) \{\nabla_{\mathbf{U}} h\}(\mathbf{v}, \mathbf{w}) &\\ = \kappa g(\mathbf{U}, \mathbf{v} \wedge \mathbf{e}) h(\mathbf{w}, \mathbf{e}) &\\ + \kappa g(\mathbf{U}, \mathbf{w} \wedge \mathbf{e}) h(\mathbf{v}, \mathbf{e}), & \end{aligned}$$

which is exactly the equation for ∇h obtained within the formal theory of five-vectors.

D. *Operator ∇ and matrix g with five-vector indices*

Above I have introduced the covariant derivative operator, $\nabla_{\mathbf{U}}$, which differentiates five-vector fields in the direction specified by its argument—by the four-vector $\mathbf{U}$. As a consequence, the corresponding connection coefficients, $G^A_{B\mu}$, have indices of two kinds: two five-vector indices A and B and one four-vector index μ. This is not very convenient in those cases where indices of different kinds have to be permuted, for any relation with such permutations is valid only if the four- and five-vector bases have been chosen accordingly.

This inconvenience can be easily eliminated if instead of $\nabla_{\mathbf{U}}$ one considers the operator $\nabla_{\mathbf{u}}$, defined by the relation

$$\nabla_{\mathbf{u}} = \nabla_{\mathbf{U}} \quad \text{for} \quad \mathbf{u} \in \mathbf{U}. \qquad (59)$$

It is obvious that $\nabla_{\mathbf{u}}$ is absolutely equivalent to $\nabla_{\mathbf{U}}$. However, unlike the latter, it formally depends on a *five*-vector. It is evident that $\nabla_{\mathbf{u}} = \nabla_{(\mathbf{u}^{\mathcal{Z}})}$ for any five-vector $\mathbf{u}$, so for any $\mathbf{e} \in \mathcal{E}$ one has $\nabla_{\mathbf{e}} = 0$. Operator $\nabla_{\mathbf{u}}$ is the analog of the operator $\partial_{\mathbf{u}}$ that acts upon scalar functions, and relation (59) is the analog of the relation

$$\partial_{\mathbf{u}} = \partial_{\mathbf{U}} \;\text{ for }\; \mathbf{u} \in \mathbf{U}.$$

It is natural to introduce the notation $\nabla_A \equiv \nabla_{\mathbf{e}_A}$. Then, in any standard five-vector basis one has $\nabla_5 = 0$ and $\nabla_\mu^{\text{(with a five-vector index)}} = \nabla_\mu^{\text{(with a four-vector index)}}$. In view of this, I will use the same carrier letter 'G' to denote the connection coefficients corresponding to ∇_A:

$$\nabla_A \mathbf{e}_B = \mathbf{e}_C \, G^C_{\;BA}.$$

Then $G^A_{\;B\mu}$ with a five-vector μ will equal $G^A_{\;B\mu}$ with a four-vector μ in any standard basis, and rules (47), (48), (50), and (51) will apply to $G^A_{\;BC}$ without any changes. In addition, one will have a fifth rule: that in any standard five-vector basis,

$$G^A_{\;B5} = 0.$$

In the usual manner one can derive the transformation formula for $G^A_{\;BC}$, corresponding to the basis transformation $\mathbf{e}'_A = \mathbf{e}_B L^B_{\;A}$:

$$G'^A_{\;BC} = (L^{-1})^A_{\;D} G^D_{\;EF} L^E_{\;B} L^F_{\;C} + (L^{-1})^A_{\;D} (\partial_F L^D_{\;B}) L^F_{\;C}.$$

If both bases are standard, one will have $G'^A_{\;B5} = G^A_{\;B5} = 0$ and

$$G'^A_{\;B\mu} = (L^{-1})^A_{\;D} G^D_{\;E\nu} L^E_{\;B} L^\nu_{\;\mu} + (L^{-1})^A_{\;D} (\partial_\nu L^D_{\;B}) L^\nu_{\;\mu},$$

which is the usual formula for transformation of connection coefficients.

In a similar manner one can deal with four-vector indices in $g_{\mu\nu}$. Actually, I have already defined the corresponding five-vector quantity in subsection 1.F, where it has been denoted as h'. From now on, instead of $h'(\mathbf{u}, \mathbf{v})$ I will use the notation $g(\mathbf{u}, \mathbf{v})$, so formulae (5) and (6) will acquire the form:

$$g(\mathbf{u}, \mathbf{v}) = g(\mathbf{U}, \mathbf{V})$$

for $\mathbf{u} \in \mathbf{U}$ and $\mathbf{v} \in \mathbf{V}$, and

$$h(\mathbf{u}, \mathbf{v}) = g(\mathbf{u}, \mathbf{v}) + \xi \cdot \lambda_{\mathbf{u}} \lambda_{\mathbf{v}}. \tag{60}$$

It is evident that $g(\mathbf{u}, \mathbf{v}) = g(\mathbf{u}^{\mathcal{Z}}, \mathbf{v}^{\mathcal{Z}})$ for any five-vectors $\mathbf{u}$ and $\mathbf{v}$, so for any $\mathbf{e} \in \mathcal{E}$ one has $g(\mathbf{u}, \mathbf{e}) = 0$. If one now introduces the notation $g_{AB} \equiv g(\mathbf{e}_A, \mathbf{e}_B)$, then in any standard five-vector basis one will have

$$g_{55} = g_{\alpha 5} = g_{5\alpha} = 0$$

and

$$g_{\alpha\beta}^{\text{(with five-vector indices)}} = g_{\alpha\beta}^{\text{(with four-vector indices)}}.$$

From these formulae and equations (47) and (48) of subsection A it follows that in any standard five-vector basis

$$\partial_\mu g_{AB} - g_{CB} G^C_{\;A\mu} - g_{AC} G^C_{\;B\mu} = 0,$$

which means that g *regarded as a five-tensor* satisfies the equation $\nabla g = 0$.

The latter equation and formula (60) enable one to obtain the following expression for the first covariant derivative of the inner product h regarded as a five-tensor:

$$\{\nabla_{\mathbf{u}} h\}(\mathbf{v}, \mathbf{w}) = \xi \, \{\nabla_{\mathbf{u}} \lambda\}_{\mathbf{v}} \lambda_{\mathbf{w}} + \xi \, \lambda_{\mathbf{w}} \{\nabla_{\mathbf{u}} \lambda\}_{\mathbf{v}},$$

where $\{\nabla_{\mathbf{u}} \lambda\}_{\mathbf{v}} \equiv \partial_{\mathbf{u}} \lambda_{\mathbf{v}} - \lambda_{(\nabla_{\mathbf{u}} \mathbf{v})}$. Comparing this expression with equation (58), one can see that the latter is equivalent to the following simpler equation:

$$\{\nabla_{\mathbf{u}} \lambda\}_{\mathbf{v}} = g(\mathbf{u}, \mathbf{v}). \tag{61}$$

E. *Forms associated with five-vectors*

As in the case of any other type of vectors, one can consider linear forms corresponding to five-vectors. Such forms will be denoted with lower-case boldface Roman letters with a tilde: $\widetilde{\mathbf{a}}$, $\widetilde{\mathbf{b}}$, $\widetilde{\mathbf{c}}$, etc., and their space will be denoted as $\widetilde{V}_5$. To distinguish a p-form associated with five-vectors from a p-form associated with four-vectors I will call the former a *five-vector* p-form and the latter a *four-vector* p-form.

Five-vector 1-forms have all the properties common to linear forms in general. In addition, they have several specific features, which are due to their association with five-vectors, and it is these latter properties I will now consider.

The existence of two distinguished subspaces in V_5 results in the existence of two distinguished subspaces in $\widetilde{V}_5$. The first of these subspaces is made up by all those 1-forms from $\widetilde{V}_5$ whose contraction with any five-vector from $\mathcal{E}$ is zero. It is evident that this subspace is four-dimensional, and I will denote it as $\widetilde{\mathcal{Z}}$. The other distinguished subspace is made up by all those 1-forms that have a zero contraction with any five-vector from $\mathcal{Z}$. This subspace is one-dimensional, and I will denote it as $\widetilde{\mathcal{E}}$. It is easy to see that $\widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{E}}$ have only one common element—the zero 1-form, and that $\widetilde{V}_5$ is the direct sum of $\widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{E}}$. The components of an arbitrary five-vector 1-form $\widetilde{\mathbf{w}}$ in these two subspaces will be denoted as $\widetilde{\mathbf{w}}^{\widetilde{\mathcal{Z}}}$ and $\widetilde{\mathbf{w}}^{\widetilde{\mathcal{E}}}$, respectively.

If $\mathbf{e}_A$ is a standard five-vector basis and $\widetilde{\mathbf{o}}^A$ is the corresponding dual basis of five-vector 1-forms, then $\widetilde{\mathbf{o}}^\alpha \in \widetilde{\mathcal{Z}}$ for all α. The fifth basis 1-form will not necessarily be an element of $\widetilde{\mathcal{E}}$: this will be the case only if all $\mathbf{e}_\alpha \in \mathcal{Z}$. The same conclusions follow from the transformation formulae for the dual basis of 1-forms, corresponding to the transformation $\mathbf{e}'_A = \mathbf{e}_B L^B_A$ from one standard five-vector basis to another. Since in this case $(L^{-1})^\alpha_{\ 5} = 0$, one has

$$\widetilde{\mathbf{o}}'^{\,\alpha} = (L^{-1})^\alpha_{\ B}\, \widetilde{\mathbf{o}}^B = (L^{-1})^\alpha_{\ \beta}\, \widetilde{\mathbf{o}}^\beta,$$

but

$$\widetilde{\mathbf{o}}'^{\,5} = (L^{-1})^5_{\ 5}\, \widetilde{\mathbf{o}}^5 + (L^{-1})^5_{\ \beta}\, \widetilde{\mathbf{o}}^\beta.$$

If $\mathbf{e}_A$ is a passive regular basis, then $\widetilde{\mathbf{o}}^5 \in \widetilde{\mathcal{E}}$ and $< \widetilde{\mathbf{o}}^5, \mathbf{1} > = 1$. This particular five-vector 1-form will be denoted as $\widetilde{\mathbf{j}}$.

The fact that $\mathcal{Z}$ is isomorphic to V_4 enables one to establish a natural isomorphism between $\widetilde{\mathcal{Z}}$ and the space of four-vector 1-forms, which will be denoted as $\widetilde{V}_4$. Namely, to each five-vector 1-form $\widetilde{\mathbf{w}}$ from $\widetilde{\mathcal{Z}}$ one can put into correspondence such a four-vector 1-form $\widetilde{\mathbf{W}}$ that for any five-vector $\mathbf{u} \in \mathcal{Z}$ one will have $< \widetilde{\mathbf{w}}, \mathbf{u} > = < \widetilde{\mathbf{W}}, \mathbf{U} >$, where $\mathbf{u} \in \mathbf{U}$. It is evident that this isomorphism can be extended to a map of $\widetilde{V}_5$ onto $\widetilde{V}_4$, which will be a homomorphism but will not be a one-to-one correspondence. In the standard way, this latter map defines an equivalence relation on $\widetilde{V}_5$:

$$\widetilde{\mathbf{u}} \equiv \widetilde{\mathbf{v}} \ \text{ iff their images in } \widetilde{V}_4 \text{ are equal.} \tag{62}$$

This enables one to regard $\widetilde{V}_4$ as a quotient set and four-vector 1-forms as equivalence classes. It is not difficult to see that the equality of the images of $\widetilde{\mathbf{u}}$ and $\widetilde{\mathbf{v}}$ in $\widetilde{V}_4$ is equivalent to $\widetilde{\mathbf{u}} - \widetilde{\mathbf{v}} \in \widetilde{\mathcal{E}}$. The relation between $\widetilde{V}_4$ and $\widetilde{V}_5$ is thus similar to the relation between V_4 and V_5, however, unlike the latter, it is not preserved by parallel transport, as it will be shown below.

The parallel transport of five-vector 1-forms is defined in the standard way: by requiring that it conserve the contraction. Consequently, if $G^A_{\ B\mu}$ are connection coefficients for a standard five-vector basis, then for the corresponding dual basis of 1-forms one has

$$\nabla_\mu \widetilde{\mathbf{o}}^A = -\, G^A_{\ B\mu} \widetilde{\mathbf{o}}^B, \tag{63}$$

and from formulae (47) and (48) one obtains that

$$\nabla_\mu \widetilde{\mathbf{o}}^\alpha = -\, G^\alpha_{\ B\mu} \widetilde{\mathbf{o}}^B = -\, G^\alpha_{\ \beta\mu} \widetilde{\mathbf{o}}^\beta = -\, \Gamma^\alpha_{\ \beta\mu} \widetilde{\mathbf{o}}^\beta.$$

This means that 1-forms from $\widetilde{\mathcal{Z}}$ are transported into 1-forms from $\widetilde{\mathcal{Z}}$ and that the isomorphism between $\widetilde{\mathcal{Z}}$ and $\widetilde{V}_4$ is preserved by parallel transport. From formula (63) it also follows that

$$\nabla_\mu \widetilde{\mathbf{o}}^5 = -\, G^5_{\ 5\mu} \widetilde{\mathbf{o}}^5 - G^5_{\ \beta\mu} \widetilde{\mathbf{o}}^\beta,$$

which shows that in the general case, 1-forms from $\widetilde{\mathcal{E}}$ are not transported into 1-forms from $\widetilde{\mathcal{E}}$, so equivalence relation (62) is not invariant under parallel transport.

As in the case of any other vector space, each inner product on V_5 defines a certain correspondence between five-vectors and five-vector 1-forms. Since one has two inner products on V_5 — g and h, there are two such correspondences, which I will denote as ϑ_g and ϑ_h, respectively. By definition, $\vartheta_g(\mathbf{u})$ is such a five-vector 1-form that

$$< \vartheta_g(\mathbf{u}), \mathbf{v} > = g\,(\mathbf{u}, \mathbf{v}) \quad \text{for any} \ \ \mathbf{v} \in V_5. \tag{64}$$

The definition of the 1-form $\vartheta_h(\mathbf{u})$ is similar. It is evident that both ϑ_g and ϑ_h are linear maps of V_5 into $\widetilde{V}_5$. If u^A are components of some five-vector $\mathbf{u}$ in a certain five-vector basis, then the components of $\vartheta_g(\mathbf{u})$ and $\vartheta_h(\mathbf{u})$ in the corresponding dual basis of 1-forms are $g_{AB}u^B$ and $h_{AB}u^B$, respectively. Since the matrix h_{AB} is nondegenerate, this means that ϑ_h is a one-to-one correspondence and is a map of V_5 *onto* $\widetilde{V}_5$. It is also easy to see that $\vartheta_h(\mathcal{Z}) = \widetilde{\mathcal{Z}}$ and $\vartheta_h(\mathcal{E}) = \widetilde{\mathcal{E}}$. By contrast, ϑ_g is neither a one-to-one correspondence nor a surjection. It is evident that $\vartheta_g(\mathbf{u}) = \vartheta_g(\mathbf{u}^{\mathcal{Z}}) = \vartheta_h(\mathbf{u}^{\mathcal{Z}})$, so $\vartheta_g(\mathcal{Z}) = \widetilde{\mathcal{Z}}$, but $\vartheta_g(\mathcal{E}) = \{\widetilde{\mathbf{0}}\}$. Consequently, one can use g_{AB} only to lower five-vector indices. Raising indices with g_{AB} is possible only if one confines oneself to five-vectors from $\mathcal{Z}$ and to 1-forms from $\widetilde{\mathcal{Z}}$.

All this is in agreement with the general theorem that asserts that the following three statements are equivalent: (*i*) the correspondence between vectors and linear forms induced by a given inner product is injective; (*ii*) this correspondence is surjective; (*iii*) the inner product is nondegenerate.

Another general theorem states that the correspondence between vectors and linear forms is invariant under parallel transport if and only if the corresponding inner product is covariantly constant. Since g, as a five-tensor, satisfies the equation $\nabla g = 0$, one has

$$[\vartheta_g(\mathbf{u})]^{\text{transported}} = \vartheta_g(\mathbf{u}^{\text{transported}})$$

for any $\mathbf{u}$. Alternatively, this can be expressed as

$$\nabla_{\mathbf{v}}[\vartheta_g(\mathbf{u})] = \vartheta_g(\nabla_{\mathbf{v}}\mathbf{u})$$

for all $\mathbf{u}$ and $\mathbf{v}$, which means that the lowering of five-vector indices with g_{AB} commutes with covariant differentiation.

As it has been discussed earlier, the nondegenerate inner product h is *not* covariantly constant, and so in the general case, $[\vartheta_h(\mathbf{u})]^{\text{transported}}$ does not coincide with $\vartheta_h(\mathbf{u}^{\text{transported}})$. Consequently, the lowering and raising of five-vector indices with h_{AB} does not commute with covariant differentiation, and one should take special care whenever these two operations are performed on the same five-tensor.

In section 5 of part I I have introduced the five-vector 1-form $\widetilde{\mathbf{x}}$, which by definition coincides with the fifth element of the 1-form basis dual to an active regular five-vector basis. Comparing this with the definition of the 1-form $\widetilde{\mathbf{j}}$, one finds that $\widetilde{\mathbf{x}} = \varsigma^{-1} \cdot \widetilde{\mathbf{j}}$. Furthermore, it is easy to see that for any five-vector $\mathbf{v}$,

$$\lambda_{\mathbf{v}} = < \widetilde{\mathbf{x}}, \mathbf{v} > .$$

Substituting this expression for $\lambda_{\mathbf{v}}$ into the definition of $\nabla\lambda$, one finds that

$$\begin{aligned} \{\nabla_{\mathbf{u}}\lambda\}_{\mathbf{v}} &= \partial_{\mathbf{u}} < \widetilde{\mathbf{x}}, \mathbf{v} > - < \widetilde{\mathbf{x}}, \nabla_{\mathbf{u}}\mathbf{v} > \\ &= < \nabla_{\mathbf{u}}\widetilde{\mathbf{x}}, \mathbf{v} > . \end{aligned}$$

Substituting this latter expression and definition (64) into equation (61), one obtains that

$$< \nabla_{\mathbf{u}}\widetilde{\mathbf{x}} - \vartheta_g(\mathbf{u})\,, \mathbf{v} > = 0$$

for any five-vector $\mathbf{v}$, which means that

$$\nabla_{\mathbf{u}}\widetilde{\mathbf{x}} = \vartheta_g(\mathbf{u}) \qquad (65)$$

for any $\mathbf{u}$, which is nothing but equation (38) of part I. In equation (65) the 1-form $\vartheta_g(\mathbf{u})$ can be presented as a contraction of g regarded as a five-tensor of rank $(0, 2)$, with the five-vector $\mathbf{u}$. Considering also that $\nabla_{\mathbf{u}}\widetilde{\mathbf{x}} = < \nabla\widetilde{\mathbf{x}}, \mathbf{u} >$, one can present equation (65) as

$$\nabla\widetilde{\mathbf{x}} = g.$$

Let me finally say a few words about five-vector p-forms with p other than 1. It is a simple matter to see that any five-vector p-form $\widetilde{\mathbf{s}}$ with $p > 1$ can be uniquely presented as a sum of two terms: (i) a p-form made only of 1-forms from $\widetilde{\mathcal{Z}}$ and (ii) a wedge product of the type $\widetilde{\mathbf{t}} \wedge \widetilde{\mathbf{j}}$, where $\widetilde{\mathbf{t}}$ is a $(p-1)$-form. In the following, these two terms will be referred to as the $\widetilde{\mathcal{Z}}$- and $\widetilde{\mathcal{E}}$-components of $\widetilde{\mathbf{s}}$, respectively, and will be denoted as $\widetilde{\mathbf{s}}^{\widetilde{\mathcal{Z}}}$ and $\widetilde{\mathbf{s}}^{\widetilde{\mathcal{E}}}$. It is easy to see that at $p = 1$ this definition agrees with the definition of the $\widetilde{\mathcal{Z}}$- and $\widetilde{\mathcal{E}}$-components of a 1-form given above. It is obvious that a five-vector 5-form has only the $\widetilde{\mathcal{E}}$-component, and it is convenient to take that for any 0-form f,

$$f^{\widetilde{\mathcal{Z}}} = f \text{ and } f^{\widetilde{\mathcal{E}}} = 0.$$

The application of five-vector forms in exterior differential calculus will be discussed in detail in part IV.