Reflected BSDE with stochastic Lipschitz coefficient

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Abstract

In this paper, we deal with a class of one-dimensional reflected backward stochastic differential equations with stochastic Lipschitz coefficient. We derive the existence and uniqueness of the solutions for those equations via Snell envelope and the fixed point theorem.

Keywords: Reflected backward stochastic differential equation; stochastic Lipschitz coefficient; Snell envelope

AMS 2000 Subject Classification: 60H10

1 Introduction

Nonlinear backward stochastic differential equations (BSDE in short) were firstly introduced by Pardoux and Peng (1990). Since then, a lot of work have been devoted to the study of BSDEs as well as to their applications. This is due to the connections of BSDEs with mathematical finance (see e.g. El Karoui et al. (1997c)) as well as to stochastic optimal control (see e.g. Peng (1993)) and stochastic games (see e.g. Hamadène and Lepeltier (1995)). El Karoui et al. (1997a) introduced the notion of one barrier reflected BSDE (RBSDE in short), which is actually a backward equation but the solution

*Support by the National Basic Research Program of China (973 Program) grant No. 2007CB814900 and The Youth Fund of Yantai University (SX08Z9).

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is forced to stay above a given barrier. This type of BSDEs is motivated by pricing American options (see El Karoui et al. (1997b)) and studying the mixed game problems(see e.g. Cvitanic and Karatzas (1996), Hamadène and Lepeltier (2000)).

The existence and uniqueness of solution of BSDE in Pardoux and Peng (1990) and of RBSDE in El Karoui et al. (1997a) are both proved under the Lipschitz assumption on the coefficient. However, the Lipschitz condition is too restrictive to be assumed in many applications. For instance, the pricing problem of an American claim is equivalent to solving the linear BSDE

$$\mathrm{d}Y_t = [r(t)Y_t + \theta(t)Z_t]\mathrm{d}t + Z_t\mathrm{d}B_t,\ Y_T = \xi,$$

where $r(t)$ is the interest rate and $\theta(t)$ is the risk premium vector. In general, both of them may be unbounded, therefore Pardoux and Peng's result may be invalid. And so is it in the case of RBSDE.

Consequently, many papers have devoted to relax the Lipschitz condition in both cases of BSDE and RBSDE (see e.g. Lepeltier and Martin (1997), El Karoui and Huang (1997), Bender and Kohlmann (2000), Wang and Huang (2009), Matoussi (1997), Lepeltier et al. (2005) and the references therein). During them, El Karoui and Huang (1997) established a general result of existence and uniqueness for BSDEs driven by a general cadlag martingale with stochastic Lipschitz coefficient. Later, Bender and Kohlmann (2000) showed the same result for BSDEs driven by a Brownian motion. Motivated by the above works, the purpose of the present paper is to consider a class of one-dimensional RBSDEs with stochastic Lipschitz coefficient. We try to get the existence and uniqueness of solutions for those RBSDEs by means of the Snell envelope and the fixed point theorem.

The rest of the paper is organized as follows. In Section 2, we introduce some notations including some spaces. Section 3 is devoted to prove the existence and uniqueness of solutions to RBSDEs with stochastic Lipschitz coefficient.

2 Notations

Let $(B_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We denote $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration of $(B_t)_{t\geq 0}$, augmented by all $\mathbf{P}$-null sets of $\mathcal{F}$. The Euclidean norm of a vector $y \in \mathbf{R}^n$ will be defined by $|y|$.

Let $T > 0$ be a given real number. Let's introduce some spaces:

- $\mathbf{L}^2$ the space of $\mathcal{F}_T$-measurable random variables ξ such that

$$\mathbf{E}[|\xi|^2] < +\infty.$$

- $\mathbf{S}^2$ the space of predictable processes $\{\psi_t : t \in [0,T]\}$ such that

$$\mathbf{E}[\sup_{0\le t\le T} |\psi_t|^2] < +\infty.$$

- $\mathbf{H}^2$ the space of predictable processes $\{\psi_t : t \in [0,T]\}$ such that

$$\mathbf{E}\int_0^T |\psi_t|^2 \mathrm{d}t < +\infty.$$

Let $\beta > 0$ and $(a_t)_{t\ge 0}$ be a nonnegative $\mathcal{F}_t$-adapted process. Define

$$A(t) = \int_0^t a^2(s)\mathrm{d}s, \quad 0 \le t \le T.$$

We further introduce the following spaces:

- $\mathbf{L}^2(\beta, a)$ the space of $\mathcal{F}_T$-measurable random variables ξ such that

$$\mathbf{E}[e^{\beta A(T)}|\xi|^2] < +\infty.$$

- $\mathbf{S}^2(\beta, a)$ the space of predictable processes $\{\psi_t : t \in [0,T]\}$ such that

$$\mathbf{E}[e^{\beta A(T)} \sup_{0\le t\le T} |\psi_t|^2] < +\infty.$$

- $\mathbf{H}^2(\beta, a)$the space of predictable processes $\{\psi_t : t \in [0,T]\}$ such that

$$\mathbf{E}\int_0^T e^{\beta A(t)}|\psi_t|^2 \mathrm{d}t < +\infty.$$

In this paper, we consider the following RBSDE:

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}B_s, \\ Y_t \ge S_t,\ 0 \le t \le T \text{ a.s. and } \int_0^T (Y_t - S_t)\mathrm{d}K_t = 0, \text{a.s.} \end{cases} \tag{1}$$

where the coefficient $f : \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{R} \to \mathbf{R}$ satisfies the following assumptions:
(**H1**) $\forall t \in [0,T], (y_i, z_i) \in \mathbf{R} \times \mathbf{R}^d, i = 1,2$, there are two nonnegative $\mathcal{F}_t$-adapted processes $\mu(t)$ and $\gamma(t)$ such that

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq \mu(t)|y_1 - y_2| + \gamma(t)|z_1 - z_2|; \tag{2}$$

(**H2**) $\exists \epsilon > 0$ such that $a^2(t) := \mu(t) + \gamma^2(t) \geq \epsilon$;
(**H3**) For all $(y,z) \in \mathbf{R} \times \mathbf{R}^d$, the process $f(\cdot, \cdot, y, z)$ is progressively measurable and such that $\forall t \in [0,T], \dfrac{f(t,0,0)}{a} \in \mathbf{H}^2(\beta, a)$.

Furthermore, we make the following assumptions:
(**H4**) The terminal value $\xi \in \mathbf{L}^2(\beta, a)$;
(**H5**) The "obstacle" $\{S_t, 0 \leq t \leq T\}$ is a continuous progressively measurable real-valued process satisfying $\mathbf{E}[\sup_{0 \leq t \leq T} e^{2\beta A(t)} (S_t^+)^2] < \infty$ and $S_T \leq \xi$ a.s.

We now give the definition of solution to RBSDE (1).

Definition 2.1 *Let $\beta > 0$ and a a nonnegative $\mathcal{F}_t$-adapted process. A solution to RBSDE (1) is a triple (Y,Z,K) satisfying (1) such that $(Y,Z) \in \mathbf{S}^2(\beta,a) \times \mathbf{H}^2(\beta,a)$ and $K \in \mathbf{S}^2$ is continuous and increasing with $K_0 = 0$.*

3 Main results

3.1 A priori estimate

We first give a priori estimate of the solution of RBSDE (1).

Lemma 3.1 *Let $(Y_t, Z_t, K_t)_{0 \leq t \leq T}$ be a solution of RBSDE (1) with data (ξ, f, T). Then there exists a constant C_β depending only on β such that*

$$\begin{aligned} & \mathbf{E}\left[\sup_{0 \leq t \leq T} |Y_t|^2 e^{\beta A(t)} + \int_0^T e^{\beta A(s)} |Z_s|^2 ds + \int_0^T e^{\beta A(s)} a^2(s) |Y_s|^2 ds + K_T^2\right] \\ \leq \;& C_\beta \mathbf{E}\left[|\xi|^2 e^{\beta A(T)} + \int_0^T e^{\beta A(s)} \frac{|f(s,0,0)|^2}{a^2(s)} ds + \sup_{0 \leq t \leq T} e^{2\beta A(t)} (S_t^+)^2\right]. \end{aligned}$$

Proof. Applying Itô's formula to $e^{\beta A(t)}|Y_t|^2$, we have

$$
\begin{aligned}
& e^{\beta A(t)}|Y_t|^2+\int_t^T e^{\beta A(s)}|Z_s|^2\mathrm{d}s+\beta\int_t^T a^2(s)e^{\beta A(s)}|Y_s|^2\mathrm{d}s \\
= \;& e^{\beta A(T)}|\xi|^2+2\int_t^T e^{\beta A(s)}Y_s f(s,Y_s,Z_s)\mathrm{d}s+2\int_t^T e^{\beta A(s)}Y_s\mathrm{d}K_s-2\int_t^T e^{\beta A(s)}Y_sZ_s\mathrm{d}B_s \\
\leq \;& e^{\beta A(T)}|\xi|^2+\frac{\beta}{2}\int_t^T a^2(s)e^{A(s)}|Y_s|^2\mathrm{d}s+2\int_t^T e^{\beta A(s)}\frac{1}{\beta a^2(s)}|f(s,Y_s,Z_s)|^2\mathrm{d}s \\
& +2\int_t^T e^{\beta A(s)}Y_s\mathrm{d}K_s-2\int_t^T e^{\beta A(s)}Y_sZ_s\mathrm{d}B_s \\
\leq \;& e^{\beta A(T)}|\xi|^2+\frac{\beta}{2}\int_t^T a^2(s)e^{A(s)}|Y_s|^2\mathrm{d}s+\frac{6}{\beta}[\int_t^T e^{\beta A(s)}a^2(s)|Y_s|^2\mathrm{d}s+\int_t^T e^{\beta A(s)}|Z_s|^2\mathrm{d}s] \\
& +\frac{6}{\beta}\int_t^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}\mathrm{d}s+2\int_t^T e^{\beta A(s)}Y_s\mathrm{d}K_s-2\int_t^T e^{\beta A(s)}Y_sZ_s\mathrm{d}B_s.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
e^{\beta A(t)}|Y_t|^2 \;&+\; (1-\frac{6}{\beta})\int_t^T e^{\beta A(s)}|Z_s|^2\mathrm{d}s+(\frac{\beta}{2}-\frac{6}{\beta})\int_t^T a^2(s)e^{\beta A(s)}|Y_s|^2\mathrm{d}s \\
&\leq\; e^{\beta A(T)}|\xi|^2+\frac{6}{\beta}\int_t^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}\mathrm{d}s \\
&\quad +2\int_t^T e^{\beta A(s)}S_s\mathrm{d}K_s-2\int_t^T e^{\beta A(s)}Y_sZ_s\mathrm{d}B_s. \qquad (3)
\end{aligned}
$$

where we have used the fact that $\mathrm{d}K_s=\mathbf{I}_{[Y_s=S_s]}\mathrm{d}K_s$ and the stochastic Lipschitz property of f. For a sufficient large $\beta>0$, taking expectation on both sides above, we get

$$
\begin{aligned}
& \mathbf{E}[\int_t^T a^2(s)e^{\beta A(s)}|Y_s|^2\mathrm{d}s+\int_t^T e^{\beta A(s)}|Z_s|^2\mathrm{d}s] \\
\leq \;& c_\beta\mathbf{E}\left[e^{\beta A(T)}|\xi|^2+\int_t^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}\mathrm{d}s+2\int_t^T e^{\beta A(s)}S_s^+\mathrm{d}K_s\right]. \quad (4)
\end{aligned}
$$

Moreover, by the Burkholder-Davis-Gundy's inequality, one can derive that

$$
\begin{aligned}
& \mathbf{E}[\sup_{0\le t\le T}|\int_t^T e^{\beta A(s)}Y_sZ_s\mathrm{d}B_s|] \\
\le\ & \mathbf{E}[|\int_0^T e^{\beta A(s)}Y_sZ_s\mathrm{d}B_s|]+\mathbf{E}[\sup_{0\le t\le T}|\int_0^t e^{\beta A(s)}Y_sZ_s\mathrm{d}B_s|] \\
\le\ & 2c\mathbf{E}\left\{\left[\int_0^T e^{2\beta A(s)}|Y_s|^2|Z_s|^2\mathrm{d}s\right]^{\frac{1}{2}}\right\} \\
\le\ & 2c\mathbf{E}\left[(\sup_{0\le t\le T}e^{\beta A(t)}|Y_t|^2)^{\frac{1}{2}}(\int_0^T e^{\beta A(s)}|Z_s|^2\mathrm{d}s)^{\frac{1}{2}}\right] \\
\le\ & \frac{1}{2}\mathbf{E}[(\sup_{0\le t\le T}e^{\beta A(t)}|Y_t|^2)]+2c^2\mathbf{E}[\int_0^T e^{\beta A(s)}|Z_s|^2\mathrm{d}s].
\end{aligned}
$$

Combining this with (3) and (4), we have

$$
\begin{aligned}
& \mathbf{E}\left[\sup_{0\le t\le T}|Y_t|^2e^{\beta A(t)}+\int_0^T a^2(s)e^{\beta A(s)}|Y_s|^2\mathrm{d}s+\int_0^T e^{\beta A(s)}|Z_s|^2\mathrm{d}s\right] \\
\le\ & k_\beta\mathbf{E}\left[e^{\beta A(T)}|\xi|^2+\int_0^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}\mathrm{d}s+2\int_0^T e^{\beta A(s)}S_s^+\mathrm{d}K_s\right]. \quad (5)
\end{aligned}
$$

We now give an estimate of K_T^2. From the equation

$$
K_T=Y_0-\xi-\int_0^T f(s,Y_s,Z_s)\mathrm{d}s+\int_0^T Z_s\mathrm{d}B_s
$$

and (5), we have

$$
\begin{aligned}
&\mathbf{E}[K_T^2] \\
\leq\ & d_\beta \mathbf{E}\left[\sup_{0\leq t\leq T}|Y_t|^2 e^{\beta A(t)} + |\xi|^2 + \int_0^T |Z_s|^2 \mathrm{d}s \right.\\
&\left. + \int_0^T a^2(s)e^{-\beta A(s)}\mathrm{d}s \int_0^T e^{\beta A(s)}\frac{|f(s,Y_s,Z_s)|^2}{a^2(s)}\mathrm{d}s\right] \\
\leq\ & d_\beta \mathbf{E}\left[e^{\beta A(T)}|\xi|^2 + \int_0^T a^2(s)e^{\beta A(s)}|Y_s|^2\mathrm{d}s + \int_0^T e^{\beta A(s)}|Z_s|^2\mathrm{d}s \right.\\
&\left. + \int_0^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}\mathrm{d}s\right] \\
\leq\ & d_\beta \mathbf{E}\left[e^{\beta A(T)}|\xi|^2 + \int_0^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}\mathrm{d}s + 2\int_0^T e^{\beta A(s)}S_s^+\mathrm{d}K_s\right] \\
\leq\ & d_\beta \mathbf{E}\left[e^{\beta A(T)}|\xi|^2 + \int_0^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}\mathrm{d}s + \sup_{0\leq t\leq T} e^{2\beta A(t)}(S_t^+)^2\right] + \frac{1}{2}\mathbf{E}[K_T^2].
\end{aligned}
$$

Hence,

$$
\mathbf{E}[K_T^2] \leq d_\beta \mathbf{E}\left[e^{\beta A(T)}|\xi|^2 + \int_0^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}\mathrm{d}s + \sup_{0\leq t\leq T} e^{2\beta A(t)}(S_t^+)^2\right], \quad (6)
$$

where we use the notation d_β for a constant depending only on β and whose value could be changing from line to line. We get the desired result by estimates (6) and (5). $\Box$

3.2 Existence and uniqueness of solution

We first consider the special case that is the coefficient does not depend on (Y, Z), i.e. $f(\omega, t, y, z) \equiv g(\omega, t)$. We have the following result.

Theorem 3.1 *Let $\beta > 0$ large enough and a a nonnegative $\mathcal{F}_t$-adapted process. Assume $\frac{g}{a} \in \mathbf{H}^2(\beta, a)$ and* ***(H4)****-****(H5)*** *hold. Then RBSDE (1) with data (ξ, g, S) has a solution.*

Proof. For $0 \leq t \leq T$, we define

$$
\widetilde{Y}_t = \operatorname*{ess\,sup}_{\nu\geq t} \mathbf{E}[\int_0^\nu g(s)ds + S_\nu \mathbf{I}_{\{\nu<T\}} + \xi\mathbf{I}_{\{\nu=T\}}|\mathcal{F}_t]
$$

where ν is a $\mathcal{F}_t$-stopping time. The process $\widetilde{Y}_t$ is called the Snell envelope of the process which is inside ess sup.

By assumptions of the theorem, one can easily to check that $\xi \in \mathbf{L}^2$, $S_t^+ \in \mathbf{S}^2$ and $(\int_0^t |g(s)|\mathrm{d}s)_{0\leq t\leq T} \in \mathbf{L}^2$. Indeed, for given $\beta > 0$, by Hölder inequality, we have

$$\begin{aligned}\mathbf{E}\left[\left(\int_0^T |g(s)|\mathrm{d}s\right)^2\right] &= \mathbf{E}\left[\left(\int_0^T |\frac{g(s)}{a(s)}||a(s)|\mathrm{d}s\right)^2\right] \\ &\leq \mathbf{E}\left[\left(\int_0^T |\frac{g(s)}{a(s)}|^2 e^{\beta A(s)}\mathrm{d}s\right)\left(\int_0^T a(s)^2 e^{-\beta A(s)}\mathrm{d}s\right)\right] \\ &\leq \frac{1}{\beta}\mathbf{E}\left[\left(\int_0^T |\frac{g(s)}{a(s)}|^2 e^{\beta A(s)}\mathrm{d}s\right)\right] < +\infty.\end{aligned}$$

Consequently, by Doob-Meyer decomposition theorem in Dellacherie and Meyer (1980), there exists an increasing continuous process $(K_t)_{0\leq t\leq T}$ which belongs to $\mathbf{S}^2$ $(K_0 = 0)$ and a martingale $M_t \in \mathbf{S}^2$ such that

$$\widetilde{Y}_t = M_t - K_t. \quad \forall t \in [0, T]$$

Since $M_t \in \mathbf{S}^2$, there exists $Z_t \in \mathbf{H}^2$ such that

$$M_t = M_0 + \int_0^t Z_s \mathrm{d}B_s. \quad \forall t \in [0, T]$$

Let $Y_t = \widetilde{Y}_t - \int_0^t f(s)\mathrm{d}s$, by Proposition 5.1 of El Karoui et al. (1997a), we derive that the triple (Y, Z, K) verities

$$Y_t = \xi + \int_t^T g(s)\mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}B_s.$$

Moreover, $Y_t \geq S_t$ and $\int_0^T (Y_t - S_t)\mathrm{d}K_t = 0$. By Lemma 3.1, $(Y_t, Z_t, K_t)_{0\leq t\leq T}$ is a solution of RBSDE (1). $\square$

Furthermore, we have the following uniqueness result.

Proposition 3.1 *With the same assumptions of Theorem 3.1, the RBSDE (1) with data (ξ, g, S) has at most one solution.*

Proof. Let (Y, Z, K) and (Y', Z', K') be two solutions of RBSDE (1). Let

$$\Delta Y = Y - Y', \ \Delta Z = Z - Z', \ \Delta K = K - K'.$$

For $0 \leq t \leq T$, we have

$$\Delta Y_t = \Delta K_T - \Delta K_t - \int_t^T \Delta Z_s \mathrm{d}B_s.$$

Applying Itô's formula to $e^{\beta A(t)}|\Delta Y_t|^2$, we obtain

$$-\mathbf{E}[e^{\beta A(t)}|\Delta Y_t|^2] = -2\,\mathbf{E}[\int_t^T e^{\beta A(s)}\Delta Y_s \mathrm{d}(\Delta K_s)] + \mathbf{E}[\int_t^T e^{\beta A(s)}|\Delta Z_s|^2 \mathrm{d}s].$$

Noting that $\int_t^T e^{\beta A(s)}\Delta Y_s \mathrm{d}(\Delta K_s) \leq 0$, it follows that $\Delta Y_t = \Delta Z_t = 0$ and then $\Delta K_t = 0$, $0 \leq t \leq T$ a.s. $\Box$

We can now state and prove our main result.

Theorem 3.2 *Assume (H1)-(H5) hold for a sufficient large β. Then RBSDE (1) with data (ξ, f, S) has a unique solution.*

Proof. Let $\mathcal{H}(\beta, a) = \mathbf{S}^2(\beta, a) \times \mathbf{H}^2(\beta, a)$. Given $(U, V) \in \mathcal{H}(\beta, a)$, consider the following RBSDE:

$$Y_t = \xi + \int_t^T f(s, U_s, V_s)\mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}B_s. \tag{7}$$

By Young's inequality, we have

$$\frac{|f(t, U_t, V_t)|^2}{a^2(t)} \leq 3[a^2(t)|U_t|^2 + |V_t|^2 + \frac{|f(t, 0, 0)|^2}{a^2(t)}],$$

it follows from (H3) and Theorem 3.1 that the RBSDE (7) has a unique solution.

Define a mapping Φ from $\mathcal{H}(\beta, a)$ to itself. Let (U', V') be another element in $\mathcal{H}(\beta, a)$, set

$$(Y, Z) = \Phi(U, V),\ (Y', Z') = \Phi(U', V'),$$

where (Y, Z, K) (resp. (Y', Z', K')) is the unique solution of the RBSDE associated with data $(\xi, f(t, U_t, V_t), S)$ (resp.$(\xi, f(t, U'_t, V'_t), S)$).

Let

$$\Delta Y = Y - Y', \Delta Z = Z - Z', \Delta U = U - U', \Delta V = V - V',$$
$$\Delta f_s = f(s, U_s, V_s) - f(s, U'_s, V'_s), \Delta K = K - K'.$$

For $0 \leq t \leq T$, we have

$$\Delta Y_t = \int_t^T \Delta f_s \mathrm{d}s + \Delta K_T - \Delta K_t - \int_t^T \Delta Z_s \mathrm{d}B_s$$

Applying Itô's formula to $e^{\beta A(t)}|\Delta Y_t|^2$, using (**H1**) and the fact that $\mathrm{d}K_s = \mathbf{I}_{[Y_s=S_s]}\mathrm{d}K_s$ and $\mathrm{d}K'_s = \mathbf{I}_{[Y'_s=S_s]}\mathrm{d}K'_s$, we get

$$\begin{aligned}
& e^{\beta A(t)}|\Delta Y_t|^2 + \beta \int_t^T a(s)^2 e^{\beta A(s)}|\Delta Y_s|^2 \mathrm{d}s + \int_t^T e^{\beta A(s)}|\Delta Z_s|^2 \mathrm{d}s \\
\leq\ & 2\int_t^T e^{\beta A(s)}\Delta Y_s \Delta f_s \mathrm{d}s + 2\int_t^T e^{\beta A(s)}\Delta Y_s \mathrm{d}(\Delta K_s) - \int_t^T 2e^{\beta A(s)}\Delta Y_s \Delta Z_s \mathrm{d}B_s \\
\leq\ & 2\int_t^T e^{\beta A(s)}\Delta Y_s \Delta f_s \mathrm{d}s - \int_t^T 2e^{\beta A(s)}\Delta Y_s \Delta Z_s \mathrm{d}B_s \\
\leq\ & \frac{\beta}{2}\int_t^T a(s)^2 e^{\beta A(s)}|\Delta Y_s|^2 \mathrm{d}s + \frac{6}{\beta}\int_t^T e^{\beta A(s)}|(a(s)^2|\Delta U_s|^2 + |\Delta V|^2)\mathrm{d}s \\
& - \int_t^T 2e^{\beta A(s)}\Delta Y_s \Delta Z_s \mathrm{d}B_s,
\end{aligned}$$

it follows that

$$\begin{aligned}
& \mathbf{E}[\int_t^T a(s)^2 e^{\beta A(s)}|\Delta Y_s|^2 \mathrm{d}s] + \mathbf{E}[\int_t^T e^{\beta A(s)}|\Delta Z_s|^2 \mathrm{d}s] \\
\leq\ & (\frac{12}{\beta^2} + \frac{6}{\beta})\left\{\mathbf{E}[\int_t^T a(s)^2 e^{\beta A(s)}|\Delta U_s|^2] + \mathbf{E}[\int_t^T e^{\beta A(s)}|\Delta V|^2 \mathrm{d}s]\right\}.
\end{aligned}$$

For $\beta > 0$ large enough, one can easily to check that Φ is a contraction mapping with the norm

$$\|(Y,Z)\|_\beta^2 = \mathbf{E}\left[\int_0^T e^{\beta A(s)}(a(s)^2|Y_s|^2 + |Z_s|^2)ds\right].$$

Thus, ϕ has a unique fixed point and the theorem is proved. □

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