Nonlinear Phenomena in Complex Systems, vol. **23**, no. 1 (2020), pp. 33 - 53

QFT in Flat Space and Measurability II. Perturbation Theory for a Scalar Field Model

A. Shalyt-Margolin*
Research Institute for Nuclear Problems, Belarusian State University,
11 Bobruiskaya Str, 220040 Minsk, BELARUS
(Received 20 July, 2019)

In the present paper the quantum theory for a scalar field model has been considered by the author in terms of the **measurability** concept introduced in his previous works. This theory is studied within the scope of the lattice approach. It is shown that the perturbation theory for such a model at the energies $E \ll E_p$ leads to the completely finite contributions into the loop amplitudes of all physical processes, and also to finite values of both **bare** and experimental (i.e. renormalized) quantities associated with the above-mentioned energy region. Besides, it is demonstrated that a continuum limit of the lattice model includes no ultra-violet divergences.

PACS numbers: 03.70.+k,04.60.Nc
Keywords: quantum field theory, lattice, measurability, scalar model, perturbation theory
DOI: https://doi.org/10.33581/1561-4085-2020-23-1-33-53

1. Introduction

This paper is a continuation of the previous works by the author [1–7] and, specifically, of the paper [7].

In [7] free quantum fields at low energies $E \ll E_p$ have been studied in terms of the **measurability** concept [1–5] within the scope of a scalar field model.

In the present paper a perturbation theory for the indicated model at the same energies is subjected to a **measurable** consideration. The principal objective is to construct at low energies $E \ll E_p$ the *correct* perturbation theory for a scalar field model in the **measurable** picture.

In this case the *correct* perturbation theory is understood as a quantum theory considered within the lattice approach and

(a) including only the integrals making contributions into the corresponding amplitudes in QFT the integration domain of which lies within the indicated energy range;

(b) containing only finite values of all available parameters, both experimental (i.e. renormalized) and *bare*, expressed in terms of quantities from the same energy range;

(c) having a continuum limit without ultraviolet divergences.

The structure of the paper is as follows.

Section 2 briefly presents earlier obtained results which are significant in this consideration, with all required refinements and additions.

In Section 3 the principal difference in the occurrence of ultraviolet divergences in canonical QFT and its **measurable** analog is revealed.

Section 4 that is of major importance presents the results associated with the above points (a), (b), and (c).

It should be noted that, because in [7] QFT in the **measurable** consideration has been presented as a lattice theory, we can adequately use the mathematical apparatus of this theory for derivation of key results in this new work.

*E-mail: a.shalyt@mail.ru

2. Important preliminary information and some refinements

2.1. Measurability concept in quantum theory and gravity

In this Section we briefly consider some of the results from [1–7] which are essential for subsequent studies. Without detriment to further consideration, in initial definitions we lift some unnecessary restrictions and make important specifications.

Presently, many researchers are of the opinion that at very high energies (Planck's or trans-Planck's) the ultraviolet cutoff exists that is determined by some maximal momentum.

Therefore, it is further assumed that there is a maximal bound for the measurement momenta $p = p_{max}$ represented as follows:

$$p_{max} \doteq p_\ell = \hbar/\ell, \tag{1}$$

where ℓ is some small length and $\tau = \ell/c$ is the corresponding time. Let us call ℓ the *primary* length and τ the *primary* time.

Without loss of generality, we can consider ℓ and τ at Planck's level, i.e. $\ell \propto l_p, \tau = \kappa t_p$, where the numerical constant κ is on the order of 1. Consequently, we have $E_\ell \propto E_p$ with the corresponding proportionality factor, where $E_\ell \doteq p_\ell c$.

Explanation.
In the theory under study it is not assumed from the start that there exists some minimal length l_{min} and that ℓ is such. In fact, the minimal length is defined with the use of Heisenberg's Uncertainty Principle (HUP) $\Delta x \cdot \Delta p \geq \frac{1}{2}\hbar$ or its generalization to high (Planck) energies – Generalized Uncertainty Principle (GUP) [8–16], for example, of the form [8]

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_p^2 \frac{\Delta p}{\hbar}, \tag{2}$$

where α' is a constant on an order of 1. Evidently the formula (2) initially leads to the minimal length $\widetilde{\ell}$ on an order of the Planck length $\widetilde{\ell} \doteq 2\sqrt{\alpha'} l_p$. Besides, other forms of GUP [16] also lead to the minimal length.

Thus, we should note that in all the works l_{min} is actually (but not explicitly) introduced on the basis of some measuring procedure (different forms of the Generalized Uncertainty Principle (GUP)). In any form GUP in turn is a high-energy generalization of HUP. But in the original proof of HUP a planar geometry of the initial space-time was actively used [17]. Extension of this principle to other pairs of conjugate variables is also valid only for quantum mechanics in zero-curvature spaces [18]. As HUP is a local principle, at low energies in the curved space-time, by virtue of Einstein's Equivalence Principle, we can consider that in a fairly small neighborhood of an any point the geometry is planar an hence HUP is valid too. But all the results obtained point to the fact that l_{min} should be at a level of l_p,i.e. $l_{min} \propto l_p$, or even should be smaller. As noted in the Section 2 of [7], at the Planck scales Einstein's Equivalence Principle is obviously inapplicable, and there is no way to use the measuring procedure ignoring the space geometry at these scales. Meantime, none of the GUP forms [16] makes an effort to include it and is hardly completely correct. Moreover, there are some serious arguments against GUP as demonstrated in Section IX of the review paper [16]. The foregoing considerations support argumentation against the introduction of l_{min} from the start.

Because of this, in the present work the validity of this principle is not implied from the start too. GUP is given merely as an example. As p_{max} (1) is taken at Planck's level, it is clear that HUP is inapplicable. Taking this into consideration, the existence of a certain minimal length $\widetilde{\ell}$ is not mandatory. So, we start from the *primary* length ℓ and the *primary* time τ. The whole formalism, developed in [1–5] on condition that ℓ is the minimal length, is valid for the case when ℓ is the *primary* length but now we can lift the formal requirement for involvement of l_{min} in the theory from the start.

There is one more barrier for the use of l_{min} in the theory as indicated in [15] and other works (for example, [16]). In the above-mentioned papers, it has been noted that there is a nonzero minimal uncertainty in position, i.e. l_{min} implies that there is no physical state which is a position eigenstate since an eigenstate would, of course, have zero uncertainty in position. So, in this case in a quantum theory we have the momentum representation rather than the position representation, and the quantum theory becomes very depleted.

The question arises whether the introduction of p_{max} is naturally associated with the involvement of a minimal length. But this is the case only when at the energies E_{max} corresponding to p_{max} we have the substantiated measuring procedure. Unfortunately, this is not the case.

Note that in the canonical QFT in continuous space-time (i.e. without l_{min})[19–22] measurements of the contributions in the loop amplitudes involve the standard cut-off procedure for some large (maximal) momentum $p_{cut} \doteq p_{max}$. Then it is demonstrated that the theory at low energies $p \ll p_{cut}$ is in fact independent of the selection of $p_{cut} \doteq p_{max}$. Of course, the theory still remains to be continuous [19–22]. In this case we make another step forward, relating the corresponding length $\ell = \hbar/p_{max}$ to p_{max} and constructing on its basis a low-energy theory very close to the initial continuous theory. Now, we have the naturally derived parameter ℓ for the construction of a high-energy deformation of this theory at the energies $E \approx E_{max}$ within the scope of determining the physical theory deformation [23]. So, we start from the *primary* length ℓ and the *primary* time τ. The whole formalism, developed in [1–5] on condition that ℓ is the minimal length, is valid for the case when ℓ is the *primary* length but now we can lift the formal requirement for involvement of l_{min} in the theory from the start.

In what follows we mainly make references to [5] and [7]. In particular, the basic definitions of **Primary Measurability**, **Generalized Measurability**, **Primarily Measurable Quantities** (**PMQ**), **Primarily Measurable Momenta** (**PMM**), **Generalized Measurable Quantities** (**GMQ**) and the like are given in Section II of [5].

Besides, in Section III from [5] it has been demonstrated how, at low energies $E \ll E_p$, the arbitrary metric $g_{\mu\nu}(x)$ may be derived in terms of **measurable** quantities. In the process it is important to note the following.

Remark 2.1
According to the present approach, there is no relativistic invariance (RI) from the canonical statement (for example, [22]). Specifically, for momenta **PMM** this is obvious just from the start. Indeed, as at low energies $E \ll E_p$ a set of four-momenta **PMM** is limited and is not space, it is not retained by the Lorentz Group (LG) or Poincare Group (PG). As a matter of fact, for the scalar

$$m^2c^2 = \eta_{ab}p^a p^b = \frac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2, \qquad (3)$$

the quantities E, p_x^2, p_y^2, p_z^2 have no limits because boosts may result in values of any highness, conflicting with the condition $E \ll E_p$. Because of this, in the case under study we can speak only about the transformations from LG or PG which retain **PMM** at low energies $E \ll E_p$ (or same $E \ll E_\ell$).

However, as noted in Section 2.2, RI should be violated at high energies not only in the model proposed but also in the general case of the well-known QFT. In what follows, unless the contrary is stated, QFT is considered at the energies $E \ll E_p$ (same $E \ll E_\ell$).

2.2. Relativistic invariance, equivalence-principle applicability boundary, and QFT in flat space

The canonical quantum field theory (QFT) [19–22] is a local relativistically-invariant theory considered in continuous space-time with a plane geometry, i.e with the local Minkowski metric $\eta_{\mu\nu}(\overline{x})$. And this assumption is valid for all the energy range. Still, it is quite clear that the quantum processes associated with QFT (particle collisions, decay, ...) can introduce perturbations into the space-time geometry, varying its curvature. But as QFT is a local theory, a strong Equivalence Principle (EP) [24] enables one, in a **sufficiently small** region $\mathcal{V}_r$ of the fixed point, to consider space-time as a flat space in this case too. Consequently, we naturally think about the applicability boundary of this principle. In Section 2 of [7] this problem has been thoroughly studied.

In essence, **sufficiently small** $\mathcal{V}_r$ means that the region $\mathcal{V}^{'}$, for which $\overline{x} \in \mathcal{V}^{'}_{r'} \subset \mathcal{V}_r$ with $r' < r$ (here r, r' are characteristic spatial sizes of $\mathcal{V}_r$ and $\mathcal{V}'_r$, respectively), satisfies the condition $g_{\mu\nu}(\overline{x}) \equiv \eta_{\mu\nu}(\overline{x})$, where $\eta_{\mu\nu}(\overline{x})$ is Minkowski metric. In this way we can construct the sequence

$$\begin{gathered} \ldots \subset \mathcal{V}^{''}_{r''} \subset \mathcal{V}^{'}_{r'} \subset \mathcal{V}_r, \\ \ldots < r^{''} < r' < r. \end{gathered} \tag{4}$$

The problem arises: is there any lower limit for the sequence in formula (4)?

The answer is positive. Currently, there is no doubt that at very high energies (on the order of Planck energies $E \approx E_p$), i.e. on Planck scales, $l \approx l_p$ quantum fluctuations of any metric $g_{\mu\nu}(\overline{x})$ are so high that in this case the geometry determined by $g_{\mu\nu}(\overline{x})$ is replaced by the "geometry" following from **quantum foam** that is defined by great quantum fluctuations of $g_{\mu\nu}(\overline{x})$,i.e. by the characteristic spatial sizes of the quantum-gravitational region (for example, [26–31]). The above-mentioned geometry is drastically differing from the locally smooth geometry of continuous space-time and EP in it is no longer valid [32–39]. Actually, the **quantum foam** is not geometry in a common sense as locally it is determined by a set of different metrics, each of which is taken into consideration with its statistical weight [29].

From this it follows that the region $\mathcal{V}_{\overline{r},\overline{t}}$ with the characteristic spatial size $\overline{r} \approx l_p$ (and hence with the temporal size $\overline{t} \approx t_p$) is the lower (approximate) limit for the sequence in (4).

In this way EP has the applicability boundary that, at least, lies in the region of Planck energies and hence the relativistic invariance must be violated at the same energy scales because its applicability necessitates space-time with the locally flat geometry, just supported by EP.

It should be noted that initially strong EP has been formulated for the macroscopic case (i.e. for the space-time domains of great size) that is beyond quantum consideration. On extension of this principle to microscopic domains, the problem of its applicability boundaries is absolutely natural.

It is difficult to find the exact lower limit for the sequence in formula (4) – it seems to be dependent on the processes under study. Section 2 of [7] presents the arguments that it should be associated with the energy scales $E \ll E_p$.

Therefore, it is assumed that the Equivalence Principle is valid for the locally smooth space-time and this suggests that all the energies E of the particles in the most general form meet the necessary condition

$$E \ll E_p. \tag{5}$$

As validity of RI requires the applicability of EP, we can consider the condition (5) a necessary condition for the validity of RI. Then, if not stipulated otherwise, we can assume that the condition (5) is valid.

The canonical quantum field theory (QFT) [19–22] is a local theory considered in continuous space-time with a local plane geometry, i.e with the Minkowski metric $\eta_{\mu\nu}(\overline{x})$. In addition, it is assumed that all objects in QFT are point-like. However, as noted above, this assumption will be true to a certain limit: the assumptions that (a) even local space-time geometry is plane and (b) all objects in QFT are point-like have natural applicability boundaries directly specifying the EP applicability boundary.

Within the scope of the canonical QFT, the process of passage to more higher energies without a change in the local curvature has no limits [19–22], just this fact is the reason for ultraviolet divergences in QFT.

However, on passage to the Planck energies

$E \approx E_p$ (Planck scales $l \approx l_p$), the space in the Planck neighborhood $\mathcal{V}_{\overline{r},\overline{t}}$ of the point $\overline{x}$ one cannot consider flat even locally and in this case (as noted above) EP is not valid.

Then we introduce the following assumption:

Assumption 2.2
In the canonical QFT in calculations of the quantities it is wrong to sum (or same consider within a single sum) the contributions corresponding to space-time manifolds with locally nonzero or zero curvatures since these contributions are associated with different processes: (1) with the existence of a gravitational field that, in principle, can hardly be excluded; (2) in the absence of a gravitational field.
From the start, we can isolate the case when EP is valid and hence RI takes place (at sufficiently low energies, specifically satisfying the condition (5)) from the cases when EP becomes invalid (for example, Planck energies $E \approx E_p$).

Remark 2.3
According to **Assumption 2.2**, we should consider two limiting cases:
(a) low energies $E \ll E_p$ and
(b) very high (essentially maximal) energies $E \approx E_p$.
Then it should be noted that, as all the experimentally involved energies E are low, they satisfy condition a). Specifically, for LHC maximal energies are $\sim$ 10 TeV $= 10^4$ GeV, that is by 15 orders of magnitude lower than the Planck energy $\sim 10^{19}$ GeV.
Moreover, the characteristic energy scales of all fundamental interactions also satisfy condition a). Indeed, in the case of strong interactions this scale is $\Lambda_{QCD} \sim 200$ MeV; for electroweak interactions this scale is determined by the vacuum average of a Higgs boson and equals $v \approx 246$ GeV; finally, the scale of the (Grand Unification Theory (GUT)) M_{GUT} lies in the range of $\sim 10^{14}$–10^{16} GeV. It is obvious that all the above figures satisfy condition a).
Thus, only the expected characteristic energy scale of quantum gravity satisfies condition b).

From **Remark 2.3** it directly follows that even very high energies arising on unification of all the interaction types $M_{GUT} \approx 10^{14}$–10^{16} GeV (except gravitational), satisfy the condition (5). At the same time, it is clear that the RI validity requirement in canonical QFT [19–22], due to the action of Lorentz boost (or same hyperbolic rotations) (formula (3) in [25]), results in however high momenta and energies. But it has been demonstrated that unlimited growth of the momenta and energies is impossible because in this case we fall within the energy region, where the conventional quantum field theory is invalid. Just this has been indicated in **Remark 2.1** for a **measurable** consideration. This section supports the validity of the fact in the general case of the canonical QFT in continuous space-time as well.
Note that at the present time there are experimental indications that RI is violated in QFT on transition to higher energies (for example, [40]). Besides, one should note important recent works associated with EP applicability boundaries and violation in nuclei and atoms at low energies (for example [41]). We can mention other works indicating the applicability boundaries of EP for specific processes, especially associated with the context of this paper (for example, [42, 43]). Proceeding from the above, the requirement for RI and EP is possible only within the scope of the condition (5).
Due to the condition $E \ll E_p$ and to the results of Section 2 in [7], all conclusions made in this section are valid both for the canonical Quantum theory in continuous space-time, [19–22], and for its **measurable** analog in Section 2 of [7] (Subsection 2.1 in the present paper).

Remark 2.4
Why in canonical QFT it is so important to never forget about the fact that space-time has a flat geometry, or the same possesses the Minkowski metric $\eta_{\mu\nu}(\overline{x})$? Simply, in the contrary case we should refuse from some fruitful methods and

from the results obtained by these methods in canonical QFT, in particular from Wick rotation [22]. In fact, in this case the time variable is replaced by $t \mapsto it \doteq t_E$, and the Minkowski metric $\eta_{\mu\nu}(\overline{x})$ is replaced by the four-dimensional Euclidean metric

$$ds^2 = dt_E^2 + dx^2 + dy^2 + dz^2. \tag{6}$$

Clearly, such replacement is possible only in the case when from the start space-time (locally) has a flat geometry, i. e. possesses the Minkowski metric $\eta_{\mu\nu}(\overline{x})$. This is another argument supporting the key role of the EP applicability boundary. Otherwise, when we go beyond this boundary, Wick rotation becomes invalid. Naturally, some other methods of canonical QFT will lose their power too.

2.2.1 *In this paper we consider two limiting energy scales:*$E \ll E_p$ *and* $E \approx E_p$. *Of course, the whole energy range* $0 < E \leq E_p$ *is not reduced to these scales. But, assuming that the onset of the Universe had started from the energies close to the Planck energies* E_p *and its expansion was very fast, the above boundary is reasonable. An additional argument in support is the fact that, as noted in* **Remark 3.2**, *the energy ranges for all the fundamental models combining various interactions are associated with these scales.*

2.2.2 It is clear that the equivalence-principle applicability boundaries (EPAB) in each specific case are dependent on the particular processes studied in particle physics. *In what follows we consider only the energy range* $E \ll E_p$ *assuming that the common EPAB lies within* $0 < E \leq\approx 10^{-2}E_p$.

2.3. Quantum field theory in measurable format

In Section 4 of [7] it has been shown that at low energies $E \ll E_p$ (same $E \ll E_\ell$) we can construct a **measurable** QFT variant, very close to canonical QFT in continuous space-time [19]. It is important that all the principal components of the mathematical apparatus for canonical QFT have their direct analogs in **measurable** QFT.

In particular, the d'Alembertian **measurable** in this case is represented as (formula (15) in [7])

$$\Box_{\mathbf{N}_{\mathbf{x}_\mu}} = \frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}} \cdot \frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}^\mu}}}. \tag{7}$$

where $\frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}}, \frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}^\mu}}}$ are corresponding piecewise-differential analogs of the usual derivatives (the formula (9) in [5]).

The paper [5] presents in detail a measurable form of the Least Action Principle. In this case in all the formulae on passage from QFT in continuous consideration to the **measurable** form of QFT, in accordance with (8) and (9) in [5], the substitution is performed (formula (71) in [5]):

$$\int \mapsto \sum; \partial_\mu \mapsto \frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}}; d^4x \mapsto \prod_{\mu=0}^{3} \frac{\ell}{N_{x_\mu}}, ... \tag{8}$$

The Dirac's Delta Function in position representation has the **measurable** form (formula (17) in [7])

$$\sum_{-\infty}^{\infty} \delta_{meas}(x_\mu)\frac{\ell}{N_{x_\mu}}, |N_{x_\mu}| \gg 1. \tag{9}$$

and in momentum representation (formula (23) in [7])

$$\sum_{p_{N^*}}^{p_{N_*}} \delta_{meas}(p_i)p_{N_i(N_{i+1})}, |N_i| \gg 1 \tag{10}$$

where **GMQ** ℓ/N_{x_μ} and **PMQ** $p_{N^*}, p_{N_*}, p_{N_i(N_{i+1})}$ are taken from Section II of [5].

In a similar way Section 4 of [7] presents definitions of all other analogs for the principal components of the mathematical apparatus of canonical QFT in continuous space-time [19] including the Fourier transformations. At the same time, we should take into consideration that at low energies a set of **Primarily Measurable Momenta** (**PMM**), being a discrete finite set, is not a space.

3. General remarks on ultraviolet divergences in canonical QFT and its measurable analog

It is known that an ultraviolet divergence (UVD) in QFT can be described as one that comes from

(1) the region in the integral where all particles in the loop have large energies and momenta;

(2) very short wavelengths and high-frequencies fluctuations of the fields, in the path integral for the field;

(3) very short proper-time between particle emission and absorption, if the loop is thought of as a sum over particle paths.

That is, all of them are short-distance, short-time phenomena.

As noted in Section 2.1, the available energies E satisfy the condition $E \ll E_\ell$ (or same $E \ll E_p$), E takes the form $E = E_\ell/N'$ and momenta p take the form $p_{N'} = p_\ell/N', |N'| \gg 1$. So, in the case under study "large energies and momenta" in point (1) will always be bound and hence there will be no UVD associated with point (1). We can speak only about some minimal values of N and $|N'|$, characterized by the property $N \gg 1, |N'| \gg 1$, and the corresponding maximal values of E_N and p'_N.

Similarly, due to the condition $E \ll E_\ell$, we will not have "very short wavelengths" in point (2) and "very short proper-time" in point (3). Proceeding from the definition of *observables* (p. 140 in Section 4 of [7]), at the above-mentioned energy scales variations of the *observable* quantities are limited to $L = N_L\ell$ and $t = N_t\tau$ with $N_L \gg 1, N_t \gg 1$. So, in this case we can also speak about some minimal values of N_L, L and N_t, t having the indicated property. Because of this, in the case under study there are no UVD associated with points (2) and (3). This means that the present paradigm in a **measurable** consideration includes no UVD.

Then the problem arises, how to derive to a high accuracy all the principal results (PCCT) (Section 2 in [7]) of canonical QFT in continuous space-time [19–22] in accordance with the **Principle of Correspondence to Continuous Theory** within **measurable** QFT, by definition *finite* in all orders of a perturbation theory, i.e. in the absence of UVD.

Since ℓ is chosen at a level of Planck length, i.e. $\ell \propto l_p \approx 10^{-33}$ cm, the quantities ℓ/N, where $|N| \gg 1$ from **Remark 2.3**, are of the order of $10^{-33-\lg|N|}$ cm. It is clear that for all calculations these quantities may be considered infinitesimal to any accuracy. Similar statement is possible for **PMM** p_N as well. On the face of it, at low energies $E \ll E_p$ the difference between **measurable** QFT and canonical QFT in continuous space-time is negligible. But we should take into consideration the above-mentioned definition of *observables* in Section 4 of [7]. Actually, the meaning of this definition is as follows.

Remark 3.1
When at low energies, $E \ll E_\ell$, the *observable quantity* $\mathcal{A}$ has the space-time coordinates $\{x_\mu\}$, all *real* variations of these coordinates for $\mathcal{A}$, provided $\mathcal{A}$ remains an *observable quantity*,take the form $x_\mu \mapsto x_\mu + \Delta(x_\mu)$, where $\Delta(x_\mu) = N_{x_\mu}\ell, |N_{x_\mu}| \gg 1$ or $\Delta(x_\mu) = 0$. In this case, at least, for one index we ν,$\Delta(x_\nu) \neq 0$. Other variations of space-time coordinates we regard as *nonobservable* shifts $\mathcal{A}$. For example, such shifts as $\mathcal{A}(x_\mu) \to \mathcal{A}(x_\mu + \frac{\ell}{N_{x_\mu}})$.
Similarly, in the momentum representation: if coordinates of the *observable quantity* $\mathcal{A}$ are $\{p_{N_{x_i}}\}$ in the momentum representation, then all variations of these coordinates for $\mathcal{A}$, provided $\mathcal{A}$ remains an *observable quantity*, are of the form $p_{N_{x_i}} \mapsto p_{N_{x_i}} + \Delta p_{N_{x_i}} = p_{N^*_{x_i}}$, where $|N_{x_i} - N^*_{x_i}| \gg 1$. Other variations in the momentum representation lead to the *non-observable* shifts $\mathcal{A}$, in particular, $\mathcal{A}(p_{N_{x_i}}) \mapsto \mathcal{A}(p_{N_{x_i}} \pm 1)$.

As the theory is studied in a **measurable** representation, the numbers N_{x_μ} must be integers [1–7]. Considering **Remark 4*** in [7], due to the condition $|N_{x_\mu}| \gg 1$, this fact seems to be insignificant. Still, note that the **measurable** picture is associated with a lattice model, for which both RI and translational invariance is violated. As indicated below, spacing (or pitch) of the emerging lattice is very small, the proposed lattice model may be considered continuous to a high accuracy. Then at low energies $E \ll E_p$ the translational invariance is retained to a high accuracy when the condition $|N_{x_\mu} - N'_{x_\mu}| \gg 1$ is imposed at $|N_{x_\mu}| \gg 1, |N'_{x_\mu}| \gg 1$. But in this case a role of the condition N_{x_μ} is significant. Otherwise, nothing prevents taking noninteger numbers $N_{x_\mu}, |N_{x_\mu}| \gg 1$ together with their integer part $[N_{x_\mu}]$ to obtain $|\Delta(x_\mu)| < 1$. in **Remark 3.1**.

As noted above, in the **measurable** picture at low energies $E \ll E_p$ UVD are nonexistent. This is true for a classical consideration too. To illustrate, the electromagnetic mass m_{em} of a classical particle with the charge q, uniformly distributed over the surface of the sphere having the radius a, equals $m_{em} = q^2/6\pi ac^2$ (section 1.3 in [22]) and $m_{em} \to \infty$ if $a \to 0$. At the same time, in the **measurable** picture, due to **Remark 3.1**, for *observable* values we always have $a = N\ell$, $N \gg 1$ and hence the condition $a \to 0$ is not fulfilled.

Remark 3.2
Taking into account the above information, without loss of generality, at low energies $E \ll E_\ell$ we, to a high accuracy, can replace a **measurable** variant of QFT by canonical QFT in continuous space-time [19–22] with due regard for two important moments:
3.2.1 Remark 3.1 points to the fact that definitions of the *observable* in canonical QFT [19–22] and in its **measurable** variant are different;

3.2.2 In canonical QFT we have problems with UVD, whereas in a **measurable** form of QFT we have no UVD.

Obviouslly, point **3.2.1** is technically insignificant and is related to different theoretical views on the *observable*.
Point **3.2.2** is very important. Naturally, the question arises: how close a **measurable** picture is to QFT in continuous space-time?
This problem is considered in the following section in connection with the scalar quantum-field model φ^4, where it is shown that we have no UVD on the limiting transition of this model in a **measurable** form to canonical QFT.

4. Scalar quantum field model φ^4 in measurable form

4.1. Free fields

As follows from the results of Section 2, in a **measurable** consideration the usual derivatives are replaced by their corresponding piecewise-differential analogs. In this way we can derive **measurable** analogs of the known Lagrangians. In particular canonical Lagrangian for model φ^4 in continuous space-time has the form [19]

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu\varphi)^2 - \frac{m_0^2}{2}\,\varphi^2 - \frac{g_0}{4!}\varphi^4, \qquad (11)$$

where $\mathcal{L}_0 \doteq \frac{1}{2}((\partial_\mu\varphi)^2 - m_0^2\varphi^2)$ is free fields Lagrangian and $\mathcal{L}_I \doteq -\frac{g}{4!}\varphi^4$ is interaction Lagrangian and g_0 is a dimensionless constant (in four dimensions).

Using operator definition $\frac{\boldsymbol{\Delta}}{\boldsymbol{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}}$ from formula (7), we can easily obtain, instead of $\mathcal{L}$ its **measurable** form

$$\begin{aligned}\mathcal{L}_{meas,\{N\}} = &\frac{1}{2}\,(\frac{\boldsymbol{\Delta}}{\boldsymbol{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}}\varphi_{meas})^2 - \frac{1}{2}m_0^2\varphi_{meas}^2\\ &-\frac{g_0}{4!}\varphi_{meas}^4\end{aligned} \qquad (12)$$

and instead $\mathcal{L}_0$ with the corresponding *Klein–Gordon equation* or *KGE*

$$(\Box + m_0^2)\,\phi = 0 \qquad (13)$$

their **measurable** forms

$$\mathcal{L}_{meas,\{N\},0} = \frac{1}{2}\,(\frac{\mathbf{\Delta}}{\mathbf{\Delta}_{\mathbf{N}_{\mathbf{x}_\mu}}}\phi_{meas})^2 - \frac{m_0^2}{2}\,\phi_{meas}^2 \quad (14)$$

and

$$(\Box_{\mathbf{N}_{\mathbf{x}_\mu}} + m_0^2)\,\phi_{meas} = 0\ . \quad (15)$$

Similarly, using the replacement from formula (8), a solution of the equation (13) in terms of a complete set of the plane waves $e^{\pm ikx}$

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 k}{2k^0}\,[a(k)\,e^{-ikx} + a^\dagger(k)\,e^{ikx}\,] \quad (16)$$

in the **measurable** form should be written as follows:

$$\begin{aligned}\phi(x, N^*, N_*)_{meas} &\doteq \frac{1}{(2\pi)^{3/2}} \sum_{N_i=N^*}^{N_*} \frac{\Delta^3 k}{2k^0}\,[a(k)\,e^{-ikx} + a^\dagger(k)\,e^{ikx}] \\ &= \frac{1}{(2\pi)^{3/2}} \sum_{p_{N^*}}^{p_{N_*}} \frac{\Delta^3 k}{2k^0}\,[a(k)\,e^{-ikx} + a^\dagger(k)\,e^{ikx}].\end{aligned} \quad (17)$$

Here $N^* \doteq \{N_i^*\};\ N_* \doteq \{N_{i*}\},\ i = 1,2,3$ are integer set, $x = \{x_i\},\ x_i = N_{x_i}\ell,\ N_{x_i}$ – integers with the property $|N_{x_i}| \gg 1,\ k \doteq \{k_i\}$, $k_i \doteq p_{N_i} = \hbar/(N_i\ell),\ \Delta k_i \doteq k_i - k_{i+1} = k_{i(i+1)}$, $\Delta^3 k \doteq \prod_{i=1}^{3}\Delta k_i, k^0 = \sqrt{\vec{k_i}^{\,2} + m_0^2},\ N_i$ are integer numbers too, and condition $|N^*| \geq |N_i| \geq |N_*| \gg 1$ is satisfied.

Which conditions should be satisfied by the lower N^* and upper N_* bounds of the summation in formula (17)?

Clearly, in the **measurable** case the function $\phi(x, N^*, N_*)_{meas}$ from this formula is not a complete analog of the function $\phi(x)$ from formula (17). It is only an analog of the function $\phi(x, N^*, N_*)$:

$$\phi(x, N^*, N_*) \doteq \frac{1}{(2\pi)^{3/2}} \int_{p_{N^*}}^{p_{N_*}} \frac{\mathrm{d}^3 k}{2k^0}\,[a(k)\,e^{-ikx} + a^\dagger(k)\,e^{ikx}\,] \quad (18)$$

that seems to be a certain low-energy part of $\phi(x)$. It is natural that $\phi(x, N^*, N_*)$ is a summand in the general solution *KGE*. Similarly, $\phi(x, N^*, N_*)_{meas}$ (17) is a summand in the general solution for the **measurable** *KGE* analog of (15).

In the proposed paradigm for propagators in momentum and position representations with a **measurable** picture we have the same formalism as of the wave functions. Specifically, in canonical QFT in continuous space-time [19–22] the propagators in the momentum and position representations $G(k)$ and $G(x-y)$ are related by the Fourier transformation

$$G(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\,\tilde{G}(k)\,e^{-ik(x-y)}, \quad (19)$$

and the Green function $G(x-y)$ is a solution of the inhomogeneous field equation

$$(\Box + m_0^2)\, G(x-y) = -\delta^4(x-y)\,. \tag{20}$$

Similar to formula (18), at low energies $E \ll E_p$ in the canonical case of the well-known QFT in $G(x-y)$ one can emphasize the summand (or cutoff) $G(x-y, N^*, N_*)$:

$$G(x-y, N^*, N_*) \doteq \int\limits_{p_{N^*}}^{p_{N_*}} \frac{\mathrm{d}^4 k}{(2\pi)^4}\, \tilde{G}(k)\, e^{-ik(x-y)}, \tag{21}$$

where N^*, N_* – sets of the numbers from formula (17) with the added corresponding bound for p_0. $G(x-y, N^*, N_*)$ – low-energy term dependent on N^*, N_* in the full propagator $G(x-y)$.

In a **measurable** consideration at low momenta in the domain of $E \ll E_\ell$ formula (21) is replaced by

$$G(x-y, N^*, N_*)_{meas} \doteq \sum_{p_{N^*}}^{p_{N_*}} \frac{\Delta^4 k}{(2\pi)^4}\, \tilde{G}(k)_{meas}\, e^{-ik(x-y)}, \tag{22}$$

where $\Delta^4 k$ is defined in the same way as $\Delta^3 k$ in formula (17) with the addition of the zero-index coordinate.

It should be noted that since *only* free fields are considered in this subsection, the point **3.2.2** of the **Remark 3.2** is not applicable

Therefore as follows from the **Remark 3.2**, to a high accuracy, we have

$$\tilde{G}(k)_{meas} = \tilde{G}(k), \tag{23}$$

for $\tilde{G}(k)$ from formula (21).

For the wave functions in continuous space-time describing free scalar particles for the *propagator* in the momentum space, we have well-known formula [19–22]

$$(k^2 - m_0^2)\, \tilde{G}(k) = 1 \tag{24}$$

and a solution for (24) is of the form

$$i\,\tilde{G}(k) = \frac{i}{k^2 - m_0^2 + i\epsilon}. \tag{25}$$

Remark 4.1
In this way by virtue of **Remark 3.2** the above-mentioned formulae point to the fact that at low energies $E \ll E_p$ in a **measurable** consideration for free scalar fields we can use the results of the corresponding continuous theory, but taking into account **Remark 3.1**.

The Feynman diagram associated with a scalar propagator in this case is of the standard form

$$i\,\tilde{G}(k)_{meas} \qquad \bullet\text{- - - }\underset{k}{>}\text{- - -}\bullet$$

Of particular importance are the following remarks.

Remark 4.2
(1) If we assume that $|N^*|$ are large enough (and the corresponding momenta p_N^* are thus small) so that, without any detriment for the performed calculations, we can consider $|N^*| = \infty$, (and, consequently, $p_N^* = 0$).
Then the above-mentioned functions $\phi(x, N^*, N_*)_{meas}$, $\phi(x, N^*, N_*)$, $G(x-y, N^*, N_*)_{meas}$, $G(x-y, N^*, N_*)$, ... should be dependent on the single parameter N_*

$$\phi(x, \infty, N_*)_{meas} \equiv \phi(x, N_*)_{meas},$$
$$G(x-y, \infty, N_*)_{meas} \equiv G(x-y, N_*)_{meas}, ... \tag{26}$$

It is clear that $|N_*|$–integer cutoff parameter at the upper bound of momenta: the lower the parameter, the larger the momenta $p_{|N_*|}$, the closer the quantities $\phi(x, N_*)_{meas}$, $\phi(x, N_*)$, $G(x-y, N_*)_{meas}$, $G(x-y, N_*)$, ... to the initial ones $\phi(x)$, $G(x-y)$, ... But, considering that (see Section 2.2) the Einstein Equivalence Principle (EP) should have the applicability boundary, it is assumed:

(2) The parameter $|N_*|$ has the *minimum* $|\widetilde{N}_*|$ determined by linear dimensions of the *minimal* neighborhood $|\widetilde{N}_*|\ell$, for which (EP)

remains valid in the process under study. The fact that such a neighborhood should be the case, with $|N_*| \gg 1$, was noted in Section 2.2.

4.2. Perturbation theory

Within the scope of a perturbation theory, let us consider examples of Feynman diagrams, which give UVD for the φ^4-model in canonical QFT in continuous space-time [19–22], to find what are the correspondences with a **measurable** picture.

Now, we consider one-loop corrections for the two- and four-vertex functions given by the following diagrams in Fig. 1.

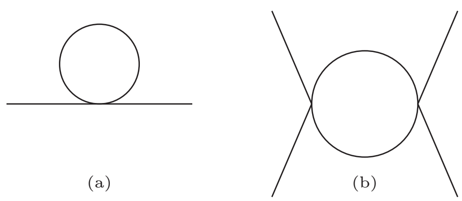

FIG. 1. Diagrams for one-loop corrections to the two-vertex (a) and four-vertex (b) Green functions.

Then the quantity $G(0)$, quadratically divergent over the momentum k (associated with the diagram (a) in Fig.1, formula (9.1) in [19])

$$G(0) = \mathrm{g}_0 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \tilde{G}(k) = \mathrm{g}_0 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{k^2 - m_0^2} \tag{27}$$

corresponds in a **measurable** picture to the integral, finite over k, from formula (21) with $|N^*| = \infty$

$$G(0, N_*) \doteq \mathrm{g}_0 \int\limits_{-p_{N_*}}^{p_{N_*}} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{k^2 - m_0^2}. \tag{28}$$

Similarly, another divergent diagram–graph of the order $O(\mathrm{g}^2)$, whose contribution is represented by the logarithmically divergent integral (formula (9.2) in [19])

$$\mathrm{g}_0^2 \int \frac{\mathrm{d}^4k}{(2\pi)^8} \frac{1}{(k^2 - m_0^2)((p_1 + p_2 - k)^2 - m_0^2)} \tag{29}$$

in a **measurable** consideration will be associated with the finite quantity

$$\mathrm{g}_0^2 \int\limits_{-p_{N'_*}}^{p_{N'_*}} \frac{\mathrm{d}^4k}{(2\pi)^8} \frac{1}{(k^2 - m_0^2)((p_1 + p_2 - k)^2 - m_0^2)}. \tag{30}$$

(Here in the **measurable** case the right-hand sides of formulae (28),(30) should have the corresponding sums instead of the integrals but, due to formula (23), the sums may be replaced by the corresponding integrals).

Obviously, this is the case for all the loop Feynman diagrams in canonical QFT – no wonder, because, as noted at the beginning of Section 3, UVD disappear in a **measurable** variant of canonical QFT.

Then we should understand how, with the use of this approach, we can obtain to a high accuracy the correspondence of a **measurable** finite quantum-field theory to renormalized QFT in continuous space-time.

From Section 2 in [5] it follows directly that the **measurable** approach generates a lattice quantum-field theory, with the lattice that in the momentum representation differs from the canonical lattice involved in the Lattice Quantum Field Theory (LQFT). Hereinafter we use the symbols, terms, and results from LQFT [44, 45].

Then it is assumed that the theory under study is considered in a sufficiently large hypercubic box with the edge length L and space-time size L^4, where $L = N_L\ell, N_L \gg 1$. In general, not necessarily N_L is an integer number.

For convenience, let us introduce the following:

$$\Omega \doteq L^4. \tag{31}$$

Assuming that t varies over the interval $0 \leq t \leq T, T \neq L$, (31) will take the form (formula (2.3) in [45])

$$\Omega \doteq VT = L^3T. \tag{32}$$

In what follows, when not stated otherwise, we assume $T = L$ and hence formula (32) takes the form (31).

Without loss of generality, we can assume that all integer N_{x_μ} are equal to each other and are equal to some integer $N \gg 1$ maximally high in the absolute value. Then, according to the present consideration, in the **measurable** form there arises a lattice model of the position representation with $a = \ell/N$, where a is the lattice distance or same lattice spacing (section 2.5 from [44]).

In line with the general approach, in LQFT we have [44]

$$L = aM = \frac{\ell}{N}M, \tag{33}$$

i.e. $i.e.\frac{L}{M} = \frac{\ell}{N}$, where $M \gg 1$ is an integer number. It is obvious that $M/N = N_L$, i.e. $M \gg N_L$.

As L is great, also without loss of generality, it is assumed that the periodic boundary conditions (formula (2.58) in [44])) are valid

$$\phi(x+L) = \phi(x). \tag{34}$$

Then all formulae of LQFT in the position representation (Sections 2.5, 2.6 in [44]) are valid for the **measurable** form of a continuous theory. And formula (2.54) from [44]

$$\sum_x f(x) \to \int_0^L d^4x f(x), M \to \infty,\, a = \frac{L}{M} \tag{35}$$

(L fixed) may be rewritten for such consideration with substitution of $f(x) \to \mathcal{L}$ under the integration sign for $f(x) \to \mathcal{L}_{meas,\{N\}}$ within the summation, and $a \to \ell/N$, where $\mathcal{L}$ is taken from formula (11) and $\mathcal{L}_{meas,\{N\}}$ – from formula (12), respectively.

Since ℓ is also a fixed quantity, it is clear that the conditions $M \to \infty$ and $N \to \infty$ in the case under study are equivalent, representing the thermodynamic limit that gives a continuous pattern. Note that in this case we can use the results from Sections 2.5 and 2.6 of [44],assigning a_t as the temporal lattice distance $a_t \doteq \tau/N_t$, where τ/N_t is taken from formula (4) in [5].

Thus, in the coordinate representation the studied lattice of **measurable** quantities may be regarded as a canonical space-time lattice of LQFT, with the spacing $a = \ell/N$ and temporal distance $a_t = \tau/N_t$.

In this case all the basic operators in Sections 2.5 and 2.6 of [44] have their analogs in the present work. Specifically, piecewise difference operators finite-differences operators $\partial_\mu\varphi_x, \partial'_\mu\varphi_x$ from formulae (2.55), (2.56) in [44] and formulae of Section 2.3 in [46] in the present paper correspond to the operators $\frac{\mathbf{\Delta}}{\mathbf{\Delta_N}}$ for positive and negative values of $\mathbf{N}$. The transfer-operator $\hat{T}$ may be constructed for the lattice of interest, with the spacing $a = \ell/N$ and temporal distance $a_t = \tau/N_t$, in accordance with formulae (2.71), (2.74) of [44], so all the formulae from Section 2.6 in [44] are valid for this case. We assume that $a_t = a$.

It should be noted that we can pass to Euclidean space-time by means of Wick rotation (**Remark 2.4**) for better convergence of the integrals. Then, with the help of an analytical extension, we can return to Minkowski space-time. This is a standard method both for QFT and LQFT [44, 45].

The continuum action of the theory (11) in Euclidean space-time is of the form (formula (2.17) from [46])

$$S = \int d^4x (\frac{1}{2}(\partial_\mu\varphi)^2 + \frac{m_0^2}{2}\varphi^2 + \frac{g_0}{4!}\varphi^4), \tag{36}$$

and the corresponding lattice action has the following form:

$$S_{meas,\{N\}} = a^4 \sum_x (\frac{1}{2}(\frac{\mathbf{\Delta}}{\mathbf{\Delta_N}}\varphi_{meas})^2 + \frac{1}{2}m_0^2\varphi_{meas}^2 + \frac{g_0}{4!}\varphi_{meas}^4). \tag{37}$$

For the lattice values of momenta, in the momentum representation, according to formula (2.81) in [44], we have

$$p_\mu(latt) = n_\mu \frac{2\pi}{L}, \tag{38}$$

where n_μ are integers.
Consequently, the lattice edge in the momentum representation $\Delta p_\mu(latt)$ adopts the value

$$\Delta p_\mu(latt) = \frac{2\pi}{L} \propto \frac{1}{N_L}, \tag{39}$$

where it is assumed that $\hbar = 1$.
At the same time, the integer numbers n_μ are varying in magnitude over the interval $[0, N_L N]$,, where $N_L N = L/a$ (formula (2.82) in [44]). As a result, in the case of interest a maximum value of the momentum along any axis will be given by

$$p_{latt,max} = \frac{\pi}{a} = \frac{\pi}{\ell/N} = \frac{\pi N}{\ell} \doteq \Lambda. \tag{40}$$

However, here the difficulty arises – the corresponding lattice in the momentum representation on L^4 is uniform with the lattice spacing in formula (39).
In the considered case the lattice of **measurable** momenta is nonuniform with the lattice spacing

$$\triangle p_\mu(meas) = \frac{1}{(N^* - \kappa)(N^* - \kappa \mp 1)\ell}, \tag{41}$$

where κ is an integer number, $|\kappa| \ll |N^*| \gg 1$.
As shown in [7], in order to use the results from [44], it is required that the condition

$$\triangle p_\mu(latt) \approx \triangle p_\mu(meas) \tag{42}$$

be fulfilled.
As follows from formula (41) and [7] this is the case when

$$N_L \approx (N^*)^2. \tag{43}$$

This condition is quite natural considering that L may be chosen no matter how large but finite.

Formula (40) gives an explicit expression for a maximal lattice momentum $p_{latt,max} = \Lambda$. To be more exact, the momenta are restricted to the so-called first Brillouin zone (**BZ**) $\mathcal{B}$ (formula (1.218) from [45])

$$\mathcal{B} \doteq \left\{ p \middle| \frac{-\pi}{a} < p_\mu \leq \frac{\pi}{a} \right\}. \tag{44}$$

It is clear that $p_{latt,max} = \Lambda \gg p_\ell$. As follows from formula (40), $\Lambda \propto N p_\ell, N \gg 1$, i.e. the boundary of **BZ** Λ passes far beyond the region of the physical energy values.

But due to the condition $E \ll E_p$, we consider only a low-energy part of the lattice, the momenta of which, in line with (41)–(43), are given as $p \approx \frac{\hbar}{N^* \ell}$ with $|N^*| \gg 1$. Because of this, in the case under study only particular momenta may be maximal (so-called "maximally reachable" momentum) $p_{max,reach}$ and $p_{max,reach} \ll p_{latt,max}$.

In this way **BZ** in formula (44) is narrowed significantly

$$-p_{max,reach} \leq p_\mu \leq p_{max,reach}, \tag{45}$$

where $p_{max,reach} \ll p_\ell$.

As $a = \ell/N$,where $N \gg 1$, when the mass m is fixed, am is close to zero and hence the correlation length ξ (formula (1.224) in [45])

$$\xi \equiv \frac{1}{am} = \frac{N}{\ell m} \tag{46}$$

is finite but very great. Transition to a continuum limit $\xi \to \infty$ means going to $N \to \infty$. In this case, within the constant factor m^{-1}, we have

$$\xi = \frac{N}{\ell} \propto N p_\ell \approx N p_{pl} \propto p_{latt,max} = \Lambda. \tag{47}$$

From formulae (40), (45) it follows directly that

$$p_{max,reach} = \frac{p_l}{\tilde{N}} = \frac{\Lambda}{N\tilde{N}}, N \gg 1, \tilde{N} \gg 1. \tag{48}$$

Then, proceeding from the formulae above, in the case of interest (**BZ**) $\mathcal{B}$ (44) is narrowed to $\mathcal{B}_\mathcal{N}$

$$\mathcal{B}_\mathcal{N} \doteq \{p| \frac{-\pi}{N\tilde{N}a} < p_\mu \leq \frac{\pi}{N\tilde{N}a}\}, N \gg 1, \tilde{N} \gg 1. \tag{49}$$

Lattice summation in the general case is given by formula (2.7) from [45]

$$\int\limits_{p\in\mathcal{B}} \doteq \int\limits_{\mathcal{B}} \equiv \frac{1}{a^4\Omega}\sum_{p\in\mathcal{B}}. \qquad (50)$$

In the case under study the lattice summation takes the form

$$\int\limits_{p\in\mathcal{B}_{\mathcal{N}}} \doteq \int\limits_{\mathcal{B}_{\mathcal{N}}} \equiv \frac{1}{a^4\Omega}\sum_{p\in\mathcal{B}_{\mathcal{N}}}. \qquad (51)$$

Respectively, on transition to the thermodynamic limit $L\rightarrow\infty, T\rightarrow\infty$, in the general case we arrive at formula (2.8) in [46]

$$\int\limits_{p\in\mathcal{B}} = \frac{1}{(2\pi)^4}\int_{\frac{-\pi}{a}}^{\frac{\pi}{a}} d^4p. \qquad (52)$$

In the case of interest (52) is transformed to

$$\int\limits_{p\in\mathcal{B}_{\mathcal{N}}} = \frac{1}{(2\pi)^4}\int_{\frac{-\pi}{N\tilde{N}a}}^{\frac{\pi}{N\tilde{N}a}} d^4p. \qquad (53)$$

Now in the same way we consider the momentum representation and Fourier transformation of the above mentioned lattice (formula (1.171) in [45])

$$G(x-y;a)=\int\limits_{-\pi/a}^{\pi/a}\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\tilde{G}(p;a)=\int\limits_{p\in\mathcal{B}}\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\tilde{G}(p;a). \qquad (54)$$

Then we can use the results of [45] to find, how well a continuous propagator of the momentum representation is approximated by the "lattice" propagator in this representation. As it has been noted, all calculations in [45] are first performed in Euclidean space-time and followed by the analytical extension to Minkowski space.

In virtue, using formula (1.173) from [45], we have

$$\tilde{G}(p;a)=\left\{\sum_{\mu=1}^{4}a^{-2}4\sin^2\frac{ap_\mu}{2}+m^2\right\}^{-1}. \qquad (55)$$

But it has been shown that in the case under study the momenta p are taken only from the subset $\mathcal{B}_{\mathcal{N}}$. Consequently, $p_\mu\propto 1/N_\mu, |N_\mu|\gg 1$. As $a=\ell/N, N\gg 1$, the argument of the function $\sin^2$ is $\propto 1/(NN_\mu)$, i.e. it is very close to zero. Further we use a simple property: $\sin x\approx x$ for x close to 0. Immediately, within a high accuracy, by formula (55) we can obtain

$$\tilde{G}(p;a)=\left\{\sum_{\mu=1}^{4}a^{-2}4\frac{a^2p_\mu^2}{4}+m^2\right\}^{-1}=\left\{\sum_{\mu=1}^{4}p_\mu^2+m^2\right\}^{-1}=(p^2+m^2)^{-1} \qquad (56)$$

in a good agreement with the corresponding formula in a continuous picture, i.e. for $a\rightarrow 0$ ([45], formula (1.178)).

So, in a **measurable** form at low energies

$E \ll E_p$ the theory studied is to a high accuracy coincident with the corresponding theory in the continuous case.

However, we are interested not in a continuum limit of the theory but in its **measurable** *variant. We are most interested not in the lattice as an approximation and regularization means in a continuous consideration but in the lattice as it is.*

The lattice operation (37) may be rewritten in the *dimensionless* form (formulae (1.220), (1.221) in [45]) as follows:

$$S_{\{N\}} \doteq \sum_x \left\{ -2\kappa \sum_{\mu=1}^{4} \varphi(x)\varphi(x+a\hat{\mu}) + \varphi(x)^2 + \lambda[\varphi(x)^2-1]^2 - \lambda \right\}, \tag{57}$$

where for $a = \ell/N$ we have $a\varphi_{meas}(x) = 2\kappa\varphi(x)$, $a^2 m_0^2 = \frac{1-2\lambda}{\kappa} - 8$, $\mathrm{g}_0 = \frac{6\lambda}{\kappa^2}$.

Obviously, (57) includes only *dimensionless* quantities, and m_0 is a *bare* value of mass, $\varphi_{meas}(x)$ and g_0 are *bare* values for the field and coupling constant, respectively.

Perturbation theory and Feynman rules for this lattice are analogous to a continuous theory but they have the interaction term

$$S^{(1)} = a^4 \sum_x \frac{\mathrm{g}_0}{4!}\varphi_{meas}^4. \tag{58}$$

Remark 4.3

As a rule, in the literature devoted to LQFT it is assumed that the lattice edge a is equal to 1. Then the formula for the first Brillouin zone $\mathcal{B}$ (44) is of the form

$$\mathcal{B} \doteq \left\{ p \middle| -\pi < p_\mu \leq \pi \right\}. \tag{59}$$

Whereas for the "short-cut" Brillouin zone $\mathcal{B}_{\mathcal{N}}$ (49) we have

$$\mathcal{B}_{\mathcal{N}} \doteq \{p| \frac{-\pi}{N\tilde{N}} < p_\mu \leq \frac{\pi}{N\tilde{N}}\}, N \gg 1, \tilde{N} \gg 1, \tag{60}$$

with the corresponding changes in all other formulae.

As distinct from a continuous consideration, by the lattice approach all Feynman graphs satisfy the following properties in momentum space ([45], p. 64) in the general case:

- each line is associated with the propagator $\tilde{\Delta}(q) \equiv (m_0^2 + \hat{q}^2)^{-1}$;
- each vertex is an end point of four lines and is associated with the factor $-\mathrm{g}_0$;
- at in inner vertices momentum conservation holds modulo 2π;
- loop momenta should to be integrated over the first Brillouin zone $\mathcal{B}$ with the integration measure $\int\limits_{p\in\mathcal{B}}$;
- there is in overall factor $(2\kappa)^{-n/2}$ resulting from our normalization of the lattice scalar fields;
- UVD appear only in the continuum limit, i.e. when $a \to 0$

Note that in the second point $-\mathrm{g}_0$ should be replaced by $-\mathrm{g}_{0,\mathcal{B}_{\mathcal{N}}}$, and it seems that the fourth item should be replaced by:

- loop momenta should be integrated over the short-cut Brillouin zone,$\mathcal{B}_{\mathcal{N}}$ with the integration measure $\int\limits_{p\in\mathcal{B}_{\mathcal{N}}}$.

Now let us consider these points as applied to the particular lattice that is under study in this work.

(1) As, for $N \gg 1$, the lattice edge $a = \ell/N$ is very small and hence the correlation length ξ (formula (46)) is very great but not infinite, the indicated lattice in the space-time and momentum representation is actually not distinct from a continuous consideration for the momenta satisfying **BZ** $\mathcal{B}$ (44).

(2) It has been already mentioned (e.g., formula (45)) that, due to the condition $E \ll E_p$, we need only a low-energy part of the lattice, i.e. $\mathcal{B}_{\mathcal{N}}$ (formula (49)), and we should take this into consideration when calculating the contributions into amplitudes of the corresponding processes which generate the loop Feynman diagrams.

Thus, as directly follows from formula (49), we should include the contributions made *only* by very small momenta p in $\mathcal{B}$,i.e. for $p \in \mathcal{B}_{\mathcal{N}}$. Taking this into account, further we use the known formulae of LQFT for small momenta (Section 2 in [45]).

First, we consider the field $\varphi(x)$ in a *symmetric phase*

$$\langle\varphi(x)\rangle = 0, \tag{61}$$

i.e. Z_2-symmetry of $\varphi(x) \mapsto -\varphi(x)$ is the case, whereas Green's functions with an odd number of arguments vanish.

As it has been correctly noted in Section 2 of [20] "...Renormalization has its own intrinsic physical basis and is not brought about solely by the necessity to expurgate infinities. Even in a totally finite theory we would still have to renormalize physical quantities". This is associated with the fact that the theoretical initial (bare) quantities (mass m_0,charge q_0 and so on) can differ drastically from the real (physical) quantities (m_R, q_R and so on).But because in this case in the **measurable** *picture at energies $E \ll E_p$ a low-energy part of the lattice is involved, very close to continuous space-time, there is a possibility to derive QFT without infinities, when renormalization of the theory is understood as a passage from some finite quantities to the other.*

Note that the "infinities" (i.e. LQFT) are understood as all the energies lying beyond the initial physical boundary $E \ll E_p$ (same $E \ll E_p$). The principal objective of this work is to derive

Main target: **a completely finite quantum theory within the scope of the proposed paradigm when all quantities are at the energy scales $E \ll E_p$ meeting all the above-mentioned restrictions**.

Let us revert to one-loop diagrams (a) and (b) in Fig. 1. Using the designations from Section 2 in [45], we have

$$\begin{aligned}\tilde{\Delta}(q) &\equiv (m_0^2 + \hat{q}^2)^{-1},\\ J_n(m_0) &\equiv \int\limits_{\mathcal{B}(q)} \tilde{\Delta}(q)^n,\\ I_3(m_0,p) &\equiv \int\limits_{\mathcal{B}(q_1)}\int\limits_{\mathcal{B}(q_2)} \tilde{\Delta}(q_1)\tilde{\Delta}(q_2)\tilde{\Delta}(p-q_1-q_2),\end{aligned} \tag{62}$$

where $\mathcal{B}(\tilde{q})$ is **BZ** for the variable $\tilde{q}$.

In the general case a one-loop correction to the two-vertex function (diagram (a)) takes the form ([46], p. 53):

$$\Gamma^{(2)}(p,-p) = -(\hat{p}^2 + m_0^2) - \frac{\mathrm{g}_0}{2}J_1(m_0), \tag{63}$$

where, as a rule, the term $\mathcal{O}(\mathrm{g}_0^2)$ in the right-hand side is omitted.

But, proceeding from the earlier results, in this case it follows that $\Gamma^{(2)}(p,-p)$ should be replaced by

$$\begin{aligned}\Gamma^{(2)}(p,-p,\mathcal{B}_{\mathcal{N}}) = &-(\hat{p}^2 + m^2_{0,\mathcal{B}_{\mathcal{N}}})\\ &- \frac{\mathrm{g}_{0,\mathcal{B}_{\mathcal{N}}}}{2}J_1(m_0,\mathcal{B}_{\mathcal{N}}),\end{aligned} \tag{64}$$

where $p \in \mathcal{B}_{\mathcal{N}}$,

$$J_n(m_0,\mathcal{B}_{\mathcal{N}}) \equiv \int\limits_{\mathcal{B}_{\mathcal{N}}(q)} \tilde{\Delta}(q)^n, \tag{65}$$

and $m_{0,\mathcal{B}_{\mathcal{N}}}$, $\mathrm{g}_{0,\mathcal{B}_{\mathcal{N}}}$ are corresponding *bare* mass and coupling constant within $\mathcal{B}_{\mathcal{N}}$. Here, similar to formula (62), $\mathcal{B}_{\mathcal{N}}(\tilde{q})$ is the narrowed **BZ** $\mathcal{B}_{\mathcal{N}}$ for the variable $\tilde{q}$, and in the right side (64) there is no term $\mathcal{O}(\mathrm{g}^2_{0,\mathcal{B}_{\mathcal{N}}})$.

Generally, formula (63) may be given as

$$\Gamma^{(2)}(p,-p) = -(\hat{p}^2+m_0^2) - \frac{\mathrm{g}_0}{2}J_1(m_0) \equiv -(\hat{p}^2+m_R^2), \tag{66}$$

where m_R is the renormalized mass in the general case.

In a similar way formula (64) is given as

$$\Gamma^{(2)}(p,-p,\mathcal{B}_{\mathcal{N}}) = -(\hat{p}^2+m^2_{0,\mathcal{B}_{\mathcal{N}}}) - \frac{\mathrm{g}_{0,\mathcal{B}_{\mathcal{N}}}}{2}J_1(m_0,\mathcal{B}_{\mathcal{N}}) \equiv -(\hat{p}^2+m^2_{R,\mathcal{B}_{\mathcal{N}}}), \tag{67}$$

where $m_{R,\mathcal{B}_{\mathcal{N}}}$ are the experimental values of mass obtained for the energies on the order of $\mathcal{B}_{\mathcal{N}}$. Naturally, we can suppose that the renormalized (i.e. experimental) values of mass m_R and coupling constant g_R at energies $E \ll E_p$ *should not* depend on the whole domain of $\mathcal{B}$, the limiting values of which are much greater than E_p. Besides, in any region satisfying the condition $E \ll E_p$ they are independent of this domain and hence we have $m_{R,\mathcal{B}_{\mathcal{N}}} = m_R, \mathrm{g}_{\mathrm{R},\mathcal{B}_{\mathcal{N}}} = \mathrm{g}_R$.

Due to the condition $m_{R,\mathcal{B}_{\mathcal{N}}} = m_R$ and considering the terms $\mathcal{O}(\mathrm{g}_0^2), \mathcal{O}(\mathrm{g}^2_{0,\mathcal{B}_{\mathcal{N}}})$, we can rewrite formula (66) as (formula (2.93) in [45])

$$m_R^2 = m_0^2 + \frac{\mathrm{g}_0}{2}J_1(m_0) + \mathcal{O}(\mathrm{g}_0^2), \tag{68}$$

and formula (67) as

$$m^2_{R,\mathcal{B}_{\mathcal{N}}} = m_R^2 = m^2_{0,\mathcal{B}_{\mathcal{N}}} + \frac{\mathrm{g}_{0,\mathcal{B}_{\mathcal{N}}}}{2}J_1(m_{0,\mathcal{B}_{\mathcal{N}}},\mathcal{B}_{\mathcal{N}}) + \mathcal{O}(\mathrm{g}^2_{0,\mathcal{B}_{\mathcal{N}}}). \tag{69}$$

Similar calculations may be performed for the coupling constant too. Specifically, let $\Gamma_R^{(4)}(p_1,p_2,p_3,p_4)$ be the renormalized four-point function. Then, for the renormalized coupling constant g_R, we have ([45], formula (2.96))

$$\mathrm{g}_R = -\Gamma_R^{(4)}(0,0,0,0) = \mathrm{g}_0 - \frac{3}{2}\mathrm{g}_0^2 J_2(m_0) + \mathcal{O}(\mathrm{g}_0^3), \tag{70}$$

and, since $g_{R,\mathcal{B}_{\mathcal{N}}} = g_R$, we have

$$\mathrm{g}_{\mathrm{R},\mathcal{B}_{\mathcal{N}}} = \mathrm{g}_R = -\Gamma^{(4)}_{R,\mathcal{B}_{\mathcal{N}}}(0,0,0,0) = \mathrm{g}_{0,\mathcal{B}_{\mathcal{N}}} - \frac{3}{2}\mathrm{g}^2_{0,\mathcal{B}_{\mathcal{N}}}J_2(m_{0,\mathcal{B}_{\mathcal{N}}},\mathcal{B}_{\mathcal{N}}) + \mathcal{O}(\mathrm{g}^3_{0,\mathcal{B}_{\mathcal{N}}}). \tag{71}$$

As follows from the four last equations, since left sides of each pair of these equations are equal, whereas the integrals $J_1(m_0)$ and $J_1(m_0,\mathcal{B}_{\mathcal{N}})$ and hence $J_2(m_0)$ and $J_2(m_0,\mathcal{B}_{\mathcal{N}})$ are greatly differing (because in the second case the integration domain is drastically narrowed), the quantities $m_0, m_{0,\mathcal{B}_{\mathcal{N}}}$ and $\mathrm{g}_0, \mathrm{g}_{0,\mathcal{B}_{\mathcal{N}}}$ should also differ from each other. And this really is the case.

According to formulae (2.110), (2.111) from [45] in the general case, for *bare* quantities in the one-loop order we have

$$\begin{aligned} m_0^2 &= m_R^2 + \frac{\mathrm{g}_R}{2}J_1(m_R) + \mathcal{O}(\mathrm{g}_R^2), \\ \mathrm{g}_0 &= \mathrm{g}_R + \frac{3}{2}\mathrm{g}_R^2 J_2(m_R) + \mathcal{O}(\mathrm{g}_R^3). \end{aligned} \tag{72}$$

Then, considering the equalities, we can rewrite $m_{R,\mathcal{B}_{\mathcal{N}}} = m_R, \mathrm{g}_{\mathrm{R},\mathcal{B}_{\mathcal{N}}} = \mathrm{g}_R$ (72) in the one-loop order in the **measurable** picture under study as follows:

$$\begin{aligned} m^2_{0,\mathcal{B}_{\mathcal{N}}} &= m_R^2 + \frac{\mathrm{g}_R}{2}J_1(m_R,\mathcal{B}_{\mathcal{N}}) + \mathcal{O}(\mathrm{g}_R^2), \\ \mathrm{g}_{0,\mathcal{B}_{\mathcal{N}}} &= \mathrm{g}_R + \frac{3}{2}\mathrm{g}_R^2 J_2(m_R,\mathcal{B}_{\mathcal{N}}) + \mathcal{O}(\mathrm{g}_R^3). \end{aligned} \tag{73}$$

BZ $\mathcal{B}_{\mathcal{N}}$ is a narrow low-energy (in fact central) part of the total **BZ** $\mathcal{B}$. From this it follows that the integrals $J_1(m_R, \mathcal{B}_\mathcal{N})$, $J_2(m_R, \mathcal{B}_\mathcal{N})$ are low-energy components of the integrals $J_1(m_R)$, $J_2(m_R)$, respectively, and hence they are small.

As it has been noted above, by the lattice approach UVD in QFT appear on passage to a theory in continuous space-time, i.e. for $a \to 0$. However, in this **measurable** picture we study the lattice per se rather than the continuum limit. As this takes place, UVD of a continuous theory in this case are associated with the quantities lying beyond the boundary of E_p and, in particular, beyond that of the narrowed **BZ**, i.e. $\mathcal{B}_\mathcal{N}$.

Because we are most interested in the experimental (renormalized) quantities of m_R, g_R which are coincident in the cases $\mathcal{B}_\mathcal{N}$ and $\mathcal{B}$ and defined within the energy range $E \ll E_p$, formula (73) demonstrates that *bare* quantities can be also defined at low energies $E \ll E_p$ and in terms of "narrow" **BZ** $\mathcal{B}_\mathcal{N}$. Just this solution is the principal objective of this paper – above *Main target* for a scalar field theory when using the **measurable** picture in the one-loop order.

For the two-loop order the foregoing algorithm remains valid, excepting greater complexity of the formulae. To illustrate, in the two-loop order formula (2.85) in [45] for the general case is of the form

$$-\frac{1}{2\kappa}\widetilde{G}(p)^{-1} = -(\hat{p}^2 + m_0^2) - \frac{\mathrm{g}_0}{2} J_1(m_0) + \frac{\mathrm{g}_0^2}{4} J_1(m_0) J_2(m_0) + \frac{\mathrm{g}_0^2}{6} I_3(m_0, p) + \mathcal{O}(\mathrm{g}_0^3), \tag{74}$$

where $\widetilde{G}(p) = (2\kappa)^{-1}\tilde{\Delta}(p)$.

In the **measurable** picture within the boundaries of **BZ** $\mathcal{B}_\mathcal{N}$ equation (74) may be rewritten as

$$\begin{aligned} -\frac{1}{2\kappa}\widetilde{G}(p, \mathcal{B}_\mathcal{N})^{-1} = {} & -(\hat{p}^2 + m_{0,\mathcal{B}_\mathcal{N}}^2) - \frac{\mathrm{g}_{0,\mathcal{B}_\mathcal{N}}}{2} J_1(m_{0,\mathcal{B}_\mathcal{N}}, \mathcal{B}_\mathcal{N}) + \frac{\mathrm{g}_{0,\mathcal{B}_\mathcal{N}}^2}{4} J_1(m_{0,\mathcal{B}_\mathcal{N}}, \mathcal{B}_\mathcal{N}) J_2(m_{0,\mathcal{B}_\mathcal{N}}, \mathcal{B}_\mathcal{N}) \\ & + \frac{\mathrm{g}_{0,\mathcal{B}_\mathcal{N}}^2}{6} I_3(m_{0,\mathcal{B}_\mathcal{N}}, p, \mathcal{B}_\mathcal{N}) + \mathcal{O}(\mathrm{g}_{0,\mathcal{B}_\mathcal{N}}^3), \end{aligned} \tag{75}$$

where $\widetilde{G}(p, \mathcal{B}_\mathcal{N}) = (2\kappa)^{-1}\tilde{\Delta}(p, \mathcal{B}_\mathcal{N})$, $(2\kappa)^{-1}\tilde{\Delta}(p, \mathcal{B}_\mathcal{N}) = (2\kappa(m_{0,\mathcal{B}_\mathcal{N}}^2 + \hat{p}^2))^{-1}$ and $\mathrm{g}_{0,\mathcal{B}_\mathcal{N}}$, $m_{0,\mathcal{B}_\mathcal{N}}$ are taken from formula (73). It is important that all formulae of a perturbation theory in the two-loop order in a **measurable** consideration can be derived in the same way as in the one-loop order by substitution of the short-cut Brillouin zone $\mathcal{B}_\mathcal{N}$ for the corresponding integrals around loop momenta over the first Brillouin zone $\mathcal{B}$, as well as in formulae (62)–(65).

It should be noted that the case of symmetry violation (61), i.e. $\langle\varphi(x)\rangle \neq 0$ (Section 2.2.3 in [45]) has no principal differences from our consideration. We can derive all the basic formulae in the **measurable** picture at low energies $E \ll E_p$ replacing the Brillouin zone $\mathcal{B}$ by the short-cut Brillouin zone $\mathcal{B}_\mathcal{N}$ in all the relevant formulae in Section 2.2.3 from [45].

Next we consider the limiting transition of this LQFT in the general case to a theory in continuous space-time, i.e. when $a \to 0$. As $a = \ell/N, N \gg 1$, we get $N \to \infty$, and from formula (44) it is inferred that full (**BZ**) $\mathcal{B} \to \infty$. It is obvious that the right and left sides of formulae

(63), (72), ..., where we have full (**BZ**) $\mathcal{B}$, tend to infinity. Precisely this is demonstration of UVD in canonical QFT in continuous space-time.

Since we are interested particularly in the short-cut Brillouin zone $\mathcal{B}_{\mathcal{N}}$ that is invariable, due to formulae (49) (or same (60)), the left and right sides of the corresponding formulae (64),(73),... for $N \to \infty$ always are finite limited quantities and hence we have no UVD on passage to the continuum limit in the present consideration.

The principal distinction of the earlier results, e.g. [45, 46], from those obtained in this paper is the fact that in the previous works *bare* quantities m_0 and g_0 take infinite values on passage to the continuum limit, as is accepted by canonical QFT in continuous space-time (for example, Section 10.2 in [21]), whereas in this paper they are finite quantities obtained within the energy range $E \ll E_p$.

5. Conclusion

In conclusion, it should be noted:

- Quantum scalar field model φ^4 in the **measurable** form is a lattice field model with a very short edge length of the space-time lattice $a = \ell/N, N \gg 1$ and hence it is very close to the corresponding model in continuous space-time;
- Smallness of a results in a large variability domain of the momenta, i.e. the first Brillouin zone $\mathcal{B}$ with this model. But the restrictions imposed by the Equivalence Principle Applicability Boundary (Section 2.2) lead to drastic narrowing of $\mathcal{B}$ to the domain of very small momenta in $\mathcal{B}$– short-cut Brillouin zone $\mathcal{B}_{\mathcal{N}}$;
- Within the domain of the momenta belonging to $\mathcal{B}_{\mathcal{N}}$, the quantum scalar model φ^4 in the **measurable** form presents a completely finite theory, with all the parameters (quantities) obtained within the energy range $E \ll E_p$.
  The limiting transition to a continuous theory in this case involves no UVD.

5.1 As follows from the restrictions imposed on the existent energies E at the very end of Section 2.2, $\tilde{N}$ from formula (49) should meet the condition $\tilde{N} \geq 10^2$.

5.2 Though in this paper of the author the part of a perturbation theory associated with the renormgroup methods (Section 1.7 in [45]) has not been used, for similar studies of more complex theories (electrodynamics, electroweak, non-Abelian gauge theories) the renormgroup will be involved without fail.

5.3 Considering the contents of Section 2.2, selection of $p_{max} = p_\ell$ (formula (1)) at the level of p_{pl} or (same *primary* length ℓ at the level l_p) is quite natural because at such a level of energies the space-tine geometry is drastically changed and hence there is no way to describe it, even locally, in terms of any special metric $g_{\mu\nu}(x)$.

References

[1] A.E. Shalyt-Margolin. Minimal Length and the Existence of Some Infinitesimal Quantities in Quantum Theory and Gravity. Adv. High Energy Phys. **8**, 195157 (2014).

[2] A.E. Shalyt-Margolin. Minimal Length, Measurability and Gravity. Entropy. **18**, 80 (2016).

[3] A.E. Shalyt-Margolin. Uncertainty Principle at All Energies Scales and Measurability Conception for Quantum Theory and Gravity.

Int. J. Nonlinear Phenomena in Complex Systems. **19**, 166–181 (2016).

[4] A. E. Shalyt-Margolin. Minimal Length, Minimal Inverse Temperature, Measurability and Black Hole. Electronic Journal of Theoretical Physics. **14**, 35–54 (2018).

[5] A. E. Shalyt-Margolin. Minimal Quantities and Measurability. Gravity in Measurable Format and Natural Transition to High Energies. Int. J. Nonlinear Phenomena in Complex Systems. **21**, 138–163 (2018).

[6] A. E. Shalyt-Margolin. The Equivalence Principle Applicability Boundaries, Gravity at Low and High Energies, Measurability and Dark Energy Problem. In: *A Closer Look at Dark Energy*. Adances in Astronomy and Astrophysics. (Nova Science Publishers, New York, 2019). Pp. 57–78.

[7] A. Shalyt-Margolin.The Equivalence Principle Applicability Boundaries, QFT in Flat Space and Measurability I. Free Quantum Fields. Int. J. Nonlinear Phenomena in Complex Systems. **22**, 135–150 (2019).

[8] R. J. Adler, D. I. Santiago. On gravity and the uncertainty principle. Mod. Phys. Lett. A. **14**, 1371–1378 (1999).

[9] M. Maggiore. Black Hole Complementarity and the Physical Origin of the Stretched Horizon. Phys. Rev. D. **49**, 2918–2921 (1994).

[10] M. Maggiore. Generalized Uncertainty Principle in Quantum Gravity. Phys. Lett. B. **304**, 65–69 (1993).

[11] M. Maggiore. The algebraic structure of the generalized uncertainty principle. Phys. Lett. B. **319**, 83–86 (1993).

[12] E. Witten. Reflections on the fate of spacetime. Phys. Today. **49**, 24–28 (1996).

[13] D. Amati, M. Ciafaloni, G. A. Veneziano. Can spacetime be probed below the string size? Phys. Lett. B. **216**, 41–47 (1989).

[14] S. Capozziello, G. Lambiase, G. Scarpetta. The Generalized Uncertainty Principle from Quantum Geometry. Int. J. Theor. Phys. **39**, 15–22 (2000).

[15] A. Kempf, G. Mangano, R. B. Mann. Hilbert space representation of the minimal length uncertainty relation. Phys. Rev. D. **52**, 1108–1118 (1995).

[16] A. N. Tawfik, A. M. Diab. Generalized Uncertainty Principle: Approaches and Applications. Int. J. Mod. Phys. D. **23**, 1430025 (2014).

[17] W. Heisenberg. Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. Z. Phys. **43**, 172–198 (1927).

[18] A. Messiah. *Quantum Mechanics.* (North Holland Publishing Company, Amsterdam, 1981).

[19] L.H. Rayder. *Quantum Field Theory.* (Cambridge University Press, New York, 1996).

[20] T.-P. Cheng, L.-F. Li.*Gauge Theory of elementary particle physics.* Oxford Scince Publications. (Clarendon Press, Oxford University Press, 2014).

[21] M.E. Peskin, D.V. Schroeder. *An Introduction to Quantum Field Theory.* (Addison-Wesley Publishing Company, Barbara Holland, 1995).

[22] S. Weinberg. *The Quantum Theory of Fields.* Vol.1, 2. (Cambridge University Press, 1995).

[23] L. Faddeev. Mathematical View on Evolution of Physics. Nature (Priroda). **5**, 11 (1989). (in Russian).

[24] S. Weinberg. *Gravitation and Cosmology. Principles and Applications of General Theory of Relativity.* (Wiley, New York, 1972).

[25] E. T. Akhmedov. *Lectures on General Theory of Relativity.* ArXiv:1601.04996 [gr-qc].

[26] J.A. Wheeler. Geons. Phys. Rev. **97**, 511 (1955).

[27] J.A. Wheeler. *Geometrodynamics.* (Academic Press, New York and London, 1962).

[28] C. W. Misner, K. S. Thorne, J. A. Wheeler. *Gravitation.* (Freeman, San Francisco, 1973).

[29] S. W. Hawking. Space-time foam. Nuclear Phys. B. **114**, 349 (1978).

[30] Y. J. Ng. Selected topics in Planck-scale physics, Mod. Phys. Lett. A. **18**, 1073–1098 (2003).

[31] F. Scardigli. Black Hole Entropy: a spacetime foam approach, Class. Quant. Grav. **14**, 1781–1793 (1997).

[32] R. Garattini. A Spacetime Foam approach to the cosmological constant and entropy. Int. J. Mod. Phys. D. **4**, 635–652 (2002).

[33] R. Garattini. A Spacetime Foam Approach to the Schwarzschild-de Sitter Entropy. Entropy. **2**, 26–38 (2000).

[34] R. Garattini. Entropy and the cosmological constant: a spacetime-foam approach. Nucl. Phys. Proc. Suppl. **88**, 297–300 (2000).

[35] R. Garattini. Entropy from the foam. Phys. Lett. B. **459**, 461–467 (1999).

[36] F. Scardigli. Generalized Uncertainty Principle in Quantum Gravity from Micro-Black Hole Gedanken Experiment. Phys. Lett. B. **452**, 39–

44 (1999).

[37] F. Scardigli. Gravity coupling from micro-black holes. Nucl. Phys. Proc. Suppl. **88**, 291–294 (2000).

[38] L. J. Garay. Thermal properties of spacetime foam. Phys. Rev. D. **58**, 124015 (1998).

[39] L. J. Garay. Spacetime foam as a quantum thermal bath. Phys. Rev. Lett. **80**, 2508–2511 (1998).

[40] V. A. Kostelecky, N. Russell. Data tables for Lorentz and CPT violation. Rev. Mod. Phys. **83**, 11–31 (2011).

[41] V. V. Flambaum. Enhanced violation of the Lorentz invariance and Einstein's equivalence principle in nuclei and atoms. Phys. Rev. Lett. **117**, 072501 (2016).

[42] D.-C. Dai. Serious limitations of the strong equivalence principle. Int. J. Mod. Phys. A. **32**, 1750068 (2017).

[43] R. Lafrance, R. C. Myers. Gravity's Rainbow: Limits for the applicability of the equivalence principle. Phys.Rev. D. **51**, 2584–2590 (1995).

[44] J. Smit. *Introduction to Quantum Fields on a Lattice*. Cambridge Lecture Notes in Physics. (Cambridge University Press, Cambridge, 2002).

[45] I. Montvay, G. Munster. *Quantum Fields on a Lattice*. (Cambridge University Press, Cambridge, 1994).

[46] P. Hernandez. *Introduction to Lattice Field Theory*. (Universidad de Valencia and IFIC, 2010).