**Discrete Nature of Space and Time, Measurability, and Another Paradigm**

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PACS: 03.65, 05.20
Keywords: continuous theory, discrete theory, measurability

**Abstract**

This paper is a continuation of the previous author's works on the subject sometimes overlapping over them. Additional arguments supporting the idea that a minimal length and minimal time should exist in nature and should be considered at all energy scales are given. Compared with the previous works, apart from the notion of **measurability** (**measurability in principle**), the author first introduces the notion of **measurability in relation to the energy**. The principal objective of the paper is framing of discrete analogues for the well-known theories (Quantum Theory, Gravity, and so on) in terms of **measurable quantities only**. A program of further studies to this end is presented in the last section of this work.

# 1 Introduction.

In his previous related works the author has discussed the idea that for the adequate and correct understanding of the physical processes at all the energy scales, one should exclude infinitesimal space-time variations (increments) from all physical theories. The reasoning is that the mathematical apparatus currently used is inadequate necessitating its replacement by the mathematical apparatus based on the **initial** introduction of the minimal length $l_{min}$ and of the minimal time $t_{min}$ into the above-mentioned theories

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not only at maximal energies $E_{max} \propto E_P$ but at all the energy scales. Then these theories become discrete, though approaching the starting continuous theories at low energies which are far from the energies associated with the Big Bang. Actually, discreteness is revealed only at the energies close to those of the Big Bang.

This paper is a continuation of the previous author's works and is aimed at elaboration of the program of further studies on the subject. The first step is to introduce into a theory the minimal length $l_{min}$ and the minimal time $t_{min}$ at all energy scales together with the associated **measurability** notion.

One of the key problems of the modern fundamental physics (Quantum Theory (QT) and Gravity (GR)) is framing of a correct theory associated with the ultraviolet region, i. e. the region of the highest (apparently Planck's) energies approaching those of the Big Bang.

However, it is well known that at high energies (on the order of the quantum gravity energies) the minimal length $l_{min}$ to which the indicated energies are «sensitive», as distinct from the low ones, should inevitably become apparent in the theory. But if $l_{min}$ is really present, it must be present at all the «Energy Levels» of the theory, low energies including.

What follows from the existence of the minimal length $l_{min}$? When the minimal length is involved, any nonzero **measurable** quantity having the dimensions of length should be a multiple of $l_{min}$. Otherwise, its **measurement** with the use of $l_{min}$ would result in the **measurable** quantity $\varsigma$, so that $\varsigma < l_{min}$, and this is impossible.

This suggests that the spatial-temporal quantities $dx_\mu$ are **nonmeasurable** quantities because the latter lead to the infinitely small length $ds$ [1]

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu \qquad (1)$$

**nonmeasurable** because of $l_{min}$.

And this has been indicated in my previous work [2].

Of course, as a mathematical notion, the quantities $dx_\mu, ds$ are naturally existent but one should realize that there is no way to express them in terms of the minimal possible measuring unit $l_{min}$.

So, trying to frame a theory (QT and GR) correct at all the energy levels using only the **measurable** quantities, one should realize that then the

mathematical formalism of the theory should not involve any infinitesimal spatial-temporal quantities. Besides, proceeding from the acknowledged results associated with the Planck scales physics [3]–[11], one can infer that certain new parameters dependent on $l_{min}$ should be involved.
What are the parameters of interest in the case under study? It is obvious that, as the quantum-gravitational effects will be revealed at very small (possibly Planck's) scales, these parameters should be dependent on some limiting values, e.g., $l_P \propto l_{min}$ and hence Planck's energy $E_P$.
**This means that in high-energy QT and GR the energy- or, what is the same, measuring scales-dependent parameters should be necessarily introduced.**
But, on the other hand, these parameters could hardly disappear totally at low energies both in QT and in GR.
But, provided $l_{min}$ exists, it must be involved at all the energy levels, both **high** and **low**.
The fact that $l_{min}$ is omitted in the formulation of low-energy QT and GR and the theories give perfect results leads to two different inferences:

1.1. The influence of the above-mentioned new parameters associated with $l_{min}$ in low-energy QT and GR is so small that it may be disregarded at the modern stage in evolution of the theory and of the experiment.

1.2. The modern mathematical apparatus of conventional QT and GR has been derived in terms of the infinitesimal spatial-temporal quantities $dx_\mu$ which, as noted above, are **nonmeasurable quantities** in the formalism of $l_{min}$ and $l_{min}$.

The main reasoning for the introduction of the notion of **measurability** based on the **Uncertainty Principle at All Scales Energies** is substantiated in the following Section.

## 2 Uncertainty Principle at All Scales Energies and ≪Principle of bounded space-time variations (increments) ≫

Let us begin with Heisenberg's Uncertainty Principle (HUP) [14] and with the ≪**Principle of bounded variations (increments)** ≫ on its basis that is hereinafter referred to as the **Principle 1**:

**Principle 1.** ≪**Principle of bounded variations (increments)** ≫

*Any small variation (increment) $\widetilde{\Delta} x_\mu$ of any spatial coordinate $x_\mu$ of the arbitrary point $x_\mu, \mu = 0, ..., 3$ in some space-time system* R *may be realized in the form of the uncertainty $\Delta x_\mu$ when this coordinate is measured within the scope of Heisenberg's Uncertainty Principle (HUP)*

$$\widetilde{\Delta} x_\mu = \Delta x_\mu, \Delta x_\mu \simeq \frac{\hbar}{\Delta p_\mu}, \mu = 1, 2, 3 \tag{2}$$

*for some $\Delta p_\mu \neq 0$.*
*Similarly, for $\mu = 0$ and for the arbitrary value of $\widetilde{\Delta} x_0 = \widetilde{\Delta} t$ we have*

$$\widetilde{\Delta} t = \Delta t, \Delta t \simeq \frac{\hbar}{\Delta E} \tag{3}$$

*for some $\Delta E \neq 0$.*
Here HUP is given for the nonrelativistic case. In the relativistic case HUP has the distinctive features [39] which, however, are of no significance for the general formulation of **Principle 1**, being associated only with particular alterations in the right-hand side of the second relation (2)as shown later.
It is clear that at low energies $E$ (momentums $P$) **Principle 1** sets a lower bound for the variations (increments) $\widetilde{\Delta} x_\mu$ of any space-time coordinate $x_\mu$. At high energies $E$ (momentums $P$) this is not the case if $E$ ($P$) have no upper limit. But, according to the modern knowledge, $E$ ($P$) are bounded by some maximal quantities $E_{max}$, ($P_{max}$)

$$E \leq E_{max}, P \leq P_{max}, \tag{4}$$

where in general $E_{max}, P_{max}$ may be on the order of Planck's quantities $E_{max} \propto E_P, P_{max} \propto P_pl$ and also may be the trans-Planck's quantities.
In any case the quantities $P_{max}$ and $E_{max}$ lead to the introduction of the minimal length $l_{min}$ and of the minimal time $t_{min}$.
Because of this, it is natural to complete **Principle 1** with
**Principle 2**:
*In nature the minimal length $l_{min}$ is used as a minimal measuring unit for all quantities having the dimension of length, whereas the minimal time $t_{min} = l_{min}/c$ -- as a minimal measuring unit for all quantities having the dimension of time, where c is the speed of light.*
$l_{min}$ and $t_{min}$ are naturally introduced as $\Delta x_\mu, \mu = 1, 2, 3$ and $\Delta t$ in (2),(3) for $\Delta p_\mu = P_{max}$ and $\Delta E = E_{max}$.
For definiteness, we consider that $E_{max}$ and $P_{max}$ are the quantities on the order of the Planck quantities, then $l_{min}$ and $t_{min}$ are also on the order of Planck's quantities $l_{min} \propto l_P$, $t_{min} \propto t_P$.
In this case it is more convenient to start not with Heisenberg's Uncertainty Principle (HUP) [14]

$$\Delta x \geq \frac{\hbar}{\Delta p} \tag{5}$$

but with its widely known high-energy generalization — the Generalized Uncertainty Principle (GUP) [15] [27]:

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_P^2 \frac{\Delta p}{\hbar}. \tag{6}$$

Here $\alpha'$ is the model-dependent dimensionless numerical factor and $l_P$ is the Planckian length. (Note that the normalization $\Delta x \Delta p \geq \hbar$ is used rather than $\triangle x \Delta p \geq \hbar/2$.)
Note also that initially GUP (6) was derived within a string theory [15]– [18] and, subsequently, in a series of works far from this theory [19] – [25] it has been demonstrated that on going to high (Planck's) energies in the right-hand side of HUP (5) an additional «high-energy» term $\propto l_P^2 \frac{\triangle p}{\hbar}$ appears. Of particular interest is the work [19], where by means of a simple gedanken experiment it has been demonstrated that with regard to the gravitational interaction (6) is the case.
As (6) – quadratic inequality, then it naturally leads to the minimal length

$l_{min} = \xi l_P = 2\sqrt{\alpha'} l_P$.
This means that the theory for the quantities with a particular dimension has a **minimal measurement unit**. At least, all the quantities with such a dimension should be ≪quantized≫, i. e. be measured by an integer number of these **minimal units** of measurement.
Specifically, if $l_{min}$ – **minimal unit** of length, then for any length $L$ we have the ≪**Integrality Condition**≫ **(IC)**

$$L = N_L l_{min}, \tag{7}$$

where $N_L > 0$ – integer.
What are the consequences for GUP (6) and HUP (5)?
Assuming that HUP is to a high accuracy derived from GUP on going to low energies or that HUP is a special case of GUP at low values of the momentum, we have

$$(GUP, \Delta p \rightarrow 0) = (HUP). \tag{8}$$

By the language of $N_L$ from(7), (8) is nothing else but a change-over to the following:

$$(N_{\Delta x} \approx 1) \rightarrow (N_{\Delta x} \gg 1). \tag{9}$$

The assumed equalities in (5) and (6) may be conveniently rewritten in terms of $l_{min}$ with the use of the deformation parameter $\alpha_a$. This parameter has been introduced earlier in the papers [28]–[36] as a **deformation parameter** on going from the canonical quantum mechanics to the quantum mechanics at Planck's scales (early Universe) that is considered to be the quantum mechanics with the minimal length (QMML):

$$\alpha_a = l_{min}^2 / a^2, \tag{10}$$

where $a$ is the measuring scale.

**Definition 1**.
*Deformation is understood as an extension of a particular theory by inclusion of one or several additional parameters in such a way that the initial theory appears in the limiting transition* [37].

Then with the equality ($\Delta p \Delta x = \hbar$) (6) is of the form

$$\Delta x = \frac{\hbar}{\Delta p} + \frac{\alpha_{\Delta x}}{4} \Delta x. \tag{11}$$

In this case due to formulae (7) and (9) the equation (11) takes the following form:

$$N_{\Delta x} l_{min} = \frac{\hbar}{\Delta p} + \frac{1}{4 N_{\Delta} x} l_{min} \tag{12}$$

or

$$(N_{\Delta x} - \frac{1}{4 N_{\Delta} x}) l_{min} = \frac{\hbar}{\Delta p}. \tag{13}$$

That is

$$\Delta p = \frac{\hbar}{(N_{\Delta x} - \frac{1}{4 N_{\Delta} x}) l_{min}}. \tag{14}$$

From (12)–(14) it is clear that HUP (5) in the case of the equality appears to a high accuracy in the limit $N_{\Delta} x \gg 1$ in conformity with (9).
It is easily seen that the parameter $\alpha_a$ from (10) is discrete as it is nothing else but

$$\alpha_a = l_{min}^2 / a^2 = \frac{l_{min}^2}{N_a^2 l_{min}^2} = \frac{1}{N_a^2}. \tag{15}$$

At the same time, from (15) it is evident that $\alpha_a$ is irregularly discrete.
It is clear that from formula (14) at low energies ($N_{\Delta} x \gg 1$), up to a constant

$$\frac{\hbar^2}{l_{min}^2} = \frac{\hbar c^3}{4 \alpha' G} \tag{16}$$

we have

$$\alpha_{\Delta x} = (\Delta p)^2. \tag{17}$$

But all the above computations are associated with the nonrelativistic case. As known, in the relativistic case, when the total energy of a particle with the mass $m$ and with the momentum $p$ equals [38]:

$$E = \sqrt{p^2 c^2 + m^2 c^4}, \tag{18}$$

a minimal value for $\Delta x$ takes the form [39]:

$$\Delta x \approx \frac{c\hbar}{E}. \tag{19}$$

And in the **ultrarelativistic case**

$$E \approx pc \tag{20}$$

this means simply that

$$\Delta x \approx \frac{\hbar}{p}. \tag{21}$$

Provided the minimal length $l_{min}$ is involved and considering the ≪**Integrality Condition**≫ **(IC)** (7), in the general case for (19) at the energies considerably lower than the Planck energies $E \ll E_P$ we obtain the following:

$$\Delta x = N_{\Delta x} l_{min} \approx \frac{c\hbar}{E},$$
$$or$$
$$E \approx \frac{c\hbar}{N_{\Delta x}}. \tag{22}$$

Similarly, at the same energy scale in the ultrarelativistic case we have

$$p \approx \hbar / N_{\Delta x}. \tag{23}$$

Note that all the foregoing results associated with GUP and with its limiting transition to HUP for the pair $(\Delta x, \Delta p)$, as shown in [30], may be in **ultrarelativistic case** easily carried to the ≪energy - time≫ pair $(\Delta t, \Delta E)$. Indeed (6) gives [30]:

$$\frac{\Delta x}{c} \geq \frac{\hbar}{\Delta pc} + \alpha' l_P^2 \frac{\Delta p}{c\hbar}, \tag{24}$$

then

$$\Delta t \geq \frac{\hbar}{\Delta E} + \alpha' \frac{l_p^2}{c^2} \frac{\Delta pc}{\hbar} = \frac{\hbar}{\Delta E} + \alpha' t_p^2 \frac{\Delta E}{\hbar}. \tag{25}$$

where according to (20) the difference between $\Delta E$ and $\Delta(pc)$ can be neglected and $t_P$ is the Planck time $t_P = L_P/c = \sqrt{G\hbar/c^5} \simeq 0,54\ 10^{-43} sec.$

From whence it follows that we have a maximum energy of the order of Planck's:

$$E_{max} \sim E_P$$

Then the foregoing formulae (5)–(13) are rewritten by substitution as follows:

$$\Delta x \to \Delta t; \Delta p \to \Delta E; l_{min} \to t_{min}; N_L \to N_{t=L/c} \quad (26)$$

Specifically, (13) takes the form

$$(N_{\Delta t} - \frac{1}{4N_{\Delta}t})t_{min} = \frac{\hbar}{\Delta E}. \quad (27)$$

As shown, for the ultrarelativistic case there is $t_{min}$.
Next we assume that for **all cases** there is a minimal measuring unit of time

$$t_{min} = l_{min}/v_{max} = l_{min}/c. \quad (28)$$

Then, similar to (7), we get the ≪**Integrality Condition**≫ **(IC)** for any time $t$:

$$t \equiv t(N_t) = N_t t_{min}, \quad (29)$$

for certain $|N_t| \geq 0$ – integer.
According to (27), let us define the corresponding energy $E$

$$E \equiv E(N_t) = \frac{\hbar}{|N_t - \frac{1}{4N_t}|t_{min}}. \quad (30)$$

Note that at low energies $E \ll E_P$, that is for $|N_t| \gg 1$, the formula (30) naturally takes the following form:

$$E \equiv E(N_t) = \frac{\hbar}{|N_t|t_{min}} = \frac{\hbar}{|t(N_t)|}. \quad (31)$$

**Definition 2.1**.
*1) Let us define the quantity having the dimensions of length L or time t*

**measurable in principle**, *when it satisfies the relation (7) and, respectively, (29).*

*2) Let us define the quantity having the dimensions of momentum p or energy E* **measurable in principle**, *when it satisfies in the corresponding cases (nonrelativistic and relativistic) the foregoing formulas (14),(22),(23),(30) for the momentums and energies. At low energies ($E \ll E_P$) this means that p and E , within the known multiplicative constants and sign, are coincident with $1/N_L$,$1/N_t$, where $|N_L| \gg 1$,$|N_t| \gg 1$ – integers.*

*3) Let us define any physical quantity* **measurable in principle**, *when its value is consistent with points 1) and 2) of this Definition.*

**Definition 2.2**.
*1) Let us define any small quantity* **measurable in principle** *having the dimensions of length L or time t and* **measurable with respect to the energy** *E if it is associated with the value E within the scope of the Uncertainty Principle at All Energies Scales.*

*2)Let us define any physical quantity* **measurable with respect to the energy** *E when its value is consistent with point 1) of this Definition.*

Specifically, the minimal length $l_{min}$ and the minimal time $t_{min}$ are **measurable** only with respect to the energy $E_{max} \propto E_P$ and **nonmeasurable** with respect to the energies $E < E_{max}$.

Besides, **measurable in principle** and hence **measurable with respect to any energy** $E$ infinitesimal changes in length (and in time) are **impossible**, such changes being dependent on the existing energies.
In particular, a minimal possible **measurable** change of length is $l_{min}$. It corresponds to some maximal value of the energy $E_{max}$ or momentum $P_{max}$, If $l_{min} \propto l_P$, then $E_{max} \propto E_P$,$P_{max} \propto P_{Pl}$, where $P_{max} \propto P_{Pl}$, where $P_{Pl}$ is where the Planck momentum. Then denoting in **nonrelativistic** case with $\triangle_p(w)$ a **minimal measurable** change every spatial coordinate $w$

corresponding to the energy $E$ we obtain

$$\triangle_{P_{max}}(w) = \triangle_{E_{max}}(w) = l_{min}. \qquad (32)$$

Evidently, for lower energies (momentums) the corresponding values of $\triangle_p(w)$ are higher and, as the quantities having the dimensions of length are quantized (7), for $p \equiv p(N_p) < p_{max}$, $\triangle_p(w)$ is transformed to

$$|\triangle_{p(N_p)}(w)| = |N_p| l_{min}. \qquad (33)$$

where $|N_p| > 1$-integer so that we have

$$|N_p - \frac{1}{4N_p}| l_{min} = \frac{\hbar}{|p(N_p)|}. \qquad (34)$$

In the relativistic case the formula (32) holds, whereas (33) and (34) for $E \equiv E(N_E) < E_{max}$ are replaced by

$$|\triangle_{E(N_E)}(w)| = |N_E| l_{min}, \qquad (35)$$

where $|N_E| > 1$-integer.
Next we assume that at high energies $E \propto E_P$ there is a possibility only for the **nonrelativistic** case or **ultrarelativistic** case.
Then for the **ultrarelativistic** case, with regard to (20)–(27), formula (34) takes the form

$$|N_E - \frac{1}{4N_E}| l_{min} = \frac{\hbar c}{E(N_E)} = \frac{\hbar}{|p(N_p)|}, \qquad (36)$$

where $N_E = N_p$.
In the relativistic case at low energies we have

$$E \ll E_{max} \propto E_P. \qquad (37)$$

In accordance with (18),(19) formula (33) is of the form

$$|\triangle_{E(N_E)}(w)| = |N_E| l_{min} = \frac{\hbar c}{E(N_E)}, |N_E| \gg 1 - integer. \qquad (38)$$

In the nonrelativistic case at low energies (37) due to (34) we get

$$|\triangle_{p(N_p)}(w)| = |N_p| l_{min} = \frac{\hbar}{|p(N_p)|}, |N_p| \gg 1 - integer. \tag{39}$$

In a similar way for the time coordinate $t$, by virtue of formulas (29)–(31), at the same conditions we have similar formulas (32),(33),(34)

$$\triangle_{E_{max}}(t) = t_{min}. \tag{40}$$

For $E \equiv E(N_t) < E_{max}$

$$|\triangle_{E(N_t)}(t)| = |N_t| t_{min}, \tag{41}$$

where $|N_E| > 1$-integer, so that we obtain

$$|N_t - \frac{1}{4N_t}| t_{min} = \frac{\hbar c}{E(N_t)}. \tag{42}$$

In the relativistic case at low energies

$$E \ll E_{max} \propto E_P, \tag{43}$$

in accordance with (18),(19), formula (33) takes the form

$$|\triangle_{E(N_t)}(w)| = |N_t| l_{min} = \frac{\hbar c}{E(N_t)}, |N_t| \gg 1 - integer. \tag{44}$$

Now we consider a very simple but important example of the **nonmeasurable quantity** from [2]:
**The infinitesimal increment of entropy** $dS$ of the spherically symmetric holographic screen $\mathcal{S}$ with the radius $R$ and with the surface area $A$ is a **nonmeasurable quantity**.
Really, it is obvious that infinitesimal variations of the screen surface area $dA$ are possible only in a continuous theory involving no $l_{min}$.
When $l_{min} \propto l_P$ is involved, the minimal variation $\triangle A$ is evidently associated with a minimal variation in the radius $R$

$$R \rightarrow R \pm l_{min} = R \pm \triangle_{E_{max}}(R) \tag{45}$$

it is dependent on $R$ and growing with $\sim R$ for $R \gg l_{min}$ (possible only at the maximum energy $E_{max} \propto E_P$)

$$\triangle_{\pm} A(R) = (A(R \pm l_{min}) - A(R)) \propto (\pm 2R l_{min} + l_{min}^2) \propto (\pm 2N_R + 1), \quad (46)$$

where $N_R = R/l_{min}$, as indicated above in (7).
But if $E \ll E_{max} \propto E_P$, then a minimal variation in the radius $R$ is obviously greater than $l_{min}$

$$R \to R \pm \triangle_{E(N_E)}(R) = R \pm |N_E| l_{min}, \quad (47)$$

and in this case in the right-hand side of (46), within the constant $l_{min}^2$, we have the number quickly growing at low energies as well:

$$\triangle_{\pm} A(R) = (A(R \pm l_{min}) - A(R)) \propto (\pm 2R N_E l_{min} + N_E^2 l_{min}^2) \propto N_E(\pm 2N_R + N_E). \quad (48)$$

In any case from this it follows that $dA$ has no chance to be a **measurable quantity**, as its measurability suggests measurability of the quantity $dR$, and this is impossible.
Since $dS$, within a multiplicative constant, equals $dA$ [45],[46]: $dS \propto dA/4$, $dS$ is also a **nonmeasurable quantity**.
Because of this, the ≪main instrument≫ in the well-known paper [47] that is the infinitesimal variation $dN$ in the bit numbers $N$ on the holographic screen $\mathcal{S}$ is also a **nonmeasurable quantity** [2] as $dN \propto dS$ to within an integer factor.
It is easily seen that the infinitesimal variation $dV$ in the volume $V$ of $\mathcal{S}$ is also a **nonmeasurable quantity**.
The following comments are of particular importance.
**Comm 1.1.** It should be noted that the lattice is usually understood as a uniform discrete structure with one and the same constant parameter $a$ (lattice pitch). But in this case we have a nonuniform discrete structure (lattice in its nature), where the analogous parameter is variable, is a multiple of $l_{min}$,i. e.$a = N_a l_{min}$, and also is dependent on the energies. Only in the limit of high (Planck's) energies we get a (nearly) uniform lattice with (nearly) constant pitch $a \approx l_{min}$ or $a = \kappa l_{min}$ where $\kappa$ is on the order of 1.

**Comm 1.2** As it has been already noted above, the parameter $\alpha_a$ from (15) is first derived from (10) but without the additional constraint $0 < \alpha_a \leq 1/4$. This is due to the fact that for $\alpha_x$ from Section 2 this additional constraint is quite naturally arising from the density matrix deformation in a quantum mechanics at Planck scales [28]–[34]. In this Section the constraint is redundant at the initial stage. Possibly it may arise later during the elaboration of the proposed approach.

**Comm 1.3** Obviously, when $l_{min}$ is involved, the foregoing formulas for the momentums $p(N_p)$ and for the energies $E(N_E), E(N_t)$ may **certainly** give the highly accurate result that is close to the experimental one only at the verified low energies: $|N_p| \gg 1, |N_E| \gg 1, |N_t| \gg 1$. In the case of high energies $E \propto E_{max} \propto E_P$ or, what is the same $|N_p| \rightarrow 1, |N_E| \rightarrow 1, |N_t| \rightarrow 1$, we have a certain, experimentally unverified, model with a correct low-energy limit.

**Comm 1.4** It should be noted that dispersion relations (18) are valid only at low energies $E \ll E_P$. In the last few years in a series of works [40]–[44] it has been demonstrated that within the scope of GUP the high-energy generalization of (18)–Modified Dispersion Relations (MDRs)–is valid. Specifically, in its most general form the Modified Dispersion Relation (formula (9) in [44]) may be given as follows:

$$p^2 = f(E, m; l_p) \simeq E^2 - \mu^2 + \alpha_1 l_p E^3 + \alpha_2 l_p^2 E^4 + O\left(l_p^3 E^5\right), \tag{49}$$

where in the notation of [44] the fundamental constants are $c = \hbar = k_B = 1$, $f$ is the function that gives the exact dispersion relation, and in the right-hand side the applicability of the Taylor-series expansion for $E \ll 1/l_P$ is assumed. The coefficients $\alpha_i$ can take different values in different quantum-gravity proposals. $m$ is the rest energy of a particle, and the mass parameter $\mu$ in the right-hand side is directly related to the rest energy but $\mu \neq m$ if not all the coefficients $\alpha_i$ are vanishing.

The general case of (MDRs) (49) in terms of the considerations given in this section is yet beyond the scope of this paper and necessitates further studies of the transition from low $E \ll E_P$ to high $E \approx E_P$ energies.

For now it is assumed that at low energies formula (18) is valid to within a high accuracy, whereas at high energies, i.e. for $|N_p| \rightarrow 1, |N_E| \rightarrow 1, |N_t| \rightarrow 1$, (18) should be replaced by (49). Besides, it is important that in this paper, as distinct from [40]–[44], the author uses the simplest (earlier) variant of GUP [15]–[27], involving a minimal length but not a minimal momentum. Also note that references [40]–[44] give not nearly so complete a list of the publications devoted to GUP (and, in particular, MDR) – a very complete and interesting survey may be found in [41].

**Comm 1.5** In what follows, within the scope of the above definitions, we consider, unless stated otherwise, **only measurable** increments (variations) of the space-time quantities and the corresponding momentums and energies.

Proceeding from all the above, this simply means that all minimal increments (variations) of the space-time quantities are dependent on the present energies and coincident with the corresponding **minimal uncertainties** from the **Uncertainty Principle at All Energy Scales.**

# 3 Space-Time Lattice of Measurable Quantities and Dual Lattice

So, provided the minimal length $l_{min}$ exists, two lattices are naturally arising.

I.Lattice of the **space-time variation** – $Lat_{S-T}$ representing, to within the known multiplicative constants, the sets of nonzero integers $N_w \neq 0$ and $N_t \neq 0$ in the corresponding formulas from the set (33)–(44) for each of the three space variables $w \doteq x; y; z$ and the time variable $t$

$$Lat_{S-T} \doteq (N_w, N_t), N_w \neq 0, N_t \neq 0 - integers. \qquad (50)$$

Which restrictions should be initially imposed on these sets of nonzero integers?

It is clear that in every such set all the integers $(N_w, N_t)$ should be sufficiently ≪close≫, because otherwise, for one and the same space-time point, variations in the values of its different coordinates are associated with principally different values of the energy $E$ which are ≪far≫ from each other.

Note that the words ≪close≫ and ≪far≫ will be elucidated further in this text.

Thus, at the admittedly low energies (Low Energies) $E \ll E_{max} \propto E_P$ the low-energy part (sublattice) $Lat_{S-T}[LE]$ of $Lat_{S-T}$ is as follows:

$$Lat_{S-T}[LE] = (N_w, N_t) \equiv (|N_x| \gg 1, |N_y| \gg, |N_z| \gg 1, |N_t| \gg 1). \quad (51)$$

At high energies (High Energies) $E \to E_{max} \propto E_P$ we, on the contrary, have the sublattice $Lat_{S-T}[HE]$ of $Lat_{S-T}$

$$Lat_{S-T}[HE] = (N_w, N_t) \equiv (|N_x| \approx 1, |N_y| \approx 1, |N_z| \approx 1, |N_t| \approx 1). \quad (52)$$

II. Next let us define the lattice **momentums-energies variation** $Lat_{P-E}$ as a set to obtain
$(p_x(N_{x,p}), p_y(N_{y,p}), p_z(N_{z,p}), E(N_t))$ in the nonrelativistic and ultrarelativistic cases for all energies, and as a set to obtain
$(E_x(N_{x,E}), E_y(N_{y,E}), E_z(N_{z,E}), E(N_t))$ in the relativistic (but not ultrarelativistic) case for low energies $E \ll E_P$, where all the components of the above sets conform to the space coordinates $(x, y, z)$ and time coordinate $t$ and are given by the corresponding formulas(32)–(44) from the previous Section.
Note that, because of the suggestion made after formula (37) in the previous Section, we can state that the foregoing sets exhaust all the collections of momentums and energies possible for the lattice $Lat_{S-T}$.
From this it is inferred that, in analogy with point I of this Section, within the known multiplicative constants, we have

$$Lat_{P-E} \doteq (\frac{1}{N_w - \frac{1}{1/4N_w}}, \frac{1}{N_t - \frac{1}{1/4N_t}}), \quad (53)$$

where $N_w \neq 0, N_t \neq 0$-integers from (50). Similar to (51), we obtain the low-energy (Low Energy) part or the sublattice $Lat_{P-E}[LE]$ of $Lat_{P-E}$

$$Lat_{P-E}[LE] \approx (\frac{1}{N_w}, \frac{1}{N_t}), |N_w| \gg 1, |N_t| \gg 1. \quad (54)$$

In accordance with (52), the high-energy (High Energy) part (sublattice) $Lat_{P-E}[HE]$ of $Lat_{P-E}$ takes the form

$$Lat_{P-E}[HE] \approx (\frac{1}{N_w - \frac{1}{1/4N_w}}, \frac{1}{N_t - \frac{1}{1/4N_t}}), |N_w| \to 1, |N_t| \to 1. \quad (55)$$

Considering **Comment 1** from the previous Section, it should be noted that, as currently the low energies $E \ll E_{max} \propto E_P$ are verified by numerous experiments and thoroughly studied, the sublattice $Lat_{P-E}[LE]$ (54) is correctly defined and rigorously determined by the sublattice $Lat_{S-T}[LE]$ (51).
However, at high energies $E \to E_{max} \propto E_P$ we can't be so confident that the sublattice $Lat_{P-E}[HE]$ may be defined more exactly.
Specifically, $\alpha_a$ is obviously a small parameter. And, as demonstrated in [49],[50], in the case of GUP we have the following:

$$[\vec{x}, \vec{p}] = i\hbar(1 + a_1\alpha + a_2\alpha^2 + ...). \quad (56)$$

But, according to (15), $|1/N_a| = \sqrt{\alpha_a}$, then, due to (56), the denominators in the right-hand side of (55) may be also varied by adding the terms $\propto 1/N_w^2, \propto 1/N_w^3 ..., \propto 1/N_t^2, \propto 1/N_t^3 ...$, that is liable to influence the final result for $|N_w| \to 1, |N_t| \to 1$.
The notions ≪close≫ and ≪far≫ for $Lat_{P-E}$ will be completely determined by the dual lattice $Lat_{S-T}[LE]$ and by formulas (33)– (44).
It is important to note the following.
**In the low-energy sublattice $Lat_{P-E}[LE]$ all elements are varying very smoothly enabling the approximation of a continuous theory**.

# 4 Conclusion. Physical Theory in Terms of Measurable Quantities

In [2] we have considered a simple example of gravity for the static spherically-symmetric space with horizon, earlier treated by T. Padmanabhan (for example, [48]). It has been demonstrated that, provided all variations (increments) of space-time variables are given only in terms of **measurable quantities**, at low energies the theory is close to the starting continuous one due to $|N_L| \gg 1$ from formula (7).
More precisely, in [2] for the above example it has been shown that
**... despite the absence of infinitesimal spatial-temporal increments, owing to the existence of $l_{min}$ and the essential "discreteness" of a theory, this discreteness at low energies is not "felt",**

**the theory being actually continuous. The indicated discreteness is significant only in the case of high (Planck) energies.**
A similar conclusion may be drawn for **heuristic Markov's Model** [13].This model already considered by the author in his previous paper [50] is treated from the standpoint of the above-mentioned arguments. In [13], it is assumed that ≪by the universal decree of nature a quantity of the material density $\varrho$ is always bounded by its upper value given by the expression that is composed of fundamental constants≫ ([13], p.214):

$$\varrho \leq \varrho_p = \frac{c^5}{G^2 \hbar}, \tag{57}$$

with $\varrho_p$ as ≪Planck's density≫.
Then the quantity

$$\wp_\varrho = \varrho / \varrho_p \leq 1 \tag{58}$$

is the **deformation parameter** as it is used in [13] to construct the following **of Einstein's equations deformation or $\wp_\varrho$-deformation** ([13],formula (2)):

$$R_\mu^\nu - \frac{1}{2} R \delta_\mu^\nu = \frac{8\pi G}{c^4} T_\mu^\nu (1 - \wp_\varrho^2)^n - \Lambda \wp_\varrho^{2n} \delta_\mu^\nu, \tag{59}$$

where $n \geq 1/2$, $T_\mu^\nu$–energy-momentum tensor, $\Lambda$– cosmological constant.
The case of the parameter $\wp_\varrho \ll 1$ or $\varrho \ll \varrho_p$ correlates with the classical Einstein equation, and the case when $\wp_\varrho = 1$ – with the de Sitter Universe. In this way (59) may be considered as $\wp_\varrho$-deformation of the General Relativity.
As shown in [50], $\wp_\varrho$-of Einstein's equations deformation (59) is nothing else but $\alpha$-deformation of GR for the parameter $\alpha_l$ at $x = l$ from (10).
If $\varrho = \varrho_l$ is the average material density for the Universe of the characteristic linear dimension $l$, i.e. of the volume $V \propto l^3$, we have

$$\wp_{l,\varrho} = \frac{\varrho_l}{\varrho_p} \propto \alpha_l^2 = \omega \alpha_l^2, \tag{60}$$

where $\omega$ is some computable factor.
Then it is clear that $\alpha_l$-representation (59) is of the form

$$R_\mu^\nu - \frac{1}{2} R \delta_\mu^\nu = \frac{8\pi G}{c^4} T_\mu^\nu (1 - \omega^2 \alpha_l^4)^n - \Lambda \omega^{2n} \alpha_l^{4n} \delta_\mu^\nu, \tag{61}$$

or in the general form we have

$$R_\mu^\nu - \frac{1}{2}R\delta_\mu^\nu = \frac{8\pi G}{c^4}T_\mu^\nu(\alpha_l) - \Lambda(\alpha_l)\delta_\mu^\nu. \quad (62)$$

But, as r.h.s. of (62) is dependent on $\alpha_l$ of any value and particularly in the case $\alpha_l \ll 1$, i.e. at $l \gg l_{min}$, l.h.s of (62) is also dependent on $\alpha_l$ of any value and (62) may be written as

$$R_\mu^\nu(\alpha_l) - \frac{1}{2}R(\alpha_l)\delta_\mu^\nu = \frac{8\pi G}{c^4}T_\mu^\nu(\alpha_l) - \Lambda(\alpha_l)\delta_\mu^\nu. \quad (63)$$

Thus, in this specific case we obtain the explicit dependence of GR on the available energies $E \sim 1/l$, that is insignificant at low energies or for $l \gg l_{min}$ and, on the contrary, significant at high energies, $l \to l_{min}$.

(M.1.1)**Low energies. Nonmeasurable case**. In this case at low energies, using formula (10) in the limit $l_{min} = 0$ for $a = l$, we get a **continuous theory** coincident with the General Relativity.

(M.1.2)**Low energies. Measurable case**. In this case at low energies, using formulas (10), (15) for $l_{min} \neq 0$, for $a = l$ (and hence for $N_l \gg 1$), we get a **discrete theory** which is a ≪**nearly continuous theory**≫, practically similar to the General Relativity with the slowly (smoothly) varying parameter $\alpha_{l(t)}$, where $t$ – time.

So, due to low energies and momentums ($E \ll E_P, p \ll P_{Pl}$), the ≪**continuous case**≫ M.1.1) (General Relativity) and the ≪**discrete case**≫ M.1.2) that is actually a ≪**nearly continuous case**≫.

(M.2)**At high energies we consider the measurable case** only. Then it is clear that at high energies the parameter $\alpha_{l(t)}$ is discrete and for the limiting value of $\alpha_{l(t)} = 1$ we get a discrete series of equations of the form (62)(or a single equation of this form met by a discrete series of solutions) corresponding to $\alpha_{l(t)} = 1; 1/4; 1/9; ...$
As this takes place, $T_\mu^\nu(\alpha_l) \approx 0$, and in both cases M.1.2) and M.2) $\Lambda(\alpha_l)$ is not longer a cosmological constant, being a dynamical cosmological term.

Note that because of formula (17) in Section 2, $\sqrt{\alpha_{l(t)}}$ in cases (M.1.2) and (M.2) is an element of the lattice $Lat_{P-E}$ from Section 3. And in case (M.1.2) it is an element of the sublattice $Lat_{P-E}[LE]$, whereas case M.2) is associated with the sublattice $Lat_{P-E}[HE]$.

The main idea of the author is to demonstrate **the existence of the correct limiting high-energy transition**:

$$(M.1.2) \stackrel{High\ Energy}{\Rightarrow} (5.2) \tag{64}$$

and **the nonexistence of the correct limiting high-energy transition**:

$$(M.1.1) \stackrel{High\ Energy}{\Rightarrow} (5.2). \tag{65}$$

In the general case, based on the parameter $\alpha_a$ from the formula (10) this means that **there exists the correct limiting high-energy transition**:

$$\lim_{l_{min}\neq 0,|N_a|\gg 1} \alpha_a \stackrel{High\ Energy}{\Rightarrow} \lim_{l_{min}\neq 0,|N_a|\approx 1} \alpha_a \tag{66}$$

and **there is no correct limiting high-energy transition**

$$\lim_{l_{min}=0} \alpha_a \stackrel{High\ Energy}{\Rightarrow} \lim_{l_{min}\neq 0,|N_a|\approx 1} \alpha_a. \tag{67}$$

However, the whole theoretical physics, in which presently at low energies $E \ll E_P$ the minimal length $l_{min}$, is not involved (i. e. $l_{min} = 0$), is framed around the search for **nonexistent limits** (65) in a case of the **heuristic Markov's Model** [13], and also in the general case (67) in terms of the parameter $\alpha_a$, respectively.

Thus, the main "ideology" of the proposed approach is as follows:

Provided the minimal length $l_{min}$ is involved, its existence must be taken into consideration not only at high but also at low energies in a any Physical Theory (PH). This becomes apparent by rejection of the infinitesimal quantities associated with the spatial-temporal variations $dx_\mu, ...$. In other words, with the involvement of $l_{min}$, the "continuous" (PH) must be replaced by a (still unframed) minimal-length theory that may be denoted as $PH^{l_{min}}$. In low energies $E \ll E_{max} \propto E_P$ their results (PH) and $PH^{l_{min}}$ should be very close but, as regards their mathematical apparatus (instruments), these theories are absolutely different.

Besides, $PH^{l_{min}}$ should offer a rather natural transition from high to low energies

$$[N_L \approx 1] \rightarrow [N_L \gg 1] \tag{68}$$

and vice versa

$$[N_L \gg 1] \rightarrow [N_L \approx 1], \tag{69}$$

where $N_L$ – integer from formula (7) determining the characteristics scale of the lengths $L$ (energies $E \sim 1/L \propto 1/N_L$).
In this way the results of [2] and this paper may be summarized as follows.

4.1.When in the theory the minimal length $l_{min} \neq 0$ is actualized (involved) at all the energy scales, a mathematical apparatus of this theory must be changed considerably: no infinitesimal space-time variations (increments) must be involved, the key role being played by the definitions of **measurability in principle** and **measurability in relation to the energy** (**Definition 2.1** and **Definition 2.2** from Section 2).

4.2.As this takes place the theory becomes **discrete** at all the energy scales but, as shown by the example gravity for the static spherically-symmetric space with horizon [48], considered in [2] and example for **heuristic Markov's Model** [13], considered in this Section, at low energies (far from the Planck energies) the sought for theory must be very close in its results to the starting continuous theory (with $l_{min} = 0$). In the process a real **discreteness** is exhibited only at high energies which are close to the Planck energies.

4.3. By this approach the theory at low and high energies is associated with a common single set of the parameters ($N_L$ from formulas (68), (69)) or with the dimensionless small parameters ($1/N_L = \sqrt{\alpha_L}$) which are lacking if at low energies the theory is continuous, i.e. when $l_{min} = 0$.

The principal objective of my further studies is to develop for quantum theory and gravity, within the scope of the considerations given in points 4.1–4.3, the corresponding discrete models (with $l_{min} \neq 0$) for all the energy scales and to meet the following requirements:

4.4.At low energies the models must, to a high accuracy, represent the results of the corresponding continuous theories.

4.5. The models should not have the problems of transition from low to high energies and, specifically, the ultraviolet divergence problem.

4.6. By author's opinion, the problem associated with points 4.4. and 4.5. is as follows.
4.6.1.It is interesting to know why, with the existing $l_{min} \neq 0, t_{min} \neq 0$ and discreteness of nature, at low energies $E \ll E_{max} \propto E_P$ the apparatus of mathematical analysis based on the use of infinitesimal space-time quantities ($dx_\mu, \frac{\partial \varphi}{\partial x_\mu}$, and so on)is very efficient giving excellent results. The answer is simple: in this case $l_{min}$ and $t_{min}$ are very far from the available scale of $L$ and $t$, the corresponding $N_L \gg 1, N_t \gg 1$ being in general true but insufficient. There is a need for rigorous calculations.

4.6.2. What is a dynamics of any physical quantity **measureable with respect to the energy** $E$ within the scope of the **Definition 2.2** in Section 2 when going from low to high energies? It seems that for solving of this problem we have to correct the **Definition 2.2** itself.

The author is hopeful that the correct construction of low-energy $Grav^{l_{min}}$ close to GR allows for a more natural transition to quantum (Planck's) gravity. Besides, within the notion of **measurability**, gravity could be saved from some odd solutions, from wormholes in particular.

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