New classes of exact solutions of three-dimensional Navier–Stokes equations

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Abstract

New classes of exact solutions of the three-dimensional unsteady Navier–Stokes equations containing arbitrary functions and parameters are described. Various periodic and other solutions, which are expressed through elementary functions are obtained. The general physical interpretation and classification of solutions is given.

Keywords: Navier–Stokes equations, exact solutions, periodic solutions, three-dimensional equations, unsteady equations.

1 Class of motions of the viscous incompressible fluid under consideration

Self-similar, invariant, partially invariant, and certain other exact solutions of the Navier–Stokes equations including those with generalized separation o f variables were considered, for example, in [1–15]. Below, the term ”exact solutions” is used according to the definition given in [14, p. 10].

Three-dimensional unsteady motions of a viscous incompressible fluid are described by the Navier–Stokes and continuity equations:

$$\frac{\partial V_1}{\partial t}+V_1\frac{\partial V_1}{\partial x}+V_2\frac{\partial V_1}{\partial y}+V_3\frac{\partial V_1}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 V_1}{\partial x^2}+\frac{\partial^2 V_1}{\partial y^2}+\frac{\partial^2 V_1}{\partial z^2}\right), \quad (1)$$

$$\frac{\partial V_2}{\partial t}+V_1\frac{\partial V_2}{\partial x}+V_2\frac{\partial V_2}{\partial y}+V_3\frac{\partial V_2}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\left(\frac{\partial^2 V_2}{\partial x^2}+\frac{\partial^2 V_2}{\partial y^2}+\frac{\partial^2 V_2}{\partial z^2}\right), \quad (2)$$

$$\frac{\partial V_3}{\partial t}+V_1\frac{\partial V_3}{\partial x}+V_2\frac{\partial V_3}{\partial y}+V_3\frac{\partial V_3}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\nu\left(\frac{\partial^2 V_3}{\partial x^2}+\frac{\partial^2 V_3}{\partial y^2}+\frac{\partial^2 V_3}{\partial z^2}\right), \quad (3)$$

$$\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z}=0. \quad (4)$$

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Here x, y, and z are the Cartesian coordinates, t is time; V_1, V_2, and V_3 are the fuid-velocity components; P is pressure; and ρ and ν are the fluid density and kinematic viscosity, respectively. When writing Eqs. (1)–(4), it was assumed that the mass forces are potential and included in the pressure.

We consider the flow of a viscous incompressible fluid when the fluid-velocity vector on the z axis is directed along this axis. Near the z axis, the transverse velocity components are small and can be expanded in a Taylor series in terms of the transverse x and y coordinates. If we restrict ourselves to the main terms of the expansion in x and y for the velocity components, it is possible to obtain the following representation for the desired values after the corresponding analysis:

$$\begin{gathered} V_1 = x\Big(-\frac{1}{2}\frac{\partial F}{\partial z} + w\Big) + yv, \quad V_2 = xu - y\Big(\frac{1}{2}\frac{\partial F}{\partial z} + w\Big), \quad V_3 = F, \\ \frac{P}{\rho} = p_0 - \frac{1}{2}\alpha x^2 - \frac{1}{2}\beta y^2 - \gamma xy - \frac{1}{2}F^2 + \nu\frac{\partial F}{\partial z} - \int \frac{\partial F}{\partial t}\,dz, \end{gathered} \tag{5}$$

where p_0, α, β, and γ are arbitrary functions of time t setting the transverse pressure distribution; F, u, v, and w are unknown functions dependent on the coordinate z and t. The substitution of Eqs. (5) into Navier–Stokes Eq. (3) and continuity Eq. (4) results in identities, and Eqs. (1) and (2) become $A_n x + B_n y = 0$ $(n = 1,\, 2)$, where A_n and B_n represent certain differential expressions dependent on the variables z and t. The split in the variables x and y results in four equations $A_n = 0$ and $B_n = 0$ $(n = 1,\, 2)$, which can be transformed to the following form:

$$\frac{\partial^2 F}{\partial t \partial z} + F\frac{\partial^2 F}{\partial z^2} - \frac{1}{2}\Big(\frac{\partial F}{\partial z}\Big)^2 = -(\alpha+\beta) + \nu\frac{\partial^3 F}{\partial z^3} + 2(uv + w^2), \tag{6}$$

$$\frac{\partial u}{\partial t} + F\frac{\partial u}{\partial z} - u\frac{\partial F}{\partial z} = \gamma + \nu\frac{\partial^2 u}{\partial z^2}, \tag{7}$$

$$\frac{\partial v}{\partial t} + F\frac{\partial v}{\partial z} - v\frac{\partial F}{\partial z} = \gamma + \nu\frac{\partial^2 v}{\partial z^2}, \tag{8}$$

$$\frac{\partial w}{\partial t} + F\frac{\partial w}{\partial z} - w\frac{\partial F}{\partial z} = \frac{\alpha - \beta}{2} + \nu\frac{\partial^2 w}{\partial z^2}. \tag{9}$$

It is important to emphasize that solution (5) precisely satisfies Eqs. (1)–(4) for the viscous fluid motion by virtue of Eqs. (6)–(9).

For $\gamma = 0$, the structure of exact solution (5) and system (6)–(9) was obtained in [12] from other reasons by the investigation of the class of partially invariant solutions (the case of $\alpha = \beta = \gamma = 0$ was considered in [7]). In [12], the group classification of system (6)–(9) was carried out for $\gamma = 0$, which resulted in singling out two types of time dependences for the determining functions: (i) α and β are constant, and (ii) α and β are proportional to t^{-2} (the exact solutions of the Navier–Stokes equations with a reasonably simple structure correspond to these dependences).

In this work, we obtained new classes of exact solutions of system (6)–(9), when the determining functions contain a functional arbitrariness. The basic idea of the following analysis is that we can obtain a single isolated equation for the longitudinal velocity component $V_3 = F$ from system (6)–(9).

2 Reduction of system (6)–(9) to a single equation

We consider first the special class of exact solutions described by a single equation.

In Eqs. (6)–(9), we put

$$u = m\frac{\partial F}{\partial z} + A, \quad v = n\frac{\partial F}{\partial z} + B, \quad w = k\frac{\partial F}{\partial z} + C, \tag{10}$$

where m, n, k, A, B, and C are the desired functions of time t. We require that four Eqs. (6)–(9) coincide after the substitution of Eq. (10) in them. As a result, for determining the desired functions, we obtain the nonlinear system consisting of one algebraic equation and six ordinary differential equations:

$$mn + k^2 = \frac{1}{4}, \tag{11}$$

$$\frac{A - m'}{m} = \frac{B - n'}{n} = \frac{C - k'}{k} = 2(An + Bm + 2Ck), \tag{12}$$

$$\frac{\gamma - A'}{m} = \frac{\gamma - B'}{n} = \frac{\alpha - \beta - 2C'}{2k} = -\alpha - \beta + 2AB + 2C^2. \tag{13}$$

This system contains seven equations for nine functions—six functions m, n, k, A, B, and C from Eq. (10) and three functions α, β, and γ from Eqs. (6)–(9) (in this case, they are also treated as desired). It is possible to show that the last equation in Eq. (12) is the consequence of three other equations (11), (12). Therefore, three desired functions in system (11), (13) can, in general, be chosen arbitrarily.

Taking into account Eqs. (10)–(13), we reduce system (6)–(9) to a single equation

$$\frac{\partial^2 F}{\partial t \partial z} + F\frac{\partial^2 F}{\partial z^2} - \left(\frac{\partial F}{\partial z}\right)^2 = \nu\frac{\partial^3 F}{\partial z^3} + q\frac{\partial F}{\partial z} + p, \tag{14}$$

the functions $p = p(t)$ and $q = q(t)$ are defined by the relations

$$p = \frac{\gamma - A'}{m}, \quad q = \frac{A - m'}{m}. \tag{15}$$

The general property of Eq. (14). Suppose $F_0(z,t)$ is a solution of this equation. Then the function

$$F = F_0(z + \psi(t), t) - \psi'_t(t), \tag{16}$$

where $\psi(t)$ is an arbitrary function, is also a solution of Eq. (14).

For constructing solutions of system (11)–(13), it is necessary to distinguish two cases.

1°. *Case of* $m = n$. In this case, the general solution of system (11)–(13) can be represented as

$$
\begin{aligned}
& m = n = \tfrac{1}{2}\sin\varphi, \quad k = \tfrac{1}{2}\cos\varphi, \\
& A = B = \tfrac{1}{2}(q\sin\varphi + \varphi'_t\cos\varphi), \quad C = \tfrac{1}{2}(q\cos\varphi - \varphi'_t\sin\varphi), \\
& \alpha = \tfrac{1}{4}q^2 + \tfrac{1}{4}(\varphi'_t)^2 - \tfrac{1}{2}p(1-\cos\varphi) + C'_t, \\
& \beta = \tfrac{1}{4}q^2 + \tfrac{1}{4}(\varphi'_t)^2 - \tfrac{1}{2}p(1+\cos\varphi) - C'_t, \\
& \gamma = \tfrac{1}{2}p\sin\varphi + A'_t,
\end{aligned} \tag{17}
$$

where $p = p(t)$, $q = q(t)$, and $\varphi = \varphi(t)$ are arbitrary functions. For convenience, the free functions p and q in Eq. (17) are chosen so that system (6)–(9) is reduced to a single equation (14) with the same functions $p = p(t)$ and $q = q(t)$ as a result of the transformation (10), (17).

Thus, this important statement is proved. An arbitrary solution of Eq. (14) for arbitrary functions $p = p(t)$ and $q = q(t)$ generates an exact solution of the Navier–Stokes equations (1)–(4). This solution is described by the function $F = F(z,t)$ and Eqs. (5), (10), (17).

2°. *Case of* $m \neq n$. In this case, the general solution of system (11)–(13) can be obtained as follows. The functions $m = m(t)$, $k = k(t)$, and $q = q(t)$ are set arbitrarily under the condition $m^2 + k^2 \neq 1/4$. The remaining functions included in system (11)–(13) and Eq. (14) are calculated sequentially from the formulas

$$
\begin{gathered}
n = \frac{1-4k^2}{4m}, \\
A = mq + m'_t, \quad B = nq + n'_t, \quad C = kq + k'_t, \\
p = \frac{A'_t - B'_t}{n-m}, \quad \alpha = AB + C^2 + C'_t - \frac{1}{2}p(1-2k), \\
\beta = AB + C^2 - C'_t - \frac{1}{2}p(1+2k), \quad \gamma = pm + A'_t.
\end{gathered} \tag{18}
$$

In this case, the coefficient $p = p(t)$ in Eq. (14) is determined through the functions $m = m(t)$, $k = k(t)$, and $q = q(t)$ and their derivatives (instead of being set arbitrarily as in the case of $m = n$). An attempt to set $p = p(t)$ directly instead of the function m (or k) results in a nonlinear ordinary differential equation of the second order for the function m (or k) with an arbitrary function $q = q(t)$.

We consider how to choose the function $q = q(t)$ so that the identity $p \equiv 0$ is satisfied. From the expression for p in Eq. (18), we have $A = B + s_0$, where s_0 is an arbitrary constant. From here, taking into account Eqs. (18) for A, B, and n, we find the function

$$
q = \frac{4s_0 m - 8(mm'_t + kk'_t)}{4(m^2+k^2)-1} + \frac{m'_t}{m} \qquad (\text{for } p = 0). \tag{19}
$$

3 Exact solutions of Eq. (14) for various $p = p(t)$ and $q = q(t)$

1°. Functional separable solution:

$$F = -a'_t(t) + b(t)[z + a(t)] - \frac{6\nu}{z + a(t)}, \quad q = -4b, \quad p = b'_t + 3b^2,$$

where $a = a(t)$ and $b = b(t)$ are arbitrary functions.

2°. Periodic solutions in the form of the product of functions of different arguments:

$$\begin{aligned} F &= a(t)\sin(\sigma z + B), \quad a(t) = C\exp\Bigl[-\nu\sigma^2 t + \int q(t)\,dt\Bigr], \\ p &= -\sigma^2 a^2(t), \quad q = q(t) \text{ is an arbitrary function,} \end{aligned} \tag{20}$$

where B, C, and σ are arbitrary constants. Putting in Eq. (20) that $q(t) = \nu\sigma^2 + \varphi'_t(t)$, where $\varphi(t)$ is a periodic function, we obtain a solution periodic in both arguments z and t.

Example. Consider the stationary case. In Eqs. (17), (20), we put

$$\varphi = 0, \quad a = -\frac{a_1 + a_2}{\sigma}, \quad q = \nu\sigma^2 = 2a_1, \quad p = -a^2\sigma^2, \quad \sigma = (2a_1/\nu)^{1/2}.$$

As a result, using Eqs. (5) and (10), we obtain the solution

$$V_1 = a_1 x, \quad V_2 = [(a_1 + a_2)\cos(\sigma z) - a_1]y, \quad V_3 = -\frac{a_1 + a_2}{\sigma}\sin(\sigma z),$$

which describes the three-dimensional flow of a fluid layer between two flat elastic films (the film position depends on the values of $z = 0$ and $z = 2\pi/\sigma$), the surfaces of which are stretched according to the law $V_1 = a_1 x$ and $V_2 = a_2 y$.

3°. Generalized separable solutions exponential in z:

$$F = a(t)e^{-\sigma z} + b(t), \quad p = 0, \quad q = \frac{a'_t}{a} - \sigma b - \sigma^2\nu, \tag{21}$$

where $a = a(t)$ and $b = b(t)$ are arbitrary functions. Choosing $a(t)$ and $b(t)$ to be periodic functions, we obtain a solution periodic in time.

Formulas (20) and (21) together with Eqs. (5), (10), (17) define two classes of solutions of the Navier–Stokes equations dependent on several arbitrary functions.

4°. Solution (21) can be represented in the form

$$F = a_0 \exp\Big[-\sigma z + \sigma^2 \nu t + \int (q + \sigma b)\,dt\Big] + b(t), \quad p = 0, \tag{22}$$

where $b = b(t)$ and $q = q(t)$ are arbitrary functions and a_0 is an arbitrary constant. Formula (22) defines a new class of exact solutions of the Navier–Stokes equations with the help of Eqs. (5), (18), for $p = 0$ and Eq. (19).

5°. Exact solution in the form of the product of functions of different arguments:

$$F = a(t)(C_1 e^{\sigma z} + C_2 e^{-\sigma z}), \quad p = 4C_1 C_2 \sigma^2 a^2(t), \quad q = \frac{a'_t}{a} - \sigma^2 \nu,$$

where $a = a(t)$ is an arbitrary function, C_1, C_2, and σ are arbitrary constants.

6°. Monotonic traveling-wave solution:

$$F = -6\nu\sigma \tanh[\sigma(z - \lambda t) + B] + \lambda, \quad p = 0, \quad q = 8\nu\sigma^2.$$

7°. Unbounded periodic traveling-wave solution:

$$F = 6\nu\sigma \tan[\sigma(z - \lambda t) + B] + \lambda, \quad p = 0, \quad q = -8\nu\sigma^2.$$

8°. Functional separable solution:

$$F = \frac{a(t)}{\lambda(t)} \exp[-\lambda(t) z] + b(t) + c(t) z,$$

where the functions $a = a(t)$, $b = b(t)$, $c = c(t)$, and $\lambda = \lambda(t)$ satisfy the system of ordinary differential equations

$$\lambda'_t = -c\lambda, \quad a'_t = (\nu\lambda^2 + q + 2c + b\lambda)a, \quad c'_t = c^2 + qc + p.$$

Here, three of the six functions $a(t)$, $b(t)$, $c(t)$, $\lambda(t)$, $p(t)$, and $q(t)$ can be set arbitrarily.

9°. Functional separable solution:

$$F = \omega(t) z + \frac{\xi(t)}{\theta(t)} \sin[\theta(t) z + a], \tag{23}$$

where a is an arbitrary constant, and the functions $\omega = \omega(t)$, $\xi = \xi(t)$, and $\theta = \theta(t)$ are described by the system of ordinary differential equations

$$\theta'_t = -\omega\theta, \quad \omega'_t = \omega^2 + q(t)\omega + p(t) + \xi^2, \quad \xi'_t = [2\omega - \nu\theta^2 + q(t)]\xi. \tag{24}$$

In this system, it is possible to treat the functions $\theta(t)$ and $\xi(t)$ as arbitrary, whereas the functions $\omega(t)$, $p(t)$, and $q(t)$ are elementarily determined (without quadratures). A periodic solution (23) corresponds to periodic functions $\theta(t)$ and $\xi(t)$.

10°. Functional separable solution:

$$F = \omega(t)z + \frac{\xi(t)}{\theta(t)}\Big[C_1 e^{\theta(t)z} + C_2 e^{-\theta(t)z}\Big], \tag{25}$$

where C_1 and C_2 are arbitrary constants, and the functions $\omega = \omega(t)$, $\xi = \xi(t)$, and $\theta = \theta(t)$ are described by the system of ordinary differential equations

$$\theta'_t = -\omega\theta, \quad \omega'_t = \omega^2 + q(t)\omega + p(t) - 4C_1C_2\xi^2, \quad \xi'_t = [2\omega + \nu\theta^2 + q(t)]\xi. \tag{26}$$

Remark. See also [8, 13, 15] for exact solutions of Eq. (14) with $q = 0$.

4 Reduction of system (6)–(9) to two equations

We describe two cases of reducing system (6)–(9) to a single isolated nonlinear equation for the longitudinal velocity F and a second equation for determining a new auxiliary function.

1°. *First case*. By letting

$$u = a^2G, \quad v = -b^2G, \quad w = abG, \quad \alpha = \beta, \quad \gamma = 0, \tag{27}$$

where a and b are arbitrary constants, we reduce system (6)–(9) to an isolated equation for the longitudinal velocity F and an additional equation for the function $G = G(z,t)$:

$$\frac{\partial^2 F}{\partial t\partial z} + F\frac{\partial^2 F}{\partial z^2} - \frac{1}{2}\Big(\frac{\partial F}{\partial z}\Big)^2 = -2\alpha + \nu\frac{\partial^3 F}{\partial z^3}, \tag{28}$$

$$\frac{\partial G}{\partial t} + F\frac{\partial G}{\partial z} - G\frac{\partial F}{\partial z} = \nu\frac{\partial^2 G}{\partial z^2}. \tag{29}$$

2°. *Second case*. In Eqs. (6)–(9), let

$$\begin{aligned}
u &= \frac{1}{2}\sin\varphi\Big(\frac{\partial F}{\partial z} + q\Big) + a^2\Theta,\\
v &= \frac{1}{2}\sin\varphi\Big(\frac{\partial F}{\partial z} + q\Big) - b^2\Theta,\\
w &= \frac{1}{2}\cos\varphi\Big(\frac{\partial F}{\partial z} + q\Big) + ab\Theta,\\
\alpha &= \frac{1}{4}q^2 - \frac{1}{2}p(1-\cos\varphi) + \frac{1}{2}q'_t\cos\varphi,\\
\beta &= \frac{1}{4}q^2 - \frac{1}{2}p(1+\cos\varphi) - \frac{1}{2}q'_t\cos\varphi,\\
\gamma &= \frac{1}{2}p\sin\varphi + \frac{1}{2}q'_t\sin\varphi,
\end{aligned} \tag{30}$$

where $p = p(t)$ and $q = q(t)$ are arbitrary functions, and a and b are arbitrary constants, $\Theta = \Theta(z,t)$ is an unknown function, and φ is the constant determined from the transcendental equation

$$(a^2 - b^2)\sin\varphi + 2ab\cos\varphi = 0. \tag{31}$$

As a result, system (6)–(9) is reduced to two equations:

$$\frac{\partial^2 F}{\partial t\partial z} + F\frac{\partial^2 F}{\partial z^2} - \Big(\frac{\partial F}{\partial z}\Big)^2 = \nu\frac{\partial^3 F}{\partial z^3} + q\frac{\partial F}{\partial z} + p, \tag{32}$$

$$\frac{\partial \Theta}{\partial t} + F\frac{\partial \Theta}{\partial z} - \Theta\frac{\partial F}{\partial z} = \nu\frac{\partial^2 \Theta}{\partial z^2}. \tag{33}$$

The nonlinear equation (32) for F coincides with Eq. (14) and can be treated independently (some of its exact solutions were described previously), and Eq. (33) is linear with respect to the desired function Θ.

For stationary solutions of Eqs. (32) and (28) (for constant p, q, and α), the nonstationary equations (33) and (29) are linear separable equations, whose solutions can be obtained using the Laplace transform in time.

Equation (32) (and Eq. (28)) admits an obvious degenerate solution $F = a(t)z + b(t)$; in this case, the corresponding Eq. (33) (and Eq. (29)) can be reduced to the linear heat equation.

System (28), (29) for an arbitrary function $\alpha = \alpha(t)$ has a solution in the form

$$\begin{aligned} F &= az^2 + b(t)z + \frac{1}{4a}[b^2(t) - 2b'_t(t) - 4\alpha(t)], \\ w &= A(t)z^2 + B(t)z + C(t), \end{aligned}$$

where a is an arbitrary constant ($a \neq 0$), $b(t)$ is an arbitrary function, and the functions $A = A(t)$, $B = B(t)$, and $C = C(t)$ are described by the system of ordinary differential equations, which is not presented here.

5 Interpretation and classification of the flows under consideration

Arbitrary fluid flows having two symmetry planes admit a representation of the type of Eq. (5) in the vicinity of the line of intersection of these planes (in the adopted notation, the planes intersect in the z axis). Such flows include the axisymmetric flows, combinations of axisymmetric flows with rotation around of the z axis (in particular, the von Karman type flows), plane flows symmetric with respect to a straight line, flows in rectilinear impenetrable and porous pipes with elliptic and rectangular cross sections, fluid jets flowing from orifices of elliptic and rectangular shapes, etc. (see also [10, 11]).

It is convenient to treat the axial flows described by Eqs. (5) as a nonlinear superposition of a translatory (nonuniform) flow along the z axis and a linear shear flow of a special type. In the vicinity of the point $z = z_0$ lying on the axis, the components of fluid velocity taking into account Eq. (5) can be represented as

$$\begin{gathered} V_k = F\delta_{k3} + G_{km}X_m; \\ G_{11} = -\tfrac{1}{2}F_z + w, \quad G_{12} = v, \quad G_{21} = u, \quad G_{22} = -\tfrac{1}{2}F_z - w, \\ G_{13} = G_{23} = G_{31} = G_{32} = 0, \quad G_{33} = F_z; \\ X_1 = x, \quad X_2 = y, \quad X_3 = z - z_0. \end{gathered} \tag{34}$$

Here $k,\, m = 1,\, 2,\, 3$; the G_{km} are shear matrix components; the summation is assumed over the repeated subscript m; δ_{km} is the Kronecker delta; and F_z is the partial derivative with respect to z. All values in Eq. (34) are taken for $z = z_0$. The vanishing of the sum $G_{11} + G_{22} + G_{33} = 0$ of diagonal elements is a consequence of fluid incompressibility.

An arbitrary matrix $\|G_{km}\|$ can be represented in the form of the sum of a symmetric and an asymmetric matrix

$$\begin{gathered} \|G_{km}\| = \|E_{km}\| + \|\Omega_{km}\|, \\ E_{km} = E_{mk} = \tfrac{1}{2}(G_{km} + G_{mk}), \quad \Omega_{km} = -\Omega_{mk} = \tfrac{1}{2}(G_{km} - G_{mk}). \end{gathered} \tag{35}$$

In turn, the symmetric matrix $\|E_{km}\|$ (in this case, it can be identified with the strain-rate tensor) can be reduced to a diagonal form with diagonal elements E_1, E_2, and E_3, which are roots of the cubic equation $\det \|E_{km} - \lambda\delta_{km}\| = 0$ for λ, by appropriately rotating the system of coordinates.

For this flow (34), the diagonal elements determining the intensity of the tension (compression) motion along the respective axes are calculated from the formulas

$$E_{1,2} = -\frac{1}{2}F_z \pm \frac{1}{2}\sqrt{4w^2 + (u+v)^2}, \quad E_3 = F_z. \tag{36}$$

The splitting of the shear coefficient matrix $\|G_{km}\|$ into symmetric and asymmetric parts (35) corresponds to the representation of the velocity field of the linear shear flow of the fluid as a superposition of a linear deformational flow with the tension coefficients E_1, E_2, and E_3 along the principal axes and rotations of the fluid as a solid body with the angular velocity $\vec{\omega} = (\Omega_{32}, \Omega_{13}, \Omega_{21})$.

For this flow (34), we have $\Omega_{32} = \Omega_{13} = 0$ and the fluid rotates around the z axis with the angular velocity

$$\Omega_{21} = \frac{1}{2}(u - v). \tag{37}$$

It is easy to show that Eqs. (36) and (37) remain valid for an arbitrary point (x_0, y_0, z_0) of the flow (5) under consideration.

The analysis of Eqs. (36), (37) enables us to single out certain characteristic types of flows indicated in the classification table.

Table 1: Classification of axial flows described by Eqs. (5)

Type of flow	Desired functions	Functions included in pressure
Axisymmetric	$u=v=w=0$	$\alpha=\beta,\ \gamma=0$
Combination of axisymmetric flow and rotation around of the z axis	$w=0,\ v=-u$	$\alpha=\beta,\ \gamma=0$
Pure deformational (without rotation)	$u=v$	α,β,γ are arbitrary functions
General axial	$u\neq v$	α,β,γ are arbitrary functions

6 Some generalizations

Let $V_1(x,y,z,t)$, $V_2(x,y,z,t)$, $V_3(x,y,z,t)$, and $P(x,y,z,t)$ be a certain solution of Navier–Stokes equations (1)–(4). Then the set of functions

$$\begin{aligned}\bar V_1 &= V_1(x-x_0,y-y_0,z-z_0,t)+x_0',\\ \bar V_2 &= V_2(x-x_0,y-y_0,z-z_0,t)+y_0',\\ \bar V_3 &= V_3(x-x_0,y-y_0,z-z_0,t)+z_0',\\ \bar P &= P(x-x_0,y-y_0,z-z_0,t)-\rho(x_0''x+y_0''y+z_0''z),\end{aligned} \tag{38}$$

where $x_0=x_0(t)$, $y_0=y_0(t)$, and $z_0=z_0(t)$ are arbitrary functions (primes denote the derivatives with respect to t), also gives the solution of Eqs. (1)–(4) [4, 15]. The combination of Eqs. (5) and (38) for $z_0=0$ determines an exact solution of the Navier–Stokes equations, which can be treated as the generalized axial flow with the z axis moving along the plane x, y, according to the law $x=x_0(t)$, $y=y_0(t)$. The indicated solution can be used for the mathematical simulation of destructive atmospheric phenomena such as waterspouts and tornados.

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