ON THE LOCAL THEORY OF VERONESE WEBS

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Abstract. This work is an introduction to the local geometric theory of Veronese webs developed in the last twenty years. Among the different possible approach, here one has chosen the point of view of differential forms. Moreover, in order to make its reading easier, this text is self-contained in which directly regards Veronese webs.

Introduction.

The aim of this work is to provide an introduction to the local theory of Veronese webs from the geometric viewpoint. Although the classical theory is only developed on real manifolds there is no difficulty for extending it to complex ones as well, so both case will be considered here. In our approach differential forms play a crucial role, which will allow us to benefit from the advantages of Cartan exterior differential calculus.

The notion of Veronese web, due to Gelfand and Zakharevich for the case of codimension one [3, 4, 5] and some years later extended to any codimension by Panasyuk and Turiel [9], [17] (see [18] as well), is a tool for the study of generic bihamiltonian structures in odd dimension and more generally of Kronecker bihamiltonian structures. As it is well known bihamiltonian structures, introduced by Magri in [6], are related to some differential equations many of them with a physical meaning. Therefore it seems interesting to describe this geometrical objects.

With respect to the local aspect of this subject here, among other results, one shows that:

1) giving a generic bihamiltonian structure in odd dimension is like giving a codimension one Veronese web (theorem 3.2),

2) in the analytic category Kronecker bihamiltonian structures and Veronese webs are locally equivalent (theorem 3.2 again; to point out that in codimension

one we may utilize the theorem on symmetric hyperbolic systems, therefore on real manifold the C^∞ class is enough, while in codimension two or more the Cauchy-Kowalewsky theorem and the analyticity are needed).

Moreover a completely classification of 1-codimensional Veronese webs is exhibited (theorem 6.1). In higher codimension no local classification is known but, in the analytic category, one gives a versal model for Veronese webs.

On the other hand a link between classical 3-webs and Veronese webs is established in the example at the end of section 2 (see [1] by Bouetou-Dufour too).

For the global aspect of the question, still widely open, the reader may consult the papers by Rigal [11, 12, 13].

The present text consists of six sections and, in order to make its reading easier, it is largely self-contained in which directly regards Veronese webs. The first paragraph is devoted to the algebraic theory including the classification of pairs of bivectors (proposition 1.4). In the second one the notion of Veronese web, illustrated with different examples, and its main properties are discussed.

In the third section one associates a Veronese web to every Kronecker bihamiltonian structure and conversely; moreover the local equivalence between Kronecker bihamiltonian structures and Veronese webs is established. The fourth and fifth paragraphs, rather technical, are aimed to solve some exterior differential systems needed elsewhere. Finally the sixth section contains the local classification of 1-codimensional Veronese webs and the versal models for higher codimension.

1. Algebraic theory

The first part of this section is devoted to the study of the algebraic properties of Veronese webs; in particular one gives a method for constructing any Veronese web by means of an endomorphism of the support vector space. The second part contains the classification of pairs of bivectors.

All vector spaces considered here are real or complex.

1.1. Algebraic Veronese webs.

Given an endomorphism J and a subset A of a vector space V, the vector subspace spanned by (A, J) will mean that one spanned by $A \cup J(A) \cup J^2(A) \cup ...$.

When A itself is a vector subspace and (A, J) spans V we will say that the couple (A, J) is *admissible.*

Lemma 1.1. *If (W, J) is admissible and $1 \leq dimV < \infty$ then there exist $H \in End(V)$ and a basis $\{e_1, ..., e_r\}$ of W such that:*
(a) H is nilpotent and $Im(H - J) \subset W$.
(b) $V = \bigoplus_{j=1}^{r} U_j$ where each U_j is the vector subspace spanned by (e_j, H).
(c) The number of vector subspaces U_j of dimension $\geq \ell$ equals $dim(W + JW + ... + J^{\ell-1}W) - dim(W + JW + ... + J^{\ell-2}W)$ if $\ell \geq 2$ and r if $\ell = 1$.

Therefore the family of natural numbers $\{dimU_j\}$, $j = 1, ..., r$, only depends, up to permutation, on J and W.

Proof. First remark that $W + JW + ... + J^k W = W + \tilde{J}W + ... + \tilde{J}^k W$ when $\tilde{J} = J + \tilde{G}$ and $Im\tilde{G} \subset W$. Therefore it is enough to prove lemma 1.1 for some $\tilde{J}$; moreover (c) directly follows from (a) and (b) because H is a particular case of $\tilde{J}$.

We will prove (a) and (b) by induction on $r = dimW$. Let ℓ be the first natural number such that $dim\left(\dfrac{W + JW + ... + J^{\ell}W}{W + JW + ... + J^{\ell-1}W}\right) < r$. Then there exists $e \in W - \{0\}$ such that $J^{\ell}e$ belongs to $W + JW + ... + J^{\ell-1}W$; that is to say $J^{\ell}e = v_0 + ... + v_{\ell-1}$ where each $v_k \in J^k W$:

Given a basis $\{d_1, ..., d_r\}$ of W set $\tilde{G} = \sum_{j=1}^{r} d_j \otimes \alpha_j$ with $\alpha_1, ..., \alpha_r \in V^*$. Then $(J+\tilde{G})^{\ell} = J^{\ell} + J^{\ell-1} \circ \tilde{G} + A$ where $ImA \subset W + JW + ... + J^{\ell-2}W$. Hence $(J + \tilde{G})^{\ell}e = v_{\ell-1} + \sum_{j=1}^{r} \alpha_j(e) J^{\ell-1} d_j + v'$ where v' belongs to $W + JW + ... + J^{\ell-2}W$, which allows us to choose $\alpha_1 ..., \alpha_r$ in such a way that $(J + \tilde{G})^{\ell}e = v'$. So by considering $J + \tilde{G}$ instead of J and calling it J, we can suppose that $J^{\ell}e$ belongs to $W + JW + ... + J^{\ell-2}W$.

Starting the process again with another $\tilde{G} = \sum_{j=1}^{r} d_j \otimes \alpha_j$, where this time $\alpha_1(W) = ... = \alpha_r(W) = 0$, one has $(J+\tilde{G})^{\ell}e = v_{\ell-2} + \sum_{j=1}^{r} \alpha_j(Je) J^{\ell-2} d_j + v''$ with $v'' \in (W + JW + ... + J^{\ell-3}W)$ and we may suppose that $J^{\ell}e$ belongs to $W + JW + ... + J^{\ell-3}W$. Then we choose $\alpha_1, ..., \alpha_r$ such that $\alpha_j(W) = \alpha_j(JW) = 0$, $j = 1, ..., r$, and so one. In short we can assume $J^{\ell}e = 0$ without loss of generality.

Let U denote the vector subspace spanned by (e, J). By the choice of e the set $\{e, Je, ..., J^{\ell-1}e\}$ is a basis of U and $dim(W \cap U) = 1$. Let $\pi : V \to \frac{V}{U}$ be the canonical projection and $\bar{J}$ the endomorphism of $\frac{V}{U}$ induced by J. By the induction hypothesis, applied to $\frac{V}{U}$, $\frac{W}{U}$ and $\bar{J}$, there exist vectors $e_1, ..., e_{r-1} \in W$ and an endomorphism $\bar{G} = \sum_{j=1}^{r-1} \pi(e_j) \otimes \beta_j$ of $\frac{V}{U}$ such that $\{\pi(e_1), ..., \pi(e_{r-1})\}$ and $\bar{H} = \bar{J} + \bar{G}$ are as in lemma 1.1. Let ℓ_j, $j = 1, ...r-1$, be the dimension of the vector subspace spanned by $(\pi(e_j), \bar{H})$. Since $J+\tilde{G}$ where $\tilde{G} = \sum_{j=1}^{r-1} e_j \otimes (\beta_j \circ \pi)$ projects into $\bar{H}$ and $\tilde{G}(U) = 0$, by calling J to $J + \tilde{G}$, one may directly assume $\bar{H} = \bar{J}$. Thus each $\bar{J}^{\ell_j} \pi(e_j) = 0$ whence $J^{\ell_j} e_j = \sum_{k=0}^{\ell-1} a_{kj} J^k e$.

Now suppose $\ell_j < \ell$ for some j. Let m be the biggest $k > \ell_j$, if any, such that $a_{kj} \neq 0$. Then $J^m e$ belongs to $W + JW + ... + J^{m-1}W$, which contradicts the definition of of ℓ; so $a_{kj} = 0$ when $k > \ell_j$. But in this case $J^{\ell_j}(e_j - a_{\ell_j j} e)$ belongs to $W + JW + ... + J^{\ell_j - 1}W$ which again contradicts the definition of ℓ because $\{e_1, ..., e_{r-1}, e\}$ is a basis of W and $e_j - a_{\ell_j j} e \neq 0$. In short $\ell \leq \ell_j$, $j = 1, ..., r-1$.

Let V'_j, $j = 1, ..., r-1$, the vector subspace spanned by $\{e_j, ..., J^{\ell_j - 1} e_j\}$. As $\pi : V'_j \to \bar{U}_j$ is an isomorphism, $\{e_j, ..., J^{\ell_j - 1} e_j\}$ is a basis of V'_j and $V = V'_1 \oplus ... \oplus V'_{r-1} \oplus U$. Set $G = e \otimes \alpha$ with $\alpha(U) = 0$. Then $(J + G)^{\ell_j} e_j = \sum_{k=0}^{\ell-1} (a_{kj} + \alpha(J^{\ell_j - k - 1} e_j)) J^k e$, which allows us to choose α in such a way that $(J + G)^{\ell_j} e_j = 0$. For finishing it suffices considering the basis $\{e_1, ..., e_{r-1}, e\}$ of W and the endomorphism $H = J + G$. $\square$

Lemma 1.2. *If (W, J) is admissible, $dim V = n \geq 1$ and $\{w_1, ..., w_r\}$ is a basis of W then:*

(a) The curve $\gamma(t) = \varphi(t)((J + tI)^{-1} w_1) \wedge ... \wedge ((J + tI)^{-1} w_r)$ in $\Lambda^r V$, where $\varphi(t)$ is the characteristic polynomial of $-J$, is polynomial of degree $n - r$.

More precisely there exists a basis $\{e_{ij}\}$, $i = 1, ..., n_j$ and $j = 1, ..., r$, of V such that $\gamma(t) = \gamma_1(t) \wedge ... \wedge \gamma_r(t)$ where every $\gamma_j(t) = \sum_{i=1}^{n_j} t^{i-1} e_{ij}$ and $e_{n_1 1} \wedge ... \wedge e_{n_r 1} = w_1 \wedge ... \wedge w_r$.

(b) Let $(W, \tilde{J})$ be a second admissible couple. If $Im(\tilde{J} - J) \subset W$ then $\tilde{\gamma}(t) = \gamma(t)$ where $\tilde{\gamma}(t) = \tilde{\varphi}(t)((\tilde{J} + tI)^{-1} w_1) \wedge ... \wedge ((\tilde{J} + tI)^{-1} w_r)$ and $\tilde{\varphi}(t)$ is the characteristic polynomial of $-\tilde{J}$.

Proof. Consider $H \in End(V)$ and a basis $\{e_1, ..., e_r\}$ of W like in lemma 1.1. Set $n_j = dim U_j$ were U_j is the vector subspace spanned by (e_j, H). By multiplying e_1 by a suitable scalar one can suppose that $w_1 \wedge ... \wedge w_r = e_1 \wedge ... \wedge e_r$, so $((J+tI)^{-1}e_1) \wedge ... \wedge ((J+tI)^{-1}e_r) = ((J+tI)^{-1}w_1) \wedge ... \wedge ((J+tI)^{-1}w_r)$, which allows us to work with $e_1, ..., e_r$ instead of $w_1, ..., w_r$.

Note that $\{e_{ij} = (-1)^{n_j-i}H^{n_j-i}e_j\}$, $i = 1, ..., n_j$, $j = 1, ..., r$, is a basis of V. Set $\rho(t) = t^n((H+tI)^{-1}e_1) \wedge ... \wedge ((H+tI)^{-1}e_r)$. As $t^{n_j}(H+tI)^{-1} = \sum_{i=1}^{n_j}(-1)^{n_j-i}t^{i-1}H^{n_j-i}$ on U_j, then $\rho(t) = \rho_1(t) \wedge ... \wedge \rho_r(t)$ where every $\rho_j(t) = \sum_{i=1}^{n_j} t^{i-1}e_{ij}$.

Let us see that $\gamma(t) = \rho(t)$. Since $Im(J-H) \subset W$ one has $((J+tI) \wedge ... \wedge (J+tI))\rho(t) = \psi(t)e_1 \wedge ... \wedge e_r$ while the action of $J+tI$ on $\lambda = e_{11} \wedge ... \wedge e_{n_1-1,1} \wedge ... \wedge e_{1r} \wedge ... \wedge e_{n_r-1,r}$ equals $t^{n-r}\lambda + \sum_{j=1}^{r} e_j \wedge \mu_j$ where each $\mu_j \in \Lambda^{n-r-1}V$. The n-vector $\rho(t) \wedge \lambda = t^{n-r}e_1 \wedge ... \wedge e_r \wedge \lambda$ is transformed in $det(J+tI)t^{n-r}e_1 \wedge ... \wedge e_r \wedge \lambda$ by $J+tI$. But calculating its action on $\rho(t)$ and λ separately shows that $\rho(t) \wedge \lambda$ is transformed in $\psi(t)t^{n-r}e_1 \wedge ... \wedge e_r \wedge \lambda$ as well; whence $\psi(t) = det(J+tI)$, which is the characteristic polynomial of $-J$. Thus $((J+tI) \wedge ... \wedge (J+tI))\rho(t) = \varphi(t)e_1 \wedge ... \wedge e_r = ((J+tI) \wedge ... \wedge (J+tI))\gamma(t)$ and $\rho(t) = \gamma(t)$.

A similar argument shows that $\tilde{\gamma}(t) = \rho(t)$. $\square$

A polynomial curve γ in $\Lambda^r V$, $r \geq 1$, is named a *Veronese curve* if there exists a basis $\{e_{ij}\}$, $i = 1, ..., n_j$, $j = 1, ..., r$, of V such that $\gamma(t) = \gamma_1(t) \wedge ... \wedge \gamma_r(t)$ where each $\gamma_j(t) = \sum_{i=1}^{n_j} t^{i-1}e_{ij}$. When $r = 1$ one obtains the classical notion of Veronese curve.

For convenience one will set $\gamma(\infty) = lim\dfrac{\gamma(t)}{t^{n-r}}$, when $t \to \infty$.

Lemma 1.2 provides us a method for constructing Veronese curve for which $\gamma(\infty) = w_1 \wedge ... \wedge w_r$. Conversely given a Veronese curve γ in $\Lambda^r V$ and a basis like in the definition, let H and W be the nilpotent endomorphism of V defined by $He_{ij} = -e_{i-1,j}$, $i \geq 2$, $He_{1j} = 0$, and the vector subspace of basis $\{w_1 = e_{n_1 1}, ..., w_r = e_{n_r r}\}$ respectively. Then (W, H) is admissible, $n_1, ..., n_r$ are the natural numbers associated to (W, H) by lemma 1.1, and $\{w_1, ..., w_r\}$, H give rise to γ. Thus any Veronese curve can be constructed through lemma 1.2.

Every $\gamma(t) \in \Lambda^r V$ is decomposable and defines a r-dimensional vector subspace of V. The union of all these vector subspaces spans V since each $\gamma_j(\mathbb{K})$ spans the vector subspace of basis $\{e_{ij}\}$, $i = 1, ..., n_j$. Now assume that $\gamma(t) = \varphi(t)((J+tI)^{-1}w_1) \wedge ... \wedge ((J+tI)^{-1}w_r) = \tilde{\varphi}(t)((\tilde{J}+tI)^{-1}\tilde{w}_1) \wedge ... \wedge ((\tilde{J}+tI)^{-1}\tilde{w}_r)$;

then $\gamma(\infty) = w_1 \wedge ... \wedge w_r = \tilde{w}_1 \wedge ... \wedge \tilde{w}_r$.

On the other hand the action of $\tilde{J} - J = (\tilde{J} + tI) - (J + tI)$ on $\gamma(t)$ equals $(\tilde{\varphi}(t) - \varphi(t))w_1 \wedge ... \wedge w_r$; so $\tilde{J} - J$ maps the vector subspace defined by $\gamma(t)$ into W. Hence $Im(\tilde{J} - J) \subset W$.

Obviously if $w_1 \wedge ... \wedge w_r = \tilde{w}_1 \wedge ... \wedge \tilde{w}_r$ and $Im(\tilde{J} - J) \subset W$ then $w_1 \wedge ... \wedge w_r$, J and $\tilde{w}_1 \wedge ... \wedge \tilde{w}_r$, $\tilde{J}$ define the same Veronese curve.

Two admissible couples (W, J) and $(\tilde{W}, \tilde{J})$ are named *equivalent* if $W = \tilde{W}$ and $Im(\tilde{J} - J) \subset W$. Clearly the family of natural numbers given by lemma 1.1 is the same for equivalent couples. From all that said previously follows:

Proposition 1.1. *(a) Giving a Veronese curve in $\Lambda^r V$, $r \geq 1$, is like giving a class of equivalent admissible couples (W, J), where $dim W = r$, and an element $w_1 \wedge ... \wedge w_r \in \Lambda^r W - \{0\}$, by setting $\gamma(t) = \varphi(t)((J + tI)^{-1}w_1) \wedge ... \wedge ((J + tI)^{-1}w_r)$, where $\varphi(t)$ is the characteristic polynomial of $-J$.*
(b) Consider a Veronese curve $\gamma(t) = \gamma_1(t) \wedge ... \wedge \gamma_r(t)$ in $\Lambda^r V$ and a basis $\{e_{ij}\}$, $i = 1, ..., n_j$, $j = 1, ..., r$, of V such that $\gamma_j(t) = \sum_{i=1}^{n_j} t^{i-1}e_{ij}$, $j = 1, ..., r$. Then, up to permutation, the family of natural numbers $\{n_1, ..., n_r\}$ only depends on γ and corresponds to the family $\{dim U_j\}$, $j = 1, ..., r$, given by lemma 1.1 applied to (W, J).
(c) Two Veronese curves in $\Lambda^r V$ are isomorphic (through an isomorphism of V) if and only if they have the same family of natural numbers $\{n_1, ..., n_r\}$ up to permutation.

Remark. Any vector subspace of $\Lambda^r V$ containing a Veronese curve γ is at least of dimension $n - r + 1$ since $\gamma(0), \gamma^{(1)}(0), ..., \gamma^{(n-r)}(0)$ are linearly independent. Indeed if $n_1 = ... = n_r = 1$ it is obvious; otherwise assume, for example, $n_1 \geq 2$ and consider a linear combination $\sum_{\ell=0}^{n-r} a_\ell \gamma^{(\ell)}(0) = 0$.

Let $\bar{\gamma}$ denote the projection of γ into $\Lambda^r V'$ where V' is the quotient of V by the line spanned by $e_{n_1 1}$. Then $\bar{\gamma}$ is a Veronese curve in $\Lambda^r V'$ of degree $n - r - 1$. As $\bar{\gamma}(0), \bar{\gamma}^{(1)}(0), ..., \bar{\gamma}^{(n-r)}(0)$ are the projections of $\gamma(0), \gamma^{(1)}(0), ..., \gamma^{(n-r)}(0)$ and $\bar{\gamma}^{(n-r)}(0) = 0$, the induction hypothesis implies that $a_0 = ... = a_{n-r-1} = 0$. So $a_{n-r}\gamma^{(n-r)}(0) = 0$ whence $a_{n-r} = 0$.

Now we will introduce the notion of Veronese web on a n-dimensional vector space V with $n \geq 1$. A family $w = \{w(t) \mid t \in \mathbb{K}\}$ of $(n - r)$-planes is called *a*

Veronese web of codimension r if there exists a Veronese curve γ in $\Lambda^r V^*$ such that $w(t) = Ker\gamma(t)$, $t \in \mathbb{K}$. The curve γ will be named a *representative of* w.

If $\tilde{\gamma}$ is another representative of w then $\tilde{\gamma}(t) = f(t)\gamma(t)$ for any $t \in \mathbb{K}$. As γ and $\tilde{\gamma}$ are polynomial curves of degree $n-r$ and never lie into a $(n-r-1)$-plane of $\Lambda^r V^*$, f is constant and $\tilde{\gamma}(t) = a\gamma(t)$, $a \in \mathbb{K} - \{0\}$. This allows us to define $w(\infty) = Ker\gamma(\infty)$, which does not depend on the representative. Moreover if $\{\beta_{ij}\}$, $i = 1, ..., n_j$, $j = 1, ..., r$, is a basis of V^* such that $\gamma(t) = \gamma_1(t) \wedge ... \wedge \gamma_r(t)$ where each $\gamma_j(t) = \sum_{i=1}^{n_j} t^{i-1}\beta_{ij}$, then $w(\infty) = Ker(\beta_{n_1 1} \wedge ... \wedge \beta_{n_r r})$.

In view of lemma 1.2 and proposition 1.1 one has:

Proposition 1.2. *Consider on a n-dimensional vector space V and a natural number $1 \leq r \leq n$.*
(a) Given a r-codimensional vector subspace W and an endomorphism J both two of V, if (W', J^*) spans V^* where W' is the annihilator of W in V^* then $\gamma(t) = \varphi(t)((J+tI)^{-1})^*\beta$, where φ is the characteristic polynomial of $-J$ and β a r-form such that $Ker\beta = W$, represents a Veronese web w of codimension r.

Moreover $lim_{t\to\infty} t^{r-n}\gamma(t) = \beta$, $w(\infty) = W$ and $(J+tI)w(\infty) = w(t)$ for any $t \in \mathbb{K}$.
(b) Any Veronese web on V of codimension r may be represented in this way.
(c) Assume that $\gamma(t) = \varphi(t)((J+tI)^{-1})^*\beta$ and $\tilde{\gamma}(t) = \tilde{\varphi}(t)((\tilde{J}+tI)^{-1})^*\tilde{\beta}$ represent two Veronese webs w and $\tilde{w}$ respectively. Then $w = \tilde{w}$ if and only if $\tilde{\beta} = a\beta$, $a \in \mathbb{K} - \{0\}$, and $Ker(\tilde{J} - J) \supset w(\infty) = \tilde{w}(\infty)$.

In this last case $\tilde{\gamma} = \gamma$ if and only if $\tilde{\beta} = \beta$.
(d) Up to permutation the family of natural numbers $\{n_1, ..., n_r\}$, associated to a splitting of a representative of a Veronese web w, only depends on w. This family characterizes the Veronese web up to isomorphism.

By definition $n_1, ..., n_r$ will be called the the characteristic numbers of w *and their maximum* the height of w.

Remark. Often hereafter we will write $\lambda(G, ..., G)$ or $\lambda \circ G$ instead of $G^*\lambda$ when G is a morphism and λ a form.

On the other hand, note that (W', J^*) spans V^* if and only if W does not contain any non-zero J-invariant vector subspace.

By (c) of proposition 1.2 the restriction of J to $w(\infty)$ gives rise to a morphism $\ell : w(\infty) \to V$ with no ℓ-invariant vector subspace different from zero (this notion is meaningful since : $w(\infty) \subset V$) and which only depends on the Veronese web w. Moreover $(\ell + tI)w(\infty) = w(t)$, $t \in \mathbb{K}$, that is to say $\ell^*\alpha = -t\alpha_{|w(\infty)}$ for any $\alpha \in V^*$ such that $\alpha(w(t)) = 0$ and any $t \in \mathbb{K}$. This last property characterizes ℓ completely because the union of the annihilators of $w(t)$, $t \in \mathbb{K}$, spans V^*.

Conversely given a morphism $\ell : W \to V$ whose only ℓ-invariant vector subspace is zero, we may construct a Veronese web by considering an endomorphism J of V such that $J_{|W} = \ell$ and applying (a) of proposition 1.2 to it. This Veronese web only depends on ℓ. In fact $w(t) = (\ell + tI)W$. Thus:

Giving a Veronese web of codimension $r \geq 1$ is equivalent to giving a morphism $\ell : W \to V$, where W is a r-codimensional vector subspace, without non-zero ℓ-invariant vector subspaces.

Proposition 1.3. *Consider a Veronese web w of codimension $r \geq 1$, a basis $\{\alpha_1, ..., \alpha_n\}$ of V^* and scalars $a_1, ..., a_n$. Assume that $\alpha_j(w(-a_j)) = 0$, $j = 1, ..., n$. Then w can be constructed through (a) of proposition 1.2 by means of the endomorphism J defined by $J^*\alpha_j = a_j\alpha_j$, $j = 1, ..., n$.*

Proof. As $\ell^*\alpha_j = a_j\alpha_{j|W}$ then $\ell^* = (J_{|W})^*$, so J is an extension of ℓ. □

Lemma 1.3. *Consider a Veronese web w of codimension $r \geq 1$ on a n-dimensional vector space V and its characteristic numbers $n_1 \geq ... \geq n_r$. Let k_j be the number of n_ℓ greater than or equal to j. Then $r = k_1 \geq ... \geq k_{n_1} \geq 1$, $k_j = 0$ if $j > n_1$, and $k_1 + ... + k_{n_1} = n$. Moreover:*
(1) Given non-equal scalars $b_1, ..., b_{n-k}, b$, where $1 \leq k \leq r$, there exists a basis $\{\alpha_1, ..., \alpha_n\}$ of V^* such that $\alpha_j(w(b_j)) = 0$, $j = 1, ..., n-k$, $\alpha_j(w(b)) = 0$, $j = n-k+1, ..., n$.
(2) Given, this time, non-equal scalars $c_1, ..., c_{n_1}$ there exists a basis $\{\beta_{ij}\}$, $i = 1, ..., k_j$, $j = 1, ..., n_1$, of V^* such that $\beta_{ij}(w(c_j)) = 0$, $i = 1, ..., k_j$, $j = 1, ..., n_1$.

Proof. First consider a basis $\{e^*_{ij}\}$, $i = 1, ..., n_j$, $j = 1, ..., r$ and $n_1 \geq ... \geq n_r$, of V^* such that $\gamma(t) = \gamma_1(t) \wedge ... \wedge \gamma_r(t)$, where each $\gamma_j(t) = \sum_{i=1}^{n_j} t^{i-1}e^*_{ij}$, is a representative of w. Now if $\varphi : \{1, ..., n-k\} \to \{1, ..., r\}$ is a map such that $\varphi^{-1}(\ell)$ has $n_\ell - 1$ elements when $1 \leq \ell \leq k$ and n_ℓ otherwise, it suffices to set $\alpha_j = \gamma_{\varphi(j)}(b_j)$, $j = 1, .., n-k$, and $\alpha_j = \gamma_{j+k-n}(b)$, $j = n-k+1, .., n$, for proving (1).

With regard to (2) set $\beta_{ij} = \gamma_i(c_j)$, $i = 1, ..., k_j$, $j = 1, ..., n_1$. $\square$

1.2. Pairs of bivectors.

In this paragraph we will give the classification of pairs of bivectors, due to Gelfand and Zakharevich, by regarding them as quotients of symplectic pairs. Consider, on a finite dimensional vector space W, a pair of bivectors (Λ, Λ_1). One defines *the rank of* (Λ, Λ_1) as the maximum of ranks of $(1-t)\Lambda + t\Lambda_1$, $t \in \mathbb{K}$. Note that $rank((1-t)\Lambda + t\Lambda_1) = rank(\Lambda, \Lambda_1)$ except for a finite number of scalars t, which is $\leq \frac{dimW}{2}$ (they are given by the polynomial equation $((1-t)\Lambda + t\Lambda_1)^k = 0$ where $rank(\Lambda, \Lambda_1) = 2k$). We will say that (Λ, Λ_1) is *maximal (or of maximal rank)* if $rank(\Lambda) = rank(\Lambda_1) = rank(\Lambda, \Lambda_1)$. Obviously if (Λ, Λ_1) is not maximal one may choose $\Lambda' = (1-a)\Lambda + a\Lambda_1$, $\Lambda'_1 = (1-a_1)\Lambda + a_1\Lambda_1$, with $a \neq a_1$, which is maximal. Consequently it suffices classifying maximal pairs.

Recall that to any symplectic form ω defined on a vector space V of dimension $2n$ one can associate a dual bivector Λ_ω by means of the isomorphism $v \in V \to \omega(v, \quad) \in V^*$ (or $v \in V \to \omega(\quad, v) \in V^*$; the result is the same). Conversely any bivector whose rank equals $2n$ can be defined in this way. More generally when Λ is a bivector on W, considered as a bivector on $Im\Lambda = \Lambda(W^*, \quad)$ it is the dual of a symplectic form. Thus every bivector can be described by its image and a symplectic form on it; that is to say by the annihilator of $Im\Lambda$, or one of its basis, and a 2-form whose restriction to $Im\Lambda$ is symplectic.

Let V_0, $\pi : V \to \frac{V}{V_0}$ and Λ be a vector subspace of V, the canonical projection and the bivector on $\frac{V}{V_0}$ image of Λ_ω by π respectively.

Lemma 1.4. *Consider a second vector subspace V_1 such that $V = V_0 \oplus V_1$. Assume isotropic V_0. Let Λ' be the bivector on V_1 pull-back of Λ by the isomorphism $\pi : V_1 \to \frac{V}{V_0}$. Then Λ' is defined by $\omega(V_0, \quad)_{|V_1}$ and $\omega_{|V_1}$.*

Proof. Set $dimV_0 = n-k$. There exists a basis $\{e_1, ..., e_{2n}\}$ of V such that $\omega = \sum_{j=1}^n e^*_{2j-1} \wedge e^*_{2j-1}$ and $\{e_{2j-1}\}$, $j = k+1, ..., n$, is a basis of V_0. Then $\Lambda = \pi(e_1) \wedge \pi(e_2) + ... + \pi(e_{2k-1}) \wedge \pi(e_{2k})$.

On the other hand, as $V = V_0 \oplus V_1$ there exists a basis $\mathcal{B} = \{e_1 + v_1, ..., e_{2k} + v_{2k}, \{e_{2j} + v_{2j}\}_{j=k+1,...,n}\}$ of V_1 where every $v_i \in V_0$. Obviously $\Lambda' = (e_1 + v_1) \wedge (e_2 + v_2) + ... + (e_{2k-1} + v_{2k-1}) \wedge (e_{2k} + v_{2k})$.

The restriction to V_1 of the family $\{e_j^*\}$, $j = 1, ..., 2k$ and $j = 2(k+1), ..., 2n$, is the dual basis of $\mathcal{B}$. So Λ' will be defined by the restriction to V_1 of the 2-form $e_1^* \wedge e_2^* + ... + e_{2k-1}^* \wedge e_{2k}^*$, which equals that of ω, and by the basis $\{e_{2j|V_1}^* = \omega(e_{2j-1}, \quad)_{|V_1}\}$, $j = k+1, ..., n$ of the annihilator of $Im\Lambda'$. $\square$

Warning lemma 1.4 can fail if V_0 is not isotropic. For example on $\mathbb{K}^4$: $\omega = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*$, $V_0 = \mathbb{K}\{e_3, e_4\}$ and $V_1 = \mathbb{K}\{e_1 + e_3, e_2 + e_4\}$.

Remark. On a finite dimensional vector space E consider a symplectic form Ω and a 2-form Ω_1. Let K be the endomorphism defined by $\Omega_1 = \Omega(K, \quad)$, that is to say $\Omega_1(v, w) = \Omega(Kv, w)$, $v, w \in E$. Then $\Omega(K, \quad) = \Omega(\quad, K)$; thus every $\Omega(K^k, \quad)$ is a 2-form on E. By definition the characteristic polynomial, the minimal one and the elementary divisors of (Ω, Ω_1) will be those of K.

Suppose that the characteristic polynomial of (Ω, Ω_1) is the product p_1p_2 of two monic relatively prime polynomials. Then (Ω, Ω_1, E) can be identified to the product of two similar structures $(\Omega^1, \Omega_1^1, E_1) \times (\Omega^2, \Omega_1^2, E_2)$ where p_i is the characteristic polynomial of (Ω^i, Ω_1^i), $i = 1, 2$. In this way classifying (Ω, Ω_1) reduces to the case where the characteristic polynomial is a power of an irreducible polynomial. It is not difficult to see that the model of (Ω, Ω_1) is completely determined by the Jordan structure of K. Moreover every elementary divisor occurs an even number of times, so p is the square of another polynomial, and the minimal polynomial divides the square root of p.

Let us come back to the main question. Consider a second symplectic form ω_1 on V, the dual bivector Λ_{ω_1} and its image Λ_1 by π on $\frac{V}{V_0}$. Let J be the endomorphism (in fact the automorphism) of V defined by $\omega_1 = \omega(J, \quad)$.

Lemma 1.5. *Assume that V_0 is isotropic for both ω and ω_1 and (Λ, Λ_1) is maximal. Then the vector subspace spanned by (V_0, J) is ω and ω_1 isotropic.*

Proof. First note that $rank\Lambda = rank\Lambda_1 = 2r$, where $dimV_0 = n - r$, because V_0 is bi-isotropic. On the other hand if $rank(\Lambda_\omega + t\Lambda_{\omega_1}) = 2n$ then $\omega((I + tJ^{-1})^{-1}, \quad)$ is its dual symplectic form (recall that if $\Omega_1 = \Omega(K, \quad)$ then $\Lambda_1 = \Lambda((K^{-1})^*, \quad)$ when Λ and Λ_1 are regarded as 2- forms on the dual space).

Since $rank(\Lambda+t\Lambda_1) \leq 2r$ this implies that V_0 is isotropic for $\omega((I+tJ^{-1})^{-1}, \quad)$, so $\omega((I+tJ^{-1})^{-1}v, w) = 0$ for any $v, w \in V_0$.

Near $0 \in \mathbb{K}$ one has $rank(\Lambda_\omega + t\Lambda_{\omega_1}) = 2n$ so deriving at $t = 0$ successively yields, up multiplicative constant, $\omega(J^{-k}v, w) = 0$, $k \geq 0$. Hence $\omega(J^{-\ell}v, J^{-s}w) = 0$ for any $\ell, s \geq 0$ as $\omega(J, \quad) = \omega(\quad, J)$. This implies that the vector subspace spanned by (V_0, J^{-1}) is ω-isotropic; but this last one equals the vector subspace spanned by (V_0, J) since J is invertible.

Finally as our vector subspace is J-invariant it has to be ω_1-isotropic. $\square$

For the remainder of this paragraph (Λ, Λ_1) will be a maximal pair of bivectors defined on m-dimensional vector space W. Set $r = corank(\Lambda, \Lambda_1)$. Assume that Λ is defined by $\alpha_1, ..., \alpha_r, \tilde{\omega}$, and Λ_1 by $\beta_1, ..., \beta_r, \tilde{\omega}_1$, where $\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_r \in W^*$ and $\tilde{\omega}, \tilde{\omega}_1 \in \Lambda^2 W^*$.

Let V_0 be a vector space of dimension r and $\{e_1, ..., e_r\}$ one of its basis. Let $\{e_1^*, ..., e_r^*\}$ denote the extension of the dual basis of $\{e_1, ..., e_r\}$ to $V = W \oplus V_0$ by setting $e_i^*(W) = 0$, $i = 1, ..., r$. On the other hand we will regard $\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_r, \tilde{\omega}, \tilde{\omega}_1$ as forms on V such that $\alpha_i(V_0) = \beta_i(V_0) = 0$, $i = 1, ..., r$, $\tilde{\omega}(V_0, \quad) = \tilde{\omega}_1(V_0, \quad) = 0$. Now on V one considers the symplectic forms $\omega = \tilde{\omega} + \alpha_1 \wedge e_1^* + ... + \alpha_r \wedge e_r^*$ and $\omega_1 = \tilde{\omega}_1 + \beta_1 \wedge e_1^* + ... + \beta_r \wedge e_r^*$. If we identify W to $\frac{V}{V_0}$ by means of the canonical projection, by lemma 1.4 the pair (Λ, Λ_1) is just the image of the dual pair $(\Lambda_\omega, \Lambda_{\omega_1})$. Thus any maximal pair is the quotient of a symplectic pair by a bi-isotropic vector subspace.

By technical reasons we will deform ω_1 for simplifying the algebraic structure of the symplectic pair. Set $\omega_\mu = \omega_1 + \beta_1 \wedge \mu_1 + ... + \beta_r \wedge \mu_r = \tilde{\omega}_1 + \beta_1 \wedge (e_1^* + \mu_1) + ... + \beta_r \wedge (e_r^* + \mu_r)$ where $\mu_1, ..., \mu_r \in V^*$, which is symplectic if and only if $\{(e_1^* + \mu_1)_{|V_0}, ..., (e_r^* + \mu_r)_{|V_0}\}$ is still a basis of V_0^*. In this last case V_0 is ω_μ isotropic and the dual bivector Λ_μ projects into Λ_1 as well (apply lemma 1.4 again). Let J and J_μ be the endomorphisms defined by $\omega_1 = \omega(J, \quad)$ and $\omega_\mu = \omega(J_\mu, \quad)$ respectively, and let $\bar{e}_j$ be the vector defined by $\omega(\bar{e}_j, \quad) = \mu_j$, $j = 1, ..., r$. Then $J_\mu = J + \sum_{j=1}^r (\bar{e}_j \otimes \beta_j + Je_j \otimes \mu_j)$.

Therefore, since $J_{\mu|V_0} = \sum_{j=1}^r Je_j \otimes (e_j^* + \mu_j)_{|V_0}$, the form ω_μ is symplectic, that is to say J_μ is an isomorphism, if and only if $J_{\mu|V_0}$ is a monomorphism.

Let V_1 denote the vector subspace spanned by (V_0, J), and V_2 the ω-orthogonal of V_1. As $JV_1 = V_1$ the vector subspace V_2 is the ω_1-orthogonal of V_1 too. From lemma 1.5 follows that V_1 is isotropic for ω and ω_1; thus $V_1 \subset V_2$ and $\beta_j(V_2) = 0$, $j = 1, ..., r$, since $\beta_j(V_2) = -\omega_1(e_j, V_2)$. Hence $J_\mu = J + \sum_{j=1}^{r} Je_j \otimes \mu_j$ on V_2 and $(J_\mu - J)V_2 \subset JV_0$.

Hereafter assume ω_μ symplectic. Then $J_\mu V_0 = JV_0$. This implies that (V_0, J_μ) spans V_1 as well. Again lemma 1.5, this time applied to ω, ω_μ, shows that V_1 is ω_μ-isotropic; moreover V_2 is the ω_μ-orthogonal of V_1 because $J_\mu V_1 = V_1$. Obviously $JV_2 = J_\mu V_2 = V_2$ since $JV_1 = J_\mu V_1 = V_1$ and V_2 is the orthogonal of V_1 for ω, ω_1 and ω_μ.

The restricted forms $\omega_{|V_2}$ and $\omega_{1|V_2} = \omega_{\mu|V_2}$ (recall that $\beta_j(V_2) = 0$ so $(\omega_\mu - \omega_1)_{|V_2} = 0$) project into a pair $(\bar{\omega}, \bar{\omega}_1)$ of symplectic forms on $\frac{V_2}{V_1}$. As $\omega_1 = \omega(J, \quad)$ and $\omega_\mu = \omega(J_\mu, \quad)$, the endomorphism $\bar{J}$ of $\frac{V_2}{V_1}$ defined by $\bar{\omega}_1 = \bar{\omega}(\bar{J}, \quad)$ is just the projection of both $J_{|V_2}$ and $J_{\mu|V_2}$.

The next step will be to control the characteristic polynomial of J_μ, which is the product of three characteristic polynomials: that of the projection of J_μ on $\frac{V}{V_2}$, that of $J_{\mu|V_1}$ and that of the projection of J_μ on $\frac{V_2}{V_1}$. This last one is the characteristic polynomial of $\bar{J}$, therefore it does not depend on μ; it will denote by $\psi(t)$.

As J is an isomorphism V_1 is also the vector subspace spanned by (JV_0, J). Now from lemma 1.1 applied to V_1 and (JV_0, J) follows the existence of a nilpotent $H \in End(V_1)$, such that $Im(H - J_{|V_1}) \subset JV_0$, and a basis $\{d_1, ..., d_r\}$ of JV_0 such that $V_1 = \oplus_{j=1}^{r} U_j$ where each U_j is the vector subspace spanned by (d_j, H). Set $G = H + \sum_{j=1}^{r} d_j \otimes \lambda_j$ where $\lambda_1, ..., \lambda_r \in V_1^*$ and $\lambda_j(U_i) = 0$ if $i \neq j$. Then we may choose $\lambda_1, ..., \lambda_r$ in such a way that (d_j, G) spans U_j, $j = 1, ..., r$, $Im(G - J_{|V_1}) \subset JV_0$ and the characteristic polynomial of $G_{|U_j}$ is any monic polynomial whose degree equals the dimension of U_j. Moreover if G is invertible, so a monomorphism, there exist ω_μ and J_μ such that $J_{\mu|V_1} = G$ since $J_\mu = J + \sum_{j=1}^{r} Je_j \otimes \mu_j$ on V_2.

Consider non-equal and non-zero scalars $a_1, ..., a_k$, where $k = dim V_1$, which are not roots of $\psi(t)$. Then we can suppose, without loss of generality, that (d_j, J_μ) spans U_j and the characteristic polynomial of $J_{\mu|U_j}$ equals $\prod_{i \in I_j}(t - a_i)$ where $\{1, ..., k\}$ is the disjoint union of I_1,..., I_r. Thus the characteristic polynomial $\psi_\mu(t)$ of J_μ equals $\psi(t)\rho(t)\prod_{i=1}^{k}(t - a_i)$ where $\rho(t)$ is the characteristic polynomial of the projection of J_μ on $\frac{V}{V_2}$. But $\psi_\mu(t)$ has to be a square and $a_1, ..., a_k$

are not roots of $\psi(t)$, so $\psi_\mu(t) = \psi(t)\prod_{i=1}^{k}(t-a_i)^2 = \psi(t)\prod_{j=1}^{r}(\prod_{i\in I_j}(t-a_i)^2)$.

Now we may identify (ω, ω_μ, V) to a product $\prod_{j=0}^{r}(\tau_j, \tau_j', L_j)$ in such a way that $\psi(t)$ is the characteristic polynomial of $J_{\mu|L_0}$ and $\prod_{i\in I_j}(t-a_i)^2$ that of $J_{\mu|L_j}$, $j = 1, ..., r$. Then $V_0 \cap L_0 = \{0\}$, $dim(V_0 \cap L_j) = 1$, $j = 1, ..., r$, and $V_0 = \oplus_{j=1}^{r}(V_0 \cap L_j)$; indeed $J_\mu^{-1}d_j$ is a basis of $V_0 \cap U_j$, since $J_\mu V_0 = JV_0$, and $(J_\mu^{-1}d_j, J_\mu)$ spans U_j. Remark that $\prod_{i\in I_j}(t-a_i)$ is the minimal polynomial of any $v \in V_0 \cap L_j - \{0\}$, $j = 1, ..., r$. Moreover (Λ, Λ_1) is identified, in a natural way, to the product of the dual pair $(\Lambda_{\tau_0}, \Lambda_{\tau_0'})$, *called symplectic*, times the projections of the dual pairs $(\Lambda_{\tau_j}, \Lambda_{\tau_j'})$ on $\frac{L_j}{V_0\cap L_j}$, $j = 1, ..., r$, which will be called the *Kronecker elementary pairs*. The case without symplectic factor and that with no Kronecker elementary factor happen.

Let us describe the Kronecker elementary pair in dimension $2n-1$. Consider, on a $2n$-dimensional vector space E, a pair of symplectic forms (Ω, Ω_1) and the endomorphism K defined by $\Omega_1 = \Omega(K, \quad)$. Suppose that $\prod_{i=1}^{n}(t-b_i)^2$ is the characteristic polynomial of (Ω, Ω_1), where all $b_i \neq 0$ and $b_i \neq b_j$ if $i \neq j$. Let E_0 be a 1- dimensional vector subspace of E such that the minimal polynomial of its non-zero elements is $\prod_{i=1}^{n}(t-b_i)$. Then there exists a basis $\{e_1, ..., e_{2n}\}$ of E such that $\Omega = e_1^* \wedge e_2^* + ... + e_{2n-1}^* \wedge e_{2n}^*$, $\Omega_1 = b_1 e_1^* \wedge e_2^* + ... + b_n e_{2n-1}^* \wedge e_{2n}^*$ and $e = -\sum_{j=1}^{n} e_{2j}$ is a basis of E_0.

Denote by E_1, $\tilde{\Lambda}$ and $\tilde{\Lambda}_1$ the vector subspace of basis $\{e_1, ..., e_{2n-1}\}$, and the images of Λ_Ω and Λ_{Ω_1} on $\frac{E}{E_0}$ respectively. As $E = E_0 \oplus E_1$ by lemma 1.4 the bivector $\tilde{\Lambda}$, considered on E_1 identified to $\frac{E}{E_0}$ in the natural way, is given by $\tilde{\omega} = \sum_{j=1}^{n-1} e_{2j-1}^* \wedge e_{2j}^*$, $\alpha = \sum_{j=1}^{n} e_{2j-1}^*$ (obviously both of them restricted to E_1) while $\tilde{\Lambda}_1$ is described by $\tilde{\omega}_1 = \sum_{j=1}^{n-1} b_j e_{2j-1}^* \wedge e_{2j}^*$, $\beta = \sum_{j=1}^{n} b_j e_{2j-1}^*$. Moreover, since $\Lambda_\Omega + t\Lambda_{\Omega_1}$ is the dual bivector of $\Omega((I + tK^{-1})^{-1}, \quad)$ when $t \in \mathbb{K} - \{-b_1, ..., -b_n\}$, the bivector $\tilde{\Lambda} + t\tilde{\Lambda}_1$ is given by $\mu_t = \sum_{j=1}^{n-1} b_j(t+b_j)^{-1} e_{2j-1}^* \wedge e_{2j}^*$ and $\alpha_t = (\prod_{j=1}^{n}(t+b_j))\Omega((I+tK^{-1})^{-1}e, \quad) = \sum_{j=1}^{n} b_j(\prod_{i=1; i\neq j}^{n}(t+b_i))e_{2j-1}^*$.

But μ_{-b_n}, α_{-b_n} still define a bivector on E_1, which by continuity has to be equal to $\tilde{\Lambda} - b_n\tilde{\Lambda}_1$. Thus $corank(\tilde{\Lambda} + t\tilde{\Lambda}_1) = 1$ for any $t \in \mathbb{K} - \{-b_1, ..., -b_{n-1}\}$. Reasoning in the same way but with other suitable direct summands of E_0 (for example for $-b_1$ the vector subspace spanned by $\{e_2, ..., e_{2n}\}$) finally shows that $corank(\tilde{\Lambda} + t\tilde{\Lambda}_1) = 1$, $t \in \mathbb{K}$. Hence $Im(\tilde{\Lambda} + t\tilde{\Lambda}_1) = Ker\alpha_t$, $t \in \mathbb{K}$.

Therefore $E' = \cap_{t\in\mathbb{K}} Im(\tilde{\Lambda} + t\tilde{\Lambda}_1)$ is the $(n-1)$-dimensional vector subspace of basis $\{e_{2j}\}$, $j = 1, ..., n-1$, and setting $w(t) = \frac{Im(\tilde{\Lambda}+t\tilde{\Lambda}_1)}{E'}$ defines a Veronese web w of codimension one on $\frac{E_1}{E'}$. Indeed, identify $\frac{E_1}{E'}$ to the vector subspace E'' spanned by $\{e_{2j-1}\}$, $j = 1, ..., n$, and restrict α_t to it (proposition 1.2 applied to $K_{|E''}$ and $(\sum_{j=1}^{n} b_j e^*_{2j-1})_{|E''}$ just yields $\alpha_{t|E''}$).

On the other hand $(\tilde{\Lambda}, \tilde{\Lambda}_1)$ is a particular case of (Λ, Λ_1) with $r = 1$. So $(\tilde{\Lambda}, \tilde{\Lambda}_1)$ is isomorphic to a product of a possible symplectic pair in dimension $2(n-k)$ and a Kronecker elementary pair associated to scalars $a_1, ..., a_k$. As $dim(Im(\tilde{\Lambda} + t\tilde{\Lambda}_1)) = 2n-2$, $t \in \mathbb{K}$, the characteristic polynomial of the symplectic factor has no roots and in this case an elementary calculation yields $dimE' = 2n-k-1$. But $dimE' = n-1$ so $k = n$; that is to say there is no symplectic factor. In other words our pair can be constructed from any family of non-zero scalars $\{a_1, ..., a_n\}$ such that $a_i \neq a_j$ if $i \neq j$, which shows that *the Kronecker elementary pair* $(\tilde{\Lambda}, \tilde{\Lambda}_1)$ *only depends on the dimension* $2n-1$ *but not on* $\{b_1, ..., b_n\}$. *Thus, up to isomorphism, in every odd dimension there exists just one Kronecker elementary pair.*

Now we may state:

Proposition 1.4. *Consider a maximal pair of bivectors* (Λ, Λ_1) *on a finite dimensional vector space* W. *Set* $r = corank(\Lambda, \Lambda_1)$. *Let* L_0 *be the intersection of all the vector subspaces* $Im(\Lambda + t\Lambda_1)$, $t \in \mathbb{K}$, *of codimension* r. *Denote by* L'_0 *its annihilator in* W^*. *One has:*

(a) $L_0 \subset Im\Lambda_1$ *and* $\Lambda(L'_0, \quad) = \Lambda_1(L'_0, \quad)$.

In what follows set $L_1 = \Lambda(L'_0, \quad)$.

(b) The restrictions to L_0 *of the* 2*-forms associated to* Λ *and* Λ_1 *respectively, which are unique since* $L_0 \subset Im\Lambda \cap Im\Lambda_1$, *have* L_1 *as kernel.*

Therefore the projections on $\frac{L_0}{L_1}$ *of these restricted* 2*-forms,denoted by* $\bar{\omega}$ *and* $\bar{\omega}_1$ *respectively, are symplectic.*

(c) Setting $w(t) = \frac{Im(\Lambda+t\Lambda_1)}{L_0}$, $t \in \mathbb{K}$, *defines a Veronese web on* $\frac{W}{L_0}$.

(d) The elementary divisors of $(\bar{\omega}, \bar{\omega}_1)$ *and the characteristic numbers* $n_1 \geq ... \geq n_r$ *of* w *determine the algebraic structure of* (Λ, Λ_1) *completely. More precisely* (Λ, Λ_1, W) *is isomorphic to a product* $\prod_{\ell=0}^{r}(\Lambda^\ell, \Lambda_1^\ell, W^\ell)$ *where* $(\Lambda^0, \Lambda_1^0, W^0)$ *is isomorphic, in its turn, to the dual pair of* $(\bar{\omega}, \bar{\omega}_1, \frac{L_0}{L_1})$ *and every* $(\Lambda^\ell, \Lambda_1^\ell, W^\ell)$, $\ell = 1, ..., r$, *is the Kronecker elementary pair in dimension* $2n_\ell - 1$.

(e) $corank(\Lambda + a\Lambda_1) > r$ if and only if $-a$ is a root of the characteristic polynomial of $(\bar{\omega}, \bar{\omega}_1)$.

Remark. Let (ω, ω_1) be a pair of symplectic forms on a $2n$-dimensional vector space V and let V_0 be a line in V. Denote by V_1 the vector subspace spanned by (V_0, J) where $\omega_1 = \omega(J, \quad)$. Then the dimension of the symplectic factor, given by proposition 1.4, of the pair (Λ, Λ_1) induced by $(\Lambda_\omega, \Lambda_{\omega_1})$ on $\frac{V}{V_0}$ equals $2(n - dimV_1)$.

Indeed, first note that $rank\Lambda = rank\Lambda_1 = rank(\Lambda, \Lambda_1) = 2n-2$ so (Λ, Λ_1) is maximal. Let e and W be a basis and a direct summand of V_0 respectively. Then $\omega = \tilde{\omega} + \alpha \wedge e^*$, $\omega_1 = \tilde{\omega}_1 + \beta \wedge e^*$ where $Ker\alpha$, $Ker\beta$, $Ker\tilde{\omega}$ and $Ker\tilde{\omega}_1$ contain V_0, $Kere^* = W$ and $e^*(e) = 1$. Therefore, after identifying W and $\frac{V}{V_0}$, bivectors Λ and Λ_1 are given by $\tilde{\omega}, \alpha$ and $\tilde{\omega}_1, \beta$ respectively. As $V = W \oplus V_0$ we are just in the situation which allowed us splitting any maximal pair. There it was showed that the dimension of the symplectic factor equals that of $\frac{V_2}{V_1}$, where V_2 was the orthogonal of V_1; in our case $2n - 2dimV_1$.

Proposition 1.5. *Let W be a $(2n-1)$-dimensional vector space. The action of the linear group $GL(W)$ on $(\Lambda^2 W) \times (\Lambda^2 W)$ possesses one dense open orbit whose model is the elementary Kronecker pair in dimension $2n-1$.*

Proof. First let us show that any pair (Λ, Λ_1) is approachable in $(\Lambda^2 W) \times (\Lambda^2 W)$ by a Kronecker elementary one. As bivectors of rank $2n-2$ are generic in $\Lambda^2 W$ one can suppose $rank\Lambda = rank\Lambda_1 = 2n-2$. Now assume that the symplectic factor given by proposition 1.4 applied to (Λ, Λ_1) has dimension $2k \geq 2$ and minimal polynomial φ. Note that there is only one Kronecker elementary factor since $corank(\Lambda, \Lambda_1) = 1$. By constructing this Kronecker elementary factor with scalars $\{a_1, ..., a_{n-k}\}$ which are not roots of φ, the pair (Λ, Λ_1) becomes the quotient by a line V_0 of a dual symplectic pair $(\Lambda_\omega, \Lambda_{\omega_1})$ defined on a $2n$-dimensional vector space V, in such a way that the minimal polynomial of (ω, ω_1) is $\varphi \prod_{j=1}^{n-k}(t - a_j)$ and $\prod_{j=1}^{n-k}(t - a_j)$ that of each $e \in V_0 - \{0\}$. In particular (e, J), where $\omega_1 = \omega(J, \quad)$, spans a $(n-k)$-dimensional vector subspace.

Set $V = W \oplus V_0$. By lemma 1.4 (Λ, Λ_1), regarded on W, is given by $\omega_{|W}$, $\omega(e, \quad)_{|W}$, $\omega_{1|W}$ and $\omega_1(e, \quad)_{|W}$. Now consider a vector e' near e whose minimal polynomial is $\varphi\prod_{j=1}^{n-k}(t-a_j)$. Then (e',J) spans a vector subspace of V of dimension $> n-k$ and the symplectic factor of the quotient of $(\Lambda_\omega,\Lambda_{\omega_1})$ by $\mathbb{K}\{e'\}$ has dimension $< 2k$ (see the foregoing remark). Since $V=W\oplus\mathbb{K}\{e'\}$ this last pair is given on W by $\omega_{|W}$, $\omega(e', \quad)_{|W}$, $\omega_{1|W}$ and $\omega_1(e', \quad)_{|W}$; therefore we can choose it as close to (Λ,Λ_1) as desired and, after a finite number of steps, (Λ,Λ_1) will be approached by a Kronecker pair.

On the other hand if (Λ',Λ'_1) is a Kronecker elementary pair, consider scalars $\{a_1,...,a_n\}$ all of them different. Then $dim(Im(\Lambda'+a_j\Lambda'_1))=2n-2$, $j=1,...,n$, and $dim(\cap_{j=1}^n Im(\Lambda'+a_j\Lambda'_1))=n-1$. Therefore when (Λ,Λ_1) is close enough to (Λ',Λ'_1) one has $dim(Im(\Lambda+a_j\Lambda_1))=2n-2$, $j=1,...,n$, and $dim(\cap_{j=1}^n Im(\Lambda+a_j\Lambda_1))=n-1$. But by (d) of proposition 1.4 this last dimension equals $n-k-1$ where $2k$ is the dimension of the symplectic factor of (Λ,Λ_1); so $k=0$ and (Λ,Λ_1) is Kronecker elementary too. $\square$

2. Veronese webs on manifolds

This section contains the basic theory of Veronese webs of any codimension. The notion of Veronese web of codimension one was introduced by Gelfand and Zakharevich for studying the generic bihamiltonian structures on odd dimensional manifolds [3, 4, 5]. Later on Panasyuk and Turiel dealt with the case of higher codimension [9], [17]; see [18] as well. The approach given from now on, different from that of Gelfand, Zakharevich and Panasyuk, follows the Turiel's work [15, 16, 17].

Hereafter all structures considered will be real C^∞ or complex holomorphic unless another thing is stated.

Let N be a real or complex manifold of dimension n. A family $w=\{w(t)\mid t\in\mathbb{K}\}$ of involutive distributions (or foliations) on N of codimension $r\geq 1$ is named a *Veronese web of codimension* r, if for any $p\in N$ there exist an open neighborhood A of this point and a curve $\gamma(t)$ in the module of sections of $\Lambda^r T^*A$ (that is to say $\gamma(t)(q)\in\Lambda^r T_q^*A=\Lambda^r T_q^*N$ for every $q\in A$) such that:
1) $w(t)=Ker\gamma(t)$, $t\in\mathbb{K}$, on A
2) for each $q\in A$, $\gamma(t)(q)$ is a Veronese curve in $\Lambda^r T^*N$.

The curve γ is called a *(local) representative of* w.

Although curves $\gamma(t)(q)$ and $\gamma(t)(q')$ could be not isomorphic when $q\neq q'$, $\gamma(t)=\sum_{i=0}^{n-r}t^i\gamma_i$ where $\gamma_0,...\gamma_{n-r}$ are differentiable r-forms on A. On the other hand $Ker\gamma_{n-r}$ is an involutive distribution of dimension $n-r$ since each $Ker\gamma(t)$ was integrable and $lim t^{r-n}\gamma(t) = \gamma_{n-r}$, $t \to \infty$. This allows us to define $w(\infty) = Ker\gamma_{n-r}$, which does not depend on the representative because if $\tilde{\gamma}$ is another representative then $\tilde{\gamma} = f\gamma$ on the common domain (see (c) of proposition 1.2). In particular, there exists a global representative if and only if $w(\infty)$ is transversally orientable. Obviously w as map from $\mathbb{K} \cup \{\infty\} \equiv \mathbb{K}P^1$ to the Grassmann manifold of $(n-r)$-plans of TN is smooth.

Examples. 1) On S^3 regarded as a Lie group consider three left invariant contact forms ρ_1, ρ_2, ρ_3. Suppose that $\rho_1 \wedge \rho_2 \wedge \rho_3 \neq 0$ and set $\gamma(t) = (\rho_1 + t\rho_2) \wedge \rho_3$. Then γ defines a codimension two Veronese web which is not flat because $Ker\rho_3 = w(0) \oplus w(\infty)$ is a contact structure.
2) On $\mathbb{K}^4$ with coordinates (x_1, x_2, y_1, y_2) set $\gamma(t) = (dx_2 \wedge dy_2 + x_2 dx_1 \wedge dx_2) + t(x_2 dx_2 \wedge dy_1 - dx_1 \wedge dy_2) + t^2 dy_1 \wedge dy_2$. Then γ defines a Veronese web of codimension two since $d\gamma(t) = 0$ and $\gamma(t) = (-dx_1 + x_2^{-1} dy_2 + t dy_1) \wedge (-x_2 dx_2 + t dy_2)$ when $x_2 \neq 0$, while $\gamma(t) = (dx_2 - t dx_1 + t^2 dy_1) \wedge dy_2$ if $x_2 = 0$.

Note that $\gamma(t)(q)$ and $\gamma(t)(q')$ are not isomorphic as Veronese curves when $q_2 \neq 0$ and $q'_2 = 0$.
3) Let V be the 3-dimensional Lie algebra spanned by the vectors fields on $\mathbb{K}$: $X_1 = (\partial/\partial t)$, $X_2 = t(\partial/\partial t)$ and $X_3 = t^2(\partial/\partial t)$. Set $\tilde{w}(t) = \{v \in V \mid v(t) = 0\}$. As $\tilde{w}(t) = Ker\{e_1^* + te_2^* + t^2 e_3^*\}$ where $\{e_1^*, e_2^*, e_3^*\}$ is the dual basis of $\{X_1, X_2, X_3\}$, $\tilde{w} = \{\tilde{w}(t) \mid t \in \mathbb{K}\}$ is an algebraic Veronese web on V. But V is isomorphic to the Lie algebra of $SL(2, \mathbb{K})$ and each $\tilde{w}(t)$ is a subalgebra of V; therefore $\tilde{w}$ gives rise to a Veronese web w of codimension one on any 3-dimensional homogeneous space of $SL(2, \mathbb{K})$.

Now we shall give a local description of Veronese webs of codimension r by means of a $(1,1)$ tensor field and a r-form. Consider non-equal scalars $\{a_1, ..., a_{n-k}, a\}$, where $1 \leq k \leq r$, and any point $p \in N$. By lemma 1.3 there exists a basis $\{\lambda_1, ..., \lambda_n\}$ of T_p^*N such that $Ker\lambda_j \supset w(-a_j)(p)$, $j = 1, ..., n-k$, and $Ker\lambda_j \supset w(-a)(p)$, $j = n-k+1, ..., n$. Since every distribution $w(t)$ is involutive, on some open neighbourhood A of p one may construct closed 1-forms $\beta_1, ..., \beta_n$, extensions of $\lambda_1, ..., \lambda_n$, such that $Ker\beta_j \supset w(-a_j)$, $j = 1, ..., n-k$, $Ker\beta_j \supset w(-a)$, $j = n-k+1, ..., n$, and $\beta_1 \wedge ... \wedge \beta_n$ is a volume form.

Let J be the $(1,1)$ tensor field on A defined by $\beta_j \circ J = a_j \beta_j$, $j = 1, ..., n-k$, and $\beta_j \circ J = a\beta_j$, $j = n-k+1, ..., n$. In coordinates $(x_1, ..., x_n)$ such that $\beta_j = dx_j$, $j = 1, ..., n$, one has $J = \sum_{j=1}^{n-k} a_j \frac{\partial}{\partial x_j} \otimes dx_j + a \sum_{j=n-k+1}^{n} \frac{\partial}{\partial x_j} \otimes dx_j$, so J is flat and diagonalizable.

Moreover, by propositions 1.2 and 1.3, if β is a r-form on A such that $Ker\beta = w(\infty)$ then $\gamma(t) = (\prod_{j=1}^{n-k}(t+a_j))(t+a)^k((J+tI)^{-1})^*\beta$ is a representative of w.

On the other hand if n_1 is the height of $w(p)$ and $k_1 \geq ... \geq k_{n_1}$ are like in lemma 1.3, given non-equal scalars $\tilde{a}_1, ..., \tilde{a}_{n_1}$ a similar argument allows to construct closed 1-forms $\{\tilde{\beta}_{ij}\}$, $i = 1, ..., k_j$, $j = 1, ..., n_1$, linearly independent everywhere and such that $\tilde{\beta}_{ij}(w(-\tilde{a}_j)) = 0$, $i = 1, ..., k_j$, $j = 1, ..., n_1$. Then, by propositions 1.2 and 1.3, $\gamma(t) = (\prod_{j=1}^{n_1}(t+\tilde{a}_j)^{k_j})((\tilde{J}+tI)^{-1})^*\beta$ where $\tilde{J}$ is defined by $\tilde{\beta}_{ij} \circ \tilde{J} = \tilde{a}_j\tilde{\beta}_{ij}$, $i = 1, ..., k_j$, $j = 1, ..., n_1$.

Theorem 2.1. *Let N be a n-dimensional real or complex manifold.*
(1) Consider a Veronese web w on N of codimension r and non-equal scalars $a_1, ..., a_{n-k}, a$ where $1 \leq k \leq r$. Then for each $p \in N$ there exist an open set $p \in A$ and a $(1,1)$-tensor field J on A whit characteristic polynomial $\varphi(t) = (\prod_{j=1}^{n-k}(t-a_j))(t-a)^k$, which is flat and diagonalizable, such that:
(I) $(Ker(J^*-a_jI))w(-a_j) = 0$, $j = 1, ..., n-k$, and $(Ker(J^*-aI))w(-a) = 0$.
(II) For any $q \in A$, $(w(\infty)(q)', J^*(q))$ spans T_q^*A, that is to say $w(\infty)(q)$ contains no J-invariant vector subspace different from zero (as before $'$ means the annihilator).

In particular, if β is a r-form and $Ker\beta = w(\infty)$ then $\gamma(t) = (\prod_{j=1}^{n-k}(t+a_j))(t+a)^k((J+tI)^{-1})^*\beta$ represents w.

Moreover is λ is a closed 1-form such that $Ker\lambda \supset w(\infty)$ then $d(\lambda \circ J)_{|w(\infty)} = 0$.
(2) Now consider non-equal scalars $\tilde{a}_1, ..., \tilde{a}_{n_1}$ instead of $a_1, ..., a_{n-k}, a$, where n_1 is the height of $w(p)$, and numbers $k_1 \geq ... \geq k_{n_1}$ like in lemma 1.3. Then there exists a $(1,1)$-tensor field $\tilde{J}$ defined on an open neighbourhood $\tilde{A}$, which is flat and diagonalizable, with characteristic polynomial $\tilde{\varphi} = \prod_{j=1}^{n_1}(t-\tilde{a}_j)^{k_j}$ such that $(Ker(\tilde{J}^*-\tilde{a}_jI))w(-\tilde{a}_j) = 0$, $j = 1, ..., n_1$, $(w(\infty)', \tilde{J}^*)$ spans $T\tilde{A}^*$, $\tilde{\gamma}(t) = \prod_{j=1}^{n_1}(t+\tilde{a}_j)^{k_j}((\tilde{J}+tI)^{-1})^*\beta$ represents w and $\gamma = \tilde{\gamma}$ on $A \cap \tilde{A}$.

Moreover $d(\lambda \circ \tilde{J})_{|w(\infty)} = 0$ for any 1-closed form λ such that $Ker\lambda \supset w(\infty)$.
(3) Finally, on N consider a foliation $\mathcal{F}$ of codimension $r \geq 1$, a r-form $\bar{\beta}$ such

that $Ker\bar{\beta} = \mathcal{F}$ and $(1,1)$-tensor field $\bar{J}$ with characteristic polynomial $\bar{\varphi}(t)$. Suppose that:

(I) $(\mathcal{F}', \bar{J}^*)$ spans T^*N, that is to say $\mathcal{F}$ does not contain any non-zero $\bar{J}$-invariant vector subspace.

(II) $(N_{\bar{J}})_{|\mathcal{F}} = 0$, where $N_{\bar{J}}$ is the Nijenhuis torsion of $\bar{J}$, and $d(\mu \circ \bar{J})_{|\mathcal{F}} = 0$ for each closed 1-form μ such that $Ker\mu \supset \mathcal{F}$ (note that if $\mathcal{F} = Ker(\lambda_1 \wedge ... \wedge \lambda_r)$ where each λ_j is a closed 1-form, this last condition is satisfied if and only if $\lambda_1 \wedge ... \wedge \lambda_r \wedge d(\lambda_j \circ \bar{J}) = 0$, $j = 1, ..., r$).

Then $\bar{\gamma}(t) = (-1)^n \bar{\varphi}(-t)((\bar{J} + tI)^{-1})^* \bar{\beta}$ defines a Veronese web $\bar{w}$ of codimension r for which $\bar{w}(\infty) = \mathcal{F}$. This Veronese web only depends on $\mathcal{F}$ and $\bar{J}$.

In view of propositions 1.2 and 1.3 and all that said previously, for proving theorem 2.1 it suffices to show that $d(\lambda \circ J)_{|w(\infty)} = d(\lambda \circ \tilde{J})_{|w(\infty)} = 0$ and that every $\bar{w}(t)$, $t \in \mathbb{K}$, is involutive. For this purpose we need the following result:

Lemma 2.1. *Given a 1-form ρ and a $(1,1)$-tensor field G on manifold, then $d(\rho \circ G)(G\quad,\quad) + d(\rho \circ G)(\quad, G\quad) = d\rho(G\quad, G\quad) + d(\rho \circ G^2) + \rho \circ N_G$.*

Proof. Consider two vector fields X, Y. One has:

$d(\rho \circ G)(GX, Y) = (GX)\rho(GY) - Y\rho(G^2X) - \rho(G[GX, Y])$

$d(\rho \circ G)(X, GY) = X\rho(G^2Y) - (GY)\rho(GX) - \rho(G[X, GY])$.

So $d(\rho \circ G)(GX, Y) + d(\rho \circ G)(X, GY) = d\rho(GX, GY) + d(\rho \circ G^2)(X, Y) + \rho(N_G(X, Y))$. $\square$

Let λ a closed 1-form such that $Ker\lambda \supset w(\infty)$. If $t \in \mathbb{K} - \{-a_1, ..., -a_{n-k}, -a\}$ lemma 2.1 applied to $\lambda \circ (J + tI)^{-1}$ and $(J + tI)$ yields $d(\lambda \circ J) = d(\lambda \circ (J + tI)) = -d(\lambda \circ (J + tI)^{-1})((J + tI)\quad, (J + tI)\quad)$.

But $Ker(\lambda \circ (J + tI)^{-1})$ contains $w(t)$ which is involutive, so $d(\lambda \circ (J + tI)^{-1})_{|w(t)} = 0$. Hence $d(\lambda \circ J)_{|w(\infty)} = -d(\lambda \circ (J + tI)^{-1})((J + tI)\quad, (J + tI)\quad)_{|w(\infty)} = 0$ since $(J + tI)w(\infty) = w(t)$.

The case of $\tilde{J}$ is similar.

Now we shall prove the involutivity of every $\bar{w}(t)$. Consider a point $q \in N$ and $t \in \mathbb{K}$ such that $\bar{J} + tI$ is invertible around q. If μ is a closed 1-form and $Ker\mu \supset \mathcal{F}$ then $Ker(\mu \circ (\bar{J} + tI)^{-1}) \supset \bar{w}(t)$ and by lemma 2.1

$d(\mu\circ(\bar{J}+tI)^{-1})(\bar{w}(t),\bar{w}(t)) = d(\mu\circ(\bar{J}+tI)^{-1})((\bar{J}+tI)\mathcal{F},(\bar{J}+tI)\mathcal{F}) = -d(\mu\circ(\bar{J}+tI))(\mathcal{F},\mathcal{F}) - \mu(N_{(\bar{J}+tI)}(\mathcal{F},\mathcal{F})) = -d(\mu\circ\bar{J})(\mathcal{F},\mathcal{F}) - \mu(N_{\bar{J}}(\mathcal{F},\mathcal{F})) = 0.$

That is to say $d(\mu\circ(\bar{J}+tI)^{-1})_{|\bar{w}(t)} = 0$.

Around q there exist closed 1-forms $\mu_1,...,\mu_r$ such that $Ker(\mu_1\wedge...\wedge\mu_r) = \mathcal{F}$; therefore $\mu_1\circ(\bar{J}+tI)^{-1},...,\mu_r\circ(\bar{J}+tI)^{-1}$ define $\bar{w}(t)$. But $d(\mu_j\circ(\bar{J}+tI)^{-1})_{|\bar{w}(t)} = 0$, $j=1,...,r$, so $\bar{w}(t)$ is involutive near q. On the other hand if A is an open neighbourhood of q, small enough, there exists a non-empty open set $B\subset\mathbb{K}$ such that $\bar{J}+tI$ is invertible on A, and therefore $\bar{w}(t)$ involutive, for any $t\in\mathbb{K}$. As $\bar{\gamma}(t)$ is polynomial in t this implies that every $\bar{w}(t)$ is involutive on A (indeed if X,Y are vector fields belonging to $\mathcal{F}$ then $\bar{\gamma}(t)[(\bar{J}+tI)X,(\bar{J}+tI)Y]$ is polynomial in t), *which proves theorem 2.1.*

Corollary 2.1.1. *Consider a Veronese web w on N of codimension $1\leq r\leq n-1$, an immersion $f:P\to N$ and a scalar b.*
(1) If for every $p\in P$ the characteristic numbers of $w(f(p))$ are greater than or equal to 2 and $f_*(T_pP)\supset w(b)(f(p))$, then the family $\{\tilde{w}(t)=f_*^{-1}(w(t)) \mid t\in\mathbb{K}-\{b\}\}$ extends to a Veronese web $\tilde{w}$ on P of codimension r by setting $\tilde{w}(b)=lim\,\tilde{w}(t)$, $t\to b$.
(2) Now assume that the characteristic numbers of $w(f(p))$ are constant on P; let $\tilde{r}$ the number of them greater than or equal to 2. If $f_*(T_pP)=w(b)(f(p))$ for any $p\in P$, then the family $\{\tilde{w}(t)=f_*^{-1}(w(t)) \mid t\in\mathbb{K}-\{b\}\}$ extends to a Veronese web $\tilde{w}$ on P of codimension $\tilde{r}$ by setting $\tilde{w}(b)=lim\,\tilde{w}(t)$, $t\to b$.

Proof. As the problem is local we may suppose that P is a regular (imbedded) submanifold of N of codimension k and f the canonical inclusion. Consider non-equal scalars $a_1,...,a_{n-k},a$ where $a=-b$. Then in the construction of J we can take $\beta_{n-k+1},...,\beta_n$ in such a way that $Ker(\beta_{n-k+1}\wedge...\wedge\beta_n)(p)=T_pP$, $p\in P$; even more one may suppose $P=\{x \mid x_{n-k+1}=...=x_n=0\}$ when $\beta_j=dx_j$, $j=1,...,n$. On the other hand the integrability is clear since $\tilde{w}(t)=w(t)\cap TP$, $t\in\mathbb{K}-\{b\}$.

Now consider a r-form β such that $Ker\beta=w(\infty)$ and $\beta=\mu_1\wedge...\mu_r$, where $\mu_1,...,\mu_r$ are 1-forms, and set $\bar{J}=\sum_{j=1}^{n-k}a_j\frac{\partial}{\partial x_j}\otimes dx_j$ on P. As $(\mu_1,...,\mu_r,J^*)$ spans T^*N then $(\mu_{1|P},...,\mu_{r|P},\bar{J}^*)$ spans T^*P.

In the first case of the corollary $\beta_{|P}$ has no zeros and $\tilde{\gamma}(t)=(\prod_{j=1}^{n-k}(t+$

$a_j))((\bar{J}+tI)^{-1})^*(\beta_{|P})$ is a representative of $\tilde{w}$. In the second one $\{\mu_1(p)_{|P}, ..., \mu_r(p)_{|P}\}$ spans a $\tilde{r}$-dimensional vector subspace of T_p^*P at any $p \in P$, which allows us to assume, for example, that $(\mu_1 \wedge ...\mu_{\tilde{r}})_{|P}$ never vanishes (our problem is local); then $\tilde{\gamma}(t) = \prod_{j=1}^{n-k}(t+a_j)((\bar{J}+tI)^{-1})^*(\mu_1 \wedge ... \wedge \mu_{\tilde{r}})_{|P}$ is a representative of $\tilde{w}$. $\square$

A family w of r-codimensional distributions which satisfies all the conditions of Veronese web except, perhaps, for the involutivity of each $w(t)$ will be called a *Veronese distribution*.

Corollary 2.1.2. *Consider a Veronese distribution w, of codimension $r \geq 1$, on N and a point p of this manifold. Let n_1 be the height of $w(p)$. Assume that $w(\infty), w(b_1), ..., w(b_{n_1+1})$ are integrable for n_1+1 non-equal scalars $b_1, ..., b_{n_1+1}$. Then w is a Veronese web around p.*

Indeed, let $k_1 \geq ... \geq k_{n_1}$ like in lemma 1.3. Set $\tilde{a}_j = -b_j$, $j = 1, ..., n_1$. Since $w(b_1), ..., w(b_{n_1})$ are involutive, reasoning as in the construction of $\tilde{J}$ gives rise to a $(1,1)$-tensor field H defined around p, flat and diagonalizable, with characteristic polynomial $\prod_{j=1}^{n_1}(t+b_j)^{k_j}$ such that $(Ker(H^*+b_jI))w(b_j) = 0$, $j = 1, ..., n_1$, and $\rho(t) = \prod_{j=1}^{n_1}(t-b_j)^{k_j}((H+tI)^{-1})^*\beta$, where $Ker\beta = w(\infty)$, represents w.

On the other hand $d(\lambda \circ H)_{|w(\infty)} = 0$ for any 1-closed form λ such that $Ker\lambda \supset w(\infty)$, because $w(b_{n_1+1})$ is involutive and $H + b_{n_1+1}I$ invertible (do reason as in the first and the second paragraphs after the proof of lemma 2.1). Now apply (3) of the foregoing theorem.

Remark. Corollary 2.1.2, with another proof, is due to Panasyuk [10] and it was conjectured by Zakharevich [18] (see [1] by Bouetou and Dufour as well). Note that by means of a projective transformation of $\mathbb{K}P^1 \equiv \mathbb{K} \cup \{\infty\}$, one may replace the integrability of $w(\infty)$ by that of $w(b_{n_1+2})$ for some $b_{n_1+2} \in \mathbb{K} - \{b_1, ..., b_{n_1+1}\}$; in other words it suffices the involutivity of $w(t)$ for $n_1 + 2$ elements of $\mathbb{K}P^1$. Therefore if k is the maximum of the height of $w(q)$, $q \in N$, a Veronese distribution w is a Veronese web on N if and only if $w(t)$ is involutive for $k + 2$ values of $t \in \mathbb{K}P^1$.

By a similar reason, corollary 2.1.1 still holds if $b = \infty$.

Proposition 2.1. *Let $w = \{w(t) \mid t \in \mathbb{K}\}$ a family of foliations of codi-*

mension r defined on a manifold N. Assume that each $w(p)$ is an algebraic Veronese web on T_pN, which allows us to define a r-codimensional distribution $\mathcal{F}$ on N, possibly not smooth, by setting $\mathcal{F}(p) = lim w(t)(p)$, $t \to \infty$. If $\mathcal{F}$ is smooth then w is a Veronese web on N.

Proof. Note that the $(1,1)$ tensor field J may be constructed, as before, around each point of N since every $w(t)$ is a foliation. On the other hand locally there exists a r-form β such that $Ker\beta = \mathcal{F}$ because $\mathcal{F}$ is smooth. So $\gamma(t) = (\prod_{j=1}^{n-k}(t+a_j))(t+a)^k((J+tI)^{-1})^*\beta$ is a representative of w. $\square$

Example. On an open set A of $\mathbb{K}^n$ consider a $(1,1)$-tensor field $J = \sum_{j=1}^n f_j(x_j)\frac{\partial}{\partial x_j} \otimes dx_j$ where $f_j(x_j) \neq f_k(x_k)$ whenever $x = (x_1, ..., x_n) \in A$. Set $\beta = \sum_{j=1}^n dx_j$. As $N_J = 0$, (β, J^*) spans T^*A and $d(\beta \circ J) = 0$, by (3) of theorem 2.1 the curve $\gamma(t) = \prod_{j=1}^n (t+f_j)\beta \circ (J+tI)^{-1} = \sum_{j=1}^n (\prod_{i=1; i\neq j}^n (t+f_i))dx_j$ defines a Veronese web w on A of codimension one, which generally is not flat.

Indeed, when w is flat there exists a representative $\tilde{\gamma}(t) = \sum_{i=0}^{n-1} t^i\tilde{\gamma}_i$ with each $\tilde{\gamma}_i$ closed. Set $\gamma(t) = \sum_{i=0}^{n-1} t^i\gamma_i$. As $\gamma = f\tilde{\gamma}$ then $\gamma_i \wedge d\gamma_i = 0$, $i = 0, ..., n-1$. But $\gamma_{n-2} = \sum_{j=1}^n (f_1 + ... + \hat{f}_j + ... + f_n)dx_j$; so the coefficient of $dx_i \wedge dx_j \wedge dx_k$, where $i < j < k$, in the expression of $\gamma_{n-2} \wedge d\gamma_{n-2} = 0$ equals $f_i(f_k' - f_j') + f_j(f_i' - f_k') + f_k(f_j' - f_i')$ which almost never vanishes.

For obtaining a 2-codimensional Veronese web $\tilde{w}$, one may consider a second 1-form $\beta' = \sum_{j=1}^n g_j(x_j)dx_j$ such that $\beta \wedge \beta'$ never vanishes and set $\tilde{\gamma}(t) = \prod_{j=1}^n (t+f_j)((J+tI)^{-1})^*(\beta \wedge \beta') = \sum_{1\leq j<k\leq n}(\prod_{i=1; i\neq j,k}^n (t+f_i))(g_k - g_j)dx_j \wedge dx_k$.

Theorem 2.1 gives a method to construct all Veronese webs locally. Usually the scalars $a_1, ..., a_{n-k}, a$, respectively $\tilde{a}_1, ..., \tilde{a}_{n_1}$, do not determine J, respectively $\tilde{J}$, which prevent us constructing them globally. Nevertheless if the characteristic numbers are constant and equal, for example if $r = 1$, then $n_1 = \frac{n}{r}$, $k_1 = ... = k_{n_1} = r$ and $\tilde{J}$ can be constructed on all N since, now, $Ker(\tilde{J}^* - \tilde{a}_jI)$ is the annihilator of $w(-\tilde{a}_j)$.

On the other hand, in view of proposition 1.2, the restriction of J or $\tilde{J}$ to $w(\infty)$ gives rise to a morphism (of vector bundles) $\ell : w(\infty) \to TN$, which only depends on the Veronese web, without non-zero ℓ-invariant vector subspace at any point of N. Moreover $w(t) = (\ell + tI)w(\infty)$, $t \in \mathbb{K}$.

Now consider, on a manifold M, a foliation $\mathcal{F}$ and a morphism (of vector bundles) $G : \mathcal{F} \to TM$. If α is a s-form defined on an open set A of M, then $G^*\alpha$ is a section on A of $\Lambda^s \mathcal{F}^*$ and can regarded as a s-form on the leaves of $\mathcal{F}$; thus we shall say that is *closed on* $\mathcal{F}$ if it is closed on its leaves. Besides, when $\bar{G} : TM \to TM$ is a prolongation of G, then $d(\bar{G}^*\alpha)_{|\mathcal{F}}$ equals the exterior derivative of $G^*\alpha$ along the leaves of $\mathcal{F}$; thus $G^*\alpha$ is closed on $\mathcal{F}$ if and only if $d(\bar{G}^*\alpha)_{|\mathcal{F}} = 0$.

Lemma 2.2. *Assume that $G^*\alpha$ is closed on $\mathcal{F}$ for every closed 1-form α such that $Ker\alpha \supset \mathcal{F}$. Then the restriction of $N_{\bar{G}}$ to $\mathcal{F}$, which will be named* the Nijenhuis torsion of G *and denoted by N_G, does not depend on the prolongation $\bar{G}$.*

Proof. As the problem is local we may suppose that $\mathcal{F} = Ker(\alpha_1 \wedge ... \wedge \alpha_k)$ where each α_j is a closed 1-form and $k = codim\mathcal{F}$. Since the difference between two prolongations equals $\sum_{j=1}^{k} Y_j \otimes \alpha_j$, it suffices to consider the case $H = \bar{G} + Y \otimes \alpha$ with $\alpha \wedge \alpha_1 \wedge ... \wedge \alpha_k = 0$ and $d\alpha = 0$. Now given $X \in \mathcal{F}$ one has:
$N_H(X, \quad) = L_{HX}H - HL_XH = L_{\bar{G}X}(\bar{G} + Y \otimes \alpha) - \bar{G}L_X(\bar{G} + Y \otimes \alpha) - Y \otimes \alpha(L_X\bar{G} + L_X(Y \otimes \alpha))$
whence $N_H(X, \quad) - N_{\bar{G}}(X, \quad) = Y \otimes (L_{\bar{G}X}\alpha - \alpha(L_X\bar{G})) + \tilde{Y} \otimes \alpha$ because $L_X\alpha = d(\alpha(X)) = 0$.

On the other hand when $Z \in \mathcal{F}$:
$(L_{\bar{G}X}\alpha - \alpha(L_X\bar{G}))(Z) = Z\alpha(\bar{G}X) - \alpha([X, \bar{G}Z]) + \alpha(\bar{G}[X, Z]) = Z\alpha(\bar{G}X) - X\alpha(\bar{G}Z) + \alpha(\bar{G}[X, Z]) = -d(\alpha \circ \bar{G})(X, Z) = 0$
since α is closed and $\alpha \circ \bar{G}$ is closed on $\mathcal{F}$. Therefore $(N_H)_{|\mathcal{F}} = (N_{\bar{G}})_{|\mathcal{F}}$. $\square$

Note that the Nijenhuis torsion of $\ell : w(\infty) \to TN$ vanishes and $\ell^*\alpha$ is closed on $w(\infty)$ for every closed 1-form α such that $Ker\alpha \supset w(\infty)$ since J, its local prolongation given by (1) of theorem 2.1, has zero Nijenhuis torsion and $d(\alpha \circ J)_{|w(\infty)} = 0$.

Conversely, given a foliation $\mathcal{F}$ on N of codimension $1 \leq r \leq n$ and a morphism $\ell : \mathcal{F} \to TN$ with the algebraic and differentiable properties stated before, then $w(t) = (\ell + tI)\mathcal{F}$, $t \in \mathbb{K}$, defines a Veronese of codimension r for which $w(\infty) = \mathcal{F}$. Indeed apply (3) of theorem 2.1 to a prolongation $\bar{J}$ of ℓ. Thus:

Giving a Veronese web on N of codimension $r \geq 1$ is equivalent to giving *a morphism $\ell : \mathcal{F} \to TN$, where $\mathcal{F}$ is a r-codimensional foliation without non-vanishing ℓ-invariant vector subspace at any point such that:*

1) whenever α is a closed 1-form whose kernel contains $\mathcal{F}$, restricted to the domain of α, then $\ell^*\alpha$ is closed on $\mathcal{F}$,

2) $N_\ell = 0$.

Note that if $\mathcal{F} = Ker(\alpha_1 \wedge ... \wedge \alpha_r)$, where $d\alpha_1 = ... = d\alpha_r = 0$, then $\ell^*\alpha$ is closed on $\mathcal{F}$ for any 1-form α such that $d\alpha = 0$ and $Ker\alpha \supset \mathcal{F}$, if and only if $\ell^*\alpha_1, ..., \ell^*\alpha_r$ are closed on $\mathcal{F}$.

Example. On an open set A of $\mathbb{K}^{2m}$, endowed with coordinates $(x, y) = (x_1, ..., x_m, y_1, ..., y_m)$, consider the foliation $\mathcal{F}$ defined by $dy_1 = ... = dy_m = 0$ and the morphism $\ell : \mathcal{F} \to TA$ given by $\ell(\frac{\partial}{\partial x_j}) = \sum_{k=1}^{m} f_{jk} \frac{\partial}{\partial y_k}$, $j = 1, ...m$. Assume $\mid f_{jk} \mid \neq 0$ everywhere, which implies that $\ell : \mathcal{F} \to TA$ defines a m-codimensional Veronese distribution w on A with characteristic numbers $n_1 = ... = n_m = 2$. Then w is a Veronese web if and only if $d(\sum_{j=1}^{m} f_{jk} dx_j)_{|\mathcal{F}} = 0$, $k = 1, ..., m$, and $\left[\sum_{k=1}^{m} f_{jk} \frac{\partial}{\partial y_k}, \sum_{\tilde{k}=1}^{m} f_{\tilde{j}\tilde{k}} \frac{\partial}{\partial y_{\tilde{k}}}\right] = 0$, $1 \leq j < \tilde{j} \leq m$ (indeed consider the prolongation J of ℓ given by $J(\frac{\partial}{\partial y_k}) = 0$, $k = 1, ..., m$).

When $m = 1$ there are no conditions at all. If $m = 2$ one has a partial differential system of order one with four equations and four functions; for $m \geq 3$ the system is over-determined.

More generally when $n = 2m$, the m-dimensional Veronese webs on N, with characteristic numbers $n_1 = ... = n_m = 2$, are given by a morphism $\ell : \mathcal{F} \to TN$ such that $dim\mathcal{F} = m$ and $TN = \mathcal{F} \oplus Im\ell$. As ℓ is determined by its image and its graph, which may be identified to $w(1) = (\ell + I)\mathcal{F}$, from the algebraic viewpoint giving a Veronese web w with all its characteristic number equal to 2 is like giving the 3-web $\{\mathcal{F} = w(\infty), w(0), w(1)\}$. Conversely, for any 3-web $D = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ on N there exists just one Veronese distribution w_D such that $w_D(\infty) = \mathcal{D}_1$, $w_D(0) = \mathcal{D}_2$ and $w_D(1) = \mathcal{D}_3$. It is easily seen that w_D is a Veronese web if and only if the torsion of the Chern connection of D vanishes (the Chern connection of D is the only connection making $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ parallel such that $T(\mathcal{D}_1, \mathcal{D}_2) = 0$, see [8]).

For the link between k-webs, $k \geq 4$, and Veronese webs see [1] (Bouetou-Dufour).

3. Kronecker bihamiltonian structures

Consider two Poisson structures Λ, Λ_1 defined on a real or complex manifold M of dimension m. Following Magri [6] we will say that (Λ, Λ_1) is a *bihamiltonian structure* (or that Λ, Λ_1 are *compatible*) if $\Lambda + \Lambda_1$ is still a Poisson structure, which is equivalent to say that their Schouten bracket vanishes or that $\Lambda + b\Lambda_1$ is a Poisson structure for some $b \in \mathbb{K} - \{0\}$. Recall that if Λ, Λ_1 are compatible then $a\Lambda + a_1\Lambda_1$ is a Poisson structure for all $a, a_1 \in \mathbb{K}$.

A bihamiltonian structure (Λ, Λ_1) will be called *Kronecker* when there exists $r \in \mathbb{N} - \{0\}$ such that each $(\Lambda(p), \Lambda_1(p))$, $p \in M$, is the product of r Kronecker elementary pairs. In this case from the algebraic model at each point follows that $m - r = rank(\Lambda, \Lambda_1) = rank(\Lambda) = rank(\Lambda_1) = rank(\Lambda + t\Lambda_1)$ for any $t \in \mathbb{K}$; moreover $\mathcal{D} = \cap Im(\Lambda + t\Lambda_1)$, $t \in \mathbb{K}$, is a foliation of dimension $\frac{m-r}{2}$ lagrangian for both Λ and Λ_1, and $\mathcal{D} \subset Im\Lambda_1$. This foliation will be named the *soul of* (Λ, Λ_1).

Let N be the local quotient of M by the foliation $\mathcal{D}$, which is a manifold of dimension $n = \frac{m+r}{2}$, and let $\pi : M \to N$ be the canonical projection. Then $w = \{w(t) = \pi_*(Im(\Lambda + t\Lambda_1)) \mid t \in \mathbb{K}\}$ is a family of foliation on N of codimension r, whose limit when $t \to \infty$ equals $\pi_*(Im\Lambda_1)$ since $\pi_*(Im(\Lambda + t\Lambda_1)) = \pi_*(Im(s\Lambda + \Lambda_1))$ where $s = t^{-1}$. Besides *w is a Veronese web of codimension r.*

Indeed, given $p \in N$ such that $\pi(q) = p$, proposition 1.4 applied to $(\Lambda(q), \Lambda_1(q), T_qM)$ shows that $w(p)$ is an algebraic Veronese web. Now apply proposition 2.1.

In short a Veronese web of codimension r is locally associated to any Kronecker bihamiltonian structure with r factors. Our next goal is to study when this Veronese web locally determines the Kronecker bihamiltonian structure.

Recall that a Poisson structure Λ' on M of constant rank $m - r$ can be locally described by r closed 1-forms giving the foliation $Im\Lambda'$ and a 2-form whose restriction to $Im\Lambda'$ is symplectic; this last one is only defined modulo the ideal spanned by the 1-forms. Consider non-equal and non-vanishing scalars $a_1, ..., a_{n-r}, a$, any point $p \in N$ and closed 1-forms $\alpha_1, ..., \alpha_r$, defined around p, such that $Ker(\alpha_1 \wedge ... \wedge \alpha_r) = w(\infty)$. Let J be a $(1,1)$ tensor field like in part (1) of theorem 2.1; then $(\alpha_1, ..., \alpha_r, J^*)$ spans the cotangent bundle near p and $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(\alpha_j \circ J) = 0$, $j = 1, ..., r$. On the other hand one may choose coordinates $(x_1, ..., x_{n-r}, y_1, ..., y_r)$, defined on an open neighbourhood of $p \equiv 0$, such that $dx_j \circ J = a_j dx_j$, $j = 1, ..., n - r$, and $w(0) = Ker(dy_1 \wedge ... \wedge dy_r)$;

indeed the choice of $x_1,...,x_{n-r}$ is obvious and $dx_1,...,dx_{n-r}$ restricted to $w(0)$ are linearly independent everywhere since they are independent restricted to $w(-a)$ and $w(-a)=(J-aI)J^{-1}w(0)$. As $\mathcal{D}$ is Λ-lagrangian functions $x_1\circ\pi,...,x_r\circ\pi$ are in Λ-involution, so around each $p'\in\pi^{-1}(p)$ there exist functions $f_1,...,f_{n-r}$, vanishing at p', such that Λ is given by $d(y_1\circ\pi),...,d(y_r\circ\pi)$ and $d(x_1\circ\pi)\wedge df_1+...+d(x_{n-r}\circ\pi)\wedge df_{n-r}$. Now by setting $z_j=f_j$ and writing x_j and y_k instead of $x_j\circ\pi$ and $y_k\circ\pi$, for sake of simplicity, we construct a system of coordinates $(x,y,z)=(x_1,...,x_{n-r},y_1,...,y_r,z_1,...,z_{n-r})$ such that $p'\equiv 0$, $\pi(x,y,z)=(x,y)$ and Λ is given by $dy_1,...,dy_r$, $\sum_{j=1}^{n-r}dx_j\wedge dz_j$.

But $\mathcal{D}$ is Λ_1-lagrangian too, so $x_1,...,x_{n-r}$ are in Λ_1-involution. Moreover on N forms $dx_1,...,dx_{n-r}$ restricted to $w(\infty)$ are linearly independent everywhere since $w(-a)=(J-aI)w(\infty)$; therefore around p' there exist functions $g_1,...,g_{n-r}$ such that Λ_1 is given by $dx_1\wedge dg_1+...+dx_{n-r}\wedge dg_{n-r}$ and $\alpha_1,...,\alpha_r$ (more exactly $\pi^*\alpha_1,...,\pi^*\alpha_r$). On the other hand $\pi_*^{-1}(w(-a_j))=Im(\Lambda-a_j\Lambda_1)\subset Ker dx_j$ whence $(\partial/\partial z_j)=\Lambda(dx_j,\quad)=a_j\Lambda_1(dx_j,\quad)$ and $(\partial g_k/\partial z_j)=\delta_{jk}a_k$. So Λ_1 is given by $\alpha_1,...,\alpha_r$ and $\sum_{j=1}^{n-r}a_jdx_j\wedge dz_j+\omega$ where

$$\omega=\sum h_{ij}(x,y)dx_i\wedge dx_j+\sum\tilde{h}_{ik}(x,y)dx_i\wedge dy_k \text{ and } d\omega=0.$$

Thus ω may be regarded as a closed 2-form on an open neighbourhood of p in N.

Given a k-form τ, $k\geq 1$, and a $(1,1)$ tensor field H on a manifold, $\tau\circ H$ and τ_H will denote the k-forms defined by $(\tau\circ H)(X_1,...,X_k)=\tau(HX_1,...,HX_k)$ and $\tau_H(X_1,...,X_k)=\tau(HX_1,X_2...,X_k)+\tau(X_1,HX_2,...,X_k)+...+\tau(X_1,...,X_{k-1},HX_k)$ respectively.

The next proposition, proved later on, characterizes the compatibility of Λ and Λ_1.

Proposition 3.1. *The pair (Λ,Λ_1) is compatible if and only if $\alpha_1\wedge...\wedge\alpha_r\wedge d\omega_J=0$.*

The local determination of the bihamiltonian structure by the Veronese web will be established if we are able to delete the term ω in the expression of Λ_1, since $\alpha_1,...,\alpha_r$ only depend on the web. Given a function $\varphi(x,y)$ defined around p set $u_j=z_j-(\partial\varphi/\partial x_j)$, $j=1,...,n-r$. Then, in coordinates (x,y,u), $dy_1,...,dy_r$, $\sum_{j=1}^{n-r}dx_j\wedge du_j$ define Λ (the other terms belong to the ideal spanned by $dy_1, ..., dy_r$) while Λ_1 is given by $\alpha_1, ..., \alpha_r$, $\sum_{j=1}^{n-r} a_j dx_j \wedge du_j + (\omega - d(d\varphi \circ J))$; indeed each $(dy_k \circ J) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$ since $Jw(\infty) = w(0)$, so $(d\varphi \circ J - \sum_{j=1}^{n} a_j(\partial\varphi/\partial x_j)dx_j) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$. As the 2-form expressing Λ_1 is defined modulo the ideal spanned by $\alpha_1, ..., \alpha_r$, it suffices to find a function φ such that $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(d\varphi \circ J) = \alpha_1 \wedge ... \wedge \alpha_r \wedge \omega$ for deleting ω. To remark that if a such function φ exists, by adding a suitable linear function of (x, y) we may suppose $d\varphi(p) = 0$ and $u_j(p') = 0$, $j = 1, ..., n - r$.

Theorem 3.1. *On a manifold N consider closed 1-forms $\alpha_1, ..., \alpha_r$, $r \geq 1$, linearly independent everywhere and a $(1,1)$ tensor field J, which is flat and diagonalizable with characteristic polynomial $(t-a)^r \prod_{j=1}^{n-r}(t - a_j)$ where $a_1, ..., a_r, a$ are non-equal scalars. Assume that $(\alpha_1, ..., \alpha_r, J^*)$ spans T^*N and $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(\alpha_j \circ J) = 0$, $j = 1, ..., r$.*

Given a closed 2-form ω on N if $d\omega_J = 0$ then, around each point of N, there exists a function φ such that $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(d\varphi \circ J) = \alpha_1 \wedge ... \wedge \alpha_r \wedge \omega$ at least in the following three cases:
(1) on complex manifold,
(2) in the real analytic category,
(3) in the C^∞ category when $r = 1$.

This theorem will be proved in the next section.

Theorem 3.2. *From the local viewpoint the Veronese web completely determines the Kronecker bihamiltonian structure, at least, in the following four cases: complex manifold, real analytic category, C^∞ category when $r = 1$, and flat Veronese web.*

Theorem 3.2 is an obvious consequence of theorem 3.1 except for real flat webs. In this last case in some coordinates $(v_1, ..., v_n)$ the expression of $w(t)$ does not depend on the point considered, which allows us to choose $\alpha_1, ..., \alpha_r$ and J with constant coefficients. Thus in these coordinates the partial differential equation $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(d\varphi \circ J) = \alpha_1 \wedge ... \wedge \alpha_r \wedge \omega$ is homogeneous of of order two with constant coefficients and C^∞ independent term. By the Ehrenpreis-Malgrange theorem (see [7]) there exist local solutions provided that it has formal solutions.

Let ω_k be the kth term of the Taylor expansion of ω, always in coordinates $(v_1, ..., v_r)$, at point q. Then $d\omega_k = 0$ and $\alpha_1 \wedge ... \wedge \alpha_r \wedge d((\omega_k)_J) = 0$, so by theorem 3.1 the equation $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(d\varphi \circ J) = \alpha_1 \wedge ... \wedge \alpha_r \wedge \omega_k$ has a solution $\tilde{f}$ around q. Note that the $(k+2)$th term f_{k+2} of the Taylor expansion of $\tilde{f}$ at q is a solution of this equation too. Thus if f is a polynomial of degree $\ell \geq 2$ such that $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(df \circ J) = \alpha_1 \wedge ... \wedge \alpha_r \wedge (\omega_0 + ... + \omega_{\ell-2})$ then $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(d(f + f_{\ell+1}) \circ J) = \alpha_1 \wedge ... \wedge \alpha_r \wedge (\omega_0 + ... + \omega_{\ell-1})$. Therefore the equation $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(d\varphi \circ J) = \alpha_1 \wedge ... \wedge \alpha_r \wedge \omega$ is formally integrable and there exist local solutions of it around each point.

Theorem 3.2 was proved by Gelfand and Zakharevich [3, 4] for analytic Veronese web of codimension 1; the flat case, the C^∞ case of codimension 1 and the analytic one of any codimension are due to Turiel [15, 17].

Now we will prove proposition 3.1

Lemma 3.1. *If $t \notin \{-a_1, ..., -a_{n-r}, -a\}$ then $\Lambda + t\Lambda_1$ is defined by $\alpha_1 \circ (J+tI)^{-1}, ..., \alpha_r \circ (J+tI)^{-1}$ and $\sum_{j=1}^{n-r} a_j(t+a_j)^{-1} dx_j \wedge dz_j + t\omega \circ (J+tI)^{-1}$.*

Proof. First we replace coordinates $(y_1, ..., y_r)$ by coordinates $(u_1, ..., u_r)$ such that $du_k \circ J = a du_k$, thus $J = \sum_{j=1}^{n-r} a_j \frac{\partial}{\partial x_j} \otimes dx_j + \sum_{k=1}^{r} a \frac{\partial}{\partial u_k} \otimes du_k$ in coordinates $(x_1, ..., x_{n-r}, u_1, ..., u_r)$. Let V be a r-dimensional vector space and let $\{e_1, ..., e_r\}$ be a basis of V. It will be enough to prove the result for each point q. On $T_qM \oplus V$ set $\Omega = \sum_{j=1}^{n-r} dx_j \wedge dz_j + \sum_{k=1}^{r} du_k \wedge e_k^*$, $\Omega_1 = \sum_{j=1}^{n-r} a_j dx_j \wedge dz_j + \sum_{k=1}^{r} a du_k \wedge e_k^* + \omega$ where $dx_j, dz_j, dy_k, du_k, e_k^*$ and ω are extended to $T_qM \oplus V$ in the obvious way and the point q is omitted in the notation.

Let G and H be the endomorphisms of $T_qM \oplus V$ defined by $\Omega(G, \quad) = \Omega_1 - \omega$ and $\Omega(H, \quad) = \omega$ respectively. Note that $G = \sum_{j=1}^{n-r} a_j (\frac{\partial}{\partial x_j} \otimes dx_j + \frac{\partial}{\partial z_j} \otimes dz_j) + \sum_{k=1}^{r} a(\frac{\partial}{\partial u_k} \otimes du_k + e_k \otimes e_k^*)$, $dx_j \circ G = dx_j \circ J$, $dy_k \circ G = dy_k \circ J$, $du_k \circ G = du_k \circ J$, $j = 1, ..., n-r$, $k = 1, ..., r$, and $ImH \subset U \subset KerH$, so $H^2 = 0$, where U is the vector space spanned by $(\partial/\partial z_1), ..., (\partial/\partial z_{n-r}), e_1, ..., e_r$.

Let W be the r-dimensional vector subspace of $T_qM \oplus V$ whose image by Ω is the space spanned by $dy_1, ..., dy_r$ (note that this last space is the annihilator of $w(0) \oplus U$). Obviously $W \subset U$ so W is Ω-isotropic; moreover W is a direct factor of T_qM since $dx_1, ..., dx_{n-r}, dy_1, ..., dy_r$ are linearly independent. On the other hand $\Omega_1(W, \quad) = \Omega(GW, \quad)$ is spanned by $dy_1 \circ G = dy_1 \circ J, ..., dy_r \circ G = dy_r \circ J$. As $Jw(\infty) = w(0)$, $\Omega_1(W, \quad)$ is spanned by $\alpha_1, ..., \alpha_r$ too; that is to say $\Omega_1(W, \quad)$ is the annihilator of $w(\infty) \oplus U$ and W is Λ_1-isotropic too.

By lemma 1.4 bivectors Λ, Λ_1 are the projection on $\frac{T_qM \oplus V}{W} \equiv T_qM$ of the dual bivectors Λ_Ω and Λ_{Ω_1}. Therefore $\Lambda + t\Lambda_1$ is the projection of $\Lambda_\Omega + t\Lambda_{\Omega_1}$, which is the dual bivector of $\Omega((I + t(G+H)^{-1})^{-1}, \quad)$.

By lemma 1.5 the space W is isotropic for this last symplectic form, so $\Lambda + t\Lambda_1$ will be given by the restriction to T_qM of $\Omega((I + t(G+H)^{-1})^{-1}, \quad)$ and $\Omega((I + t(G+H)^{-1})^{-1}W, \quad)$.

Recall that if A is an automorphism and B an endomorphism such that $B^2 = 0$ and $A^{-1}(ImB) \subset KerB$, then $(A+B)^{-1} = A^{-1} - A^{-1}BA^{-1}$. So $(G+H)^{-1} = G^{-1} - G^{-1}HG^{-1}$ and $(I + t(G+H)^{-1})^{-1} = ((I + tG^{-1}) - tG^{-1}HG^{-1})^{-1} = (I + tG^{-1})^{-1} + t(G+tI)^{-1}H(G+tI)^{-1}$.

Hence $\Omega((I+t(G+H)^{-1})^{-1}, \quad) = \sum_{j=1}^{n-r} a_j(t+a_j)^{-1} dx_j \wedge dz_j + \sum_{k=1}^{r} a(t+a)^{-1} du_k \wedge e_k^* + t\omega \circ (J+tI)^{-1}$ and $\Omega((I + t(G+H)^{-1})^{-1}W, \quad) = \Omega((I + tG^{-1})^{-1}W, \quad)$ equals the vector space spanned by $dy_1 \circ (I + tG^{-1})^{-1}$,..., $dy_r \circ (I + tG^{-1})^{-1}$, that is to say by $\alpha_1 \circ (J+tI)^{-1}$,..., $\alpha_1 \circ (J+tI)^{-1}$, since $dy_k \circ (I+tG^{-1})^{-1} = dy_k \circ (I + tJ^{-1})^{-1} = (dy_k \circ J) \circ (J+tI)^{-1}$ and $Jw(\infty) = w(0)$. $\square$

Lemma 3.2. *Consider a k-form τ, $k \geq 1$, and a (1, 1) tensor field G on a manifold. Suppose that the Nijenhuis torsion of G vanishes. Then $(d(\tau \circ G))_G = d((\tau \circ G)_G) + (d\tau) \circ G$.*

Proof. By induction on k. The case $k = 1$ follows from lemma 2.1; on the other hand if $k \geq 2$ it suffices proving the lemma when $\beta = \beta_1 \wedge \beta_2$ and β_1 is a 1-form. Then

$$(d(\beta \circ G))_G = (d(\beta_1 \circ G) \wedge (\beta_2 \circ G))_G - ((\beta_1 \circ G) \wedge d(\beta_2 \circ G))_G = (d(\beta_1 \circ G))_G \wedge (\beta_2 \circ G) + d(\beta_1 \circ G) \wedge (\beta_2 \circ G)_G - (\beta_1 \circ G)_G \wedge d(\beta_2 \circ G) - (\beta_1 \circ G) \wedge (d(\beta_2 \circ G))_G$$

$$d((\beta \circ G)_G) = d((\beta_1 \circ G)_G \wedge (\beta_2 \circ G)) + d((\beta_1 \circ G) \wedge (\beta_2 \circ G)_G) = d((\beta_1 \circ G)_G) \wedge (\beta_2 \circ G) + d((\beta_1 \circ G)) \wedge (\beta_2 \circ G)_G - (\beta_1 \circ G)_G \wedge d(\beta_2 \circ G) - (\beta_1 \circ G) \wedge d((\beta_2 \circ G)_G)$$

$(d\beta) \circ G = ((d\beta_1) \circ G) \wedge (\beta_2 \circ G) - (\beta_1 \circ G) \wedge (d\beta_2) \circ G.$

Now take into account that the formula is true for β_1 (lemma 2.1) and β_2 (induction hypothesis), and remark that the second and third terms of the expansion of $(d(\beta \circ G))_G$ equal the second and third ones of $d((\beta \circ G)_G)$. $\square$

By lemma 3.1, Λ and Λ_1 are compatible if and only if $(\alpha_1 \circ (J+tI)^{-1}) \wedge ... \wedge (\alpha_r \circ (J+tI)^{-1}) \wedge d(\omega \circ (J+tI)^{-1}) = 0$ for some $t \notin \{-a_1, ..., -a_{n-r}, -a\}$, that is to say when $\alpha_1 \wedge ... \wedge \alpha_r \wedge (d(\omega \circ (J+tI)^{-1}) \circ (J+tI)) = 0$. Lemma 3.2 applied to $\omega \circ (J+tI)^{-1}$ and $J+tI$ yields $d(\omega \circ (J+tI)^{-1}) \circ (J+tI) = -d(\omega_{(J+tI)}) = -d\omega_J$. Therefore Λ, Λ_1 are compatible if and only if $\alpha_1 \wedge ... \wedge \alpha_r \wedge d\omega_J = 0$, *which proves proposition 3.1.*

Consider a foliation $\mathcal{F}$ of codimension s defined on a k-manifold P. Let $\mathcal{F}'$ be the foliation, on the cotangent bundle $T^*\mathcal{F}$ of the first foliation, pull-back of $\mathcal{F}$ by the canonical projection $\pi : T^*\mathcal{F} \to P$; that is to say $\mathcal{F}'(\beta) = (\pi_*(\beta)^{-1})(\mathcal{F}(\pi(\beta)))$ (until the end of this section one will write $T^*\mathcal{F}$ instead of $\mathcal{F}^*$ for pointing out that $T^*\mathcal{F}$ is regarded as a manifold itself). On the leaves of $\mathcal{F}'$ one defines the Liouville 1-form ρ by setting $\rho(\beta)(X) = \beta(\pi_*(X))$ for any $X \in \mathcal{F}'(\beta) \subset T_\beta(T^*\mathcal{F})$ and any $\beta \in T^*\mathcal{F}$, and the Liouville 2-form $\tilde{\omega} = -d\rho$; then $\tilde{\omega}$ is symplectic on the leaves of $\mathcal{F}'$ and, by duality, gives rise to a Poisson structure Λ_L such that $Im\Lambda_L = \mathcal{F}'$, which will be named the *Liouville-Poisson structure of* $T^*\mathcal{F}$. In coordinates $(\tilde{x}, \tilde{y}) = (\tilde{x}_1, ..., \tilde{x}_k, \tilde{y}_1, ..., \tilde{y}_{k-s})$, associated to coordinates $\tilde{x} = (\tilde{x}_1, ..., \tilde{x}_k)$ on P such that $\mathcal{F}$ were defined by $d\tilde{x}_{k-s+1} = ... = d\tilde{x}_k = 0$, Λ_L is given by $d\tilde{x}_{k-s+1}, ..., d\tilde{x}_k$, $\sum_{j=1}^{k-s} d\tilde{x}_j \wedge d\tilde{y}_j$; so

$$\Lambda_L = \sum_{j=1}^{k-s} \frac{\partial}{\partial \tilde{x}_j} \wedge \frac{\partial}{\partial \tilde{y}_j}.$$

Proposition 3.2. *Consider on a n-manifold N a Veronese web w of codimension r. Let Λ and Λ' be the Liouville-Poisson structures of $T^*w(0)$ and $T^*w(\infty)$ respectively, and let $\varphi_\ell : T^*w(0) \to T^*w(\infty)$ be the vector bundle isomorphism defined by $\varphi_\ell(\beta) = \beta \circ \ell$ where $\ell : w(\infty) \to w(0)$ is the canonical isomorphism attached to w. Note Λ_1 the pull-back of Λ' by φ_ℓ (regarded as a diffeomorphism).*

Then (Λ, Λ_1) is a Kronecker bihamiltonian structure on $T^*w(0)$ with r factors, whose soul $\mathcal{D}$ is given by the fibres of the canonical fibration $T^*w(0) \to N$;

therefore the quotient manifold $\frac{M}{\mathcal{D}} = N$*. Moreover* w *is the Veronese web induced by* (Λ, Λ_1) *on* N*.*

Proof. Let $\pi : T^*w(0) \to N$ and $\pi' : T^*w(\infty) \to N$ be the canonical projections. Choose non-equal and non-vanishing scalars $\{a_1, ..., a_{n-r}, a\}$. On an open neighbourhood A of a generic point consider a $(1,1)$ tensor field J like in part (1) of theorem 2.1, coordinates $(x, y) = (x_1, ..., x_{n-r}, y_1, ..., y_r)$ such that $dx_j \circ J = a_j dx_j$, $j = 1, ..., n-r$, and $Ker(dy_1 \wedge ... \wedge dy_r) = w(0)$, and closed 1-forms $\alpha_1, ..., \alpha_r$ such that $Ker(\alpha_1 \wedge ... \wedge \alpha_r) = w(\infty)$.

The restriction to $w(0)$ of $dx_1, ..., dx_{n-r}$ is a basis on A of $T^*w(0)$; so on $\pi^{-1}(A) \equiv A \times \mathbb{K}^{n-r}$ one has coordinates (x, y, u), $u = (u_1, ..., u_{n-r})$, where $(x, y)(\beta)$ are the coordinates of $\pi(\beta)$ and $\beta = \sum_{j=1}^{n-r} u_j(\beta) dx_j$ for each $\beta \in \pi^{-1}(A)$. In the same way one constructs coordinates (x, y, u'), $u' = (u'_1, ..., u'_{n-r})$, on $(\pi')^{-1}(A)$.

In this kind of coordinates, Λ is given by $dy_1, ..., dy_r$, $\sum_{j=1}^{n-r} dx_j \wedge du_j$ while $\alpha_1, ..., \alpha_r$, $\sum_{j=1}^{n-r} dx_j \wedge du'_j$ define Λ'. On the other hand

$$\varphi_\ell(x, y, u) = (x, y, a_1 u_1, ..., a_{n-r} u_{n-r})$$

since J is an extension of ℓ and each $dx_j \circ J = a_j dx_j$. Therefore $\alpha_1, ..., \alpha_r$, $\sum_{j=1}^{n-r} a_j dx_j \wedge du_j$ define Λ_1. By lemma 3.1 (here $\omega = 0$) $\Lambda + t\Lambda_1$, $t \notin \{-a_1, ..., -a_{n-r}, -a\}$, is given by $\alpha_1 \circ (J + tI)^{-1}, ..., \alpha_r \circ (J + tI)^{-1}$ and the closed 2-form $\sum_{j=1}^{n-r} a_j (t + a_j)^{-1} dx_j \wedge du_j$, which shows the compatibility of Λ and Λ_1.

The remainder statements are obvious. □

4. The equation $d(df \circ J) = \omega$ modulo $I(E)$

By technical reasons for studying the equation above we shall need parameters that will be regarded as transverse variables to a n-foliation $\mathcal{F}$ defined on a m-dimensional manifold M. Let E be an involutive vector subbundle of $\mathcal{F}$ of dimension $n - r$ where $r \geq 1$. Consider along $\mathcal{F}$ a diagonalizable (1, 1) tensor field J with characteristic polynomial $(t - a)^r \prod_{j=1}^{n-r} (t - a_j)$ where $a_1, ..., a_{n-r}, a$ are non-equal scalars. Suppose that its Nijenhuis torsion N_J vanishes.

Let E^c and $I(E)$ be the annihilator of E on $\mathcal{F}^*$ and the differential ideal spanned by the sections of E^c respectively. Assume that (E^c, J^*) spans $\mathcal{F}^*$ and that for all closed 1-form α belonging to $I(E)$ the 2-form $d(\alpha \circ J)$ belongs to $I(E)$ as well, where d is the exterior derivative along $\mathcal{F}$.

As $N_J = 0$, distributions $Im(J - a_j I)$, $j = 1, ..., n - r$, and $Im(J - aI)$ are involutive. Therefore around every point $p \in M$ there exist functions $x_1, ..., x_{n-r}, y_1, ..., y_r$ such that $dx_1 \wedge ... \wedge dx_{n-r} \wedge dy_1 \wedge ... \wedge dy_r$ is a volume form on $\mathcal{F}$, $dx_j \circ J = a_j dx_j$, $j = 1, ..., n - r$, and $dy_1 = ... = dy_r = 0$ defines E. Indeed, since (E^c, J^*) spans $\mathcal{F}^*$ one has $E \cap Ker(J - aI) = \{0\}$, so E is a direct factor of $Ker(J - aI)$ in $\mathcal{F}^*$.

On the other hand $dy_k \circ J = a dy_k + \sum_{j=1}^{n-r} f_{kj} dx_j$, $k = 1, ..., r$. As (E^c, J^*) spans $\mathcal{F}^*$, by linearly recombining functions $y_1, ..., y_r$ and considering $b_j x_j$ instead x_j for a suitable $b_j \in \mathbb{K} - \{0\}$, *from now on we may assume that every $f_{1j}(p)$, $j = 1, ..., n - r$, is a positive real number.*

Set $\tilde{\alpha}_k = \sum_{j=1}^{n-r} f_{kj} dx_j$, $k = 1, ..., r$. Since $d(dy_k \circ J)$ belongs to $I(E)$ one has $dy_1 \wedge ... \wedge dy_r \wedge d\tilde{\alpha}_k = 0$. On the other hand vector fields

$$\partial/\partial x_1, ..., \partial/\partial x_{n-r}, \partial/\partial y_1, ..., \partial/\partial y_r$$

are defined as the dual basis of $dx_1, ..., dx_{n-r}, dy_1, ..., dy_r$.

In the domain of functions $x_1, ..., x_{n-r}, y_1, ..., y_r$, we consider the submanifold S defined by $x_j - x_{n-r} = x_j(p) - x_{n-r}(p)$, $j = 1, ..., n - r - 1$ ($S = M$ if $n = r, r + 1$). Denote by $\mathcal{F} \cap S$ the $(r + 1)$-foliation induced by $\mathcal{F}$ on S.

Given a 1-form β along $\mathcal{F}$ defined on a open set $M' \subset M$, we denote by β' its restriction to $S \cap M'$ *as a section of* $\mathcal{F}^*$. That is to say β' is a section of $\mathcal{F}^*$ over $S \cap M'$ and $\beta \to \beta'$ is a linear map. Recall that if μ is a section of $\Lambda^k \mathcal{F}^*$ on $S \cap M'$, its restriction $\mu_{|\mathcal{F} \cap (S \cap M')}$ can be considered as a k-form on $\mathcal{F} \cap (S \cap M')$. In our particular case when β is closed, $\beta'_{|\mathcal{F} \cap (S \cap M')}$ is closed as well.

Hereafter *the standard case* will mean that the structures considered are complex, real analytic, or C^∞ with $r = 1$ in this last case.

Let α_0 be a 1-form on $\mathcal{F}$.

Theorem 4.1. *Suppose that each $f_{1j}(p)$, $j = 1, ..., n - r$, is a positive real number. Then in the standard case the linear map $\beta \to \beta'$ defines an injective correspondence between germs, at p, of closed 1-forms β on $\mathcal{F}$ such that*

$$(d(\beta \circ J) + \beta \wedge \alpha_0) \wedge dy_1 \wedge ... \wedge dy_r = 0$$

and germs, at p on S, of sections β' of $\mathcal{F}^*$ whose restriction to $\mathcal{F} \cap S$ are closed.

When $\alpha_0 = 0$ this correspondence becomes bijective.

We shall prove this theorem by induction on n. For $n = r, r+1$ the result is obvious since $S = M$. Now assume that the theorem holds up to $n-1$ (whichever m and $a_1, ..., a_{n-r}, a$ are).

By sake of convenience *we will suppose* $a_1 = 0$ by replacing J by $J - a_1 I$ (the equation of theorem 1 does not change because $d(\beta \circ I) = d\beta = 0$). Set $\alpha_0 = \sum_{j=1}^{n-r} h_j dx_j + \sum_{k=1}^{r} h_{n+k-r} dy_k$ and $\beta = \sum_{j=1}^{n-r} \phi_j dx_j + \sum_{k=1}^{r} \phi_{n+k-r} dy_k$. Since $d\beta = 0$ and $dy_1 \wedge ... \wedge dy_r \wedge d\tilde{\alpha}_k = 0$ we have:

$$(d(\beta \circ J) + \beta \wedge \alpha_0) \wedge dy_1 \wedge ... \wedge dy_r =$$
$$dx_1 \wedge \sum_{j=2}^{n-r} \left(a_j \frac{\partial \phi_j}{\partial x_1} + \sum_{k=1}^{r} (f_{kj} \frac{\partial \phi_1}{\partial y_k} - f_{k1} \frac{\partial \phi_j}{\partial y_k}) + h_j \phi_1 - h_1 \phi_j \right) dx_j \wedge dy_1 \wedge ... \wedge dy_r$$
$$+ \sum_{2 \leq i < j \leq n-r} \tilde{h}_{ij} dx_i \wedge dx_j \wedge dy_1 \wedge ... \wedge dy_r.$$

Therefore the part of $d(\beta \circ J) + \beta \wedge \alpha_0$ which is divisible by dx_1 modulo $dy_1, ..., dy_r$ vanishes if and only if the following system holds:

$$(1) \qquad a_j \frac{\partial \phi_j}{\partial x_1} + \sum_{k=1}^{r} (f_{kj} \frac{\partial \phi_1}{\partial y_k} - f_{k1} \frac{\partial \phi_j}{\partial y_k}) + h_j \phi_1 - h_1 \phi_j = 0, \; j = 2, ..., n-r.$$

Let S' be the submanifold defined by $x_j - x_{n-r} = x_j(p) - x_{n-r}(p)$, $j = 2, ..., n-r-1$ ($S' = M$ if $n = r+2$). By construction S is a 1-codimension submanifold of S' and the induced foliation $\mathcal{F} \cap S'$ has dimension $r+2$.

Set $z_1 = x_1$, $z_2 = x_{n-r}$, $z_3 = y_1$,..., $z_{r+2} = y_r$. Let $\partial/\partial z_1, ..., \partial/\partial z_{r+2}$ be the dual basis of the restriction of $dz_1, ..., dz_{r+2}$ to $\mathcal{F} \cap S'$. Vector fields $\partial/\partial x_1$, $\partial/\partial x_2 + ... + \partial/\partial x_{n-r}$, $\partial/\partial y_k$, $k = 1, ..., r$, are tangent to $\mathcal{F} \cap S'$; even more $\partial/\partial z_1 = \partial/\partial x_1$, $\partial/\partial z_2 = \partial/\partial x_2 + ... + \partial/\partial x_{n-r}$ and $\partial/\partial z_{k+2} = \partial/\partial y_k$, $k = 1, ..., r$, on S'. Besides $dx_1 = dz_1$, $dy_k = dz_{k+2}$, $k = 1, ..., r$, and the restriction to $\mathcal{F} \cap S'$ of each dx_j, $j = 2, ..., n-r$, equals that of dz_2.

On S' system (1) becomes:

$$(2) \qquad a_j \frac{\partial \phi_j}{\partial z_1} + \sum_{k=1}^{r} (f_{kj} \frac{\partial \phi_1}{\partial z_{k+2}} - f_{k1} \frac{\partial \phi_j}{\partial z_{k+2}}) + h_j \phi_1 - h_1 \phi_j = 0, \; j = 2, ..., n-r.$$

The restriction of β to $\mathcal{F} \cap S'$, whose expression is

$$\phi_1 dz_{1|\mathcal{F} \cap S'} + (\sum_{j=2}^{n-r} \phi_j) dz_{2|\mathcal{F} \cap S'} + \sum_{k=1}^{r} \phi_{n+k-r} dz_{k+2|\mathcal{F} \cap S'}$$

is a closed 1-form. Hence

$$\frac{\partial\phi_1}{\partial z_2} - \sum_{j=2}^{n-r} \frac{\partial\phi_j}{\partial z_1} = 0.$$

Now on S' we can consider the system:

$$(3) \qquad \begin{cases} \dfrac{\partial\phi_1}{\partial z_2} - \displaystyle\sum_{j=2}^{n-r} \dfrac{\partial\phi_j}{\partial z_1} = 0 \\ a_j \dfrac{\partial\phi_j}{\partial z_1} + \displaystyle\sum_{k=1}^{r} (f_{kj} \dfrac{\partial\phi_1}{\partial z_{k+2}} - f_{k1} \dfrac{\partial\phi_j}{\partial z_{k+2}}) + h_j\phi_1 - h_1\phi_j = 0 \ ; j = 2, ..., n-r. \end{cases}$$

Lemma 4.1. *In the standard case, given a germ at p on S of functions $(\hat{\phi}_1, ..., \hat{\phi}_{n-r})$ there exists one and only one germ, at p on S', of functions $(\phi_1, ..., \phi_{n-r})$ which is a solution to (3) and such that $\phi_{j|S} = \hat{\phi}_j$, $j = 1, ..., n-r$.*

Proof. Consider functions $u_1, ..., u_{m-n}$, on a neighbourhood of p on S', which are basic for $\mathcal{F} \cap S'$ and such that $(z_1, ..., z_{r+2}, u_1, ..., u_{m-n})$ is a system of coordinates. Since $u_1, ..., u_{m-n}$ are basic for $\mathcal{F} \cap S'$ vector fields $\partial/\partial z_1, ..., \partial/\partial z_{r+2}$ defined above equal to partial derivative vector fields, with the same name, which are associated to coordinates $(z_1, ..., z_{r+2}, u_1, ..., u_{m-n})$.

Therefore (3) can be regarded like a system on an open set of $\mathbb{K}^{m+r+2-n}$, with coordinates $(z_1, ..., z_{r+2}, u_1, ..., u_{m-n})$, while S is identify to the hypersurface defined by $z_1 - z_2 = z_1(p) - z_2(p)$. In particular $\partial/\partial z_1 - \partial/\partial z_2$ is normal to S.

In this system $\partial/\partial z_1 - \partial/\partial z_2$ is represented by an invertible triangular matrix with entries on the diagonal $-1, a_2, ..., a_{n-r}$. Therefore in the complex case or in the real analytic one, lemma 4.1 follows from the Cauchy-Kowalewsky theorem.

Now one will proves the result in the C^∞ case when $r = 1$.

Set $f_j = f_{1j}$. By adding up to the first equation the second one multiplied by a_2^{-1}, the third one multiplied by a_3^{-1}, etc..., we obtain the system:

$$(4) \qquad \begin{cases} \dfrac{\partial\phi_1}{\partial z_2} + (\displaystyle\sum_{j=2}^{n-1} a_j^{-1} f_j) \dfrac{\partial\phi_1}{\partial z_3} - \displaystyle\sum_{j=2}^{n-1} a_j^{-1} f_1 \dfrac{\partial\phi_j}{\partial z_3} + \displaystyle\sum_{j=2}^{n-1} a_j^{-1} (h_j\phi_1 - h_1\phi_j) = 0 \\ a_j \dfrac{\partial\phi_j}{\partial z_1} + f_j \dfrac{\partial\phi_1}{\partial z_3} - f_1 \dfrac{\partial\phi_j}{\partial z_3} + h_j\phi_1 - h_1\phi_j = 0 \ ; j = 2, ..., n-1. \end{cases}$$

In this system $\partial/\partial z_1$ and $\partial/\partial z_2$ are represented by diagonal matrices with entries on the diagonal $0, a_2, ..., a_{n-1}$ and $1, 0, ..., 0$ respectively.

On the other hand $\partial/\partial z_3$ is represented by the matrix:

$$\begin{pmatrix} \sum_{j=2}^{n-1} a_j^{-1}f_j & -a_2^{-1}f_1 & -a_3^{-1}f_1 & . & . & . & -a_{n-1}^{-1}f_1 \\ f_2 & -f_1 & & & & & \\ f_3 & & -f_1 & & & & \\ . & & & . & & & \\ . & & & & . & & \\ . & & & & & . & \\ f_{n-1} & & & & & & -f_1 \end{pmatrix}$$

Obviously each $\partial/\partial u_i$ is represented by the zero matrix.

If one multiplies the jth equation, $j = 2, ..., n-1$, by $-a_j^{-1}f_1f_j^{-1}$, we obtain a linear symmetric system. In this new system $\partial/\partial z_1 - \partial/\partial z_2$ is represented by a diagonal matrix with entries on the diagonal $-1, -f_1f_2^{-1}, ..., -f_1f_{n-1}^{-1}$. This matrix is negative definite around p, then the new system is symmetric hyperbolic and S is space-like.

Therefore this case of lemma 4.1 follows from the classical results on the Cauchy problem [2], [14]. $\square$

Let us come back to the proof of theorem 4.1.

Uniqueness. Let $\beta = \sum_{j=1}^{n-r} \phi_j dx_j + \sum_{k=1}^{r} \phi_{n+k-r} dy_k$ and $\gamma = \sum_{j=1}^{n-r} \varphi_j dx_j + \sum_{k=1}^{r} \varphi_{n+k-r} dy_k$ be two solutions to the equation of theorem 4.1, such that $\beta' = \gamma'$. On S' functions $\phi_1, ..., \phi_{n-r}$ and $\varphi_1, ..., \varphi_{n-r}$ are solutions to (3), which agree on S, then by lemma 4.1 we have $\phi_j = \varphi_j$, $j = 1, ..., n-r$, as germs at p on S'.

The restriction of $\beta - \gamma$ to S', which equals $\sum_{k=1}^{r} (\phi_{n+k-r} - \varphi_{n+k-r}) dy_{k|\mathcal{F}\cap S'}$, is closed. Therefore each $\phi_{n+k-r} - \varphi_{n+k-r}$, $k = 1, ..., r$, is constant on the leaves of the foliation defined by $Ker(dy_1 \wedge ... \wedge dy_r)_{|\mathcal{F}\cap S'} = E \cap S'$. But S is transverse to this foliation and $(\phi_{n+k-r} - \varphi_{n+k-r})_{|S} = 0$ then $\phi_{n+k-r} = \varphi_{n+k-r}$, $k = 1, ..., r$, on S'. In other words β and γ agree on S' as sections of $\mathcal{F}^*$.

The next step will be to regard x_1 like a new parameter. By shrinking M we may suppose that function x_1 is defined on the whole M.

Set $\mathcal{F}' = Kerdx_1 \subset \mathcal{F}$, which is a $(n-1)$-foliation, and let d' be the the exterior derivative along it. Denote by J' and α_0' the restriction to $\mathcal{F}'$ of J and α_0 respectively (recall that $dx_1 \circ J = 0$). Set $E' = E \cap \mathcal{F}'$. Let E'^c and $I(E')$ be the annihilator of E' on $(\mathcal{F}')^*$ and the differential ideal spanned by the sections of E'^c respectively. Then (E'^c, J'^*) spans $(\mathcal{F}')^*$ and, for any closed 1-form τ belonging to $I(E')$, the 2-form $d'(\tau \circ J')$ belongs to $I(E')$ as well. On the other hand $d'x_j \circ J' = a_j d'x_j$, $j = 2, ..., n-r$, $d'y_k \circ J' = \sum_{j=2}^{n-r} f_{kj} d'x_j + a d'y_k$, $k = 1, ..., r$, and $d'y_1 = ... = d'y_r = 0$ defines E'.

Since S' plays the same role with respect to $(x_2, ..., x_{n-r}, y_1, ..., y_r)$ as S does with respect to $(x_1, ..., x_{n-r}, y_1, ..., y_r)$, $\beta_{|\mathcal{F}'}$ and $\gamma_{|\mathcal{F}'}$ satisfy to the equation of theorem 4.1 for $\mathcal{F}'$, J', E' and α'_0, and $\beta_{|\mathcal{F}'} = \gamma_{|\mathcal{F}'}$ on S', from the induction hypothesis follows that $\beta_{|\mathcal{F}'} = \gamma_{|\mathcal{F}'}$ like germs at p on M, i.e. $\phi_j = \varphi_j$, $j = 2, ..., n$.

Finally, as $\beta - \gamma = (\phi_1 - \varphi_1)dx_1$ is closed, function $\phi_1 - \varphi_1$ is constant along the leaves of $\mathcal{F}'$. But S is transverse to $\mathcal{F}'$ and $(\phi_1 - \varphi_1)_{|S} = 0$ then $\phi_1 = \varphi_1$ and $\beta = \gamma$ as germs at p on M.

Existence. Now $\alpha_0 = 0$, i.e. $h_1 = ... = h_n = 0$. Given functions $\phi_1, ..., \phi_n$ on S such that the restriction of $\beta' = \sum_{j=1}^{n-r} \phi_j dx_j + \sum_{k=1}^{r} \phi_{n+k-r} dy_k$ to $\mathcal{F} \cap S$ is closed, by means of system (3) we extend functions $\phi_1, ..., \phi_{n-r}$ to S' (around p).

Since $\phi_1 dz_{1|\mathcal{F}\cap S'} + (\sum_{j=2}^{n-r} \phi_j) dz_{2|\mathcal{F}\cap S'}$ is closed modulo $dz_{k+2|\mathcal{F}\cap S'}$, $k = 1, ..., r$, (first equation of (3)), there exist functions $\hat{\phi}_{n+1-r}, ..., \hat{\phi}_n$ on S' such that the restriction to $\mathcal{F} \cap S'$ of $\phi_1 dz_1 + (\sum_{j=2}^{n-r} \phi_j) dz_2 + \sum_{k=1}^{r} \hat{\phi}_{n+k-r} dz_{k+2}$ is closed. Consequently its restriction to $\mathcal{F} \cap S$ is closed as well. On the other hand, by hypothesis, the restriction to $\mathcal{F} \cap S$ of $\phi_1 dz_1 + (\sum_{j=2}^{n-r} \phi_j) dz_2 + \sum_{k=1}^{r} \phi_{n+k-r} dz_{k+2}$ is closed. Therefore $\sum_{k=1}^{r} (\hat{\phi}_{n+k-r} - \phi_{n+k-r}) dz_{k+2|\mathcal{F}\cap S}$ is closed.

In coordinates $(z_1, ..., z_{r+2}, u_1, ..., u_{m-n})$ like in the proof of lemma 4.1, this implies the existence, on S', of a function $h(z_3, ..., z_{r+2}, u_1, ..., u_{m-n})$ such that $dh_{|\mathcal{F}\cap S} = \sum_{k=1}^{r} (\hat{\phi}_{n+k-r} - \phi_{n+k-r}) dz_{k+2|\mathcal{F}\cap S}$ on S. Obviously functions $\hat{\phi}_{n+k-r} - \partial h / \partial z_{k+2}$, $k = 1, ..., r$, have the same property as functions $\hat{\phi}_{n+k-r}$, $k = 1, ..., r$. Then by replacing each $\hat{\phi}_{n+k-r}$ by $\hat{\phi}_{n+k-r} - \partial h / \partial z_{k+2}$, we can suppose that $\hat{\phi}_{n+k-r}$ is an extension of ϕ_{n+k-r} and call it ϕ_{n+k-r} from now on.

If we consider $\mathcal{F}'$, J', E' and the section of $(\mathcal{F}')^*$ over S': $\sum_{j=2}^{n-r} \phi_j d'x_j + \sum_{k=1}^{r} \phi_{n+k-r} d'y_k$, whose restriction to $\mathcal{F}' \cap S'$ is closed, the induction hypothesis allows us to extend functions $\phi_2, ..., \phi_n$ to an open set of M containing p, in such a way that $\bar{\beta} = \sum_{j=2}^{n-r} \phi_j d'x_j + \sum_{k=1}^{r} \phi_{n+k-r} d'y_k$ is a closed 1-form along $\mathcal{F}'$ and $d'(\bar{\beta} \circ J') \wedge d'y_1 \wedge ... \wedge d'y_r = 0$.

Since $d'\bar{\beta} = 0$ there exists a function φ such that
$\rho = \varphi dx_1 + \sum_{j=2}^{n-r} \phi_2 dx_j + \sum_{k=1}^{r} \phi_{n+k-r} dy_k$ is a closed form along $\mathcal{F}$. On the other hand
$\rho_{|\mathcal{F}\cap S'} - (\phi_1 dz_1 + (\sum_{j=2}^{n-r} \phi_j) dz_2 + \sum_{k=1}^{r} \phi_{n+k-r} dz_{k+2})_{|\mathcal{F}\cap S'} = (\varphi - \phi_1) dz_{1|\mathcal{F}\cap S'}$
is closed; i.e. $\varphi - \phi_1$ is constant on the leaves of the foliation associated to $Kerdz_{1|\mathcal{F}\cap S'}$.

Around p on M consider coordinates $(x_1, ..., x_{n-r}, y_1, ..., y_r, v_1, ..., v_{m-n})$ where $v_1, ..., v_{m-n}$ are basic functions for $\mathcal{F}$. Then as $x_1 = z_1$ there exists a function $\bar{h}(x_1, v_1, ..., v_{m-n})$, around p on M, such that $\varphi - \phi_1 = \bar{h}$ on S' and, by replacing φ by $\varphi - \bar{h}$, we may suppose that φ extends ϕ_1 and call ϕ_1 this extension too.

In short we have constructed a closed 1-form, along $\mathcal{F}$,
$\beta = \sum_{j=1}^{n-r} \phi_j dx_j + \sum_{k=1}^{r} \phi_{n+k-r} dy_k$ which extends β' and such that $d(\beta \circ J) \wedge dx_1 \wedge dy_1 \wedge ... \wedge dy_r = 0$ (this is another way for writing $d'(\bar{\beta} \circ J') \wedge d'y_1 \wedge ... \wedge d'y_r = 0$). Therefore there exist closed 1-forms $\gamma_0, ..., \gamma_r$ along $\mathcal{F}$ such that

$$d(\beta \circ J) = dx_1 \wedge \gamma_0 + \textstyle\sum_{k=1}^{r} \gamma_k \wedge dy_k.$$

Set $\gamma_0 = \sum_{j=1}^{n-r} g_j dx_j + \sum_{k=1}^{r} g_{n+k-r} dy_k$. Then

$$g_j = a_j \frac{\partial \phi_j}{\partial x_1} + \sum_{k=1}^{r} (f_{kj} \frac{\partial \phi_1}{\partial y_k} - f_{k1} \frac{\partial \phi_j}{\partial y_k})\,;\; j = 2, ..., n-r$$

(recall the construction of system (1)). Therefore each g_j, $j = 2, ..., n-r$, vanishes on S' because $\phi_1, ..., \phi_{n-r}$ satisfy to system (3).

On the other hand $(d(\beta \circ J))_J$ is closed (apply lemma 2.1 along the leaves of $\mathcal{F}$). Then

$$-dx_1 \wedge d(\gamma_0 \circ J) + \textstyle\sum_{k=1}^{r} (d(\gamma_k \circ J) \wedge dy_k - \gamma_k \wedge d(\tilde{\alpha}_k + a dy_k)) = 0\ ,$$

whence $dx_1 \wedge d(\gamma_0 \circ J) \wedge dy_1 \wedge ... \wedge dy_r = 0$. That is to say $d'(\bar{\gamma}_0 \circ J') \wedge d'y_1 \wedge ... \wedge d'y_r = 0$ where $\bar{\gamma}_0 = \sum_{j=2}^{n-r} g_j d'x_j + \sum_{k=1}^{r} g_{n+k-r} d'y_k$.

On S', $\bar{\gamma}_0$ is a combination of $d'y_1, ..., d'y_r$. Since the restriction of $\bar{\gamma}_0$ to $\mathcal{F}' \cap S'$ is closed there exists a function $\ell(x_1, y_1, ..., y_r, v_1, ..., v_{m-n})$, defined near p on M, such that $\bar{\gamma}_0 = d'\ell$ on S'.

But $d'\ell$ is a closed 1-form along $\mathcal{F}'$ defined on an open set of M and $d'(d'\ell \circ J') \wedge d'y_1 \wedge ... \wedge d'y_r = 0$. Therefore the uniqueness in dimension $n-1$ implies that $\bar{\gamma}_0 = d'\ell$. In other words γ_0 is a combination of $dx_1, dy_1, ..., dy_r$.

Then $d(\beta \circ J) \wedge dy_1 \wedge ... \wedge dy_r = 0$ and *the proof of theorem 4.1 is finished.*

The following result will be needed in the next section.

Lemma 4.2. *Suppose that each $f_{1j}(p)$, $j = 1, ..., n-r$, is a positive real number. Consider 1-forms $\rho_{\ell q}$, $\ell, q = 1, ..., s$. In the standard case, given two families of s closed 1-forms, which are solution to the system*

$$(d(\beta_q \circ J) + \textstyle\sum_{\ell=1}^{s} \beta_\ell \wedge \rho_{\ell q}) \wedge dy_1 \wedge ... \wedge dy_r = 0,\ q = 1, ..., s,$$

if they agree around p on S then they agree around p on M.

Proof. Just adapt the proof of the uniqueness of theorem 4.1 (in fact the case $s = 1$ is the first assertion of this theorem). Now system (3) is replaced by a system $\mathcal{S}(\beta_1, ..., \beta_s)$ with s boxes corresponding each of them to a β_q. Note that the symbol of every box, which only depends on β_q, is similar to the symbol of system (3). Therefore lemma 4.1 extends to $\mathcal{S}(\beta_1, ..., \beta_s)$. Finally if $\beta_q = \sum_{j=1}^{n-r} \phi_{qj} dx_j + \sum_{k=1}^{r} \phi_{qn+k-r} dy_k$ and $\gamma_q = \sum_{j=1}^{n-r} \varphi_{qj} dx_j + \sum_{k=1}^{r} \varphi_{qn+k-r} dy_k$, $q = 1, ..., s$, are two solutions to the system of lemma 4.2 such that $\beta'_q = \gamma'_q$, $q = 1, ..., s$, reasoning as in the proof of the uniqueness of theorem 4.1 shows that $\beta_q = \gamma_q$, $q = 1, ..., s$. $\square$

Theorem 4.2. *Suppose that every $f_{1j}(p)$, $j = 1, ..., n-r$, is a positive real number. In the standard case given, on an open neighbourhood of p on M, a closed 1-form γ along $\mathcal{F}$ such that $d(\gamma \circ J) \wedge dx_1 \wedge dy_1 \wedge ... \wedge dy_r = 0$, then around p there exists a closed 1-form β along $\mathcal{F}$ such that $d(\beta \circ J) \wedge dy_1 \wedge ... \wedge dy_r = dx_1 \wedge \gamma \wedge dy_1 \wedge ... \wedge dy_r$.*

Proof. As above we shall suppose that $a_1 = 0$ by replacing, if necessary, J by $J - a_1 I$. Set $\gamma = \sum_{j=1}^{n-r} \varphi_j dx_j + \sum_{k=1}^{r} \varphi_{n+k-r} dy_k$.

On S' we consider the following system:

$$(3') \qquad \begin{cases} \dfrac{\partial \phi_1}{\partial z_2} - \displaystyle\sum_{j=2}^{n-r} \dfrac{\partial \phi_j}{\partial z_1} = 0 \\ a_j \dfrac{\partial \phi_j}{\partial z_1} + \displaystyle\sum_{k=1}^{r} (f_{kj} \dfrac{\partial \phi_1}{\partial z_{k+2}} - f_{k1} \dfrac{\partial \phi_j}{\partial z_{k+2}}) = \varphi_j \ ; j = 2, ..., n-r. \end{cases}$$

This system has some solution around p because its symbol is the same as that of system (3). Let $\phi_1, ..., \phi_{n-r}$ be a solution to (3'). The first equation of

(3') allows us to find functions $\phi_{n+1-r}, ..., \phi_n$, on a neighbourhood of p on S', such that $\left(\phi_1 dz_1 + (\sum_{j=2}^{n-r} \phi_j) dz_2 + \sum_{k=1}^{r} \phi_{n+k-r} dz_{k+2}\right)_{|\mathcal{F} \cap S'}$ is closed. Obviously the restriction of this form to $\mathcal{F}' \cap S'$ is closed too.

Now we apply theorem 1 to $\mathcal{F}'$, J' and E' for extending functions $\phi_2, ..., \phi_n$ to an open set of M containing p, in such a way that $d'\bar{\beta} = 0$ and $d'(\bar{\beta} \circ J') \wedge d'y_1 \wedge ... \wedge d'y_r = 0$ where $\bar{\beta} = \sum_{j=2}^{n-r} \phi_j d'x_j + \sum_{k=1}^{r} \phi_{n+k-r} d'y_k$.

The rest of the proof is very similar to that of the existence in theorem 1. First we extend function ϕ_1 to a neighbourhood of p on M in such a way that $\beta = \sum_{j=1}^{n-r} \phi_j dx_j + \sum_{k=1}^{r} \phi_{n+k-r} dy_k$ is closed. Since $d'(\bar{\beta} \circ J') \wedge d'y_1 \wedge ... \wedge d'y_r = 0$ we get $d(\beta \circ J) \wedge dx_1 \wedge dy_1 \wedge ... \wedge dy_r = 0$. Therefore $d(\beta \circ J) = dx_1 \wedge \gamma_0 + \sum_{k=1}^{r} \gamma_k \wedge dy_k$ where $\gamma_0, ..., \gamma_r$ are closed 1-forms along $\mathcal{F}$.

Set $\gamma_0 = \sum_{j=1}^{n-r} g_j dx_j + \sum_{k=1}^{r} g_{n+k-r} dy_k$ and $\bar{\gamma}_0 = \sum_{j=2}^{n-r} g_j d'x_j + \sum_{k=1}^{r} g_{n+k-r} d'y_k$. Then

$$g_j = a_j \frac{\partial \phi_j}{\partial x_1} + \sum_{k=1}^{r} (f_{kj} \frac{\partial \phi_1}{\partial y_k} - f_{k1} \frac{\partial \phi_j}{\partial y_k}) \, ; \, j = 2, ..., n-r$$

Besides $d'(\bar{\gamma}_0 \circ J') \wedge d'y_1 \wedge ... \wedge d'y_r = 0$ because $(d(\beta \circ J))_J$ is closed (lemma 2.1).

By hypothesis $d'(\bar{\gamma} \circ J') \wedge d'y_1 \wedge ... \wedge d'y_r = 0$ where $\bar{\gamma} = \sum_{j=2}^{n-r} \varphi_j d'x_j + \sum_{k=1}^{r} \varphi_{n+k-r} d'y_k$. On the other hand $\bar{\gamma} - \bar{\gamma}_0$ is a closed 1-form along $\mathcal{F}'$ which is a combination of $d'y_1, ..., d'y_r$ on S' since $(\phi_1, ..., \phi_{n-r})$ is a solution to (3'). This fact implies the existence, on an open neighbourhood of p on M, of a function $\ell(x_1, y_1, ..., y_r, v_1, ..., v_{m-n})$ such that $\bar{\gamma} - \bar{\gamma}_0 = d'\ell$ on S'.

Obviously $d'(d'\ell \circ J') \wedge d'y_1 \wedge ... \wedge d'y_r = 0$. Now from theorem 4.1 applied to $\mathcal{F}'$, J' and E' follows that $\bar{\gamma} - \bar{\gamma}_0 = d'\ell$ around p on M. Hence $(\gamma - \gamma_0) \wedge dx_1 \wedge dy_1 \wedge ... \wedge dy_r = 0$ and $d(\beta \circ J) \wedge dy_1 \wedge ... \wedge dy_r = dx_1 \wedge \gamma_0 \wedge dy_1 \wedge ... \wedge dy_r = dx_1 \wedge \gamma \wedge dy_1 \wedge ... \wedge dy_r$. $\square$

Theorem 4.3. *Let $\mathcal{F}$ be a n-foliation defined on a m-manifold M and let $E \subset \mathcal{F}$ be a second foliation of dimension $n-r$ where $r \geq 1$. On $\mathcal{F}$ we consider a diagonalizable (1, 1) tensor field J with characteristic polynomial $(t-a)^r \prod_{j=1}^{n-r} (t-a_j)$ where $a_1, ..., a_{n-r}, a$ are non-equal scalars. Suppose $N_J = 0$.*

Let E^c and $I(E)$ be the annihilator of E on $\mathcal{F}^*$ and the differential ideal *spanned by the sections of E^c respectively. Assume that (E^c, J^*) spans $\mathcal{F}^*$ and that for all closed 1-form α belonging to $I(E)$ the 2-form $d(\alpha \circ J)$ belongs to $I(E)$ as well, where d is the exterior derivative along $\mathcal{F}$.*

In the standard case, given a closed 2-form ω on $\mathcal{F}$, the following statements are equivalents:
(a) Around each point $p \in M$ there exists a function f such that $d(df \circ J) = \omega$ modulo $I(E)$.
(b) $d\omega_J$ belongs to $I(E)$.

Proof. $(a) \Rightarrow (b)$ In this case locally $\omega = d(df \circ J) + \sum_{k=1}^{r} \mu_k \wedge \alpha_k$ where $\mu_1, ..., \mu_r, \alpha_1, ..., \alpha_r$ are closed 1-forms and $\alpha_1, ..., \alpha_r$ belong to $I(E)$. Since $(d(df \circ J))_J$ is closed (lemma 2.1) and each 2-form $d(\alpha_k \circ J)$ belongs to $I(E)$, it follows that $d\omega_J$ belongs to $I(E)$.

$(b) \Rightarrow (a)$ As the problem is local we will use the concepts and notations of the proofs of theorems 4.1 and 4.2. The implication will be proved by induction on n. For $n = r, r+1$ the results is obvious. Now, assume that it holds up to $n-1$ (whichever m is).

Let $\bar{\omega}$ be the restriction of ω to $\mathcal{F}'$. Then $d'(\bar{\omega}_{J'}) \wedge d'y_1 \wedge ... \wedge d'y_r = 0$. By the induction hypothesis there exists a function $\bar{f}$ around p such that $d'(d'\bar{f} \circ J') \wedge d'y_1 \wedge ... \wedge d'y_r = \bar{\omega} \wedge d'y_1 \wedge ... \wedge d'y_r$. Hence $d(d\bar{f} \circ J) \wedge dx_1 \wedge dy_1 \wedge ... \wedge dy_r = \omega \wedge dx_1 \wedge dy_1 \wedge ... \wedge dy_r$.

Therefore $\omega - d(d\bar{f} \circ J) = dx_1 \wedge \gamma_0 + \sum_{k=1}^{r} \gamma_k \wedge dy_k$ where $\gamma_0, ..., \gamma_r$ are 1-closed forms along $\mathcal{F}$. As $(d\omega_J) \wedge dy_1 \wedge ... \wedge dy_r = 0$ and $(d(d\bar{f} \circ J))_J$ is closed (lemma 2.1) we obtain $d(\gamma_0 \circ J) \wedge dx_1 \wedge dy_1 \wedge ... \wedge dy_r = 0$.

By theorem 4.2, around p there exists a closed 1-form β, along $\mathcal{F}$, such that $d(\beta \circ J) \wedge dy_1 \wedge ... \wedge dy_r = dx_1 \wedge \gamma_0 \wedge dy_1 \wedge ... \wedge dy_r$. Now it is enough to set $f = h + \bar{f}$ where h is a primitive of β. $\square$

Finally remark that theorem 3.1 is just the implication $(b) \Rightarrow (a)$ of the foregoing theorem when $n = m$ and $E = Ker(\alpha_1 \wedge ... \wedge \alpha_r)$.

5. Another equation.

The aim of this paragraph is to establish another theorem on some system defined by differential forms, which be needed later on in the construction of versal models of Veronese webs. The objects M, $\mathcal{F}$, etc... are as in the foregoing section unless another thing is stated. Set $J_0=\sum_{j=1}^{n-r} a_j\frac{\partial}{\partial x_j}\otimes dx_j+\sum_{k=1}^{r} a\frac{\partial}{\partial y_k}\otimes dy_k$.

Theorem 5.1. *In the standard case, given a germ at p of maps $\varphi_{kj}:S\to\mathbb{K}$, $k=1,...,r$, $j=1,...,n-r$, such that every $\varphi_{1j}(p)$, $j=1,...,n-r$, is a positive real number, then there exists one and only one germ at p on M of 1-forms $\tilde\alpha_1=\sum_{j=1}^{n-r} f_{1j}dx_j$,..., $\tilde\alpha_r=\sum_{j=1}^{n-r} f_{rj}dx_j$ such that*

$$
\text{(4)}\qquad \begin{cases} d\tilde\alpha_k\wedge dy_1\wedge...\wedge dy_r=0, \quad k=1,...,r \\ \left(d(\tilde\alpha_k\circ J_0)-\sum_{\ell=1}^{r}\tilde\alpha_\ell\wedge\dfrac{\partial\tilde\alpha_k}{\partial y_\ell}\right)\wedge dy_1\wedge...\wedge dy_r=0, \quad k=1,...,r \end{cases}
$$

and that $f_{kj|S}=\varphi_{kj}$, $k=1,...,r$, $j=1,...,n-r$.

We shall prove this theorem by induction on n. For $n=r,r+1$ the result is obvious since $S=M$. Now assume that the theorem holds up to $n-1$ (whichever m and $a_1,...,a_{n-r},a$ are).

Consider 1-forms $\tilde\alpha_k=\sum_{j=1}^{n-r} f_{kj}dx_j$, $k=1,...,r$, such that $d\tilde\alpha_k\wedge dy_1\wedge...\wedge dy_r=0$.

By sake of convenience *we will suppose* $a_1=0$ by replacing J_0 by J_0-a_1I (the main equation of theorem 1 does not change). Then:

$$
\begin{aligned}
&\left(d(\tilde\alpha_k\circ J_0)-\sum_{\ell=1}^{r}\tilde\alpha_\ell\wedge\frac{\partial\tilde\alpha_k}{\partial y_\ell}\right)\wedge dy_1\wedge...\wedge dy_r\\
&= dx_1\wedge\sum_{j=2}^{n-r}\left(a_j\frac{\partial f_{kj}}{\partial x_1}+\sum_{\ell=1}^{r}(f_{\ell j}\frac{\partial f_{k1}}{\partial y_\ell}-f_{\ell 1}\frac{\partial f_{kj}}{\partial y_\ell})\right)dx_j\wedge dy_1\wedge...\wedge dy_r+\\
&\sum_{2\le 1<j\le n-r}\tilde h_{ij}dx_i\wedge dx_j\wedge dy_1\wedge...\wedge dy_r.
\end{aligned}
$$

Therefore the part of each $d(\tilde\alpha_k\circ J_0)-\sum_{\ell=1}^{r}\tilde\alpha_\ell\wedge\frac{\partial\tilde\alpha_k}{\partial y_\ell}$ that is divisible by dx_1 modulo $dy_1,...,dy_r$ vanishes if and only if the following system holds:

$$
\text{(5)}\qquad a_j\frac{\partial f_{kj}}{\partial x_1}+\sum_{\ell=1}^{r}(f_{\ell j}\frac{\partial f_{k1}}{\partial y_\ell}-f_{\ell 1}\frac{\partial f_{kj}}{\partial y_\ell})=0,\ j=2,...,n-r,\ k=1,...,r.
$$

On S' endowed with coordinates $(z_1,...,z_{r+2})$ system (5) becomes:

$$(6) \qquad a_j \frac{\partial f_{kj}}{\partial z_1} + \sum_{\ell=1}^{r} (f_{\ell j} \frac{\partial f_{k1}}{\partial z_{\ell+2}} - f_{\ell 1} \frac{\partial f_{kj}}{\partial z_{\ell+2}}) = 0,\ j = 2, ..., n-r,\ k = 1, ..., r.$$

The restriction of each $\tilde{\alpha}_k \wedge dy_1 \wedge ... \wedge dy_r$ to $\mathcal{F} \cap S'$ whose expression is

$$\left(\left[f_{k1} dz_1 + (\sum_{j=2}^{n-r} f_{kj}) dz_2 \right] \wedge dz_3 \wedge ... \wedge dz_{r+2} \right)_{|\mathcal{F} \cap S'}$$

is a closed 2-form. Hence

$$\frac{\partial f_{k1}}{\partial z_2} - \sum_{j=2}^{n-r} \frac{\partial f_{kj}}{\partial z_1} = 0,\ k = 1, ..., r.$$

Now on S' we can consider the system:

$$(7) \qquad \begin{cases} \begin{cases} \dfrac{\partial f_{k1}}{\partial z_2} - \displaystyle\sum_{j=2}^{n-r} \dfrac{\partial f_{kj}}{\partial z_1} = 0 \\ a_j \dfrac{\partial f_{kj}}{\partial z_1} + \displaystyle\sum_{\ell=1}^{r} (f_{\ell j} \dfrac{\partial f_{k1}}{\partial z_{\ell+2}} - f_{\ell 1} \dfrac{\partial f_{kj}}{\partial z_{\ell+2}}) = 0, \quad j = 2, ..., n-r, \end{cases} \\ k = 1, ..., r. \end{cases}$$

Lemma 5.1. *In the standard case, given a germ at p on S of functions φ_{kj}, $k = 1, ..., r$, $j = 1, ..., n-r$, such that every $\varphi_{1j}(p)$, $j = 1, ..., n-r$, is a positive real number, then there exists one and only one germ, at p on S', of functions f_{kj}, $k = 1, ..., r$, $j = 1, ..., n-r$, which is a solution to (7) and such that $f_{kj|S} = \varphi_{kj}$, $k = 1, ..., r$, $j = 1, ..., n-r$.*

Proof. On a neighbourhood of p on S' consider functions $u_1, ..., u_{m-n}$ basic for $\mathcal{F} \cap S'$ and such that $(z_1, ..., z_{r+2}, u_1, ..., u_{m-n})$ is a system of coordinates. Since $u_1, ..., u_{m-n}$ are basic for $\mathcal{F} \cap S'$ vector fields $\partial/\partial z_1, ..., \partial/\partial z_{r+2}$ defined above equal to partial derivative vector fields, with the same name, which are associated to coordinates $(z_1, ..., z_{r+2}, u_1, ..., u_{m-n})$.

Therefore (7) can be regarded like a system on an open set of $\mathbb{K}^{m+r-n+2}$ with coordinates $(z_1, ..., z_{r+2}, u_1, ..., u_{m-n})$, while S is identify to the hypersurface defined by $z_1 - z_2 = z_1(p) - z_2(p)$. In particular $\partial/\partial z_1 - \partial/\partial z_2$ is normal to S.

In this system the matrix associated to $\partial/\partial z_1 - \partial/\partial z_2$ is invertible. Indeed, it consists of r blocks $(n-r) \times (n-r)$ along the diagonal corresponding to the different values of k and zero outside of them, and every block is triangular with entries $-1, a_2, ..., a_{n-r}$ on the diagonal. So in the complex case and in the real analytic one it suffices to apply the Cauchy-Kowalewsky theorem.

On the other hand if $r = 1$, systems (3) and (7) have very similar symbols and, for the C^∞ case, it is enough to reason as in the proof of lemma 4.1. $\square$

Let us come back to the proof of the theorem 5.1.

Uniqueness. Let $\tilde{\alpha}_k = \sum_{j=1}^{n-r} f_{kj} dx_j$ and $\gamma_k = \sum_{j=1}^{n-r} g_{kj} dx_j$, $k = 1, ..., r$, be two solutions to (4) such that $f_{kj|S} = g_{kj|S} = \varphi_{kj}$, $k = 1, ..., r$, $j = 1, ..., n-r$. On S' functions f_{kj} and g_{kj} are solutions to (7) which agree on S, so by lemma 5.1 we have $f_{kj} = g_{kj}$, $k = 1, ..., r$, $j = 1, ..., n-r$, as germs at p on S'.

Now, like in the proof of theorem 4.1, we consider x_1 as a new parameter. Let J_0' be the restriction of J_0 to $\mathcal{F}'$ (recall that $dx_1 \circ J_0 = 0$); then $d'x_j \circ J_0' = a_j d'x_j$, $j = 2, ..., n-r$, $d'y_k \circ J_0' = ad'y_k$, $k = 1, ..., r$.

Since S' plays the same role with respect to $(x_2, ..., x_{n-r}, y_1, ..., y_r)$ as S does with respect to $(x_1, ..., x_{n-r}, y_1, ..., y_r)$, $\tilde{\alpha}_k' = \sum_{j=2}^{n-r} f_{kj} d'x_j$ and $\gamma_k' = \sum_{j=2}^{n-r} g_{kj} d'x_j$ satisfy to system (4) of theorem 5.1 for $\mathcal{F}'$ and J_0', and $\tilde{\alpha}' = \gamma'$ on S', from the induction hypothesis follows that $f_{kj} = g_{kj}$, $k = 1, ..., r$, $j = 2, ..., n-r$, like germs at p on M.

Finally, as each $(\tilde{\alpha}_k - \gamma_k) \wedge dy_1 \wedge ... \wedge dy_r = (f_{k1} - g_{k1}) dx_1 \wedge dy_1 \wedge ... \wedge dy_r$ is closed, function $f_{k1} - g_{k1}$ is constant along the leaves of the foliation $Ker(d'y_1 \wedge ... \wedge d'y_r) \subset \mathcal{F}'$. But S is transverse to this foliation and $(f_{k1} - g_{k1})_{|S} = 0$ then $f_{k1} = g_{k1}$ and $\tilde{\alpha}_k = \gamma_k$, $k = 1, ..., r$, as germs at p on M.

For the existence we will need the following result.

Lemma 5.2. *Consider* 1*-forms* $\beta_1, ..., \beta_r$ *functional combination of* $dx_1, ..., dx_{n-r}$. *Let* G *be the* $(1,1)$ *tensor field along* $\mathcal{F}$ *defined by* $dx_j \circ G = a_j dx_j$, $j = 1, ..., n-r$, $dy_k \circ G = \beta_k + a dy_k$, $k = 1, ..., r$. *Assume that* $d\beta_k \wedge dy_1 \wedge ... \wedge dy_r = 0$, $k = 1, ..., r$. *Then* $N_G = 0$ *if and only if*

$$\left(d(\beta_k \circ J_0) - \sum_{\ell=1}^{r} \beta_\ell \wedge \frac{\partial \beta_k}{\partial y_\ell} \right) \wedge dy_1 \wedge ... \wedge dy_r = 0, \ k = 1, ..., r.$$

Proof. By lemma 2.1 one has $dx_j \circ N_G = 0$ and

$dy_k \circ N_G = (d\beta_k)_G - d(\beta_k \circ J_0 + a\beta_k) = \left(\sum_{\ell=1}^{r} dy_\ell \wedge \frac{\partial \beta_k}{\partial y_\ell} \right)_G - d_x(\beta_k \circ J_0) - \sum_{\ell=1}^{r} dy_\ell \wedge (\frac{\partial \beta_k}{\partial y_\ell} \circ J_0) - \sum_{\ell=1}^{r} a dy_\ell \wedge \frac{\partial \beta_k}{\partial y_\ell} = \sum_{\ell=1}^{r} \beta_\ell \wedge \frac{\partial \beta_k}{\partial y_\ell} - d_x(\beta_k \circ J_0)$

where d_x denotes the exterior derivative in variables $(x_1, ..., x_{n-r})$ only. $\square$

Existence. Given functions φ_{kj}, $k = 1, ..., r$, $j = 1, ..., n-r$, on S such that every $\varphi_{1j}(p)$ is a positive real number, by means of system (7) we extend them to S', around p, with the same name.

If we consider $\mathcal{F}'$ and J_0', the induction hypothesis allows us to find functions f_{kj}, $k = 1, ..., r$, $j = 2, ..., n-r$, defined on an open neighbourhood of p on M, in such a way that $d'\tilde{\alpha}_k' \wedge d'y_1 \wedge ... \wedge d'y_r = 0$, $k = 1, ..., r$,

$$\left(d'(\tilde{\alpha}_k' \circ J_0') - \sum_{\ell=1}^{r} \tilde{\alpha}_\ell' \wedge \frac{\partial \tilde{\alpha}_k'}{\partial y_\ell} \right) \wedge d'y_1 \wedge ... \wedge d'y_r = 0, \quad k = 1, ..., r,$$

and $f_{kj} = \varphi_{kj}$ on S', $k = 1, ..., r$, $j = 2, ..., n-r$, where $\tilde{\alpha}_k' = \sum_{j=2}^{n-r} f_{kj} d'x_j$ (note that $\partial/\partial x_2, ..., \partial/\partial x_{n-r}, \partial/\partial y_1, ..., \partial/\partial y_r$ is the dual basis of $d'x_2, ..., d'x_{n-r}$, $d'y_1, ..., d'y_r$ as well).

Since each $d'\tilde{\alpha}_k' \wedge d'y_1 \wedge ... \wedge d'y_r = 0$ there exist functions f_k such that $\rho_k \wedge dy_1 \wedge ... \wedge dy_r$ is closed where $\rho_k = f_k dx_1 + f_{k2} dx_2 + ... + f_{kn-r} dx_{n-r}$. On the other hand the first equations of (7) means that every $((\varphi_{k1} dx_1 + ... + \varphi_{kn-r} dx_{n-r}) \wedge dy_1 \wedge ... \wedge dy_r)_{|\mathcal{F} \cap S'}$ is closed. Therefore $(\rho_k \wedge dy_1 \wedge ... \wedge dy_r)_{|\mathcal{F} \cap S'} - ((\varphi_{k1} dx_1 + ... + \varphi_{kn-r} dx_{n-r}) \wedge dy_1 \wedge ... \wedge dy_r)_{|\mathcal{F} \cap S'} = ((f_k - \varphi_{k1}) dx_1 \wedge dy_1 \wedge ... \wedge dy_r)_{|\mathcal{F} \cap S'}$ has to be closed.

Consider coordinates $(x_1, ..., x_{n-r}, y_1, ..., y_r, v_1, ..., v_{m-n})$, around p on M, where $v_1, ..., v_{m-n}$ are basic functions for $\mathcal{F}$. Then, always around p on M, there exist functions $\bar{h}_k(x_1, y_1, ..., y_r, v_1, ..., v_{m-n})$ such that $f_k - \varphi_{k1} = \bar{h}_k$ on S'. Now by setting $f_{k1} = f_k - \bar{h}_k$, we construct 1-form $\tilde{\alpha}_k = \sum_{j=1}^{n-r} f_{kj} dx_j$, $k = 1, ..., r$, along $\mathcal{F}$ such that $f_{kj|S'} = \varphi_{kj}$, $j = 1, ..., n-r$, $d\tilde{\alpha}_k \wedge dy_1 \wedge ... \wedge dy_r = 0$ and

$$\left(d(\tilde{\alpha}_k \circ J_0) - \sum_{\ell=1}^{r} \tilde{\alpha}_\ell \wedge \frac{\partial \tilde{\alpha}_k}{\partial y_\ell} \right) \wedge dx_1 \wedge dy_1 \wedge ... \wedge dy_r = 0.$$

Therefore we can find 1-forms $\gamma_k, \gamma_{k1}, ...\gamma_{kr}$, $k = 1, ..., r$, along $\mathcal{F}$, where each γ_k is closed because $\left(d(\tilde{\alpha}_k \circ J_0) - \sum_{\ell=1}^{r} \tilde{\alpha}_\ell \wedge \frac{\partial \tilde{\alpha}_k}{\partial y_\ell} \right) \wedge dy_1 \wedge ... \wedge dy_r$ is closed since $d\tilde{\alpha}_k \wedge dy_1 \wedge ... \wedge dy_r = 0$, such that

$$\left(d(\tilde{\alpha}_k \circ J_0) - \sum_{\ell=1}^{r} \tilde{\alpha}_\ell \wedge \frac{\partial \tilde{\alpha}_k}{\partial y_\ell} \right) = dx_1 \wedge \gamma_k + \gamma_{k1} \wedge dy_1 + ... + \gamma_{kr} \wedge dy_r.$$

Hence

$$(8) \qquad \begin{cases} d(\tilde{\alpha}_k \circ J_0) = dx_1 \wedge \gamma_k + \gamma_{k1} \wedge dy_1 + ... + \gamma_{kr} \wedge dy_r + \displaystyle\sum_{\ell=1}^{r} \tilde{\alpha}_\ell \wedge \frac{\partial \tilde{\alpha}_k}{\partial y_\ell} \\ k = 1, ..., r \end{cases}$$

Set $\gamma_k = \sum_{j=1}^{n-r} g_{kj} dx_j + \sum_{\ell=1}^{r} g_{kn-r+\ell} dy_\ell$. Then

$$g_{kj} = a_j \frac{\partial f_{kj}}{\partial x_1} + \sum_{\ell=1}^{r} (f_{\ell j} \frac{\partial f_{k1}}{\partial y_\ell} - f_{\ell 1} \frac{\partial f_{kj}}{\partial y_\ell}) = 0, \; j = 2, ..., n-r, \; k = 1, ..., r.$$

(recall the construction of system (5)). So each g_{kj}, $k = 1, ..., r$, $j = 2, ..., n-r$, vanishes on S' because functions $f_{kj|S'} = \varphi_{kj}$ satisfy to system (7).

Deriving (8) with respect to y_s yields

$$(9) \qquad d\left(\frac{\partial \tilde{\alpha}_k}{\partial y_s} \circ J_0\right) = dx_1 \wedge \frac{\partial \gamma_k}{\partial y_s} + \sum_{\ell=1}^{r} \frac{\partial \gamma_{k\ell}}{\partial y_s} \wedge dy_\ell + \sum_{\ell=1}^{r} \left(\frac{\partial \tilde{\alpha}_\ell}{\partial y_s} \wedge \frac{\partial \tilde{\alpha}_k}{\partial y_\ell} + \tilde{\alpha}_\ell \wedge \frac{\partial^2 \tilde{\alpha}_k}{\partial y_s \partial y_\ell}\right)$$

On the other hand

$$(10) \qquad \begin{aligned} (d(\tilde{\alpha}_k \circ J_0))_{J_0} &= dx_1 \wedge (\gamma_k \circ J_0) + \sum_{\ell=1}^{r} (\gamma_{k\ell} \circ J_0 + a\gamma_{k\ell}) \wedge dy_\ell \\ &+ \sum_{\ell=1}^{r} \left((\tilde{\alpha}_\ell \circ J_0) \wedge \frac{\partial \tilde{\alpha}_k}{\partial y_\ell} + \tilde{\alpha}_\ell \wedge (\frac{\partial \tilde{\alpha}_k}{\partial y_\ell} \circ J_0)\right) \end{aligned}$$

By lemma 2.1 applied along the leaves of $\mathcal{F}$ we have $d((d(\tilde{\alpha}_k \circ J_0))_{J_0}) \wedge dy_1 \wedge ... \wedge dy_r = 0$, whence by calculating $d((d(\tilde{\alpha}_k \circ J_0))_{J_0})$ from (10) and taking into account (8) and (9) follows

$$(11) \qquad \begin{cases} \left(d(\gamma_k \circ J_0) + \displaystyle\sum_{\ell=1}^{r} \frac{\partial \gamma_k}{\partial y_\ell} \wedge \tilde{\alpha}_\ell - \sum_{\ell=1}^{r} \gamma_\ell \wedge \frac{\partial \tilde{\alpha}_k}{\partial y_\ell}\right) \wedge dx_1 \wedge dy_1 \wedge ... \wedge dy_r = 0 \\ k = 1, ..., r. \end{cases}$$

Set $\gamma'_k = \sum_{j=2}^{n-r} g_{kj} d'x_j + \sum_{\ell=1}^{r} g_{kn-r+\ell} d'y_\ell$. Obviously $d'\gamma'_k = 0$ because $d\gamma_k = 0$. Consider the $(1,1)$ tensor field J' on $\mathcal{F}'$ defined by $d'x_j \circ J' = a_j d'x_j$, $j = 2, ..., n-r$, and $d'y_\ell \circ J' = \tilde{\alpha}'_\ell + a d'y_\ell$, $\ell = 1, ..., r$ (recall that $\tilde{\alpha}'_\ell = \sum_{j=2}^{n-r} f_{\ell j} d'x_j$).

Since

$$\left(d'(\tilde{\alpha}'_k \circ J'_0) - \sum_{\ell=1}^{r} \tilde{\alpha}'_\ell \wedge \frac{\partial \tilde{\alpha}'_k}{\partial y_\ell}\right) \wedge d'y_1 \wedge ... \wedge d'y_r = 0, \quad k = 1, ..., r,$$

by lemma 5.2, applied to $\mathcal{F}'$ and J', the Nijenhuis torsion of J' vanishes. Set $\rho_{\ell k} = -\dfrac{\partial \tilde{\alpha}'_k}{\partial y_\ell}$. Now system (11) becomes (note that $d' g_{kn-r+\ell} = \dfrac{\partial \gamma'_k}{\partial y_\ell}$ because γ'_k is closed)

$$\left(d'(\gamma'_k \circ J') + \sum_{\ell=1}^{r} \gamma'_\ell \wedge \rho_{\ell k} \right) \wedge d' y_1 \wedge ... \wedge d' y_r = 0, \quad k = 1, ..., r. \tag{12}$$

On S', $\gamma'_k \wedge dy_1 \wedge ... \wedge dy_r = 0$ as $g_{kj|S'} = 0$, $k = 1, ..., r$, $j = 2, ..., n-r$. Since the restriction of γ'_k to $\mathcal{F}' \cap S'$ is closed, around p on M there exist functions $\phi_{k\ell}(x_1, y_1, ..., y_r, v_1, ..., v_{m-n})$, $k, \ell = 1, ..., r$, such that every $\gamma'_k = \sum_{\ell=1}^{r} \phi_{k\ell} d' y_\ell$ on S'.

Set $\lambda_k = \sum_{\ell=1}^{r} \phi_{k\ell} d' y_\ell$. Then each λ_k is a closed 1-form along $\mathcal{F}'$ defined on an open neighbourhood of p on M and $(d'(\lambda_k \circ J') + \sum_{\ell=1}^{r} \lambda_\ell \wedge \rho_{\ell k}) \wedge d' y_1 \wedge ... \wedge d' y_r = 0, \quad k = 1, ..., r$. Now lemma 4.2 applied to $\mathcal{F}'$ and J' implies that $\gamma'_k = \lambda_k$, $k = 1, ..., r$. In other words every γ_k is a functional combination of $dx_1, dy_1, ..., dy_r$. Therefore

$$\left(d(\tilde{\alpha}_k \circ J_0) - \sum_{\ell=1}^{r} \tilde{\alpha}_\ell \wedge \frac{\partial \tilde{\alpha}_k}{\partial y_\ell} \right) \wedge dy_1 \wedge ... \wedge dy_r = 0, \; k = 1, ..., r,$$

and *the proof of theorem 5.1 is finished.*

6. Local classification of codimension one Veronese webs.

On a real or complex manifold N of dimension n consider a Veronese web w of codimension $r \geq 1$. Given non-equal scalars $a_1, ..., a_{n-r}, a$ and any point $p \in N$, let J be a $(1,1)$ tensor field like in part (1) of theorem 2.1 and let $(x_1, ..., x_{n-r}, y_1, ..., y_r)$ be a system of coordinates, around p, such that $dx_j \circ J = a_j dx_j$, $j = 1, ..., n-r$, and $Ker(dy_1 \wedge ... \wedge dy_r) = w(\infty)$. Then $dy_k \circ J = a dy_k + \tilde{\alpha}_k$, $k = 1, ..., r$, where each $\tilde{\alpha}_k = \sum_{j=1}^{n-r} f_{kj} dx_j$. As $(w(\infty)', J^*)$ spans the cotangent bundle around p, by linearly recombining functions $y_1, ..., y_r$ and considering $b_j x_j$ instead x_j for a suitable $b_j \in \mathbb{K} - \{0\}$, we assume that each $f_{1j}(p)$, $j = 1, ..., n-r$, is a positive real number (see the beginning of section 4).

On the other hand $d(dy_k \circ J) \wedge dy_1 \wedge ... \wedge dy_r = 0$ and $N_J = 0$; by lemma 5.2 these last two conditions are equivalent to system

$$
(13) \qquad \begin{cases} d\tilde{\alpha}_k \wedge dy_1 \wedge ... \wedge dy_r = 0, \quad k = 1, ..., r \\ \left(d(\tilde{\alpha}_k \circ J_0) - \sum_{\ell=1}^{r} \tilde{\alpha}_\ell \wedge \dfrac{\partial \tilde{\alpha}_k}{\partial y_\ell} \right) \wedge dy_1 \wedge ... \wedge dy_r = 0, \quad k = 1, ..., r \end{cases}
$$

where $J_0 = \sum_{j=1}^{n-r} a_j \frac{\partial}{\partial x_j} \otimes dx_j + \sum_{\ell=1}^{r} a \frac{\partial}{\partial y_\ell} \otimes dy_\ell$.

Moreover $\gamma(t) = (\prod_{j=1}^{n-r}(t+a_j))(t+a)^r((J+tI)^{-1})^*(dy_1 \wedge ... \wedge dy_r)$ represents w.

Therefore, in view of (3) of theorem 2.1, locally Veronese webs correspond to those solutions of system (13) such that $f_{11}(p), ..., f_{1n-r}(p) \in \mathbb{R}^+$ (this last assumption implies that $(dy_1, ..., dy_r, J^*)$ spans the cotangent bundle near p). In turn, for the standard case, this kind of solutions to (13) are given by theorem 5.1 by setting $M = N$ and $\mathcal{F} = TN$, which means that now S is the submanifold defined by $x_j - x_{n-r} = x_j(p) - x_{n-r}(p)$, $j = 1, ..., n-r-1$.

When $r \geq 2$ the tensor field J is not unique and consequently we may associate more than one model to a same Veronese web; *thus our model of every Veronese web is versal.*

To remark that a classification in codimension≥ 2 seems rather difficult as the following example shows. Consider a field of 2-planes and a local basis of it $\{X, Y\}$. Let $\tilde{w}(t)$, $t \in \mathbb{K}$, be the 1-foliation defined by $X + tY$. Then to classify the 1-dimensional (local) Veronese web $\tilde{w} = \{\tilde{w}(t) \mid t \in \mathbb{K}\}$, roughly speaking, is like locally classifying the fields of 2-planes in any dimension; but it is well known the difficult of this problem (first dealt with by Élie Cartan in "Les systèmes de Pfaff à cinq variables" and later on by several authors).

Now let us examine the remainder case. *Assume $r = 1$ until the end of this section.* Then $a_1, ..., a_{n-1}, a$ completely determines J since $Ker(J^* - a_j I)$, $j = 1, ..., n-1$, is the annihilator of $w(-a_j)$ and $Ker(J^* - aI)$ that of $w(-a)$. The next step will be to construct an intrinsic surface S. By technical reasons *one will suppose that $a_1, ..., a_{n-1}, a$ are non-equal real numbers.*

The polynomial $\sum_{j=1}^{n-1} \prod_{k=1;k\neq j}^{n-1}(t + a_k)$ has $n-2$ different roots $b_1, ..., b_{n-2}$ since it is the derivative of $\prod_{k=1}^{n-1}(t + a_k)$, whose roots are $-a_1, ..., -a_{n-1}$; moreover $b_\ell \neq -a_j$, $\ell = 1, ..., n-2$, $j = 1, ..., n-1$ (warning this property is not true when a polynomial, even real, has some complex root, for example $t^3 - 1$ and $3t^2$; by this reason one chooses real numbers $a_1, ..., a_{n-1}, a$).

Let R be the germ at p of the leaf of the 1-foliation $w(b_1) \cap ... \cap w(b_{n-2}) \cap$

$w(\infty)$ passing through this point, and let S_0 be the germ at p of the surface containing R and to which the 1-foliation $w(-a_1) \cap ... \cap w(-a_{n-1})$ is tangent. By construction S_0 is intrinsic.

Since R is transverse to every $w(-a_j)$, $j = 1, ..., n-1$, one may take coordinates $(x_1, ..., x_{n-1}, y)$ constructed before, with two additional properties: R is defined by the equations $x_1 = ... = x_{n-1}, y = 0$, and $x_1(p) = ... = x_{n-1}(p) = y(p) = 0$; of course we write y and $\tilde{\alpha} = \sum_{j=1}^{n-r} f_j dx_j$ instead y_1 and $\tilde{\alpha}_1 = \sum_{j=1}^{n-r} f_{1j} dx_j$. In these coordinates S_0 is defined by the equations $x_1 = ... = x_{n-1}$. Moreover

$$\gamma(t) = -\sum_{j=1}^{n-1} \left(\prod_{k=1;k\neq j}^{n-1} (t + a_k) f_j \right) dx_j + \prod_{k=1}^{n-1} (t + a_k) dy$$

because a straightforward calculation shows that

$$\left(-\sum_{j=1}^{n-1} (\prod_{k=1;k\neq j}^{n-1} (t + a_k) f_j) dx_j + \prod_{k=1}^{n-1} (t + a_k) dy \right) \circ (J + tI) = \left(\prod_{k=1}^{n-1} (t + a_k)(t + a) \right) dy.$$

On the other hand $\gamma(b_\ell)(q)((\partial/\partial x_1) + ... + (\partial/\partial x_{n-1})) = 0$, $\ell = 1, ..., n-2$, for every $q \in R$ because $(\partial/\partial x_1) + ... + (\partial/\partial x_{n-1})$ is tangent to R and $T_q R = (w(b_1) \cap ... \cap w(b_{n-2}) \cap w(\infty))(q)$. Therefore $b_1, ..., b_{n-2}$ are the roots of $\sum_{j=1}^{n-1} \prod_{k=1;k\neq j}^{n-1} (t + a_k) f_j(q)$ when $q \in R$; so $f_1 = ... = f_{n-1}$ on R since $b_1, ..., b_{n-2}$ are the roots of $\sum_{j=1}^{n-1} \prod_{k=1;k\neq j}^{n-1} (t + a_k)$ too, which implies that both polynomials are equal up to multiplicative factor (conversely, if $f_1 = ... = f_{n-1}$ on R then $(\partial/\partial x_1) + ... + (\partial/\partial x_{n-1})$ is tangent to this curve and R is defined by $x_1 = ... = x_{n-1}, y = 0$).

The change of coordinates between two of such system can be regarded as a diffeomorphism $(x_1, ..., x_{n-1}, y) \to G(x_1, ..., x_{n-1}, y)$. But G has to preserve R, S_0, the foliations of dimension $n-1$ defined by dx_1,..., dx_{n-1} and dy respectively (that is to say $w(-a_1)$,..., $w(-a_{n-1})$ and $w(\infty)$), and the origin. Therefore $G(x_1, ..., x_{n-1}, y) = (h_1(x_1), ..., h_1(x_{n-1}), h_2(y))$ where h_1, h_2 are one variable functions such that $h_1(0) = h_2(0) = 0$ and $h_1'(0) \neq 0$, $h_2'(0) \neq 0$.

Denote by J' the pull-back of J by the diffeomorphism G. Then $dx_j \circ J' = a_j dx_j$, $j = 1, ..., n-1$, and $dy \circ J' = a dy + \tilde{\alpha}'$ where

$$\tilde{\alpha}' = \sum_{j=1}^{n-1} h_1'(x_j)(h_2'(y))^{-1} f_j(h_1(x_1), ..., h_1(x_{n-1}), h_2(y))dx_j.$$

Now we may take h_1, h_2 in such a way that

$$h_1'(x_1)(h_2'(y))^{-1} f_1(h_1(x_1), ..., h_1(x_{n-1}), h_2(y)) = 1$$

on the curves $x_1 = ... = x_{n-1}$, $y = 0$, and $x_1 = ... = x_{n-1} = 0$. Indeed, first consider the function h_2 defined by $(h_2'(t))^{-1} f_1(0, ..., 0, h_2(t)) = 1$, $h_2(0) = 0$, and then the function h_1 defined by $h_1'(t)(h_2'(0))^{-1} f_1(h_1(t), ..., h_1(t), 0) = 1$, $h_1(0) = 0$; note that $h_1'(0) = 1$ since

$$h_1'(0)(h_2'(0))^{-1} f_1(0, ..., 0, 0) = (h_2'(0))^{-1} f_1(0, ..., 0, 0) = 1.$$

In other words, there exist coordinates $(x_1, ..., x_{n-1}, y)$ as before with a third additional property: $f_1 = ... = f_{n-1} = 1$ on the curve $x_1 = ... = x_{n-1}$, $y = 0$, and $f_1 = 1$ on the curve $x_1 = ... = x_{n-1} = 0$.

In turn, a change of coordinates between two system with this last property is given by two functions h_1, h_2 such that $h_1'(x_1)(h_2'(y))^{-1} = 1$ on the curves $x_1 = ... = x_{n-1}$, $y = 0$, and $x_1 = ... = x_{n-1} = 0$. Therefore h_1', h_2' are constant. In short, the only possible change of coordinates is a homothety by some $b \in \mathbb{K} - \{0\}$, and $\tilde{\alpha}'(x_1, ..., x_{n-1}, y) = \tilde{\alpha}(bx_1, ..., bx_{n-1}, by)$.

A germ at the origin of a map $\phi = (\varphi_1, ..., \varphi_{n-1})$ from S_0 to $\mathbb{K}^{n-1}$ will be called *admissible* if $\varphi_1 = ... = \varphi_{n-1} = 1$ on the curve $x_1 = ... = x_{n-1}$, $y = 0$, and $\varphi_1 = 1$ on the curve $x_1 = ... = x_{n-1} = 0$. Two admissible germs ϕ and $\bar{\phi}$ will be named *equivalent* if there exists $b \in \mathbb{K} - \{0\}$ such that $\bar{\phi}(x_1, ..., x_{n-1}, y) = \phi(bx_1, ..., bx_{n-1}, by)$.

From theorem 2.1, theorem 5.1 and system (13), applied to the last kind of coordinates system, follows (remark that in this last step the number a does not play any role, which is due to the fact that a Veronese web is determined by $w(\infty)$ and $J_{|w(\infty)}$):

Theorem 6.1. *Consider non-equal real numbers $a_1, ..., a_{n-1}$. One has:*
(1) Given a Veronese web of codimension 1 on a real or complex n-manifold N and any point $\in N$, there exist coordinates $(x_1, ..., x_{n-1}, y)$ around p such that $x_1(p) = ... = x_{n-1}(p) = y(p) = 0$ and the Veronese web is represented by

$$\gamma(t) = -\sum_{j=1}^{n-1} \left(\prod_{k=1;k\neq j}^{n-1} (t + a_k) f_j \right) dx_j + \prod_{k=1}^{n-1} (t + a_k) dy,$$

where $\tilde{\alpha} = \sum_{j=1}^{n-r} f_j dx_j$ *satisfies to the system*

$$\begin{cases} d\tilde{\alpha} \wedge dy = 0 \\ \left(d\left(\sum_{j=1}^{n-1} a_j f_j dx_j \right) - \tilde{\alpha} \wedge \dfrac{\partial \tilde{\alpha}}{\partial y} \right) \wedge dy = 0, \end{cases}$$

$f_1 = ... = f_{n-1} = 1$ *on the curve* $x_1 = ... = x_{n-1}$, $y = 0$, *and* $f_1 = 1$ *on the curve* $x_1 = ... = x_{n-1} = 0$.

(2) Let S_0 *be the surface of equation* $x_1 = ... = x_{n-1}$ *and let* $\phi = (\varphi_1, ..., \varphi_{n-1})$ *be a germ at the origin of a map from* S_0 *to* $\mathbb{K}^{n-1}$. *Assume* ϕ *admissible. Then there exists one and only one germ at the origin of* 1*-form* $\tilde{\alpha} = \sum_{j=1}^{n-r} f_j dx_j$, *which satisfies to the system of part (1) and such that* $f_{j|S_0} = \varphi_j$, $j = 1, ..., n-1$.

Moreover

$$\gamma(t) = -\sum_{j=1}^{n-1} \left(\prod_{k=1; k\neq j}^{n-1} (t + a_k) f_j \right) dx_j + \prod_{k=1}^{n-1} (t + a_k) dy,$$

defines a Veronese web of codimension 1 *around the origin.*

(3) Finally given two admissible germs at the origin ϕ *and* $\bar{\phi}$ *of maps from* S_0 *to* $\mathbb{K}^{n-1}$, *the germs of* 1*-codimensional Veronese webs associated to them by virtue of part (2) are equivalent, by diffeomorphism, if and only if* ϕ *and* $\bar{\phi}$ *are equivalent as admissible germs.*

The local classification of Veronese webs of codimension 1 is due to Turiel (see [16] whose exposition is closely followed here).

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