THE LOCAL PRODUCT THEOREM FOR BIHAMILTONIAN STRUCTURES

Francisco-Javier Turiel

Geometría y Topología, Facultad de Ciencias, Campus de Teatinos, 29071 Málaga, Spain

e-mail: turiel@agt.cie.uma.es

Abstract

In this work one proves that, around each point of a dense open set (regular points), a real analytic or holomorphic bihamiltonian structure decomposes into a product of a Kronecker bihamiltonian structure and a symplectic one if a necessary condition on the characteristic polynomial of the symplectic factor holds. Moreover we give an example of bihamiltonian structure for showing that this result does not extend to the C^∞- category.

Thus a classical problem on the geometric theory of bihamiltonian structures is solved at almost every point.

Introduction

Given two Poisson structures Λ, Λ_1 on a real (at least C^∞) or complex (holomorphic) manifold M, following Magri [9] one will say that (Λ, Λ_1) is a bihamiltonian structure (or that Λ, Λ_1 are compatible) if $\Lambda + \Lambda_1$ is a Poisson structure as well. Bihamiltonian structures are a useful tool for dealing with some differential equations many of them with a physical meaning; besides they are interesting from the geometric viewpoint too that will be the case here.

The algebraic classification of the pairs of bivectors on a finite dimensional real or complex vector space was given by Gelfand and Zakharevich in [4]. Essentially each pair decomposes into the product of a Kronecker pair and a symplectic one (see [4, 17]). Therefore it is natural to ask whether this decomposition into a product Kronecker-symplectic holds, at least locally, for bihamiltonian structures as well, which would be an important steep toward their classification. One recalls that Kronecker bihamiltonian structures are intimately related to Veronese webs (see [17] for an exposition of the local theory of Veronese webs and its relationship with Kronecker bihamiltonian structures), whereas the local classification of symplectic bihamiltonian structures, that is of pairs of compatible symplectic forms, is known at almost every point (see [12, 13]).

The chief goal of this work is to show that, around every point of a dense open set (regular points), a real analytic or holomorphic bihamiltonian structure decomposes into a product Kronecker-symplectic if a necessary condition on the characteristic polynomial of the symplectic factor holds (theorem 7.1). Moreover we exhibit an example of C^∞- bihamiltonian structure for which this result fails. These results have been annoced in [18, 19].

As main tool for this purpose, to any bihamiltonian structure we associate a new object called a Veronese flag, which generalizes the notion of Veronese web introduced by Gelfand and Zakharevich in [4] (codimension one) and later on by others authors [10, 16] (higher codimension). Roughly speaking the crucial point is to show that, about each regular point, a Veronese flag is the product of a Veronese web and a pair of compatible symplectic forms. For that one has to prove, in a indirect way, the existence of solutions of some differential equations not explicitly formulated. In the complex case they are always ordinary whereas in some real cases we have to deal with systems of partial derivative equations, which may contain the Lewy's example [7] as sub-system; thus the result fails in the C^∞ category. In the real analytic case a method of complexification transforms the real problem on a complex one. Once the decomposition of Veronese flags established, that of bihamiltonian structures follows from it with a little extra-work.

The study of bihamiltonian structures at not regular points rather belongs to the theory of singularities and, in spite of its great interest, will be not considered here.

The present text consists of eight sections plus an appendix. In the first one the Veronese flag, as quotient of a bihamiltonian structure, and the bihamiltonian structure over a $(1,1)$-tensor field and a foliation, which gives a simple method for constructing bihamiltonian structures, are introduced.

Sections 2, 3 and 4, this last one rather technical, are devoted to prove a local decomposition theorem for Veronese flags with only one eigenvalue. In section 5 real Veronese flags, without real eigenvalue, are dealt with by reducing them to holomorphic ones through the analyticity.

In section 6 one shows that, locally, Veronese flags are the fibered product of those with just one eigenvalue, and in section 7 we prove the theorem of local decomposition for real analytic or holomorphic bihamiltonian structures. A C^∞ counter-example to this last result is given in section 8.

Finally, in the appendix one proves a well known result on $(1,1)$-tensor fields belonging to the folklore but without many accessible proofs.

1. The quotient of a bihamiltonian structure

As it well known to any Kronecker bihamiltonian structure one may associate a Veronese web on the local quotient of the support manifold (see [3, 10, 16]). Here we will associate a new structure, called a Veronese flag and defined on a local quotient of the support manifold also, to a very large class of bihamiltonian structures.

From now on all structured considered will be real C^∞ or complex holomorphic unless another thing is stated.

1.1. The main construction.

On a manifold P consider a foliation $\mathcal{F}$ (that is an involutive distribution) of positive codimension and a morphism of vector bundles $\ell : \mathcal{F} \to TP$. If α is a s-form on an open set B of P, then $\ell^*\alpha$ (we will write $\alpha \circ \ell$ as well) can be regarded as s-form with domain B on the leaves of $\mathcal{F}$. Let $G : TP \to TP$ be a prolongation of ℓ; then $(G^*\alpha)_{|\mathcal{F}}$ equals $\ell^*\alpha$. On the other hand if $\ell^*\alpha$ is closed on $\mathcal{F}$ for every closed 1-form α such that $Ker\alpha \supset \mathcal{F}$, then the restriction of the Nijenhuis torsion N_G of G to $\mathcal{F}$ does not depend on the prolongation G of ℓ (see lemma 2.2 of [17]) and it will called the *Nijenhuis torsion N_ℓ of ℓ*.

Let $\mathcal{A}(p)$, $p \in P$, be the largest ℓ-invariant vector subspace of $\mathcal{F}(p)$.

We will say that the pair $(\mathcal{F}, \ell)$ is a *weak Veronese flag* if the following three conditions hold:

1) $\ell^*\alpha$ is closed on $\mathcal{F}$ for every closed 1-form α such that $Ker\alpha \supset \mathcal{F}$,

2) $N_\ell = 0$,

3) $dim\mathcal{A}(p)$ does not depend on p.

First of all let us see that the distribution $\mathcal{A} = \cup_{p\in P}\mathcal{A}(p)$ is a foliation when $(\mathcal{F}, \ell)$ is a weak Veronese flag. Given any point $q \in P$ the morphism ℓ :

$\mathcal{F}(q) \to T_qP$ projects in a morphism $\varphi_q : \mathcal{F}(q)/\mathcal{A}(q) \to T_qP/\mathcal{A}(q)$ without non-zero φ_q-invariant vector subspace. Thus $(\mathcal{F}(q)/\mathcal{A}(q), \varphi_q)$ defines an algebraic Veronese web w_q on $T_qP/\mathcal{A}(q)$ of codimension ≥ 1 by setting $w_q(t) = (\varphi_q + tI)(\mathcal{F}(q)/\mathcal{A}(q))$, such that $w_q(\infty) = \mathcal{F}(q)/\mathcal{A}(q)$. Moreover, as it is well known, if $a_1, ..., a_k$, $k = dim(T_qP/\mathcal{A}(q))$, are non-equal scalars then $w_q(a_1) \cap ... \cap w_q(a_k) = \{0\}$, so $\mathcal{A}(q) = \cap_{j=1}^{k}((\ell + a_jI)\mathcal{F}(q))$.

On the other hand, if $Ker(\ell + aI) = \{0\}$ on an open set then $(\ell + aI)\mathcal{F}$ is involutive on this set. For showing this last assertion we need the following result transcription of lemma 2.1 of [17].

Lemma 1.1. *Consider a 1-form ρ, a $(1,1)$- tensor field H and two vector fields X, Y on a manifold, then $(d(\rho \circ H))(HX, Y) + (d(\rho \circ H))(X, HY) = d\rho(HX, HY) + d(\rho \circ H^2)(X, Y) + \rho(N_H(X, Y))$.*

Let G be a (local) prolongation of ℓ. If μ is a closed 1-form and $Ker\mu \supset \mathcal{F}$ then $Ker(\mu \circ (G + aI)^{-1}) \supset (G + aI)\mathcal{F}$ and by lemma 1.1 applied to $\mu \circ (G + aI)^{-1}$ and $(G + aI)$ one has $d(\mu \circ (G + aI)^{-1})((G + aI)\mathcal{F}, (G + aI)\mathcal{F}) = -d(\mu \circ (G + aI))(\mathcal{F}, \mathcal{F}) - \mu(N_G(\mathcal{F}, \mathcal{F})) = 0$ therefore $d(\mu \circ (G + aI)^{-1})_{|(G+aI)\mathcal{F}} = 0$, whence the involutivity of $(G + aI)\mathcal{F}$.

Since whichever $p \in P$ always there exist non-equal scalars $a_1, ..., a_k$ such that $Ker(\ell + a_jI)(p) = \{0\}$, $j = 1, ..., k$, around this point $\mathcal{A} = \cap_{j=1}^{k}((\ell + a_jI)\mathcal{F})$; so $\mathcal{A}$ is a foliation, called *the axis of the flag* $(\mathcal{F}, \ell)$ from now on.

Let $\pi : P \to N$ be a local quotient of P by $\mathcal{A}$; then $\bar{w}(t) = \pi_*((\ell + tI)\mathcal{F})$ is a foliation whose codimension equals that of $\mathcal{F}$ and $\bar{w} = \{\bar{w}(t) \mid t \in \mathbb{K}\}$ is a Veronese web on N. Indeed, given $q \in P$ and $\bar{q} \in N$ such that $\pi(q) = \bar{q}$ then $\bar{w}(\bar{q}) = \{\bar{w}(\bar{q})(t) \mid t \in \mathbb{K}\}$ is the algebraic Veronese web defined by $(\mathcal{F}(q)/\mathcal{A}(q), \varphi_q)$ when $T_qP/\mathcal{A}(q)$ is identified to $T_{\bar{q}}N$; so $\bar{w}$ is an algebraic Veronese web at each point of N. Moreover $\bar{w}(\infty) = \pi_*(\mathcal{F})$, which is a foliation; therefore by proposition 2.1 of [17] the family $\bar{w}$ is a Veronese web.

Thus if $\bar{\ell} : \bar{\mathcal{F}} \to TN$, where $\bar{\mathcal{F}} = \pi_*(\mathcal{F})$, is the morphism canonically associated to $\bar{w}$ then $(\bar{\mathcal{F}}, \bar{\ell})$ is the projection of $(\mathcal{F}, \ell)$.

Lemma 1.2. *Consider a weak Veronese flag $(\mathcal{F}, \ell)$ and for every integer $k \geq 0$ set $g_k = trace((\ell_{|\mathcal{A}})^k)$. Then $kdg_{k+1} = (k+1)dg_k \circ \ell$ on $\mathcal{F}$.*

Proof. As the problem is local one may extend $\bar{\ell}$ to a flat diagonalizable tensor field J and consider an extension G of ℓ projecting in J. Then $ImN_G \subset \mathcal{A}$ and $trace(G^k) = g_k + c_k$ where $c_k \in \mathbb{K}$. Since $trace(H_1 \circ H_2) = trace(H_2 \circ H_1)$ for every vector field X one has: $kd(trace(G^{k+1}))(X) = k(k+1)trace(G^k \circ L_X G) = k(k+1)trace(G^{k-1} \circ L_{GX}G) - k(k+1)trace(N_G(X, \quad)) = (k+1)d(trace(G^k))(GX) - k(k+1)trace(N_G(X, \quad))$.

But $trace(N_G(X, \quad)) = 0$ when $X \in \mathcal{F}$ because $N_G(\mathcal{F}, \mathcal{F}) = 0$ and $Im(N_G(X, \quad)) \subset \mathcal{A} \subset \mathcal{F}$. $\square$

Now let ω, ω_1 be a couple of 2-forms defined on $\mathcal{A}$. One will say that $(\mathcal{F}, \ell, \omega, \omega_1)$ is a *Veronese flag* on P if:

1) $(\mathcal{F}, \ell)$ is a weak Veronese flag.

2) ω is symplectic on $\mathcal{A}$, ω_1 closed and $\omega_1 = \omega(\ell, \quad)$ [that is $\omega_1(X, Y) = \omega(\ell X, Y)$].

3) Whenever f is a function on an open set of P such that $\ell^* df$ is closed on $\mathcal{F}$, then $L_{X_f}\ell = 0$ where X_f is the ω-hamiltonian of f along $\mathcal{A}$.

Remark. Given, on a manifold, a foliation $\mathcal{G}$, a tensor field $\mathcal{T}$ defined along $\mathcal{G}$ and a $\mathcal{G}$-foliate vector field X, then the Lie derivative $L_X\mathcal{T}$ is defined as a tensor field along $\mathcal{G}$; moreover the flow of Xpreserves $\mathcal{T}$ if and only if $L_X\mathcal{T} = 0$. In condition 3) above X_f is tangent to $\mathcal{A} \subset \mathcal{F}$ so $\mathcal{F}$-foliate. Obviously this condition implies $L_{X_f}\omega_1 = 0$.

When $\mathcal{A} = 0$, Veronese web and Veronese flag are equivalent notions.

By technical reasons we need the following definition. Given $p_0 \in P$ we will say that $(\mathcal{F}, \ell, \omega, \omega_1)$ is a *Veronese flag at point* p_0 when 1) and 2) hold but 3) is replaced by:

3') for any function f defined on an open set $p_0 \in B \subset P$ such that $\ell^* df$ is closed on $\mathcal{F}$, then $L_{X_f}\ell = 0$ on an open set $p_0 \in B' \subset B$.

Let us recall some facts about pairs of bivectors on real or complex vector spaces (see [4] and section 1.2 of [17]). Consider a pair of bivectors (λ, λ_1) on a finite dimensional vector space W. By definition *the rank of* (λ, λ_1) is the maximum of the ranks of $(1-t)\lambda + t\lambda_1$, $t \in \mathbb{K}$, and one has $rank(\lambda, \lambda_1) = rank((1-t)\lambda + t\lambda_1)$ except for a finite number of scalars t, which is $\leq \frac{dimW}{2}$. A pair (λ, λ_1) is called *maximal* when $rank(\lambda) = rank(\lambda_1) = rank(\lambda, \lambda_1)$. Given an odd dimensional vector space U, the action of $GL(U)$ on $(\Lambda^2 U) \times (\Lambda^2 U)$

possesses one dense open orbit, whose elements are named *Kronecker elementary pairs*; they are maximal and their rank equals $dimU - 1$. According to the classification by Gelfand and Zakharevich (see [4] and propositions 1.4 and 1.5 of [17]), every maximal pair decomposes into a product of Kronecker elementary pairs (U_j, μ_j, μ_{1j}), $j = 1, ...r$, where $r = corank(\lambda, \lambda_1)$, and a symplectic pair (U', μ', μ'_1); moreover these factors are unique up to isomorphism or change of order.

A bihamiltonian structure on a manifold is called *Kronecker* when at each point its algebraic model is a product of Kronecker elementary pairs only, and *symplectic* if at every point its algebraic model only has the symplectic factor.

On a real or complex m-manifold M consider a bihamiltonian structure (Λ, Λ_1) such that:

1) (Λ, Λ_1) is maximal, that is every $(\Lambda(p), \Lambda_1(p))$, $p \in M$, is maximal,

2) the rank of (Λ, Λ_1) and the dimension of the the symplectic factor at each point are constant.

As before set $r = corank(\Lambda, \Lambda_1)$ and let $2m'$ be the dimension of the symplectic factor. Since r is the number of Kronecker elementary factors, $m + r$ is even and one may set $m = 2m' + 2n - r$. Note that, at every point, $2n - r$ equals the sum of the dimensions of the Kronecker elementary factors (warning these last dimensions could depend on the point).

Our next aim is locally to associate a Veronese flag in dimension $2m' + n$ to (Λ, Λ_1). For each $p \in M$ let $\mathcal{A}_1(p)$ be the intersection of all vector subspaces $Im(\Lambda + t\Lambda_1)(p)$, $t \in \mathbb{K}$, such that $rank(\Lambda + t\Lambda_1)(p) = m - r$. From the algebraic model follows that $dim\mathcal{A}_1(p) = m - n = 2m' + n - r$, which defines a foliation $\mathcal{A}_1$ called *the (primary) axis of* (Λ, Λ_1). Indeed, given $p \in M$ one can chose non-equal scalars $t_1, ..., t_n$ such that $rank(\Lambda + t_j\Lambda_1)(p) = m - r$, $j = 1, ..., n$; in particular $\cap_{j=1}^n Im(\Lambda + t_j\Lambda_1)(p) = \mathcal{A}_1(p)$. By continuity $rank(\Lambda + t_j\Lambda_1) = m - r$, $j = 1, ..., n$ and $\cap_{j=1}^n Im(\Lambda + t_j\Lambda_1) = \mathcal{A}_1$ around p.

It is not hard to see that $\mathcal{A}_1 \subset Im\Lambda_1$ and $dim(Im(\Lambda + t\Lambda_1) + \mathcal{A}_1) = m - r$, $t \in \mathbb{K}$. Set $\tilde{w}(t) = Im(\Lambda + t\Lambda_1) + \mathcal{A}_1$, $t \in \mathbb{K}$; then $\tilde{w} = \{\tilde{w}(t) \mid t \in \mathbb{K}\}$ is a family of foliations of codimension r whose limit at each point, when $t \to \infty$, is $Im\Lambda_1$. Indeed, given $p \in M$ and $t_0 \in \mathbb{K}$ consider functions $f_1, ..., f_k$ and vector fields $X_1, ..., X_{m-r-k}$ tangent to $\mathcal{A}_1$, all of them defined around p, such that $\{(\Lambda +$

$t_0\Lambda_1)(df_1, \quad), ..., (\Lambda+t_0\Lambda_1)(df_k, \quad), X_1, ..., X_{m-r-k}\}$ at p is a basis of $\tilde{w}(t_0)(p)$. By continuity $\{(\Lambda + t\Lambda_1)(df_1, \quad), ..., (\Lambda + t\Lambda_1)(df_k, \quad), X_1, ..., X_{m-r-k}\}$ is a basis of $\tilde{w}(t)(q)$ when (q,t) is close to (p,t_0) on $M \times \mathbb{K}$, so $\tilde{w}$ is a family of distributions. But the set $D = \{(q,t) \in M \times \mathbb{K} \mid rank(\Lambda + t\Lambda_1)(q) = m - r\}$ is dense and open and, obviously, $\tilde{w}$ is a family of foliations on D, therefore $\tilde{w}$ is a family of foliations on $M \times \mathbb{K}$. Finally note that $\mathcal{A}_1 \subset Im\Lambda_1$ and $Im(\Lambda + t\Lambda_1) = Im(s\Lambda + \Lambda_1)$ when $s = t^{-1}$.

Let N be the local quotient of M by $\mathcal{A}_1$, which is a n-dimensional manifold, and $\pi_N : M \to N$ the canonical projection. Then $\bar{w} = \{\bar{w}(t) = (\pi_N)_*\tilde{w}(t) \mid t \in \mathbb{K}\}$ is a Veronese web on N of codimension r. Indeed, the algebraic model shows that, at each point of N, the family of foliations $\bar{w}$ is an algebraic Veronese web. On the other hand its limit when $t \to \infty$ equals $(\pi_N)_*(Im\Lambda_1)$, which is a foliation too; so $\bar{w}$ is a Veronese web (see proposition 2.1 of [17]).

The Poisson structure Λ is given by a symplectic form $\tilde{\omega}$ defined on $Im\Lambda$ while Λ_1 is given by a symplectic form form $\tilde{\omega}_1$ on $Im\Lambda_1$. Therefore the restricted 2-forms $\tilde{\omega}_{|\mathcal{A}_1}$ and $\tilde{\omega}_{1|\mathcal{A}_1}$ are closed; besides (see proposition 1.4 of [17]) $Ker(\tilde{\omega}_{|\mathcal{A}_1}) = Ker(\tilde{\omega}_{1|\mathcal{A}_1}) = \Lambda(\mathcal{A}'_1, \quad) = \Lambda_1(\mathcal{A}'_1, \quad)$ where $\mathcal{A}'_1$ is the annihilator of $\mathcal{A}_1$ and $dim(Ker(\tilde{\omega}_{|\mathcal{A}_1})) = n - r$. Thus $\mathcal{A}_2 = Ker(\tilde{\omega}_{|\mathcal{A}_1})$ is a foliation of dimension $n-r$, which will be called *the secondary axis of* (Λ, Λ_1), and $\mathcal{A}_2 \subset \mathcal{A}_1$.

Let P be the local quotient of M by $\mathcal{A}_2$ and $\pi_P : M \to P$ the canonical projection; then $dimP = 2m'+n$, $\mathcal{A}_1$ projects into a $2m'$-dimensional foliation $\mathcal{A}$ and $\tilde{\omega}_{|\mathcal{A}_1}, \tilde{\omega}_{1|\mathcal{A}_1}$ in two symplectic forms ω, ω_1 on $\mathcal{A}$. Moreover Λ projects in the Poisson structure defined by $\mathcal{A}$ and ω, whereas Λ_1 does in the Poisson structure defined by $\mathcal{A}$ and ω_1. Let $\mathcal{F}$ be the r-codimensional foliation on P projection of $Im\Lambda_1$. Obviously the local quotient of P by $\mathcal{A}$ is identified in a natural way to N and $\pi \circ \pi_P = \pi_N$ where $\pi : P \to N$ is the canonical projection. In short we have three of the four elements of a Veronese flag on P. Let us construct the fourth one.

As $\Lambda(\mathcal{A}'_1, \quad) = \Lambda_1(\mathcal{A}'_1, \quad) = \mathcal{A}_2$ and $\mathcal{A}'_1$ contains $Ker\Lambda$ and $Ker\Lambda_1$, the Poisson structures Λ, Λ_1 give rise to two isomorphisms $\tilde{\lambda}, \tilde{\lambda}_1$ from $\frac{T^*M}{\mathcal{A}'_1}$ to $\frac{Im\Lambda}{\mathcal{A}_2}$ and $\frac{Im\Lambda_1}{\mathcal{A}_2}$ respectively, by setting $\tilde{\lambda}([\alpha]) = [\Lambda(\alpha, \quad)]$ and $\tilde{\lambda}_1([\alpha]) = [\Lambda_1(\alpha, \quad)]$. Thus $\tilde{\ell} = \tilde{\lambda} \circ \tilde{\lambda}_1^{-1}$ is a monomorphism from $\frac{Im\Lambda_1}{\mathcal{A}_2}$ to $\frac{TM}{\mathcal{A}_2}$ whose image equals $\frac{Im\Lambda}{\mathcal{A}_2}$. By construction $\tilde{\ell}$ is an invariant of (Λ, Λ_1) and, for every $q \in M$, there exists a monomorphism $\varphi : (\pi_P)_*(Im\Lambda_1(q)) \to T_{\pi_P(q)}P$ with $Im\varphi = (\pi_P)_*(Im\Lambda(q))$ that is the projection of $\tilde{\ell}(q)$; moreover from the algebraic model follows that $\omega_1(u,v) = \omega(\varphi u, v)$, $u, v \in \mathcal{A}(\pi_P(q))$, and $\Lambda_1(\pi_P^*\varphi^*\beta, \quad) = \Lambda(\pi_P^*\beta, \quad)$, $\beta \in T^*_{\pi_P(q)}P$ [note that Λ_1 can be regarded as a linear map from $(Im\Lambda_1)^*$ to TM since $\Lambda_1(\tilde{\beta}, \quad) = 0$ whenever $\tilde{\beta}(Im\Lambda_1) = 0$].

For proving that, in fact, $\tilde{\ell}$ projects into a suitable morphism $\ell : \mathcal{F} \to TP$ we will need some extra-work, essentially local, which allows us to do it around the points of M. Therefore, given non-equal and non-vanishing scalars $a_1, ..., a_{n-r}, a$ we may assume the existence on N of coordinates $(x_1, ..., x_n)$, closed 1-forms $\alpha_1, ..., \alpha_r$ and a $(1,1)$-tensor field J such that $dx_j \circ J = a_j dx_j$, $j = 1, ..., n-r$, $dx_j \circ J = a dx_j$, $j = n-r+1, ..., n$, $Ker(\alpha_1 \wedge ... \wedge \alpha_r) = \bar{w}(\infty)$, $d(\alpha_k \circ J) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$, $k = 1, ..., r$, and that

$$\gamma(t) = (\textstyle\prod_{j=1}^{n-k}(t+a_j))(t+a)^k(\alpha_1 \circ (J+tI)^{-1}) \wedge ... \wedge (\alpha_r \circ (J+tI)^{-1})$$

represents $\bar{w}$ (see theorem 2.1 of [17]).

By identifying τ and $\pi_N^*\tau$, any k-form τ, defined on open set of N, can be regarded as an $\mathcal{A}_1$-basic k-form on an open set of M. Thus $dx_1, ..., dx_n$ span $\mathcal{A}'_1$, $Kerdx_j \supset Im(\Lambda - a_j\Lambda_1)$ whence $\Lambda(dx_j, \quad) = a_j\Lambda_1(dx_j, \quad)$, $j = 1, ..., n-r$, and $Kerdx_j \supset Im(\Lambda - a\Lambda_1)$ whence $\Lambda(dx_j, \quad) = a\Lambda_1(dx_j, \quad)$, $j = n-r+1, ..., n$.

On the other hand from the algebraic model at each point follows that two functions of $(x_1, ..., x_n)$ are always in involution for both Λ and Λ_1, and the families $\{dx_1, ..., dx_{n-r}, \alpha_1 \circ J^{-1}, ..., \alpha_r \circ J^{-1}\}$ and $\{dx_1, ..., dx_{n-r}, \alpha_1, ..., \alpha_r\}$ are linearly independent everywhere. Consequently around each $p \in M$ one may chose functions $y_1, ..., y_{n-r}, z_1, ..., z_{2m'}$ such that $(x_1, ..., x_n, y_1, ..., y_{n-r}, z_1, ..., z_{2m'})$ is a system of coordinates and Λ is given by $\alpha_1 \circ J^{-1}, ..., \alpha_r \circ J^{-1}$ and the closed 2-form $\tilde{\Omega} = \sum_{j=1}^{n-r} dx_j \wedge dy_j + \sum_{k=1}^{2m'} dz_{2k-1} \wedge dz_{2m'}$.

Therefore $\mathcal{A}_2$ is spanned by $\partial/\partial y_1, ..., \partial/\partial y_{n-r}$ since $\Lambda(dx_j, \quad) = \partial/\partial y_j$, $j = 1, ..., n-r$, and $\Lambda(\alpha_k \circ J^{-1}, \quad) = 0$, $k = 1, ..., r$.

By the same reason around each point $p \in M$ there exist functions $y'_1, ..., y'_{n-r}$, $z'_1, ..., z'_{2m'}$ such that $(x_1, ..., x_n, y'_1, ..., y'_{n-r}, z'_1, ..., z'_{2m'})$ is a system of coordinates while Λ_1 is given by $\alpha_1, ..., \alpha_r$ and the closed 2-form $\tilde{\Omega}_1 = \sum_{j=1}^{n-r} dx_j \wedge dy'_j + \sum_{k=1}^{2m'} dz'_{2k-1} \wedge dz'_{2m'}$.

But $\Lambda(dx_j, \quad) = \partial/\partial y_j$ and $\Lambda_1(dx_j, \quad) = \partial/\partial y'_j$ whence $\partial/\partial y_j = a_j \partial/\partial y'_j$, $j = 1, ..., n-r$. So expressing $dy'_1, ..., dy'_{n-r}$, $dz'_1, ..., dz'_{2m'}$ in terms of $dx_1, ..., dx_n$,

$dy_1, ..., dy_{n-r},\ dz_1, ..., dz_{2m'}$ yields $\tilde{\Omega}_1 = \sum_{j=1}^{n-r} a_j dx_j \wedge dy_j + \Omega'_1$ where Ω'_1 does not contain any term involving $dy_1, ..., dy_{n-r}$ and its coefficient functions do not depend on $(y_1, ..., y_{n-r})$.

Now it is clear that $\tilde{\ell}$ projects in a partial tensor field $\ell : \mathcal{F} \to TP$ since the flow of each $\partial/\partial y_j$, $j = 1, ..., n-r$, preserves $\tilde{\ell}$. For proving the remainder properties of ℓ consider the product manifold $M \times \mathbb{K}^r$ endowed with coordinates $(x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_{2m'})$, where $(y_{n-r+1}, ..., y_n)$ are the canonical coordinates of $\mathbb{K}^r$, and identify M to $M \times \{0\}$. As before forms on N, or on M, will be regarded, in the obvious way, as form on $M \times \mathbb{K}^r$ when necessary. On this last manifold set $\Omega = \sum_{j=1}^{n} dx_j \wedge dy_j + \sum_{k=1}^{m'} dz_{2k-1} \wedge dz_{2k}$ and $\Omega_1 = \sum_{j=1}^{n-r} a_j dx_j \wedge dy_j + \sum_{j=n-r+1}^{n} a dx_j \wedge dy_j + \Omega'_1$. Then $L_{\frac{\partial}{\partial y_j}} \Omega = L_{\frac{\partial}{\partial y_j}} \Omega_1 = 0$, $j = 1, ..., n$.

Let H and Ω_k be the $(1,1)$-tensor field and the 2-form defined by $\Omega_1(X,Y) = \Omega(HX, Y)$ and $\Omega_k(X,Y) = \Omega(H^k X, Y)$, $k \in \mathbb{Z}$, respectively (obviously $\Omega_0 = \Omega$). Then $H\partial/\partial y_j = a_j \partial/\partial y_j$, $j = 1, ..., n-r$, $H\partial/\partial y_j = a\partial/\partial y_j$, $j = n-r+1, ..., n$, and $\Omega_k = \sum_{j=1}^{n-r} a_j^k dx_j \wedge dy_j + \sum_{j=n-r+1}^{n} a^k dx_j \wedge dy_j + \Omega'_k$ where Ω'_k does not contain any term involving $dy_1, ..., dy_n$ and its coefficient functions do not depend on $y = (y_1, ..., y_n)$.

As $(i_X \Omega) \circ H = \Omega(X, H\quad) = \Omega(HX, \quad) = \Omega_1(X, \quad)$, one has $dx_j \circ H = a_j dx_j$ and $dy_j \circ H = a_j dy_j + \lambda_j$, $j = 1, ..., n-r$; $dx_j \circ H = a dx_j$ and $dy_j \circ H = a dy_j + \lambda_j$, $j = n-r+1, ..., n$, where $\lambda_1, ..., \lambda_n$ are functional combinations of $dx_1, ..., dx_n, dz_1, ..., dz_{2m'}$ whose coefficients do not depend on y. By the same reason each $dz_k \circ H$, $k = 1, ..., 2m'$, is a functional combination of $dx_1, ..., dx_n, dz_1, ..., dz_{2m'}$ and its coefficients do not depend on y.

Thus, if $\pi_1 : M \times \mathbb{K}^r \to M$ is the first projection, the tensor field H projects in J on N through $\pi_N \circ \pi_1$ and in a tensor field G on P through $\pi_P \circ \pi_1$. In turns G projects in J via $\pi : P \to N$.

On the other hand if $\tau = \sum_{j=1}^{n} f_j dx_j$ then its Ω_1-hamiltonian $\sum_{j=1}^{n-r} a_j^{-1} f_j \partial/\partial y_j + \sum_{j=n-r+1}^{n} a^{-1} f_j \partial/\partial y_j$ equals the Ω-hamiltonian of $\tau \circ J^{-1} = \sum_{j=1}^{n-r} a_j^{-1} f_j dx_j + \sum_{j=n-r+1}^{n} a^{-1} f_j dx_j$. Therefore the Ω_1-hamiltonians of $\alpha_1, ..., \alpha_r$, or the Ω-hamiltonians of $\alpha_1 \circ J^{-1}, ..., \alpha_r \circ J^{-1}$, define a r-dimensional foliation $\mathcal{A}_0$ on $M \times \mathbb{K}^r$ transverse to the first factor, which is Ω_1-symplecticly complete since $\alpha_1, ..., \alpha_r$ are closed and Ω-symplecticly complete because $\alpha_1 \circ J^{-1}, ..., \alpha_r \circ J^{-1}$

regarded on N define the foliation $\bar{w}(0)$ (recall that a foliation is called symplecticly complete if its symplectic orthogonal is a foliation too; see [8]).

Let $\pi' : M \times \mathbb{K}^r \to \dfrac{M \times \mathbb{K}^r}{\mathcal{A}_0}$ the canonical projection in the local quotient. Then $\pi' : M \to \dfrac{M \times \mathbb{K}^r}{\mathcal{A}_0}$ is a diffeomorphism. Moreover as $\mathcal{A}_0$ is bi-symplecticly complete, the Poisson structures Λ_Ω and Λ_{Ω_1}, associated to Ω and Ω_1 respectively, project in two Poisson structures Λ' and Λ'_1 on $\dfrac{M \times \mathbb{K}^r}{\mathcal{A}_0}$. But the restriction of of $\alpha_1 \circ J^{-1}, ..., \alpha_r \circ J^{-1}$ and Ω to M defines Λ and that of $\alpha_1, ..., \alpha_r$ and Ω_1 defines Λ_1, so $\pi' : M \to \dfrac{M \times \mathbb{K}^r}{\mathcal{A}_0}$ transforms (Λ, Λ_1) in (Λ', Λ'_1) as a straightforward algebraic calculation at each point $(p, 0)$ shows [or apply lemma 1.4 of [17] to $T_{(p,0)}(M \times \mathbb{K}^r)$, $\mathcal{A}_0(p, 0)$, $T_{(p,0)}(M \times \{0\})$, $\Omega(p, 0)$ and $\Omega_1(p, 0)$].

This last construction only needs the properties of $\alpha_1, ..., \alpha_r$ and J on N but not the compatibility of Λ, Λ_1, which may be expressed by means of Ω, Ω_1. More exactly:

Proposition 1.1. *The Poisson structures Λ, Λ_1 are compatible if and only if $\alpha_1 \wedge ... \wedge \alpha_r \wedge d\Omega_2 = 0$.*

For proving this proposition we need some auxiliary results.

Lemma 1.3. *On an even dimensional manifold $\tilde{M}$ consider a couple of 2-forms β, β_1 such that rank$\beta = dim\tilde{M}$ everywhere. Let K the $(1,1)$-tensor field defined by $\beta_1 = \beta(K, \quad)$ and set $\beta_2 = \beta(K^2, \quad)$. Then for any vector fields X_1, X_2, X_3 one has:*
$\beta(N_K(X_1, X_2), X_3) + d\beta(KX_1, KX_2, X_3) = -d\beta_2(X_1, X_2, X_3) + d\beta_1(KX_1, X_2, X_3) + d\beta_1(X_1, KX_2, X_3)$.

Proof. As the foregoing formula is tensorial, one may assume $[X_1, X_2] = [X_1, X_3] = [X_2, X_3] = 0$ without loss of generality. Then [recall that $\beta(K, \quad) = \beta(\quad, K)$]:
$d\beta_2(X_1, X_2, X_3) = X_1\beta(KX_2, KX_3) - X_2\beta(KX_1, KX_3) + X_3\beta(KX_1, KX_2)$
$d\beta_1(KX_1, X_2, X_3) = (KX_1)\beta(KX_2, X_3) - X_2\beta(KX_1, KX_3) + X_3\beta(KX_1, KX_2) - \beta(K[KX_1, X_2], X_3]) + \beta([KX_1, X_3], KX_2)$
$d\beta_1(X_1, KX_2, X_3) = X_1\beta(KX_2, KX_3) - (KX_2)\beta(KX_1, X_3) + X_3\beta(KX_1, KX_2) - \beta(K[X_1, KX_2], X_3]) - \beta([KX_2, X_3], KX_1)$.

Therefore the right side of the formula becomes:

$(KX_1)\beta(KX_2,X_3)-(KX_2)\beta(KX_1,X_3)+X_3\beta(KX_1,KX_2)-\beta(K[KX_1,X_2]+K[X_1,KX_2],X_3)+\beta([KX_1,X_3],KX_2)-\beta([KX_2,X_3],KX_1)=\beta(N_K(X_1,X_2),X_3)+d\beta(KX_1,KX_2,X_3)$. $\square$

Corollary 1.3.1. *Assume β,β_1 symplectic and set $\tau=\beta_1((K+tI)^{-1},\quad)=\beta((I+tK^{-1})^{-1},\quad)$. Then*
$d\tau((K+tI)X_1,(K+tI)X_2,(K+tI)X_3)=t\beta(N_K(X_1,X_2),X_3)=-td\beta_2(X_1,X_2,X_3)$.

Remark. At each point $(K+tI)^{-1}$ and $(I+tK^{-1})^{-1}$ are linear combination of powers of K, so τ is a 2-form on its domain of definition.

Proof. From lemma 1.3 applied to β,β_1 and K follows $d\beta_2=-\beta(N_K(\quad,\quad),\quad)$.

On the other hand, applying this lemma to τ, β_1 and $K+tI$ and taking into account that $N_K=N_{(K+tI)}$ and $\tau((K+tI)^2,\quad)=\beta_1((K+tI),\quad)$ yields:
$\tau(N_K(X_1,X_2),X_3)+d\tau((K+tI)X_1,(K+tI)X_2,X_3)=-d\beta_2(X_1,X_2,X_3)=\beta(N_K(X_1,X_2),X_3)$.

Hence by replacing X_3 by $(K+tI)X_3$ follows:
$d\tau((K+tI)X_1,(K+tI)X_2,(K+tI)X_3)=\beta(N_K(X_1,X_2),(K+tI)X_3)-\tau(N_K(X_1,X_2),(K+tI)X_3)=\beta(N_K(X_1,X_2),(K+tI-(I+tK^{-1})^{-1}(K+tI))X_3)=t\beta(N_K(X_1,X_2),X_3)=-td\beta_2(X_1,X_2,X_3)$. $\square$

Let us prove proposition 1.1. Locally always there exists $t\neq 0$ such that $I+tH^{-1}$ is invertible. Since $\Lambda_\Omega+t\Lambda_{\Omega_1}$ is the dual bivector of $\Omega((I+tH^{-1})^{-1},\quad)$, it projects in $\Lambda'+t\Lambda'_1$ and $\pi':M\to\dfrac{M\times\mathbb{K}^r}{\mathcal{A}_0}$ transforms $\Lambda+t\Lambda_1$ in $\Lambda'+t\Lambda'_1$, the bivector $\Lambda+t\Lambda_1$ is given by the restriction to M [always identified to $M\times\{0\}$] of $\Omega((I+tH^{-1})^{-1},\quad)$ and $\alpha_1\circ(H+tI)^{-1},...,\alpha_1\circ(H+tI)^{-r}$. Indeed if $\Omega_1(Y_j,\quad)=\alpha_j$, $j=1,...,r$, then $Y_1,...,Y_r$ span $\mathcal{A}_0$ and $\Omega((I+tH^{-1})^{-1}Y_j,\quad)=\Omega_1((H+tI)^{-1}Y_j,\quad)=\alpha_j\circ(H+tI)^{-1}$, $j=1,...,r$.

On the other hand each $\alpha_j\circ(H+tI)^{-1}$ is the pull-back of $\alpha_j\circ(J+tI)^{-1}$ and $\alpha_1\circ(J+tI)^{-1},...,\alpha_r\circ(J+tI)^{-1}$ define the foliation $\bar{w}(t)$. So $\alpha_1\circ(H+tI)^{-1},...,\alpha_r\circ(H+tI)^{-1}$ define a foliation on $M\times\mathbb{K}^r$ and, by restriction, on M. Thus $\Lambda+t\Lambda_1$ is a Poisson structure, that is (Λ,Λ_1) bihamiltonian, if and only if $\Omega((I+tH^{-1})^{-1},\quad)$ is closed modulo $dy_{n-r+1},...,dy_n,\alpha_1\circ(H+tI)^{-1},...,\alpha_r\circ(H+tI)^{-1}$ when $y_{n-r+1}=...=y_n=0$.

But the coefficients of H do not depend on y and $(I+tH^{-1})^{-1}\partial/\partial y_j$ equals

$(1+ta_j^{-1})^{-1}\partial/\partial y_j$ if $j \leq n-r$ and $(1+ta^{-1})^{-1}\partial/\partial y_j$ if $j \geq n-r+1$, so $\Omega((I+tH^{-1})^{-1}, \quad) = \sum_{j=1}^{n-r}(1+ta_j^{-1})^{-1}dx_j\wedge dy_j+\sum_{j=n-r+1}^{n}(1+ta^{-1})^{-1}dx_j\wedge dy_j+\Omega_t''$ where Ω_t'' do not contain any term involving $dy_1, ..., dy_n$ and its coefficients do not depend on y.

Therefore, since $d\Omega((I+tH^{-1})^{-1}, \quad) = d\Omega_t''$, the pair (Λ, Λ_1) is bihamiltonian if and only if $(\alpha_1\circ(H+tI)^{-1})\wedge...\wedge(\alpha_r\circ(H+tI)^{-1})\wedge d\Omega((I+tH^{-1})^{-1}, \quad) = 0$.

From corollary 1.3.1 applied to Ω, Ω_1 and H follows $d\Omega((I+tH^{-1})^{-1}, \quad)((H+tI) \quad, (H+tI) \quad, (H+tI) \quad) = -td\Omega_2$, so the above condition holds if and only if $\alpha_1 \wedge ... \wedge \alpha_r \wedge d\Omega_2 = 0$, *which finishes the proof of proposition 1.1.*

Lemma 1.4. *If* Λ, Λ_1 *are compatible then* $\alpha_1 \wedge ... \wedge \alpha_r \wedge N_G = 0$.

Proof. Since N_G is the projection of N_H it suffices to show that $N_H(X_1, X_2)$ is a functional combination of $\partial/\partial y_1, ..., \partial/\partial y_n$ if $X_1, X_2 \in Ker(\alpha_1 \wedge ... \wedge \alpha_r)$. By proposition 1.1, $d\Omega_2 = \sum_{j=1}^{r}\lambda_j \wedge \alpha_j$ so $d\Omega_2(X_1, X_2, \quad)$ is a functional combination of $\alpha_1, ..., \alpha_r$.

From lemma 1.3 applied to Ω, Ω_1 and H follows $\Omega(N_H(X_1, X_2), \quad) = -d\Omega_2(X_1, X_2, \quad)$, which implies that $\Omega(N_H(X_1, X_2), \quad)$ is a functional combination of $\alpha_1, ...\alpha_r$. Therefore $N_H(X_1, X_2)$ has to be a functional combination of $\partial/\partial y_1, ..., \partial/\partial y_n$. $\square$

Lemma 1.5. *Assume* Λ, Λ_1 *compatible. Then* G *is a prolongation of* ℓ.

Proof. As before M is identified to $M \times \{0\} \subset M \times \mathbb{K}^r$ and $\pi_P \circ \pi_1 = \pi_P$ on M. First note that $H(Im\Lambda_1) \subset Im\Lambda$ modulo $\partial/\partial y_{n-r+1}, ..., \partial/\partial y_n$ since on M forms $\alpha_1, ..., \alpha_r$ define $Im\Lambda_1$ and forms $\alpha_1 \circ J^{-1}, ..., \alpha_r \circ J^{-1}$ define $Im\Lambda$. On the other hand if Y belongs to $Im\Lambda$ modulo $\partial/\partial y_1, ..., \partial/\partial y_n$ then $\Lambda((i_Y\Omega)_{|TM}, \quad)$ equals $-Y$ modulo $\partial/\partial y_1, ..., \partial/\partial y_n$.

Now consider $X \in Im\Lambda_1$; then $\Lambda_1((i_X\Omega_1)_{|TM}, \quad) = -X$. But $i_X\Omega_1 = i_{HX}\Omega$ and HX belongs to $Im\Lambda$ modulo $\partial/\partial y_1, ..., \partial/\partial y_n$, so $\Lambda((i_X\Omega_1)_{|TM}, \quad)$ equals $-HX$ modulo $\partial/\partial y_1, ..., \partial/\partial y_n$; that is to say $\tilde{\ell}([X]) = [(\pi_1)_*HX]$. Finally projecting on P via π_P yields $\ell((\pi_P)_*X) = (\pi_P)_*((\pi_1)_*HX) = (\pi_P \circ \pi_1)_*(HX) = G((\pi_P \circ \pi_1)_*X) = G((\pi_P)_*X)$. $\square$

Let us see that $(\mathcal{F},\ell)$ is a weak Veronese flag. Since G projects in J, the morphism $\ell:\mathcal{F}\to TP$ projects in $\bar\ell=J_{|\bar w(\infty)}$ and $\ell\mathcal{A}\subset\mathcal{A}$. As $\bar w$ is a Veronese web there is no $\bar\ell$-invariant vector subspace of positive dimension and $\bar\ell^*\bar\alpha$ is closed on $\bar w(\infty)$ for any closed 1-form $\bar\alpha$ such that $Ker\bar\alpha\supset\bar w(\infty)$. Pulling-back via $\pi:P\to N$ shows that conditions 1) and 3) hold. Finally as $\mathcal{F}=Ker(\alpha_1\wedge...\wedge\alpha_r)$ on P, lemmas 1.4 and 1.5 imply $N_\ell=0$.

When we pointed out the existence, at each, point of an algebraic projection of $\tilde\ell$ it was showed that $\omega_1=\omega(\ell,\quad)$ [more exactly that $\omega_1=\omega(\varphi,\quad)$]. Therefore $(\mathcal{F},\ell,\omega,\omega_1)$ will be a Veronese flag if condition 3) of this second definition holds.

Lemma 1.6. *On an open set P' of P consider functions $f,f_1,...,f_k$, $k\geq 0$, such that $\alpha_1\wedge...\wedge\alpha_r\wedge...\wedge df_1\wedge...\wedge df_k$ has no zero. Assume closed ℓ^*df along the foliation $Ker(\alpha_1\wedge...\wedge\alpha_r\wedge...\wedge df_1\wedge...\wedge df_k)$. Then $(L_{X_f}\ell)(Ker(\alpha_1\wedge...\wedge\alpha_r\wedge...\wedge df_1\wedge...\wedge df_k))(q)$ is contained in the vector subspace of T_qP spanned by $X_{f_1}(q),...,X_{f_k}(q)$ whenever $q\in P'$.*

Proof. Let $\widetilde X_f,\widetilde X_{f_1},...,\widetilde X_{f_k}$ be the Ω-hamiltonians of $f,f_1,...,f_k$ regarded as functions on an open set of $M\times\mathbb{K}^r$. A straightforward calculation shows that $\widetilde X_f,\widetilde X_{f_1},...,\widetilde X_{f_k}$ project in $X_f,X_{f_1},...,X_{f_k}$; in particular $L_{\tilde X_f}H$ projects in $L_{X_f}G$.

On $M\times\mathbb{K}^r$ one has $\alpha_1\wedge...\wedge\alpha_r\wedge df_1\wedge...\wedge df_k\wedge d(df\circ H)=0$ since $\alpha_1\wedge...\wedge\alpha_r\wedge df_1\wedge...\wedge df_k\wedge d(df\circ G)=0$ on P. Therefore $\alpha_1\wedge...\wedge\alpha_r\wedge df_1\wedge...\wedge df_k\wedge L_{\tilde X_f}\Omega_1=0$ whence locally $L_{\tilde X_f}\Omega_1=\sum_{j=1}^r\lambda_j\wedge\alpha_j+\sum_{i=1}^k\mu_i\wedge df_i$. But $L_{\tilde X_f}\Omega_1=L_{\tilde X_f}(\Omega(H,\quad))=\Omega(L_{\tilde X_f}H,\quad)$ so $\Omega(L_{\tilde X_f}H,\quad)=\sum_{j=1}^r\lambda_j\wedge\alpha_j+\sum_{i=1}^k\mu_i\wedge df_i$; this implies that $L_{\tilde X_f}H=\sum_{j=1}^r X'_j\otimes\alpha_j+\sum_{j=1}^r X''_j\otimes\lambda_j+\sum_{i=1}^k Y_i\otimes df_i-\sum_{i=1}^k\widetilde X_{f_i}\otimes\mu_i$ where $X''_1,...,X''_r$ are functional combination of $\partial/\partial y_1,...,\partial/\partial y_n$. Thus the projection on P of $L_{\tilde X_f}H$ sends $Ker(\alpha_1\wedge...\wedge\alpha_r\wedge...\wedge df_1\wedge...\wedge df_k)(q)$ into the vector subspace of T_qP spanned by $X_{f_1}(q),...,X_{f_k}(q)$. $\square$

When $k=0$ from lemma 1.6 follows the third condition of the definition of Veronese flag. *Thus $(\mathcal{F},\ell,\omega,\omega_1)$is a Veronese flag.*

1.2. The bihamiltonian structure over a $(1,1)$-tensor field and a foliation.

This second part contains a kind of inverse construction of that of subsection 1.1. Here, under some assumptions detailed later on, one will associate a bihamiltonian structure defined on a quotient of the cotangent bundle to a $(1,1)$-tensor field and a foliation.

Let N be a n-manifold. Recall that on $\Lambda^r T^*N$ it is defined a r-form R, called the Liouville r-form, as follows: if $v_1, ..., v_r \in T_\mu(\Lambda^r T^*N)$ then $R(v_1, ..., v_r) = \mu(\pi_* v_1, ..., \pi_* v_r)$ where $\pi : \Lambda^r T^*N \to N$ is the canonical projection. In turn $\Omega = dR$ will be named the Liouville $(r+1)$-form of $\Lambda^r T^*N$. When $r = 1$, that is on the cotangent bundle, the Liouville forms will be denoted ρ and ω respectively.

Given a skew-symmetric $(1,r)$-tensor field H on N, in other words a section of $TN \otimes \Lambda^r T^*N$, let $\varphi_H : T^*N \to \Lambda^r T^*N$ be the morphism of vector bundles defined by $\varphi_H(\tau) = \tau \circ H$, that is $\varphi_H(\tau)(v_1, ..., v_r) = \tau(H(v_1, ..., Hv_r))$. Set $\omega_1 = \varphi_H^* \Omega$.

Lemma 1.7. *On a real or complex vector space V of dimension $2n$, consider a 2-form α of rank $2n$ and a $(r+1)$-form β. Then there exists $h \in V \otimes \Lambda^r V^*$ connecting α and β, that is to say such that $\beta(v_1, ..., v_{r+1}) = \alpha(h(v_1, ..., v_r), v_{r+1})$, $v_1, ..., v_{r+1} \in V$. Moreover h is unique and $\alpha(h(v_1, ..., v_r), v_{r+1})$ $= \alpha(v_r, h(v_1, ..., v_{r-1}, v_{r+1}))$, $v_1, ..., v_{r+1} \in V$.*

Conversely, given a 2-form α and $h \in V \otimes \Lambda^r V^*$ such that $\alpha(h(v_1, ..., v_r), v_{r+1})$ $= \alpha(v_r, h(v_1, ..., v_{r-1}, v_{r+1}))$, $v_1, ..., v_{r+1} \in V$, then setting $\beta(v_1, ..., v_{r+1}) = \alpha(h(v_1, ..., v_r), v_{r+1})$, $v_1, ..., v_{r+1} \in V$, defines a $(r+1)$-form β.

The foregoing lemma gives rise to a skew-symmetric $(1,r)$-tensor field H^* on T^*N connecting ω and $(-1)^{r+1}\omega_1$, which will be called *the prolongation of* H (to the cotangent bundle).

Given coordinates $x = (x_1, ...x_n)$ on N let $(x, y) = (x_1, ...x_n, y_1, ..., y_n)$ be the associated coordinates on T^*N. Denote by $m(r)$ the set of all the r-multi-index $K : k_1 < ... < k_r$ whereas dx_K will mean $dx_{k_1} \wedge ... \wedge dx_{k_r}$ (as usual elements of $m(1)$ will be represented by small letters). On the other hand $K(j)$, where $1 \leq j \leq r$ and $r \geq 2$, will be the element of $m(r-1)$ obtained by deleting the term k_j of K. Assume that $H = \sum_{j \in m(1), K \in m(r)} h_{jK}(\partial/\partial x_j) \otimes dx_K$, then:

$$H^* = \sum_{j\in m(1),K\in m(r)} h_{jK}\left(\frac{\partial}{\partial x_j}\otimes dx_K - \sum_{a=1}^{r}(-1)^a\frac{\partial}{\partial y_{k_a}}\otimes dy_j\wedge dx_{K(a)}\right)$$
$$-\sum_{j\in m(1),K\in m(r+1)} y_j\left(\sum_{a,b=1}^{r+1}(-1)^{a+b}\frac{\partial h_{jK(a)}}{\partial x_{k_a}}\frac{\partial}{\partial y_{k_b}}\otimes dx_{K(b)}\right)$$

Therefore one has:

(a) H^* projects in H.

(b) Let ξ be the radial vector field on TN^* [in coordinates $\xi = \sum_{j=1}^n y_j\partial/\partial y_j$] then $L_\xi H^* = 0$.

(c) If v_1, v_2 are vertical vectors then $H^*(v_1, v_2, ...) = 0$.

(d) Set $\lambda(X_1, ..., X_{r+1}) = \omega(H^*(X_1, ..., X_r), X_{r+1})$, which defines a $(0, r+1)$-tensor field. Then λ is a closed $(r+1)$-form.

These four properties characterize the prolongation of H to TN^*. More exactly:

Proposition 1.2. *If a $(1,r)$-tensor field H' defined on TN^* satisfies (a), (b), (c) and (d), then $H' = H^*$.*

Proof. The tensor field $H_1 = H' - H^*$ satisfies (b), (c) and (d), and its projection on N vanishes. So in coordinates (x, y):

$$H_1 = \sum_{j,a\in m(1),K\in m(r-1)} f_{jaK}\frac{\partial}{\partial y_j}\otimes dy_a\wedge dx_K + \sum_{j\in m(1),L\in m(r)} g_{jL}\frac{\partial}{\partial y_j}\otimes dx_L.$$

But $L_\xi H_1 = 0$ therefore $\xi\cdot f_{jaK} = 0$ and $\xi\cdot g_{jL} = g_{jL}$. In other words, each function f_{jaK} only depend on x and $g_{jL}(x,0) = 0$ for every $j\in m(1), L\in m(r)$.

Let λ_1 be the closed $(r+1)$-form defined by $\lambda_1(X_1, ..., X_{r+1})$ $= \omega(H_1(X_1, ..., X_r), X_{r+1})$. Then if K is the multi-index $k_1 < ... < k_{r-1}$ one has [recall that $\omega = \sum_{j=1}^n dy_j\wedge dx_j$]:

$\lambda_1(\partial/\partial y_a, \partial/\partial x_{k_1}, ..., \partial/\partial x_{k_{r-1}}, \partial/\partial x_j)(x,0)$

$= \omega(H_1(\partial/\partial y_a, \partial/\partial x_{k_1}, ..., \partial/\partial x_{k_{r-1}}), \partial/\partial x_j)(x,0) = f_{jaK}(x)$, whereas

$\lambda_1(\partial/\partial x_j, \partial/\partial x_{k_1}, ..., \partial/\partial x_{k_{r-1}}, \partial/\partial y_a)(x,0)$

$= \omega(H_1(\partial/\partial x_j, \partial/\partial x_{k_1}, ..., \partial/\partial x_{k_{r-1}}), \partial/\partial y_a)(x,0) = 0.$

Therefore $f_{jaK} = 0$ and $\lambda_1 = \sum_{S\in m(r+1)} h_S dx_S$ where each function h_S only depend on x since $d\lambda_1 = 0$, which implies $L_\xi\lambda_1 = 0$. But $L_\xi\omega = \omega$ and $L_\xi H_1 = 0$ so $L_\xi\lambda_1 = \lambda_1$. Thus λ_1 has to vanish and $H_1 = 0$. $\square$

Proposition 1.3. *Given a* $(1,1)$*-tensor field* H *on* N *then the prolongation of* N_H *equals* N_{H^*}.

Proof. By construction N_{H^*} satisfies (a), (b) and (c) with respect to N_H. Therefore it suffices to show that setting $\lambda(X_1, X_2, X_3) = \omega(N_{H^*}(X_1, X_2), X_3)$ defines a closed 3-form, which immediately follows from lemma 1.3 applied to ω, ω_1 and H^*. $\square$

Now suppose that H is an invertible $(1,1)$-tensor field and $\mathcal{G}$ a r-codimensional foliation both of them defined on N. Assume that:

1) $\alpha \circ H$ is closed on $\mathcal{G}$ whichever α is a closed 1-form such that $Ker\alpha \supset \mathcal{G}$,

2) the restriction of N_H to $\mathcal{G}$ vanishes.

Then $(H + tI)\mathcal{G}$, $t \in \mathbb{K}$, is a r-codimensional foliation on the open set A_t of all points of N where $H + tI$ is invertible. Indeed, reason as in the first paragraph after lemma 1.1.

Let $\mathcal{G}_0$ be the ω-orthogonal of the foliation $\pi_*^{-1}(H\mathcal{G}) = \{v \in T(T^*N) \mid \pi_* v \in H\mathcal{G}\}$, which equals the ω_1-orthogonal of the foliation $\pi_*^{-1}(\mathcal{G}) = \{v \in T(T^*N) \mid \pi_* v \in \mathcal{G}\}$ because $\omega_1 = \omega(H^*, \quad)$ and H^* projects in H. Note that $\mathcal{G}_0$ is a symplecticly complete foliation for ω and ω_1. On the other hand the quotient M of T^*N by $\mathcal{G}_0$ is globally defined and there is a projection $\pi' : M \to N$ such that $\pi' \circ \tilde{\pi} = \pi$, where $\tilde{\pi} : T^*N \to M$ is the canonical projection. In fact, M can be regarded as the quotient of T^*N by a vector sub-bundle and $\pi' : M \to N$ as its quotient vector bundle.

Since $\mathcal{G}_0$ is both ω and ω_1 symplecticly complete, the Poisson structures Λ_ω and Λ_{ω_1}, respectively associated to ω and ω_1, project in two Poisson structures Λ and Λ_1 on M.

Proposition 1.4. *The pair* (Λ, Λ_1) *is a bihamiltonian structure.*

Proof. The proof is very similar to that of proposition 1.1. As the question is local one may suppose $\mathcal{G}$ defined by closed 1-forms $\alpha_1, ..., \alpha_r$; of course we will regard $\alpha_1, ..., \alpha_r$ as forms on T^*N by identifying α_j and $\pi^*\alpha_j$, $j = 1, ..., r$. Let $\{Y_1, ..., Y_r\}$ the basis of $\mathcal{G}_0$ defined by $\omega_1(Y_j, \quad) = \alpha_j$, $j = 1, ..., r$. Given a point $p \in T^*N$ consider a scalar $t \neq 0$ and a small transversal P to $\mathcal{G}_0$, passing through this point, such that $I + t(H^*)^{-1}$ is invertible around p [that is $(I + tH^{-1})(\pi(p))$ is invertible], $\tilde{\pi}(P)$ is an open set of M and $\tilde{\pi} : P \to \tilde{\pi}(P)$

a diffeomorphism. It suffices to prove that the bivector $\Lambda + t\Lambda_1$ is a Poisson structure. Note that it is the projection of $\Lambda_\omega + t\Lambda_{\omega_1}$ which, in turns, is the dual bivector of $\omega((I + t(H^*)^{-1})^{-1}, \quad)$. But regarded on P by means of $\tilde{\pi} : P \to \tilde{\pi}(P)$, the bivector $\Lambda + t\Lambda_1$ is given by the restriction to this transversal of $\omega((I + t(H^*)^{-1})^{-1}, \quad)$ and $\omega((I + t(H^*)^{-1})^{-1}Y_j, \quad) = \alpha_j \circ (H^* + tI)^{-1}$, $j = 1, ..., r$.

On the other hand each $\alpha_j \circ (H^* + tI)^{-1}$ is the pull-back of $\alpha_j \circ (H + tI)^{-1}$, and $\alpha_1 \circ (H+tI)^{-1}, ..., \alpha_r \circ (H+tI)^{-1}$ define the foliation $(H+tI)\mathcal{G}$. So $\alpha_1 \circ (H^* + tI)^{-1}, ..., \alpha_r \circ (H^* + tI)^{-1}$ define a foliation on T^*N and, by restriction, on P as well. Thus $\Lambda + t\Lambda_1$ is a Poisson structure if and only if $\omega((I + t(H^*)^{-1})^{-1}, \quad)$ restricted to P is closed modulo $\alpha_1 \circ (H^* + tI)^{-1}, ..., \alpha_r \circ (H^* + tI)^{-1}$. Therefore for finishing the proof it is enough to show that $\omega((I + t(H^*)^{-1})^{-1}, \quad)$ is closed on T^*N modulo $\alpha_1 \circ (H^* + tI)^{-1}, ..., \alpha_r \circ (H^* + tI)^{-1}$.

From corollary 1.3.1, applied to ω, ω_1 and H^*, follows that

$$d(\omega((I + t(H^*)^{-1})^{-1}, \quad))((H^* + tI) \quad , (H^* + tI) \quad , (H^* + tI) \quad) = -td\omega_2$$

where $\omega_2 = \omega((H^*)^2, \quad)$. So the above condition holds if $\alpha_1 \wedge ... \wedge \alpha_r \wedge d\omega_2 = 0$.

By lemma 1.3 applied to ω, ω_1 and H^* one has $d\omega_2 = -\omega(N_{H^*}(\quad , \quad), \quad)$. Therefore $d\omega_2 = \tilde{\omega}_1$, where $\tilde{\omega}_1 = (\varphi_{N_H})^*\Omega$ and Ω is the Liouville 3-form of $\Lambda^2 T^*N$ since, by proposition 1.3, the prolongation of N_H is N_{H^*}.

On the other hand $\alpha_1 \wedge ... \wedge \alpha_r \wedge (\varphi_{N_H})^* R = 0$, where R is the Liouville 2-form of $\Lambda^2 T^*N$, because $\alpha_1 \wedge ... \wedge \alpha_r \wedge N_H = 0$ [calculate $(\varphi_{N_H})^* R$ on coordinates (x, y) such that $\alpha_1 = dx_1$,..., $\alpha_r = dx_r$]. Hence $\alpha_1 \wedge ... \wedge \alpha_r \wedge \tilde{\omega}_1 = 0$, as $\Omega = dR$ and $\alpha_1, ..., \alpha_r$ are closed, and finally $\alpha_1 \wedge ... \wedge \alpha_r \wedge d\omega_2 = 0$. $\square$

Examples. 1) On $N = \mathbb{K}^n$, $n \geq 1$, consider the foliation given by the closed 1-form $\alpha = \sum_{j=1}^n dx_j$ and the $(1,1)$-tensor field $H = \sum_{j=1}^n h_j(x_j)(\partial/\partial x_j) \otimes dx_j$ where the functions $h_1, ..., h_n$ never vanish. Then the associated bihamiltonian structure (Λ, Λ_1), defined on $M = T^*(\mathbb{K}^n)/\mathcal{G}_0$, has a symplectic factor of positive dimension at a point $p \in M$ if and only if $\tilde{h}(\pi'(p)) = 0$ where $\tilde{h} = \prod_{1 \leq j < k \leq n}(h_j - h_k)$. In other words (Λ, Λ_1) is Kronecker just on the open set $(\tilde{h} \circ \pi')^{-1}(\mathbb{K} - \{0\})$.

2) Now on $N = \mathbb{R}^n - \{0\}$, $n \geq 1$, consider the foliation $\mathcal{G}$ defined by $\alpha = \sum_{j=1}^n x_j^{a_j} dx_j$, where $a_1, ..., a_r$ are positive natural numbers, and the $(1,1)$-tensor field $H = \sum_{j=1}^n j(\partial/\partial x_j) \otimes dx_j$. Then the associated bihamiltonian structure

(Λ, Λ_1), defined on $M = T^*(\mathbb{R}^n - \{0\})/\mathcal{G}_0$, has non-trivial symplectic factor on the closed set $(h \circ \pi')^{-1}(0)$, where $h = x_1 \cdots x_n$, and is Kronecker on the open set $(h \circ \pi')^{-1}(\mathbb{R} - \{0\})$.

Let ϕ_t be the flow of the vector field $\xi = \sum_{j=1}^n (a_j + 1)^{-1} x_j \partial/\partial x_j$. As $L_\xi \alpha = \alpha$ and $L_\xi H = 0$, the foliation $\mathcal{G}$ and the $(1,1)$-tensor field H project in a foliation $\tilde{\mathcal{G}}$ and a $(1,1)$-tensor field $\tilde{H}$ respectively, defined on the quotient manifold $\tilde{N} = (\mathbb{R}^n - \{0\})/G$ where $G = \{\phi_k \mid k \in \mathbb{Z}\}$. Obviously $\tilde{\mathcal{G}}$ and $\tilde{H}$ satisfy 1) and 2), which gives rise to a bihamiltonian structure on $\tilde{M} = (T^*\tilde{N})/\tilde{\mathcal{G}}_0$. Moreover $\tilde{N}$ is diffeomorphic to $S^1 \times S^{n-1}$.

2. Some properties of Veronese flags

The aim of this section is to establish two results on Veronese flags useful later on. Given a vector bundle E over a manifold P and a morphism $H : E \to E$, we will say that H is 0-*deformable* if for any points $p, q \in P$ there exists an isomorphism between their fibers $\varphi : E(p) \to E(q)$ such that $H(p) = \varphi^{-1} \circ H(q) \circ \varphi$.

By technical reasons parameters are needed. Therefore consider a foliation $\mathcal{F}_1$ on a manifold P, a second foliation $\mathcal{F} \subset \mathcal{F}_1$ and a morphism $\ell : \mathcal{F} \to \mathcal{F}_1$, such that $(\mathcal{F}, \ell)$ is a weak Veronese flag along $\mathcal{F}_1$; set $r = dim\mathcal{F}_1 - dim\mathcal{F}$. Let $\mathcal{A}$ be the foliation of the largest ℓ-invariant vector subspaces (as in section 1) and $\pi : P \to N$ a local quotient of P by $\mathcal{A}$. Then N is endowed with the quotient foliations $\mathcal{F}'_1 = \mathcal{F}_1/\mathcal{A}$ and $\mathcal{F}' = \mathcal{F}/\mathcal{A}$. Unless another thing is stated, the Lie and the exterior derivatives of tensor fields defined along a foliation, for example $\mathcal{F}_1$ on P or $\mathcal{F}'_1$ on N, will be considered along this foliation. By definition (a system of) coordinates along a m-dimensional foliation $\mathcal{G}$ will mean a family of functions $y_1, ..., y_m$, on an open set of the support manifold, such that $dy_1 \wedge, ..., \wedge dy_m$ is a volume form along $\mathcal{G}$; in this case $\{\partial/\partial y_1, ..., \partial/\partial y_m\}$ will be the dual basis of $\{dy_1, ..., dy_m\}$.

Consider functions $x_1, ...x_n$ on N, such that $dx_1 \wedge ... \wedge dx_n$ is a volume form on $\mathcal{F}'_1$, and functions $a_1, ..., a_n$ constant along $\mathcal{F}'_1$. Set $J = \sum_{j=1}^n a_j (\partial/\partial x_j) \otimes dx_j$ where $\{\partial/\partial x_1, ..., \partial/\partial x_n\}$ is the dual basis of $\{dx_1, ..., dx_n\}$. One has:

Proposition 2.1. *Let $G : \mathcal{F}_1 \to \mathcal{F}_1$ be a morphism which extends ℓ and projects in J. Assume that:*

(a) $\ell\,|_{\mathcal{A}}$ *is* 0*-deformable, nilpotent and flat on each leaf of* $\mathcal{A}$*,*

(b) $a_1, ..., a_n$ *never vanish,*

then around every point of P *there exists a morphism* $G' : \mathcal{F}_1 \to \mathcal{F}_1$*, which extends* ℓ *and projects in* J*, such that* $N_{G'} = 0$*.*

From proposition 2.1 follows:

Lemma 2.1. *Consider a morphism* $\widetilde{H} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}$ *where* $\widetilde{\mathcal{F}}$ *is a* m*-dimensional foliation on a manifold* $\widetilde{P}$*. Suppose that* $\widetilde{H}$ *is* 0*-deformable and only has one eigenvalue. If* $\widetilde{H}$ *is flat on each leaf of* $\widetilde{\mathcal{F}}$ *then, around every point of* $\widetilde{P}$*, there exists a system of coordinates* $(z_1, ..., z_m)$ *along* $\widetilde{\mathcal{F}}$*such that* $\widetilde{H} = \sum_{j,k=1}^{m} a_{jk}(\partial/\partial z_j)\otimes dz_k$ *where* $a_{jk} \in \mathbb{K}$*.*

Proof. Assume $m < dim\tilde{P}$ otherwise the result is obvious. Consider coordinates $(x, y) = (x_1, ..., x_n, y_1, ..., y_m)$ defined on an open set B around a point of $\widetilde{P}$, such that $dx_1 = ... = dx_n = 0$ gives $\widetilde{\mathcal{F}}$. Let a be the eigenvalue of $\widetilde{H}$; by taking $\widetilde{H} - aI$ instead $\widetilde{H}$ we may suppose $a = 0$. Set $\widetilde{\ell} = \widetilde{H}$ and $J = \sum_{j=1}^{n} a_j(\partial/\partial x_j)\otimes dx_j$ where $a_1, ..., a_n \in \mathbb{K} - \{0\}$. By means of coordinates (x, y), J and $\widetilde{H}$ can be regarded too as tensor fields on B in an obvious way. Set $G = J + \widetilde{H}$. It easily seem that $(\widetilde{\mathcal{F}}, \widetilde{\ell})$ is a weak Veronese flag on B for which $\widetilde{\mathcal{A}} = \widetilde{\mathcal{F}}$ and the projected Veronese web is defined by J and $dx_1, ..., dx_n$.

Let G' be the $(1, 1)$-tensor field given by proposition 2.1. The characteristic polynomial of both G and G' equals $(\prod_{j=1}^{n}(t-a_j))t^m$; even more $Im(\prod_{j=1}^{n}(G'-a_jI)) = Im(\prod_{j=1}^{n}(G - a_jI)) = \widetilde{\mathcal{F}}$ [here product means composition]. On the other hand, as $N_{G'} = 0$ and $\prod_{j=1}^{n}(t - a_j)$ and t^m are relatively prime, locally B splits into a product following the foliations $\widetilde{\mathcal{F}} = Im(\prod_{j=1}^{n}(G' - a_jI)) = Ker((G')^m)$ and $Im((G')^m) = Ker(\prod_{j=1}^{n}(G' - a_jI))$. Thus one may consider coordinates $(x, u) = (x_1, ..., x_n, u_1, ..., u_m)$ such that $\widetilde{\mathcal{F}}$ is given by $dx_1 = ... = dx_n = 0$ and $Im((G')^m)$ by $du_1 = ... = du_m = 0$ respectively. Moreover $G' = J + \sum_{j,k=1}^{m} f_{jk}(u)(\partial/\partial u_j)\otimes du_k$ since $N_{G'} = 0$. But $\widetilde{H}$ is flat on the leaves of $\widetilde{\mathcal{F}}$ and $G'_{|\widetilde{\mathcal{F}}} = \widetilde{H}$, so $\sum_{j,k=1}^{m} f_{jk}(u)(\partial/\partial u_j)\otimes du_k$ is flat and one can choose functions $z_1, ..., z_m$ of u such that

$$\sum_{j,k=1}^{m} f_{jk}(u)(\partial/\partial u_j)\otimes du_k = \sum_{j,k=1}^{m} a_{jk}(\partial/\partial z_j)\otimes dz_k,\ a_{jk} \in \mathbb{K}.\ \square$$

Lemma 2.2. *Consider a* m*-dimensional foliation* $\widetilde{\mathcal{F}}$ *on a manifold* $\widetilde{P}$ *and* *a morphism $\widetilde{H} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}$. Suppose that $\widetilde{H}$ is 0-deformable and $N_{\widetilde{H}} = 0$. Then along $\widetilde{\mathcal{F}}$, given a function f such that $Kerdf \supset Ker\widetilde{H}$ and $d(df \circ \widetilde{H}) = 0$, locally there exists a function g such that $dg \circ \widetilde{H} = df$.*

Proof. As $N_{\widetilde{H}} = 0$ and $\widetilde{H}$ is 0-deformable, $Im\widetilde{H}$ is a foliation contained in $\widetilde{\mathcal{F}}$; moreover there exists a vector sub-bundle E of $\widetilde{\mathcal{F}}$ and a morphism $\rho : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}$ such that $\widetilde{\mathcal{F}} = E \oplus Ker\widetilde{H}$ and $(\rho \circ \widetilde{H})_{|E} = I$. Set $\alpha = df \circ \rho$; then $\alpha \circ \widetilde{H} = df$. From lemma 1.1 applied along $\widetilde{\mathcal{F}}$ follows that $d\alpha(Im\widetilde{H}, Im\widetilde{H}) = 0$, that is $\alpha_{|Im\widetilde{H}}$ is closed. Therefore locally there is a function g such that $(dg - \alpha)_{|Im\widetilde{H}} = 0$ so $dg \circ \widetilde{H} = \alpha \circ \widetilde{H} = df$. $\square$

One will prove proposition 2.1 by induction on $m = dim\mathcal{A}$. If $m = 0$ the result is obvious; now suppose the proposition true up to dimension $m-1$. Note that in this case lemma 2.1 is also true if $dim\widetilde{\mathcal{F}} \leq m - 1$. As the problem is local we may assume that $\mathcal{F}'$ is defined by r closed 1-forms $\alpha_1, ..., \alpha_r$ along $\mathcal{F}'_1$, that is $J, \alpha_1, ..., \alpha_r$ describe the associated Veronese web. Functions $x_1, ..., x_n$ and forms $\alpha_1, ..., \alpha_r$ can be regarded as defined on P in the obvious way (via π). This allows us to consider coordinates $(x, z) = (x_1, ..., x_n, z_1, ..., z_m)$ along $\mathcal{F}_1$ such that $dx_1 = ... = dx_n = 0$ defines $\mathcal{A}$ and, by means of (x, z), regard J and $H = \ell_{|\mathcal{A}}$ as $(1,1)$-tensor fields along $\mathcal{F}_1$. Moreover as $KerG = Ker(H_{|\mathcal{A}}) \subset \mathcal{A}$ is a foliation since $H_{|\mathcal{A}}$ is flat, coordinates (x, z) can be chosen in such a way that $KerG$ is defined by $dx_1 = ... = dx_n = dz_1 = ... = dz_{m-s} = 0$ where $s = dimKerG$. Then $G = J + H + \sum_{j=1}^{m} (\partial/\partial z_j) \otimes \beta_j$ where every β_j is a functional combination of $dx_1, ..., dx_n$ and $H = \sum_{j=1}^{m} \sum_{k=1}^{m-s} f_{jk} (\partial/\partial z_j) \otimes dz_k$.

But when $i = m - s + 1, ..., m$ one has:

$$-N_G \left(\frac{\partial}{\partial z_i}, \quad \right) = G \circ L_{\frac{\partial}{\partial z_i}} G - L_{G\left(\frac{\partial}{\partial z_i}\right)} G = G \circ L_{\frac{\partial}{\partial z_i}} G$$

$$= \sum_{j=1}^{m} \sum_{k=1}^{m-s} \frac{\partial f_{jk}}{\partial z_i} H \left(\frac{\partial}{\partial z_j} \right) \otimes dz_k + \sum_{j=1}^{m} H \left(\frac{\partial}{\partial z_j} \right) \otimes \frac{\partial \beta_j}{\partial z_i}$$

therefore $\partial f_{jk}/\partial z_i = 0$, $j, k = 1, ..., m - s$, and $\alpha_1 \wedge ... \wedge \alpha_r \wedge (\partial \beta_j / \partial z_i) = 0$, $j = 1, ..., m - s$, since $\alpha_1 \wedge ... \wedge \alpha_r \wedge N_G = 0$. Observe that it is the same proving proposition 2.1 for G or for $G + \sum_{j=1}^{r} X_j \otimes \alpha_j$ where $X_1, ..., X_r$ are vector fields tangent to $\mathcal{A}$. So, by choosing suitable vector fields $X_1, ..., X_r$, one may suppose $\partial \beta_j / \partial z_i = 0$, $j = 1, ..., m - s$, without loss of generality.

In this case $Im(L_{(\partial/\partial z_i)}G) \subset KerG$, $i = m - s + 1$, which allows us to project G in a $(1,1)$-tensor field $\bar{G}$ defined on the local quotient $\bar{P}$ of P by $KerG$. Besides $\mathcal{F}_1$, $\mathcal{F}$, $\mathcal{A}$ and ℓ project in similar objects $\bar{\mathcal{F}}_1$, $\bar{\mathcal{F}}$, $\bar{\mathcal{A}}$ and $\bar{\ell}$ on $\bar{P}$, $(x, z_1, ..., z_{m-s})$ can be regarded as coordinates along $\bar{\mathcal{F}}_1$, and N is still the quotient of $\bar{P}$ by $\bar{\mathcal{F}}$; in particular $\alpha_1, ..., \alpha_r$ may be seem as forms on $\bar{P}$. Obviously all these objects satisfy the hypothesis of proposition 2.1 and, by the induction hypothesis, there exists $\bar{G}' : \bar{\mathcal{F}}_1 \to \bar{\mathcal{F}}_1$, which extends $\bar{\ell}$ and projects in J, such that $N_{\bar{G}'} = 0$. Since $\bar{G}' - \bar{G} = \sum_{j=1}^{r} \bar{X}_j \otimes \alpha_j$ where $\bar{X}_1, ..., \bar{X}_r$ are tangent to $\bar{\mathcal{A}}$ by considering $G + \sum_{j=1}^{r} X_j \otimes \alpha_j$ instead of G, where $X_1, ..., X_r$ are tangent to $\mathcal{A}$ and project in $\bar{X}_1, ..., \bar{X}_r$, one may suppose $N_{\bar{G}} = 0$ without loss of generality.

The characteristic polynomial of $\bar{G}$ equals $(\prod_{j=1}^{n}(t-a_j))t^{m-s}$ since $= dim\bar{\mathcal{A}} = m - s$. As $\prod_{j=1}^{n}(t - a_j)$ and t^{m-s} are relatively prime and $N_{\bar{G}} = 0$, locally $\bar{\mathcal{F}}_1$ splits into a product of two foliations $\bar{\mathcal{A}} = Im(\prod_{j=1}^{n}(\bar{G} - a_j I)) = Ker(\bar{G}^{m-s})$ and $\mathcal{G} = Im(\bar{G}^{m-s}) = Ker(\prod_{j=1}^{n}(\bar{G} - a_j I))$. Thus we may consider coordinates $(v, x, u) = (v_1, ..., v_b, x_1, ..., x_n, u_1, ..., u_{m-s})$ on $\bar{P}$ such that $\bar{\mathcal{F}}_1$ is defined by $dv_1 = ... = dv_b = 0$, $\bar{\mathcal{A}}$ by $dv_1 = ... = dv_b = dx_1 = ... = dx_n = 0$, and $\mathcal{G}$ by $dv_1 = ... = dv_b = du_1 = ... = du_{m-s} = 0$; moreover

$$\bar{G} = \sum_{j=1}^{n} a_j(\partial/\partial x_j) \otimes dx_j + \sum_{j,k=1}^{m-s} f_{jk}(v, u)(\partial/\partial u_j) \otimes du_k.$$

Now from lemma 2.1, applied to coordinates (v, u) and the $(1,1)$-tensor field $\sum_{j,k=1}^{m-s} f_{jk}(v, u)(\partial/\partial u_j) \otimes du_k$ on $\bar{\mathcal{A}}$, follows the existence of coordinates $(v, \bar{z}_1, ..., \bar{z}_{m-s})$ such that

$$\sum_{j,k=1}^{m-s} f_{jk}(v, u)(\partial/\partial u_j) \otimes du_k = \sum_{j,k=1}^{m-s} a_{jk}(\partial/\partial \bar{z}_j) \otimes d\bar{z}_k, \; a_{jk} \in \mathbb{K}.$$

Thus $dx_j \circ \bar{G} = a_j dx_j$, $j = 1, ..., n$, and every $d\bar{z}_k \circ \bar{G}$, $k = 1, ..., m - s$, is a linear combination with constant coefficients of $d\bar{z}_1, ..., d\bar{z}_{m-s}$. Consequently if $x_1, ..., x_n, \bar{z}_1, ..., \bar{z}_{m-s}$ are regarded as functions on P, since G projects in $\bar{G}$, then $dx_j \circ G = a_j dx_j$, $j = 1, ..., n$, and each $dz_k \circ \bar{G}$, $k = 1, ..., m - s$, is a linear combination with constant coefficients of $d\bar{z}_1, ..., d\bar{z}_{m-s}$. On the other hand, as $N_{(H_{|\mathcal{A}})} = 0$, $Kerd\bar{z}_k \supset Ker(H_{|\mathcal{A}})$ and $d(d\bar{z}_k \circ H)_{|\mathcal{A}} = 0$, by lemma 2.2 there exists a function g_k such that $(dg_k \circ H)_{|\mathcal{A}} = (d\bar{z}_k)_{|\mathcal{A}}$.

As $Im((H_{|\mathcal{A}})^*)$ is the annihilator in $\mathcal{A}$ of $Ker(H_{|\mathcal{A}})$ and $H_{|\mathcal{A}}$ is nilpotent and 0-deformable, around any point and among $g_1, ..., g_{m-s}$, we may choose functions $\bar{z}_{m-s+1}, ..., \bar{z}_{m-\bar{s}}$, where $s - \bar{s} = dim(Im(H_{|\mathcal{A}}) \cap Ker(H_{|\mathcal{A}}))$, such that $Ker(H_{|\mathcal{A}}) = (Im(H_{|\mathcal{A}}) \cap Ker(H_{|\mathcal{A}})) \oplus Ker((d\bar{z}_1 \wedge ... \wedge d\bar{z}_{m-\bar{s}})_{|\mathcal{A}})$. Now if $\bar{z}_{m-\bar{s}+1}, ..., \bar{z}_m$ are functions such that $Kerd\bar{z}_j \supset Im(H_{|\mathcal{A}})$, $j = m - \bar{s} + 1, ..., m$, and $d\bar{z}_{m-\bar{s}+1} \wedge ... \wedge d\bar{z}_m$ restricted to $Ker((d\bar{z}_1 \wedge ... \wedge d\bar{z}_{m-\bar{s}})_{|\mathcal{A}})$ does not vanish anywhere, then $\bar{z} = (\bar{z}_1, ..., \bar{z}_m)$ is a system of coordinates on $\mathcal{A}$ and $(x, \bar{z})$ a system of coordinates on $\mathcal{F}_1$. By construction $d\bar{z}_j \circ G$, $j = m - s + 1, ..., m$, equals a linear combination with constant coefficients of $d\bar{z}_1, ..., d\bar{z}_{m-s}$ plus a functional combination of $dx_1, ..., dx_n$.

In short, naming z_k every function $\bar{z}_k$ allows us to suppose that in coordinates (x, z)

$$G = \sum_{j=1}^{n} a_j(\partial/\partial x_j) \otimes dx_j + \sum_{j=1}^{m} \sum_{k=1}^{m-s} a_{jk}(\partial/\partial z_j) \otimes dz_k + \sum_{j=m-s+1}^{m} (\partial/\partial z_j) \otimes \beta_j$$

where every $a_{jk} \in \mathbb{K}$, each β_j is a functional combination of $dx_1, ..., dx_n$ and $\{\partial/\partial z_{m-s+1}, ..., \partial/\partial z_m\}$ a basis of $KerG$.

Besides, by linearly rearranging $z_1, ..., z_m$ if necessary, one may suppose that $\{\partial/\partial z_\lambda\}_{\lambda \in L}$, for some subset L of $\{1, ..., m\}$, is a basis of $G(\mathcal{A})$.

But now $N_G(\partial/\partial z_k, \quad) = L_{G(\partial/\partial z_k)}G - G \circ L_{\partial/\partial z_k}G = L_{G(\partial/\partial z_k)}G$ and $\alpha_1 \wedge ... \wedge \alpha_r \wedge N_G = 0$, therefore $\alpha_1 \wedge ... \wedge \alpha_r \wedge (\partial \beta_j/\partial z_\lambda) = 0$, $\lambda \in L$, $j = m - s + 1, ..., m$. Thus considering $G + \sum_{j=1}^{r} X_j \otimes \alpha_j$ instead of G, where $X_1, ..., X_r$ are suitable functional combinations of $\partial/\partial z_{m-s+1}, ..., \partial/\partial z_m$, and calling it G again allows us to suppose $\partial \beta_j/\partial z_\lambda = 0$, $\lambda \in L$, $j = m - s + 1, ..., m$ without loss of generality.

By lemma 1.1, $dx_j \circ N_G = dz_k \circ N_G = 0$, $j = 1, ...n$, $k = 1, ..., m - s$. Therefore one has to study $dz_j \circ N_G$ when $j = m - s + 1, ...m$. Note that each $(\beta_j \circ J^{-1}) \circ N_G = 0$ [here $J^{-1} = \sum_{i=1}^{n} a_i^{-1}(\partial/\partial x_i) \otimes dx_i$], so $dz_j \circ N_G = (dz_j - \beta_j \circ J^{-1}) \circ N_G$, $j = m - s + 1, ...m$, and from lemma 1.1 applied to $dz_j - \beta_j \circ J^{-1}$ and G follows $(d(dz_j - \beta_j \circ J^{-1}))(G, G) + (dz_j - \beta_j \circ J^{-1}) \circ N_G = 0$ since $(dz_j - \beta_j \circ J^{-1}) \circ G$ and $(dz_j - \beta_j \circ J^{-1}) \circ G^2$ equal zero or a linear combination with constant coefficients of $dz_1, ..., dz_{m-s}$.

Hence

$$dz_j \circ N_G = (d(\beta_j \circ J^{-1}))(G, G) = (d_x(\beta_j \circ J^{-1}))(J, J) + \sum_{k=m-s+1}^{m} \beta_k \wedge \left(\frac{\partial(\beta_j \circ J^{-1})}{\partial z_k} \circ J \right) =$$
$$\left(d_x(\beta_j \circ J^{-1}) + \sum_{k=m-s+1}^{m} (\beta_k \circ J^{-1}) \wedge \frac{\partial(\beta_j \circ J^{-1})}{\partial z_k} \right)(J, J)$$

where d_x is the exterior derivative with respect to $x = (x_1, ..., x_n)$ only [recall that $\{\partial/\partial z_\lambda\}_{\lambda\in L}$ is a basis of $G(\mathcal{A})$ and $\partial\beta_j/\partial z_\lambda = 0$].

Therefore the equation $\alpha_1 \wedge ... \wedge \alpha_r \wedge N_G = 0$ is equivalent to the system

$$(1) \quad \begin{cases} \left(d_x(\beta_j \circ J^{-1}) + \displaystyle\sum_{k=m-s+1}^{m} (\beta_k \circ J^{-1}) \wedge \frac{\partial(\beta_j \circ J^{-1})}{\partial z_k} \right) \\ \qquad\qquad\qquad\qquad \wedge(\alpha_1 \circ J^{-1}) \wedge ... \wedge (\alpha_r \circ J^{-1}) = 0 \\ j = m-s+1, ..., m \end{cases}$$

By the same reason if $G' = G + \sum_{j=m-s+1}^{m} (\partial/\partial z_j) \otimes \beta'_j$, where $\beta'_{m-s+1}, ..., \beta'_m$ are functional combinations of $\alpha_1, ..., \alpha_r$ whose coefficient functions do not depend on z_λ, $\lambda \in L$, the equation $N_{G'} = 0$ is equivalent to the system

$$(2) \quad \begin{cases} d_x((\beta_j + \beta'_j) \circ J^{-1}) + \displaystyle\sum_{k=m-s+1}^{m} ((\beta_k + \beta'_j) \circ J^{-1}) \wedge \frac{\partial((\beta_j + \beta'_j) \circ J^{-1})}{\partial z_k} = 0 \\ j = m-s+1, ..., m \end{cases}$$

In other words we need to show that given forms $\beta_{m-s+1} ..., \beta_m$ satisfying system (1), there exist forms $\beta'_{m-s+1} ..., \beta'_m$ such that system (2) is satisfied too.

On N forms $\alpha_1 \circ J^{-1}, ..., \alpha_r \circ J^{-1}$ define a foliation contained in $\mathcal{F}'_1$ since $\alpha_1, ..., \alpha_r, J$ give rise to a Veronese web along $\mathcal{F}'_1$; moreover around every point of N there exist indices $1 \leq k_1 < ... < k_{n-r} \leq n$ such that $dx_{k_1} \wedge ... \wedge dx_{k_{n-r}} \wedge (\alpha_1 \circ J^{-1}) \wedge ... \wedge (\alpha_r \circ J^{-1})$ does not vanish anywhere. As the order of functions $x_1, ..., x_n$ is arbitrary, we may assume $dx_1 \wedge ... \wedge dx_{n-r} \wedge (\alpha_1 \circ J^{-1}) \wedge ... \wedge (\alpha_r \circ J^{-1})$ non-singular and consider coordinates $y = (y_1, ..., y_n)$ along $\mathcal{F}'_1$ such that $y_1 = x_1$,..., $y_{n-r} = x_{n-r}$ and $Ker(dy_{n-r+1} \wedge ... \wedge dy_n) = Ker((\alpha_1 \circ J^{-1}) \wedge ... \wedge (\alpha_r \circ J^{-1}))$; thus $\alpha_1 \circ J^{-1}, ..., \alpha_r \circ J^{-1}$ and each $\beta'_j \circ J^{-1}$ are functional combination of $dy_{n-r+1}, ..., dy_n$; in the first case the coefficients only depend on y and in the second one they do not depend on z_λ, $\lambda \in L$. Moreover one can assume that every $\beta_j \circ J^{-1}$ is only combination of $dy_1, ..., dy_{n-r}$; indeed if $\beta_j \circ J^{-1} = \gamma_j + \rho_j$ with $\gamma_j \wedge dy_1 \wedge ... \wedge dy_{n-r} = 0$ and $\rho_j \wedge dy_{n-r+1} \wedge ... \wedge dy_n = 0$, it suffices replacing G by $G - \sum_{j=m-s+1}^{m} (\partial/\partial z_j) \otimes (\rho_j \circ J)$.

On the other hand, linearly rearranging coordinates z allows us to suppose that $\{1, ..., m-s\} - L = \{1, ..., m'\}$ and $\{m-s+1, ..., m\} - L = \{m-s+1, ..., m-$

$s+s'\}$ where $m' \leq m-s$ and $s' \leq s$ (here $m'=0$ means $\{1,...,m-s\} \subset L$ and $s'=0$ that $\{m-s+1,...,m\} \subset L$). Now on P take a system of coordinates $(v,y,u,w) = (v_1,...,v_{a+m'},y_1,...,y_n,u_1,...,u_s,w_1,...,w_{m-m'-s})$ such that $dv_1 = ... = dv_a = 0$ defines $\mathcal{F}_1$, $v_{a+k} = z_k$, $k=1,...,m'$, $u_j = z_{m-s+j}$, $j=1,...,s$, and $w_k = z_{m'+k}$, $k=1,...,m-m'-s$. Set $\tau_j = \beta_{m-s+j} \circ J^{-1}$, $j=1,...,s$. Since β_{m-s+j} and β'_{m-s+j} do not depend on z_λ, $\lambda \in L$, our problem may be stated in coordinates (v,y,u), that is on a manifold P' of dimension $a+m'+s$ and along the foliation $\mathcal{G}'$ defined by $dv_1 = ... = dv_{a+m'} = 0$, as follows:

Given 1-forms $\tau_j = \sum_{k=1}^{n-r} f_{jk} dy_k$, $j=1,...,s$, where functions f_{jk} do not depend on $u_{s'+1},...,u_s$ such that

$$(3) \qquad d_{(y_1,...,y_{n-r})}\tau_j + \sum_{i=1}^{s} \tau_i \wedge \frac{\partial \tau_j}{\partial u_i} = 0, \ j=1,...,s,$$

find forms $\widetilde{\tau}_j = \tau_j + \sum_{k=n-r+1}^{n} f_{jk} dy_k$, $j=1,...,s$, where each f_{jk} does not depend on $u_{s'+1},...,u_s$ such that

$$(4) \qquad d_y \widetilde{\tau}_j + \sum_{i=1}^{s} \widetilde{\tau}_i \wedge \frac{\partial \widetilde{\tau}_j}{\partial u_i} = 0, \ j=1,...,s,$$

(here $d_{(y_1,...,y_{n-r})}$ and d_y are the exterior derivative in $(y_1,...,y_{n-r})$ or $y = (y_1,...,y_n)$ respectively).

Lemma 2.3. *Forms $\widetilde{\tau}_1,...,\widetilde{\tau}_s$ always exist locally.*

Proof. As a straightforward calculation shows, system (3) is equivalent to say that vector fields $X_k = \partial/\partial y_k + \sum_{j=1}^{s} f_{jk}\partial/\partial u_j$, $k=1,...,n-r$, commute among them.

An analogous statement holds for system (4).

In turn, functions f_{jk} do not depend on $u_{s'+1},...,u_s$ if and only if $X_1,...,X_{n-r}$ commute with vector fields $Y_1,...,Y_{s-s'}$, where each $Y_i = \partial/\partial u_{s'+i}$.

Since by hypothesis $X_1,...,X_{n-r},Y_1,...,Y_{s-s'}$ commute and are linearly independent everywhere, along $\mathcal{G}'$ and around every point, there exist coordinates $\nu_1,...,\nu_{n+s}$ such that $\nu_1 = y_1,...,\nu_n = y_n$, $X_k = \partial/\partial \nu_k$, $k=1,...,n-r$, and $Y_i = \partial/\partial \nu_{n+s'+i}$, $i=1,...,s-s'$.

Set $X_k = \partial/\partial\nu_k$ when $k = n-r+1, ..., n$. Then in coordinates (y, u) one has $X_k = \partial/\partial y_k + \sum_{j=1}^{s} f_{jk}\partial/\partial u_j$, $k = n-r+1, ..., n$. Moreover by construction $X_1, ..., X_n, Y_1, ..., Y_{s-s'}$ commute among them, so forms $\widetilde{\tau}_j = \sum_{k=1}^{n} f_{jk}dy_k = \tau_j + \sum_{k=n-r+1}^{n} f_{jk}dy_k$, $j = 1, ..., s$, satisfy system (4) and functions f_{jk} do not depend on $u_{s'+1}, ..., u_s$. $\square$

Now proposition 2.1 is proved.

The next step will be extending this result to Veronese flags. Therefore let ω, ω_1 be a symplectic form and a closed 2-form, respectively, defined on $\mathcal{A}$. Suppose that $(\mathcal{F}, \ell, \omega, \omega_1)$ is a Veronese flag on P or at some point of P both along $\mathcal{F}_1$ [in the second case by definition condition 3') has to hold on neighbourhoods on P of this point]. Set $dim\mathcal{A} = 2m$ (now the dimension of $\mathcal{A}$ has to be even since ω is symplectic).

Theorem 2.1. *Let $G : \mathcal{F}_1 \to \mathcal{F}_1$ be a morphism which extends ℓ and projects in J. Assume that:*
(a) $\ell\mid_{\mathcal{A}}$ is 0-deformable and its characteristic polynomial equals $(t-a)^{2m}$ where $a \in \mathbb{K}$,
(b) functions $a_1, ..., a_n$ never take the value a,
then around every point of P such that $(\mathcal{F}, \ell, \omega, \omega_1)$ is a Veronese flag at it there exist a morphism $G' : \mathcal{F}_1 \to \mathcal{F}_1$, which extends ℓ and projects in J, and functions $z_1, ..., z_{2m}$ such that $(x_1, ..., x_n, z_1, ..., z_{2m})$ is a system of coordinates along $\mathcal{F}_1$,

$$G' = \sum_{j=1}^{n} a_j(\partial/\partial x_j) \otimes dx_j + \sum_{j,k=1}^{2m} a_{jk}(\partial/\partial z_j) \otimes dz_k,$$

where every $a_{jk} \in \mathbb{K}$, and ω, ω_1 are expressed with constant coefficients relative to $(dz_1)_{|\mathcal{A}}, ..., (dz_{2m})_{|\mathcal{A}}$.

Again, one will prove theorem 2.1 by induction on m. If $m = 0$ the result is obvious; now suppose the theorem true up to $m-1$. Note that we may assume $\ell_{|\mathcal{A}}$ nilpotent by considering $G - aI$, $\ell - aI$ and $\omega_1 - a\omega$ instead of G, ℓ and ω_1. Then $KerG = Ker\omega_1 \subset \mathcal{A}$; so $KerG$ is a foliation since ω_1 is closed. Consider coordinates $(x, z) = (x_1, ..., x_n, z_1, ..., z_{2m})$ along $\mathcal{F}_1$ such that $dx_1 = ... = dx_n = dz_1 = ... = dz_{2(m-s)} = 0$ defines $KerG$, where $dimKerG = 2s$. Reasoning as in the proof of proposition 2.1 allows us to assume G projectable in a tensor field $\bar{G}$, defined on the local quotient $\bar{P}$ of P by $KerG$, and consider the objects $\bar{\mathcal{F}}_1$,

$\bar{\mathcal{F}}$, $\bar{\mathcal{A}}$ and $\bar{\ell}$ with the obvious difference that now $dim\bar{\mathcal{A}}$ is even. Thus $(\bar{\mathcal{F}}, \bar{\mathcal{A}})$ is a weak Veronese flag along $\bar{\mathcal{F}}_1$.

On the other hand ω_1 projects in a symplectic form $\bar{\omega}$ on $\bar{\mathcal{A}}$. By lemma 1.3 the 2-form $\omega_2(X,Y) = \omega(\ell X, Y)$ is closed and $Ker\omega_2 \supset Ker\omega_1 = KerG$; therefore it projects in a closed 2-form $\bar{\omega}_1$ on $\bar{\mathcal{A}}$ such that $\bar{\omega}_1 = \bar{\omega}(\bar{\ell},\quad)$, and $(\bar{\mathcal{F}}, \bar{\ell}, \bar{\omega}, \bar{\omega}_1)$ will be a Veronese flag if we are able to check the third condition of the definition of this notion.

Let h be a function on an open set of $\bar{P}$ such that $\bar{\ell}^* dh$ is a closed form on $\bar{\mathcal{F}}$, that is such that $\alpha_1 \wedge ... \wedge \alpha_r \wedge (d(dh \circ \bar{G})) = 0$. Regarded on P one has $dh(KerG) = 0$ and $\alpha_1 \wedge ... \wedge \alpha_r \wedge (d(dh \circ G)) = 0$. In particular, locally and along $\mathcal{F}_1$, $dh = \beta \circ G$ for some 1-form β and, by lemma 1.1, one has $d\beta(G,G) + d(dh \circ G) + \beta \circ N_G = 0$. Hence, as $\alpha_1 \wedge ... \wedge \alpha_r \wedge N_G = \alpha_1 \wedge ... \wedge \alpha_r \wedge (d(dh \circ G)) = 0$, results $\alpha_1 \wedge ... \wedge \alpha_r \wedge d\beta(G,G) = 0$, that is $(\alpha_1 \circ J^{-1}) \wedge ... \wedge (\alpha_r \circ J^{-1}) \wedge d\beta = 0$.

But $\alpha_1 \circ J^{-1}, ..., \alpha_r \circ J^{-1}$ define a foliation, therefore $\beta = dg$ modulo $\alpha_1 \circ J^{-1}, ..., \alpha_r \circ J^{-1}$ for some function g. Thus $\alpha_1 \wedge ... \wedge \alpha_r \wedge (dg \circ G - dh) = 0$, whence $\alpha_1 \wedge ... \wedge \alpha_r \wedge (d(dg \circ G)) = 0$; in other words $\ell^* dg$ is closed on $\mathcal{F}$.

Let X be the ω-hamiltonian of g. From $\omega_1(X, \quad) = \omega(GX, \quad) = -(dg \circ G)_{|\mathcal{A}} = -dh_{|\mathcal{A}}$ follows that the projection $\bar{X}$ of X on$\bar{P}$ is the $\bar{\omega}$-hamiltonian of h. But $L_X \ell = 0$ since $(\mathcal{F}, \ell, \omega, \omega_1)$ is a Veronese flag, so $L_{\bar{X}} \bar{\ell} = 0$; that is to say $(\bar{\mathcal{F}}, \bar{\ell}, \bar{\omega}, \bar{\omega}_1)$ is a Veronese flag too (everywhere or at some point).

By the induction hypothesis, there exist a morphism $\bar{G}' : \bar{\mathcal{F}}_1 \to \bar{\mathcal{F}}_1$ extending $\bar{\ell}$ and projecting in J and functions $z_1, ..., z_{2(m-s)}$, such that $(x_1, ..., x_n, z_1, ..., z_{2(m-s)})$ is a system of coordinates along $\bar{\mathcal{F}}_1$ in which $\bar{G}'$, $\bar{\omega}$ and $\bar{\omega}_1$ are written with constant coefficients. But $\bar{G}' - \bar{G} = \sum_{j=1}^r \bar{X}_j \otimes \alpha_j$ where $\bar{X}_1, ..., \bar{X}_r$ are tangent to $\bar{\mathcal{A}}$. Therefore considering $G + \sum_{j=1}^r \bar{X}_j \otimes \alpha_j$ instead of G, where $X_1, ..., X_r$ are tangent to $\mathcal{A}$ and project in $\bar{X}_1, ..., \bar{X}_r$, allows us to suppose that G projects in $\bar{G}'$; that is to say $\bar{G}' = \bar{G}$.

On the other hand, proceeding as in the proof of proposition 2.1 shows the existence of vector fields $\widetilde{X}_1, ..., \widetilde{X}_r$, tangent to $KerG$, such that the Nijenhuis torsion of $G + \sum_{j=1}^r \widetilde{X}_j \otimes \alpha_j$ vanishes; in other words one may assume $N_G = 0$. Indeed, see $(\bar{\mathcal{F}}, \bar{\ell})$, $(\mathcal{F}, \ell)$ as weak Veronese flags and $\bar{G}$, G like suitable prolongations of $\bar{\ell}$, ℓ respectively.

In short, only case to consider: in coordinates $(x_1, ..., x_n, z_1, ..., z_{2(m-s)})$ $\bar{G}$,

$\bar\omega$, $\bar\omega_1$ are written with constant coefficients following theorem 2.1 and $N_G = 0$.

Regarded like function on P every $dz_k \circ G$, $k = 1, ..., 2(m-s)$, is a linear combination with constant coefficients of $dz_1, ...dz_{2(m-s)}$; moreover $dx_j \circ G = a_j dx_j$, $j = 1, ..., n$. By lemma 2.2 there exist functions $g_1, ..., g_{2(m-s)}$ such that $dg_k \circ G = dz_k$, $k = 1, ..., 2(m-s)$.

Since $G_{|\mathcal{A}} = \ell_{|\mathcal{A}}$ is 0-deformable and nilpotent, around any point and among $g_1, ..., g_{2(m-s)}$, one can choose functions $\bar z_{2(m-s)+1}, ..., \bar z_{2(m-\bar s)}$, where $2(s-\bar s) = dim(G(\mathcal{A}) \cap KerG)$, such that

$$KerG = (G(\mathcal{A}) \cap KerG) \oplus Ker((dz_1 \wedge ... \wedge dz_{2(m-s)} \wedge d\bar z_{2(m-s)+1} \wedge ... \wedge d\bar z_{2(m-\bar s)})_{|\mathcal{A}}).$$

Now if $\bar z_{2(m-\bar s)+1}, ..., \bar z_{2m}$ are functions such that $Kerd\bar z_j \supset ImG$, $j = 2(m-\bar s)+1, ..., 2m$, and $d\bar z_{2(m-\bar s)+1} \wedge ... \wedge d\bar z_{2m}$ restricted to

$$Ker((dz_1 \wedge ... \wedge dz_{2(m-s)} \wedge d\bar z_{2(m-s)+1} \wedge ... \wedge d\bar z_{2(m-\bar s)})_{|\mathcal{A}})$$

does not vanish anywhere, then $(x_1, ...x_n, z_1, ..., z_{2(m-s)}, \bar z_{2(m-s)+1}, ..., \bar z_{2m})$ is a system of coordinates along $\mathcal{F}_1$. By construction $d\bar z_j \circ G$, $j = 2(m-s)+1, ..., 2m$, equals a linear combination with constant coefficients of $dz_1, ..., dz_{2(m-s)}$. Thus

$$G = \sum_{j=1}^{n} a_j(\partial/\partial x_j) \otimes dx_j + \sum_{j=1}^{2m} \sum_{k=1}^{2(m-s)} Z_{jk} \otimes dz_k$$

where each Z_{jk} is a linear combination with constant coefficients of

$$\partial/\partial z_1, ..., \partial/\partial z_{2(m-s)}, \partial/\partial \bar z_{2(m-s)+1}, ..., \partial/\partial \bar z_{2m}.$$

Moreover, in these coordinates, ω_1 and ω_2 are written with constant coefficients since $\bar\omega$ and $\bar\omega_1$ are in coordinates $(x_1, ..., x_n, z_1, ..., z_{2(m-s)})$.

Let X_k be the ω-hamiltonian of z_k, $k = 1, ...2(m-s)$, or $\bar z_k$, $k = 2(m-s)+1, ...2m$. Then $\omega_1(X_k, \quad) = \omega(GX_k, \quad)$ equals $-dz_k \circ G$ or $-d\bar z_k \circ G$; in both cases a linear combination with constant coefficients of $dz_1, ..., dz_{2(m-s)}$ because ω_1 projects in $\bar\omega$. In other words

$$X_k = \sum_{i=2(m-s)+1}^{2m} f_{ki}\partial/\partial \bar z_i + \sum_{j=1}^{2(m-s)} b_{kj}\partial/\partial z_j$$

where each $b_{kj} \in \mathbb{K}$.

In particular $\{z_j, z_k\}_\omega$, $j, k = 1, ..., 2(m-s)$, and $\{z_j, \bar z_k\}_\omega$, $j = 1, ..., 2(m-s)$, $k = 2(m-s)+1, ..., 2m$, are constant [here $\{\quad,\quad\}_\omega$ and Λ_ω are respectively the Poisson structure and the dual bivector on $\mathcal{A}$ associated to ω].

On the other hand, everywhere or close to some point, $L_{X_k}\ell = 0$ since $dz_k \circ G$, or $d\bar z_k \circ G$, is closed. Hence $\alpha_1 \wedge ... \wedge \alpha_r \wedge L_{X_k} G = 0$. A straightforward calculation shows that $L_{X_k} G = -\sum_{i=2(m-s)+1}^{2m} (\partial/\partial \bar z_i) \otimes (df_{ki} \circ G)$, so $\alpha_1 \wedge ... \wedge \alpha_r \wedge (d_x f_{ki} \circ J) = 0$ where d_x is the exterior derivative with respect to $x = (x_1, ..., x_n)$;

that is $(\alpha_1 \circ J^{-1}) \wedge ... \wedge (\alpha_r \circ J^{-1}) \wedge d_x f_{ki} = 0$, $i = 2(m-s)+1, ..., 2m$. In other words functions f_{ki} are basic for the foliation $\mathcal{G}'' \subset \mathcal{F}_1$ defined by $\alpha_1 \circ J^{-1}, ..., \alpha_r \circ J^{-1}, dz_1, ..., dz_{2(m-s)}, d\bar{z}_{2(m-s)+1}, ..., d\bar{z}_{2m}$.

But $\{\bar{z}_k, \bar{z}_i\}_\omega = f_{ki}$, therefore Λ_ω and by consequence ω are written with coefficients which are $\mathcal{G}''$-basic functions.

Lemma 2.3. *Along a foliation $\widetilde{\mathcal{F}}$ of dimension $2\widetilde{m}$ defined on a manifold $\widetilde{P}$, consider a symplectic form λ and functions $f_1, ..., f_k$ such that $df_1 \wedge ... \wedge df_k$ has no zeros. Assume constant every function $\{f_i, f_j\}$, $i, j = 1, ..., k$. Then locally there are functions $g_1, ..., g_{2\widetilde{m}-k}$ such that $(f_1, ..., f_k, g_1, ..., g_{2\widetilde{m}-k})$ is a system of coordinates along $\widetilde{\mathcal{F}}$ and λ is written with constant coefficients relative to it.*

Proof. It is just one of the version of Darboux theorem. $\square$

Pulling-back the functions given by lemma 2.3, applied to the projections on the local quotient $P/\mathcal{G}''$ of $\mathcal{A}$, $z_1, ..., z_{2(m-s)}$ and ω, yields $\mathcal{G}''$-basic functions $g_1, ..., g_{2s}$ such that $(x_1, ..., x_n, z_1, ..., z_{2(m-s)}, g_1, ..., g_{2s})$ is a system of coordinates along $\mathcal{F}_1$. In this system ω and ω_1 are written with constant coefficients [recall that ω_1 is a constant linear combination of $dz_j \wedge dz_k$, $1 \leq j < k \leq 2(m-s)$]; by consequence the restriction of G to $\mathcal{A}$ is written with constant coefficients too and every $(dg_i \circ G)_{|\mathcal{A}}$ is a constant linear combination of $dz_{1|\mathcal{A}}, ..., dz_{2(m-s)|\mathcal{A}}$. Therefore in coordinates $(x_1, ..., x_n, z_1, ..., z_{2(m-s)}, \bar{z}_{2(m-s)+1}, ..., \bar{z}_{2m})$ each $dg_i \circ G$ equals $d_x g_i \circ J$ plus a constant linear combination of $dz_1, ..., dz_{2(m-s)}$.

But g_i is $\mathcal{G}''$-basic, so $d_x g_i \circ J$ is a functional combination of $\alpha_1, ..., \alpha_r$. Thus in coordinates $(x, z) = (x_1, ..., x_n, z_1, ..., z_{2m})$ where $z_{2(m-s)+i} = g_i$, $i = 1, ..., 2s$, one has:

$$G = \textstyle\sum_{j=1}^{n} a_j (\partial/\partial x_j) \otimes dx_j + \sum_{j=1}^{2m} \sum_{k=1}^{2(m-s)} c_{jk} (\partial/\partial z_j) \otimes dz_k$$
$$+ \textstyle\sum_{i=1}^{2s} (\partial/\partial z_{2(m-s)+i}) \otimes \beta_i$$

where every $c_{jk} \in \mathbb{K}$ and $\alpha_1 \wedge ... \wedge \alpha_r \wedge \beta_i = 0$, $i = 1, ..., 2s$.

Now it suffices to set $G' = G - \sum_{i=1}^{2s} (\partial/\partial z_{2(m-s)+i}) \otimes \beta_i$ *for finishing the proof of theorem 2.1.*

3. The case of an eigenvalue function

In the foregoing section one has studied Veronese flags with parameters when $\ell_{|\mathcal{A}}$ is 0-deformable and nilpotent (theorem 2.1). Here we will consider Veronese flags for more general tensor field $\ell_{|\mathcal{A}}$, which will be the main tool for establishing the splitting theorem of bihamiltonian structures.

One starts introducing the notion of regular open set. Let $\mathbb{K}_P[t]$ be the polynomial algebra in one variable over the ring of differentiable functions on a manifold P. A polynomial $\varphi \in \mathbb{K}_P[t]$ is said *irreducible* if it is irreducible at every point of P. Two polynomials $\varphi, \psi \in \mathbb{K}_P[t]$ are called *relatively prime* if they are at each point. Given a vector bundle E over P, of dimension $\widetilde{m}$, and a morphism $H : E \to E$ its characteristic polynomial $\varphi = \sum_{j=0}^{\widetilde{m}} h_j t^j$ belongs to $\mathbb{K}_P[t]$. Set $g_j = trace(H^j)$. Since $h_0, ..., h_{\widetilde{m}-1}$ are, up to sign, the elementary symmetric polynomials of the roots and each g_j the sum of their j-th powers, every function g_j may be expressed as a rational polynomial of $h_0, ..., h_{\widetilde{m}-1}$, and each function h_j like a rational polynomial of $g_1, ..., g_{\widetilde{m}}$. In particular g_j when $j \geq \widetilde{m}+1$ equals a rational polynomial of $g_1, ..., g_{\widetilde{m}}$.

One will say that $H : E \to E$ *has constant algebraic type* if there exist relatively prime irreducible polynomials $\varphi_1, ..., \varphi_s \in \mathbb{K}_P[t]$ and positive integers a_{jk}, $j = 1, ..., r_k$, $k = 1, ..., s$, such that at each point $p \in P$ the family $\{\varphi_k^{a_{jk}}(p)\}$, $j = 1, ..., r_k$, $k = 1, ..., s$, is that of elementary divisors of $H(p)$. Let $f_1, ..., f_{\widetilde{n}}$ be the family of all significant coefficient functions of $\varphi_1, ..., \varphi_s \in \mathbb{K}_P[t]$; that is f when $\varphi_k = t + f$ and f, g if $\varphi_k = t^2 + ft + g$. Obviously $h_0, ..., h_{\widetilde{m}-1}$, and by consequence each g_j, are rational polynomials of $f_1, ..., f_{\widetilde{n}}$. Conversely, for every point of P there exist analytic functions $\lambda_k(u_1, ..., u_{\widetilde{n}})$ such that close to this point $f_k = \lambda_k(g_1, ..., g_{\widetilde{n}})$, $k = 1, ..., \widetilde{n}$ (note that $\widetilde{n} \leq \widetilde{m}$). Indeed, assume the degree of every φ_k equals one (otherwise complexify E and H); then $\widetilde{n} = s$ and it suffices to remark that the polynomial map $F : \mathbb{K}^s \to \mathbb{K}^s$, defined by $F(z) = (\sum_{k=1}^s b_k z_k, \sum_{k=1}^s b_k z_k^2, ..., \sum_{k=1}^s b_k z_k^s)$ where each $b_k = \sum_{j=1}^{r_k} a_{jk}$, is a local diffeomorphism on the open set $\{z \in \mathbb{K}^s \mid z_j \neq z_k \text{if} j \neq k\}$ since the determinant of its Jacobian matrix equals $c \prod_{1 \leq j < k \leq j}(z_j - z_k)$ with $c \in \mathbb{K} - \{0\}$.

Let B_H be the set of all points such that around them H has constant algebraic type.

Lemma 3.1. *The set B_H is open and dense.*

Proof. One may suppose $\mathbb{K} = \mathbb{C}$ by complexifying E and H if necessary. Given $p \in P$ let a be a root of $\varphi(p)$ of multiplicity b. Then if $\varepsilon > 0$ is small enough and q close to p, the sum of multiplicities of the roots of $\varphi(q)$ belonging to the disk $D_\varepsilon(a)$ equals b; indeed, this sum is the degree of the map $e^{i\theta} \in S^1 \to \varphi(q)(a+\varepsilon e^{i\tau}) \parallel \varphi(q)(a+\varepsilon e^{i\theta}) \parallel^{-1} \in S^1$. Therefore the number of different roots of φ is locally constant on a dense open set P' of P.

Now assume $p \in P'$. Then there exist $\varepsilon > 0$ and an open set $p \in B \subset P'$ such that $\varphi(q)$, $q \in B$, has just one root on $D_\varepsilon(a)$ and its multiplicity is b. Let λ be the $(b-1)$-derivative of φ with respect to t. Then $(\partial\lambda/\partial t)(p,a) \neq 0$ and by the implicit function theorem applied to λ and $0 \in \mathbb{C}$, shrinking B if necessary, there is a differentiable (holomorphic or C^∞) function $f : B \to \mathbb{C}$ such that $-f(q)$, $q \in B$, is the root of $\varphi(q)$ on $D_\varepsilon(a)$. Thus $\varphi = \prod_{k=1}^{s}(t+f_k)^{b_k}$ around p where $f_1, ..., f_s$ are differentiable functions, $b_1, ..., b_s$ integers ≥ 1 and $\prod_{1\leq j<k\leq s}(f_j - f_k)$ never vanishes.

Finally, remark that the functions $dimKer((H + f_k I)^j)$ are locally decreasing, so locally constant on a dense open set $B' \subset B$. $\square$

Suppose that E is a foliation and $N_H = 0$; then $jdg_{j+1} = (j+1)dg_j \circ H$. Indeed, consider (E, H) as a weak Veronese flag (if $codimE = 0$ regard the problem on $\mathbb{K} \times P$ in the obvious way) and apply lemma 1.2. Therefore $\cap_{j=1}^{\widetilde{m}} Kerdg_j(p) = \cap_{j=0}^{\widetilde{m}-1} Kerdh_j(p)$ is a H-invariant vector subspace of T_pP because each g_j, $j \geq \widetilde{m}+1$, is a function of $g_1, ..., g_{\widetilde{m}}$.

One will be say that a point $p \in P$ is *regular* if there exists an open neighbourhood B of p such that:

(1) H has constant algebraic type on B,

(2) $\cap_{j=1}^{\widetilde{m}} Kerdg_j$, restricted to B, is a vector sub-bundle of E and therefore a foliation,

(3) H restricted to $\cap_{j=1}^{\widetilde{m}} Kerdg_j$ has constant algebraic type on B.

By lemma 3.1, applied to H and its restriction to $\cap_{j=1}^{\widetilde{m}} Kerdg_j$, the set of all regular points is a dense open set P, called the *regular open set*.

To remark that if H has constant algebraic type on an open set D, then $\cap_{j=1}^{\widetilde{m}} Kerdg_j = \cap_{j=1}^{\widetilde{n}} Kerdf_j$ on it where $f_1, ..., f_{\widetilde{n}}$ are the significant coefficient functions of $\varphi_1, ..., \varphi_s$.

Consider a Veronese flag $(\mathcal{F}, \ell, \omega, \omega_1)$ on a manifold P or at some point of P. Let $\mathcal{A}$ be the foliation of the largest ℓ-invariant vector subspace (as in section 1) and $\pi : P \to N$ a local quotient of P by $\mathcal{A}$. Set $codim\mathcal{F} = r$, $dim\mathcal{A} = 2m$ and

$dimN = n$. Then N is endowed with a r-codimensional Veronese web whose limit when $t \to \infty$ equals the quotient foliation $\mathcal{F}' = \mathcal{F}/\mathcal{A}$; moreover ℓ projects in the morphism ℓ' associated to this Veronese web.

Suppose that $\varphi = (t-f)^{2m}$ is the characteristic polynomial of $\ell_{|\mathcal{A}}$; then from lemma 1.2 follows $df \circ \ell = fdf$ on $\mathcal{F}$. Now assume that $df_{|\mathcal{A}}$ never vanishes. Let X_f be the ω-hamiltonian of f; then $L_{X_f}\ell = 0$ and $\ell(X_f) = fX_f$ since $df \circ \ell = fdf$ on $\mathcal{F}$. Denoted by $\bar{P}$ and $\bar{\pi} : P \to \bar{P}$, respectively, the local quotient of P by X_f and its canonical projection. Consider coordinates $(y, z) = (y_1, ..., y_n, z_1, ..., z_{2m})$ on P such that $dy_1 = ... = dy_r = 0$ defines $\mathcal{F}$, $dy_1 = ... = dy_n = 0$ the foliation $\mathcal{A}$, $f = z_{2m}$ and $X_f = -\partial/\partial z_{2m-1}$. Thus $(y_1, ..., y_n)$ can be regarded as coordinates on N, $(y_1, ..., y_n, z_1, ..., z_{2m-2}, z_{2m})$ as coordinates on $\bar{P}$ and f as a function on this last manifold. Now it is obvious that $Kerdf$ and $\mathcal{F} \cap Kerdf$ project in two foliations $\bar{\mathcal{F}}_1$ and $\bar{\mathcal{F}}$ on $\bar{P}$, respectively, and $\ell_{|\mathcal{F}\cap Kerdf}$ projects in a morphism $\bar{\ell} : \bar{\mathcal{F}} \to \bar{\mathcal{F}}_1$; moreover $(\bar{\mathcal{F}}, \bar{\ell})$ is a weak Veronese flag along $\bar{\mathcal{F}}_1$ (locally any extension of $\bar{\ell}$ can be lifted to an extension of ℓ), whose foliation $\bar{\mathcal{A}}$ of the largest $\bar{\ell}$-invariant vector subspaces equals the projection of $\mathcal{A} \cap Kerdf$, $\bar{P}/\bar{\mathcal{A}}$ is identified to $N \times B$, where B is an open neighbourhood of $f(p)$ on $\mathbb{K}$, and $\bar{\mathcal{F}}_1$ projects in the foliation of $N \times B$ by the first factor. Moreover, the Veronese web induced by $(\bar{\mathcal{F}}, \bar{\ell})$ on each leaf $N \times \{b\}$ of this last foliation equals the pull-back, by the first projection $\pi_1 : N \times B \to N$, of that induced by $(\mathcal{F}, \ell)$.

On the other hand, since $i_{X_f}\omega = -df_{|\mathcal{A}}$ and $i_{X_f}\omega_1 = -(df \circ \ell)_{|\mathcal{A}} = -fdf_{|\mathcal{A}}$, the vector field X_f belongs to $Ker(\omega_{|\mathcal{A}\cap Kerdf})$ and $Ker(\omega_{1|\mathcal{A}\cap Kerdf})$, so $\omega_{|\mathcal{A}\cap Kerdf}$ projects in a symplectic form $\bar{\omega}$ and $\omega_{1|\mathcal{A}\cap Kerdf}$ in a closed 2-form $\bar{\omega}_1$, both on $\bar{\mathcal{A}}$; besides $\bar{\omega}_1 = \bar{\omega}(\bar{\ell}, \quad)$. The family $(\bar{\mathcal{F}}, \bar{\ell}, \bar{\omega}, \bar{\omega}_1)$ will be called the *symplectic reduction of* $(\mathcal{F}, \ell, \omega, \omega_1)$. For proving that this family is a Veronese flag it suffices to check the third condition of the definition.

On N consider coordinates $(x_1, ..., x_n)$ and a $(1,1)$-tensor field $J = \sum_{j=1}^{n} a_j(\partial/\partial x_j) \otimes dx_j$ where $a_1, ..., a_n$ are scalars.

Theorem 3.1. *Let G be a $(1,1)$-tensor field, which extends ℓ and projects in J, defined around a point p of P such that $(\mathcal{F}, \ell, \omega, \omega_1)$ is a Veronese flag at this point. Assume that:*
(a) the characteristic polynomial of $\ell_{|\mathcal{A}}$ equals $(t-f)^{2m}$ where $df_{|\mathcal{A}}$ never van-

ishes,

(b) the function f does not take the values $a_1, ..., a_n$,

(c) p is a regular point of $\ell_{|\mathcal{A}}$,

(d) the symplectic reduction of $(\mathcal{F}, \ell, \omega, \omega_1)$ is a Veronese flag at $\bar{\pi}(p)$,

then around p there exist a $(1,1)$-tensor field G' extending ℓ and projecting in J and functions $z_1, ..., z_{2m}$ such that $(x, z) = (x_1, ..., x_n, z_1, ..., z_{2m})$ is a system of coordinates,

$$G' = \sum_{j=1}^{n} a_j(\partial/\partial x_j) \otimes dx_j + \sum_{j,k=1}^{2m} h_{jk}(z)(\partial/\partial z_j) \otimes dz_k$$

and ω, ω_1 are expressed relative to $dz_{1|\mathcal{A}}, ..., dz_{2m|\mathcal{A}}$ with coefficient functions only depending on z.

Lets us prove theorem 3.1. Consider closed 1-forms $\alpha_1, ..., \alpha_r$ defining $\mathcal{F}$; by modifying the order of variables $x_1, ..., x_n$ if necessary one may suppose that $dx_1 \wedge ... \wedge dx_{n-r} \wedge \alpha_1 \wedge ... \wedge \alpha_{n-r}$ has no zeros. Since $df \circ \ell = f df$ on $\mathcal{F}$, one has $df \circ G = f df + \sum_{j=1}^{r} h_j \alpha_j$. Now consider a vector field $Y \in \mathcal{A}$ such that $Yf = 1$ and set $G_1 = G - Y \otimes (\sum_{j=1}^{r} h_j \alpha_j)$; then $df \circ G_1 = f df$, which allows us to assume $df \circ G = f df$ by considering G_1 instead of G and calling it G. On the other hand from $d(df \circ G) = 0$ follows $L_{X_f} \ell = 0$, that is $\alpha_1 \wedge ... \wedge \alpha_r \wedge L_{X_f} G = 0$, whence $L_{X_f} G = \sum_{j=1}^{r} X_j \otimes \alpha_j$; moreover $X_1, ... X_r \in \mathcal{A} \cap Ker df$. Indeed, $0 = L_{X_f}(f df) = L_{X_f}(df \circ G) = df \circ L_{X_f} G$ and the projection on N of $L_{X_f} G$ vanishes since G projects in J and X_f in zero.

Around p there exist vector fields $Y_1, ... Y_r \in \mathcal{A} \cap Ker df$ such that $[X_f, Y_j] = -X_j$, $j = 1, ..., r$. Then $L_{X_f}(G + \sum_{j=1}^{r} Y_j \otimes \alpha_j) = 0$ and, by the same reason as before, we can suppose $df \circ G = f df$ and $L_{X_f} G = 0$. Thus $G_{|Ker df}$ projects in a $(1,1)$-tensor field $\bar{G}$ defined along $\bar{\mathcal{F}}_1$, which extend $\bar{\ell}$ and projects in J (regarded along the foliation of $N \times B$ by the first factor in the obvious way).

On the other hand $(\bar{\mathcal{F}}, \bar{\ell} - fI, \bar{\omega}, \bar{\omega}_1 - f\bar{\omega})$ is a Veronese flag along $\bar{\mathcal{F}}_1$ because f is basic for this last foliation. Moreover near $\bar{\pi}(p)$ the tensor field $(\bar{\ell} - fI)_{|\bar{\mathcal{A}}}$ is nilpotent (obvious!) and 0-deformable. Indeed, p regular implies the existence of positive integers $k_1, ..., k_s$ such that, around this point, $\{t^{k_j}, t^{k_j}\}$, $j = 1, ..., s$, is the family of elementary divisors of $(\ell - fI)_{|\mathcal{A}}$ [recall that every elementary divisor of $\ell_{|\mathcal{A}}$ occurs an even number of times because $\omega_1 = \omega(\ell, \quad)$] whereas $\{\{t^{k_j}, t^{k_j}\}_{j=1,...,s-1}, t^{k_s}, t^{k_s - 1}\}$ is that of $(\ell - fI)_{|\mathcal{A} \cap Ker df}$; now a straightforward calculation shows that, close to $\bar{\pi}(p)$, the family of elementary divisors of $(\bar{\ell} -$

$fI)_{|\bar{\mathcal{A}}}$ has to be $\{\{t^{k_j}, t^{k_j}\}_{j=1,\dots,s-1}, t^{k_s-1}, t^{k_s-1}\}$.

Thus theorem 2.1 applied to $\bar{G} - fI$, $(\bar{\mathcal{F}}, \bar{\ell} - fI, \bar{\omega}, \bar{\omega}_1 - f\bar{\omega})$ and $\sum_{j=1}^{n}(a_j - f)(\partial/\partial x_j) \otimes dx_j$ yields coordinates $(x_1, ..., x_n, z_1, ..., z_{2m-2})$ along $\bar{\mathcal{F}}_1$, which become coordinates coordinates $(x_1, ..., x_n, z_1, ..., z_{2m-2}, z_{2m})$ on $\bar{P}$ by setting $z_{2m} = f$, such that $dz_{2m} = 0$ defines $\bar{\mathcal{F}}_1$,

$$\bar{G} - z_{2m}I = \sum_{j=1}^{n}(a_j - z_{2m})(\partial/\partial x_j) \otimes dx_j + \sum_{j,k=1}^{2m-2} a_{jk}(\partial/\partial z_j) \otimes dz_k + \sum_{j=1}^{r} \bar{X}_j \otimes \alpha_j$$

where $\bar{X}_1, ...\bar{X}_r \in \bar{\mathcal{A}}$ and each $a_{jk} \in \mathbb{K}$, and $\bar{\omega}, \bar{\omega}_1 - z_{2m}\bar{\omega}$ are written with constant coefficients. Moreover, considering $G - \sum_{j=1}^{r} X_j \otimes \alpha_j$ instead of G where each $X_j \in \mathcal{A} \cap Kerdf$, commutes with X_f and projects in $\bar{X}_j$, and linearly rearranging coordinates $(z_1, ..., z_{2m-2})$ allows us to suppose $\bar{\omega} = (\sum_{k=1}^{m-1} dz_{2k-1} \wedge dz_{2k})_{|\bar{\mathcal{A}}}$ and

$$\bar{G} - z_{2m}I = \sum_{j=1}^{n}(a_j - z_{2m})(\partial/\partial x_j) \otimes dx_j + \sum_{j,k=1}^{2m-2} a_{jk}(\partial/\partial z_j) \otimes dz_k.$$

If we regard $z_1, ..., z_{2m-2}, z_{2m}$ as functions on P one has $\omega = (\sum_{k=1}^{m-1} dz_{2k-1} \wedge dz_{2k} + \mu \wedge dz_{2m})_{|\mathcal{A}}$. But ω is closed so $(d\mu \wedge dz_{2m})_{|\mathcal{A}} = 0$ and one may choose a function z_{2m-1} such that $dz_{2m-1} \wedge dz_{2m}$ equals $\mu \wedge dz_{2m}$ on $\mathcal{A}$; that is to say $\omega = (\sum_{k=1}^{m} dz_{2k-1} \wedge dz_{2k})_{|\mathcal{A}}$ and $(x_1, ..., x_n, z_1, ..., z_{2m})$ is a system of coordinates around p. Now $f = z_{2m}$, $X_f = -\partial/\partial z_{2m-1}$ and $G = \sum_{j=1}^{n} a_j(\partial/\partial x_j) \otimes dx_j + (\partial/\partial z_{2m-1}) \otimes \tau + T$ where τ is a functional combination of $dx_1, ..., dx_n$ and T of $(\partial/\partial z_j) \otimes dz_k$, $j, k = 1, ..., 2m$; moreover the coefficient function of these combinations do not depend on z_{2m-1} since $L_{X_f}G = 0$.

On the other hand $\omega_1 = z_{2m}\omega + \Omega_1 + \beta \wedge (dz_{2m|\mathcal{A}})$ where Ω_1 is a constant linear combination of $(dz_j \wedge dz_k)_{|\mathcal{A}}$, $1 \leq j < k \leq 2m-2$, whose restriction to $Kerdf$ projects in $\bar{\omega}_1 - f\bar{\omega}$, and β a functional combination of $dz_{1|\mathcal{A}}, ..., dz_{2m-2|\mathcal{A}}$ whose coefficient functions do not depend on z_{2m-1} (recall that $i_{X_f}\omega_1 = -(df \circ \ell)_{|\mathcal{A}} = -fdf_{|\mathcal{A}}$; in particular $L_{X_f}\omega_1 = 0$). Therefore as $\omega_1 = \omega(\ell, \quad) = \omega(G, \quad) = \omega(T, \quad)$ one has:

$$T = z_{2m}I_z + H + (\partial/\partial z_{2m-1}) \otimes \beta^* - Z \otimes dz_{2m}$$

where $I_z = \sum_{k=1}^{2m}(\partial/\partial z_k) \otimes dz_k$, Z is is the vector field functional combination of $\partial/\partial z_1, ..., \partial/\partial z_{2m-2}$ defined by the equation $\omega(Z, \quad) = \beta$, β^* the extension of β to TP such that $\beta^*(\partial/\partial x_j) = 0$, $j = 1, ..., n$, and H the constant linear combination of $(\partial/\partial z_j) \otimes dz_k$, $j, k = 1, ..., 2m-2$, satisfying the relation $\omega(H, \quad) = \Omega_1$.

Set $\Omega = \omega - (dz_{2m-1} \wedge dz_{2m})_{|\mathcal{A}}$. Then $\Omega_1 = \Omega(H, \quad)$ and $i_Z\Omega = \beta$.

Note that any tensor field on P, partial or not, without terms including $\partial/\partial z_{2m-1}$, $\partial/\partial z_{2m}$, dz_{2m-1} or dz_{2m}, whose coefficients functions do not depend on z_{2m-1}, can be regarded as a tensor field along the foliation $Kerdz_{2m}$ on $\bar{P}$ by projecting, via $\bar{\pi}$, its restriction to $Kerdz_{2m}$. In coordinates (x, z) this is equivalent to delete coordinate z_{2m-1} and consider the tensor field along the foliation $dz_{2m} = 0$. For the sake of simplicity both tensor fields will be denoted by the same letter. Among others that will be the case of $Z, H, \Omega, \Omega_1, \beta$ already defined and $\widetilde{Z}, \widetilde{\beta}$ defined later on.

Until the end of the proof of theorem 3.1 and for making calculations easier, D will denote the exterior derivative with respect of variables $(z_1, ..., z_{2m-2})$, along $\mathcal{A}$ on P or $\bar{\mathcal{A}}$ on $\bar{P}$, and $\mathcal{L}$ the Lie derivative on $\bar{P}$. As $\omega_1 = z_{2m}(\Omega + (dz_{2m-1} \wedge dz_{2m})_{|\mathcal{A}}) + \Omega_1 + \beta \wedge (dz_{2m|\mathcal{A}})$ is closed one has $D\beta = \mathcal{L}_Z\Omega = -\Omega$.

On the other hand $N_G(\partial/\partial z_{2m}, \quad) = L_{G(\partial/\partial z_{2m})}G - G \circ L_{(\partial/\partial z_{2m})}G = -L_Z G - H + (\partial/\partial z_{2m-1}) \otimes \lambda + Z' \otimes dz_{2m}$ with $\lambda \wedge dx_1 \wedge ... \wedge dx_n \wedge dz_1 \wedge ... \wedge dz_{2m-2} = 0$ and $Z' \wedge (\partial/\partial z_1) \wedge ... \wedge (\partial/\partial z_{2m-2}) = 0$. Now from $\alpha_1 \wedge ... \wedge \alpha_r \wedge N_G = 0$ follows $L_Z G = -H + \sum_{j=1}^{r} Y_j \otimes \alpha_j + (\partial/\partial z_{2m-1}) \otimes \lambda + Z' \otimes dz_{2m}$, where $Y_1, ..., Y_r \in \mathcal{A}$ because G projects in J on N and Z in zero.

Hence projecting on $\bar{P}$, that is to say considering variables $(x_1, ..., x_n, z_1, ..., z_{2m-2})$ and parameter z_{2m}, yields $\mathcal{L}_Z\bar{G} = \mathcal{L}_Z H = -H + \sum_{j=1}^{r} \bar{Y}_j \otimes \alpha_j$ where $\bar{Y}_1, ..., \bar{Y}_r \in \bar{\mathcal{A}}$, which is the foliation defined by $dx_1 = ...dx_n = dz_{2m} = 0$.

Set $\widetilde{G} = \bar{G} - z_{2m}I = \sum_{j=1}^{n}(a_j - z_{2m})(\partial/\partial x_j \otimes dx_j) + H$. Then on $\bar{P}$ one has $\Omega(Z, \quad) = \beta$, $\Omega_1 = \Omega(\widetilde{G}, \quad) = \Omega(H, \quad)$ and $\mathcal{L}_Z\widetilde{G} = \mathcal{L}_Z H = -H + \sum_{j=1}^{r} \bar{Y}_j \otimes \alpha_j$. Moreover $\alpha_1 \wedge ... \wedge \alpha_r \wedge N_{\widetilde{G}} = \alpha_1 \wedge ... \wedge \alpha_r \wedge N_{\bar{G}} = 0$.

Given an endomorphism S of a vector space V and a vector $v \in V$, one will say that v *is S-generic* if v and S have the same minimal polynomial; in particular $v \neq 0$ if $V \neq \{0\}$.

Lemma 3.2. *Close to p on P and to $\bar{\pi}(p)$ on $\bar{P}$ the vector field Z is H-generic.*

Proof. First remark that, for any $q \in P$, $Z(q)$ is H-generic if and only if $Z(\bar{\pi}(q))$ is H-generic on $\bar{P}$.

Assume $dim\bar{\mathcal{A}} \geq 2$, otherwise the result is obvious. Along $\bar{\mathcal{A}}$ each $\Omega(H^k, \quad)$,

$k \geq 0$, is a constant 2-form and $\mathcal{L}_Z(\Omega(H^k, \quad)) = -(k+1)\Omega(H^k, \quad)$; on the other hand $\mathcal{L}_Z(\Omega(H^k, \quad)) = D(\Omega(H^k Z, \quad))$. Now suppose the minimal polynomial of H equals t^{k+1}. Then $\Omega(H^k, \quad)$ never vanishes and by consequence $H^k Z$ only does on a closed set of empty interior; otherwise $\mathcal{L}_Z(\Omega(H^k, \quad)) = 0$ on some non-empty open set. Therefore Z is H-generic almost everywhere around $\bar{\pi}(p)$ on $\bar{P}$ and p on P.

A straightforward calculation shows that the minimal polynomial of $(\ell_{|\mathcal{A}})(q)$ is $(t - f(q))^{k+2}$ if and only if $Z(q)$ is H-generic. So Z has to be H-generic close to p since this point is regular. $\square$.

Let $\widetilde{Z}$ be a second vector field defined around p, functional combination of $\partial/\partial z_1, ..., \partial/\partial z_{2m-2}$ with coefficient only depending on $(z_1, ..., z_{2m-2})$, such that $\widetilde{Z}(p) = Z(p)$, $L_{\widetilde{Z}}\Omega = -\Omega$ and $L_{\widetilde{Z}}\Omega_1 = -2\Omega_1$. Then $L_{\widetilde{Z}}H = -H$ on P and $\mathcal{L}_{\widetilde{Z}}\widetilde{G} = \mathcal{L}_{\widetilde{Z}}H = -H$ on $\bar{P}$. The existence of a such vector field is clear since Ω and Ω_1 are written with constant coefficients; for example take as $\widetilde{Z}$ a suitable linear vector field (just a linear algebra exercise) plus a constant one.

Set $\widetilde{\beta} = \Omega(\widetilde{Z}, \quad)$. Then on $\bar{P}$ one has $D\widetilde{\beta} = -\Omega$ and there is a function $g = g(x, z_1, ..., z_{2m-2}, z_{2m})$ such that $\beta = \widetilde{\beta} + Dg$ and $Dg(\bar{\pi}(p)) = 0$; moreover $Z = \widetilde{Z} - X_g$ where X_g is the Ω-hamiltonian of g (recall that Ω is symplectic on $\bar{\mathcal{A}}$).

Given a 1-form μ defined on a vector sub-bundle E by $\mu(KerH^k) = 0$ one means $\mu(v) = 0$ for every $v \in E \cap KerH^k$. It is clear that $Dg(KerH^k) = 0$ for some integer $k \geq 0$. The next step will be to show that our problem reduces to the case $Dg(KerH^{k+1}) = 0$.

Set $\beta_t = \tilde{\beta} + tDg$ and $Z_t = (1-t)\tilde{Z} + tZ$, $t \in \mathbb{K}$. Then $\Omega(Z_t, \quad) = \beta_t$, $\mathcal{L}_{Z_t}\Omega = -\Omega$ and $Z_t(\bar{\pi}(p)) = Z(\bar{\pi}(p))$, so $Z_t(\bar{\pi}(p))$ is H-generic. Moreover $(\mathcal{L}_{Z_t}\widetilde{G} + H) \wedge \alpha_1 \wedge ... \wedge \alpha_r = ((1-t)\mathcal{L}_{Z_t}\widetilde{G} + t\mathcal{L}_Z\widetilde{G} + H) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$ and $(\mathcal{L}_{X_g}\widetilde{G}) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$ since $X_g = \widetilde{Z} - Z$. By technical reasons let us decompose the manifold $\bar{P}$ into a product of a n-manifold [variables $(x_1, ..., x_n)$], a $(2m-2)$-manifold [variables $(z_1, ..., z_{2m-2})$] a 1-manifold [variable (z_{2m})], and set $\bar{\pi} = (\pi_1, \pi_2, \pi_3)$ following this decomposition.

On $\bar{P} \times \mathbb{K}$ consider coordinates $(x_1, ..., x_n, z_1, ..., z_{2m-2}, z_{2m}, t)$ and the foliation $dz_{2m} = dt = 0$, that is variables $(x_1, ..., x_n, z_1, ..., z_{2m-2})$ and parameters (z_{2m}, t). From proposition 4.1, applied to Z_t, $\widetilde{G}$, Ω, Ω_1, $\bar{\mathcal{A}}$ all of them regarded

along $dz_{2m} = dt = 0$, $\pi_1(p)$, $\pi_2(p)$, the compact set $K = \{\pi_3(p)\} \times [0,1]$ and g when $a = c = c' = -1$, follows the existence of a function f_t defined around $K' = \{\bar{\pi}(p)\} \times [0,1]$ such that:
(I) $Df_{t|K'} = 0$,
(II) $(\mathcal{L}_{X_t}\widetilde{G}) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$ where X_t is the Ω-hamiltonian of f_t,
(III) $Df_t(KerH^k) = 0$ and $D(Z_t f_t + f_t - g)(KerH^{k+1}) = 0$.

Therefore:
(1) $X_{t|K'} = 0$,
(2) X_t is tangent to ImH^k,
(3) $\mathcal{L}_{X_t}\Omega = \mathcal{L}_{X_t}\Omega_1 = 0$,
(4) $(\mathcal{L}_{X_t}\beta_t - Dg)(KerH^{k+1}) = 0$.

Indeed, assertions (1) is clear and (3) follows from he fact that $\Omega_1 = \Omega(\widetilde{G}, \quad)$. For checking (2) remark that ImH^k is the Ω-orthogonal of $\bar{\mathcal{A}} \cap KerH^k$ and $Ker\Omega(X_t, \quad) = KerDf_t \supset \bar{\mathcal{A}} \cap KerH^k$. Finally, for assertion (4) take into account that $\mathcal{L}_{X_t}\beta_t = \mathcal{L}_{X_t}(\Omega(Z_t, \quad)) = \Omega([X_t, Z_t], \quad) = -\mathcal{L}_{Z_t}(\Omega(X_t, \quad)) + (\mathcal{L}_{Z_t}\Omega)(X_t, \quad) = D(Z_t f_t + f_t)$ and apply (III).

Let Ψ_s be the flow of the time depending vector field $-X_t$ (on $\bar{P} \times \mathbb{K}$ one considers the vector field $\partial/\partial t - X_t$). As $X_{t|K'} = 0$, Ψ_1 is defined around $\bar{\pi}(p)$ and it can be regarded as a germ of diffeomorphism at this point. By construction Ψ_1 preserves $\bar{\pi}(p)$, Ω, Ω_1, $\alpha_1, ..., \alpha_r$, $\widetilde{G}_{|\bar{\mathcal{A}}} = H_{|\bar{\mathcal{A}}}$ and $\widetilde{G} \wedge \alpha_1 \wedge ... \wedge \alpha_r$.

Since X_t is tangent to $\bar{\mathcal{A}}$ one has

$\Psi_1(x, z_1, ..., z_{2m-2}, z_{2m}) = (x, \Phi(x, z_1, ..., z_{2m-2}, z_{2m}), z_{2m})$.

Thus the pull-back by Ψ_1 of $\widetilde{G}$ equals $\widetilde{G} + \sum_{j=1}^{r} \bar{Y}_j \otimes \alpha_j$ with $\bar{Y}_1, ..., \bar{Y}_r \in \bar{\mathcal{A}}$, and that of $\bar{G}$ equals $\bar{G} + \sum_{j=1}^{r} \bar{Y}_j \otimes \alpha_j$

Now we construct a germ of diffeomorphism F, at point p, by setting

$$F(x, z) = (x, \Phi(x, z_1, ..., z_{2m-2}, z_{2m}), z_{2m-1} + \varphi(x, z_1, ..., z_{2m-2}, z_{2m}), z_{2m})$$

such that $F(p) = p$ and $F^*\omega = \omega$. Indeed, $F^*\omega = \Omega + (dz_{2m-1} \wedge dz_{2m})_{|\mathcal{A}} + \rho \wedge (dz_{2m|\mathcal{A}}) + (d\varphi \wedge dz_{2m})_{|\mathcal{A}}$ where ρ is a 1-form along $\mathcal{A}$; as $0 = d(F^*\omega) = d\rho \wedge (dz_{2m|\mathcal{A}})$ one may choose φ in such a way that $(d\varphi \wedge dz_{2m})_{|\mathcal{A}} = -\rho \wedge (dz_{2m|\mathcal{A}})$.

On the other hand:

$$F^*(z_{2m}\omega + \beta \wedge (dz_{2m|\mathcal{A}})) = z_{2m}\omega + (\widetilde{\beta} + Dh_1) \wedge (dz_{2m|\mathcal{A}})$$

where $h_1 = h_1(x, z_1, ..., z_{2m-2}, z_{2m})$ and $Dh_1(KerH^{k+1}) = 0$ since $D\beta = D\widetilde{\beta}$ and Φ transforms $\widetilde{\beta}_{|KerH^{k+1}}$ in $\beta_{|KerH^{k+1}}$, while

$$F^*\Omega_1 = \Omega_1 + Dh_2 \wedge (dz_{2m|\mathcal{A}})$$

where $h_2 = h_2(x, z_1, ..., z_{2m-2}, z_{2m})$ because Ω_1 is closed.

Let us see that $Dh_2(KerH^{k+1}) = 0$. Set

$$\Psi_s(x, z_1, ..., z_{2m-2}, z_{2m}) = (x, \psi_s(x, z_1, ..., z_{2m-2}, z_{2m}), z_{2m}).$$

Since X_t is tangent to $\bar{A} \cap ImH^k = ImH^k$, the flow Ψ_s respects each leaf of the foliation $\bar{A} \cap ImH^k$. Therefore $(z_1, ..., z_{2m-2})$ and $\psi_s(x, z_1, ..., z_{2m-2}, z_{2m})$ belong to the same leaf of the foliation ImH^k regarded on the variables $(z_1, ..., z_{2m-2})$ only; by consequence $(\psi_s)_*(\partial/\partial z_{2m}) \in ImH^k$ and, in particular, $\Phi_*(\partial/\partial z_{2m}) \in ImH^k$, whence $(F_*(\partial/\partial z_{2m}) - \partial/\partial z_{2m}) \in ImH^k$; so set $F_*(\partial/\partial z_{2m}) = \partial/\partial z_{2m} + H^kV$. As F respects $\mathcal{A}$, $Kerdz_{2m}$ and $H_{|\mathcal{A} \cap Kerdz_{2m}}$ one has $F_*(\mathcal{A} \cap KerH^{k+1} \cap Kerdz_{2m}) = \mathcal{A} \cap KerH^{k+1} \cap Kerdz_{2m}$. But on P, $\Omega_1(\partial/\partial z_{2m}, \quad) = 0$, $\Omega_1 = \Omega(H, \quad)$ and $\Omega(H, \quad) = \Omega(\quad, H)$, therefore

$$\begin{aligned}
&(F^*\Omega_1)(\partial/\partial z_{2m}, \mathcal{A} \cap KerH^{k+1} \cap Kerdz_{2m}) \\
&\quad = \Omega_1(F_*(\partial/\partial z_{2m}), \mathcal{A} \cap KerH^{k+1} \cap Kerdz_{2m}) \\
&\quad\quad = \Omega(H^{k+1}V, \mathcal{A} \cap KerH^{k+1} \cap Kerdz_{2m}) \\
&\quad\quad\quad = \Omega(V, H^{k+1}(\mathcal{A} \cap KerH^{k+1} \cap Kerdz_{2m})) = 0
\end{aligned}$$

which implies $Dh_2(KerH^{k+1}) = 0$.

In short $F^*\omega_1 = z_{2m}\omega + \Omega_1 + (\widetilde{\beta} + Dh) \wedge (dz_{2m|\mathcal{A}})$ where $h = h(x, z_1, ..., z_{2m-2}, z_{2m})$ and $Dh(KerH^{k+1}) = 0$.

Set $\gamma = \widetilde{\beta} + Dh$. Let γ^* be the extension of γ to TP such that $\gamma(\partial/\partial x_j) = 0$, $j = 1, ..., n$, and U the vector field functional combination of $\partial/\partial z_1, ..., \partial/\partial z_{2m-2}$ defined by $\omega(U, \quad) = \gamma$. Since G, up to the term $(\partial/\partial z_{2m-1}) \otimes \tau$, is determined by ω, ω_1 and $\bar{G}$, its pull-back G_F by F is determined by $F^*\omega = \omega$, $F^*\omega_1 = z_{2m}\omega + \Omega_1 + \gamma \wedge (dz_{2m|\mathcal{A}})$ and $\bar{G} + \sum_{j=1}^r \bar{Y}_j \otimes \alpha_j$; therefore reasoning as before yields

$$\begin{aligned}
G_F = &\textstyle\sum_{j=1}^n a_j(\partial/\partial x_j) \otimes dx_j + (\partial/\partial z_{2m-1}) \otimes \sigma + z_{2m}I_z + H \\
&+ (\partial/\partial z_{2m-1}) \otimes \gamma^* - U \otimes dz_{2m} + \textstyle\sum_{j=1}^r Y_j \otimes \alpha_j
\end{aligned}$$

where $Y_1, ..., Y_r \in \mathcal{A}$ and γ is a functional combination of $dx_1, ..., dx_n$ whose coefficients do not depend on z_{2m-1}.

Clearly it suffices proving theorem 3.1 for G_F, $F^*\omega$ and $F^*\omega_1$; even more the term $\sum_{j=1}^r Y_j \otimes \alpha_j$ is irrelevant and may be deleted. Thus a change of notation (denote G_F by G, γ by β etc...) allows us to assume $\beta = \widetilde{\beta} + Dh$ where $h = h(x, z_1, ..., z_{2m-2}, z_{2m})$ and $Dh(KerH^{k+1}) = 0$.

Now we start the process again with $\widetilde{Z} + (Z - \widetilde{Z})(p)$. Finally, after a finite number of steps, we may suppose $Z = \widetilde{Z} + W$ where W is a constant vector field linear combination of $\partial/\partial z_1, ..., \partial/\partial z_{2m-2}$. Thus Z only depend on $(z_1, ..., z_{2m-2})$.

Lemma 3.3. *On an open set of $\mathbb{K}^{k+1}$ endowed with coordinates $(v, u) = (v_1, ..., v_k, u)$ consider a point $q = (q_1, ..., q_k, \bar{q})$ and a tensor field $\widetilde{T} = uI + \widetilde{H} - U \otimes du$ where $\widetilde{H} = \sum_{i,j=1}^{k} a_{ij}(\partial/\partial v_i) \otimes dv_j$, with each a_{ij} constant, and $U = \sum_{j=1}^{k} \varphi_j(v)(\partial/\partial v_j)$. Assume that $U(q)$ is $\widetilde{H}$-generic, $\widetilde{H}$ nilpotent and $L_U \widetilde{H} = c\widetilde{H}$, $c \in \mathbb{K}$. Then around q there exist functions $h_1, ..., h_k$ of v such that $d(dh_j \circ \widetilde{T}) = 0$, $j = 1, ..., k$, and $(dh_1 \wedge ... \wedge dh_k \wedge du)(q) \neq 0$.*

Proof. Given $h = h(v)$ one has $d(dh \circ \widetilde{T}) = d(d \circ \widetilde{H}) - d(Uh + h) \wedge du$. Close to q and for every $j = 1, ..., k$, consider a function g_j of v such that $g_j(q_1, ..., q_k) = 0$ and $dg_j = d(Uv_j + v_j)$. Then $d(dg_j \circ \widetilde{H}) = 0$; indeed, $d(dv_j \circ \widetilde{H}) = 0$ and $d(Uv_j) \circ \widetilde{H} = (L_U dv_j) \circ \widetilde{H} = L_U(dv_j \circ \widetilde{H}) - c dv_j \circ \widetilde{H}$. By proposition 4.2, applied in coordinates $v = (v_1, ..., v_k)$ with a zero dimensional space of parameters, close to $(q_1, ..., q_k)$ there exists $f_j = f_j(v)$ such that $df_j(q_1, ..., q_k) = 0$, $d(df_j \circ \widetilde{H}) = 0$ and $Uf_j = -f_j + g_j$.

Set $h_j = v_j + f_j$; then $d(dh_j \circ \widetilde{T}) = 0$, $j = 1, ..., k$, and $(dh_1 \wedge ... \wedge dh_k \wedge du)(q) = (dv_1 \wedge ... \wedge dv_k \wedge du)(q)$. $\square$

Let us come back to the proof of theorem 3.1. If $h = h(z_1, ..., z_{2m-2}, z_{2m})$ one has $dh \circ G = z_{2m} dh + dh \circ H - (Zh) dz_{2m}$. Thus we can apply lemma 3.3, in variables $(z_1, ..., z_{2m-2}, z_{2m})$ when $\widetilde{T} = z_{2m} I + H - Z \otimes dz_{2m}$, for concluding the existence close to p of functions $h_j(z_1, ..., z_{2m-2})$, $j = 1, ..., 2m-2$, such that $d(dh_j \circ G) = 0$ and $(dh_1 \wedge ... \wedge dh_{2m-2} \wedge dz_{2m})(p) \neq 0$.

Denote by X_j the ω-hamiltonian of h_j, $j = 1, ..., 2m-2$. From the third condition of Veronese flag follows $(L_{X_j} G) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$ (everywhere or close to point p), which in turn implies $(L_{X_j} \tau) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$, $j = 1, ..., 2m-2$. Now set $\tau = \sum_{k=1}^{n-r} \varphi_k dx_j + \sum_{k=n-r+1}^{n} \varphi_k \alpha_{k+r-n}$; then $X_j \varphi_k = 0$, $j = 1, ..., 2m-2$, $k = 1, ..., n-r$. But $(X_1 \wedge ... \wedge X_{2m-2} \wedge (\partial/\partial z_{2m-1}))(p) \neq 0$ because $(dh_1 \wedge ... \wedge dh_{2m-2} \wedge dz_{2m})(p) \neq 0$, so each φ_k, $k = 1, , .n - r$, only depends on

(x, z_{2m}). Besides the term $(\partial/\partial z_{2m-1}) \otimes (\sum_{k=n-r+1}^{n} \varphi_k \alpha_{k+r-n})$ is irrelevant for our purpose and it may be deleted. In short, one can suppose that τ only depends on (x, z_{2m}).

On the other hand

$$N_G(\partial/\partial x_i, \partial/\partial x_j) = [a_i\partial/\partial x_i + \tau(\partial/\partial x_i)\partial/\partial z_{2m-1}, a_j\partial/\partial x_j + \tau(\partial/\partial x_j)\partial/\partial z_{2m-1}]$$
$$-G[\partial/\partial x_i, a_j\partial/\partial x_j + \tau(\partial/\partial x_j)\partial/\partial z_{2m-1}] - G[a_i\partial/\partial x_i + \tau(\partial/\partial x_i)\partial/\partial z_{2m-1}, \partial/\partial x_j]$$
$$= \left((a_i - z_{2m})\frac{\partial(\tau(\partial/\partial x_j)}{\partial x_i} - (a_j - z_{2m})\frac{\partial(\tau(\partial/\partial x_i)}{\partial x_j}\right)\frac{\partial}{\partial z_{2m-1}}$$
$$= d_x(\tau \circ (J - z_{2m}I_x)^{-1})((J - z_{2m}I_x)(\partial/\partial x_i), (J - z_{2m}I_x)(\partial/\partial x_j))\frac{\partial}{\partial z_{2m-1}}$$

where $I_x = \sum_{j=1}^{n}(\partial/\partial x_j) \otimes dx_j$ and d_x is the exterior derivative with respect to x.

Therefore $N_G \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$ implies $d_x(\tau \circ (J - z_{2m}I_x)^{-1}) \wedge (\alpha_1 \circ (J - z_{2m}I_x)^{-1}) \wedge ... \wedge (\alpha_r \circ (J - z_{2m}I_x)^{-1}) = 0$. In other words, $\tau \circ (J - z_{2m}I_x)^{-1}$ is closed along the foliation in variables (x, z_{2m}) defined by $\alpha_1 \circ (J - z_{2m}I_x)^{-1}, ..., \alpha_r \circ (J - z_{2m}I_x)^{-1}, dz_{2m}$, and near p there exists a function $h = h(x, z_{2m})$ such that $d_x h$ equals $\tau \circ (J - z_{2m}I_x)^{-1}$ modulo $\alpha_1 \circ (J - z_{2m}I_x)^{-1}, ..., \alpha_r \circ (J - z_{2m}I_x)^{-1}$. Thus adding a suitable functional combination of $\alpha_1, ..., \alpha_r$ to τ allows us to suppose $\tau \circ (J - z_{2m}I_x)^{-1} = d_x h$. Then
$d(z_{2m-1} - h) \circ G = z_{2m}dz_{2m-1} + \tau - z_{2m}(\partial h/\partial z_{2m})dz_{2m} - (d_x h) \circ J + \beta^* = z_{2m}dz_{2m-1} + d_x h \circ (J - z_{2m}I_x) - z_{2m}(\partial h/\partial z_{2m})dz_{2m} - (d_x h) \circ J + \beta^* = z_{2m}d(z_{2m-1} - h) + \beta^*$.

Finally, if the coordinate z_{2m-1} is replaced by $\widetilde{z}_{2m-1} = z_{2m-1} - h$ and next $\widetilde{z}_{2m-1}$ is called z_{2m-1}, as h only depends on (x, z_{2m}), then the expression of ω and that of ω_1 are not modified whereas

$$G = \sum_{j=1}^{n} a_j(\partial/\partial x_j) + z_{2m}I_z + H + (\partial/\partial z_{2m-1}) \otimes \beta^* - Z \otimes dz_{2m}$$

which proves theorem 3.1.

4. The equation $Zf = af + g$

In this section, rather technical, one will establish the results on the foregoing equation needed in the proof of theorem 3.1. Consider three open sets $A \subset \mathbb{K}^n$, $A' \subset \mathbb{K}^{2m}$ and $B \subset \mathbb{K}^{\bar{s}}$, their product $A \times A' \times B \subset \mathbb{K}^{n+2m+\bar{s}}$ endowed with product coordinates $(x, z, u) = (x_1, ..., x_n, z_1, ..., z_{2m}, u_1, ..., u_{\bar{s}})$ and the following objects on it:

$\mathcal{F}_1$: foliation defined by setting $u_1, ...u_{\bar{s}}$ constant,

d: exterior derivative along $\mathcal{F}_1$,

$\mathcal{A}$: foliation contained in $\mathcal{F}_1$ defined by $dx_1 = ... = dx_n = 0$,

D: exterior derivative along $\mathcal{A}$,

ω, ω_1: couple of 2-forms on $\mathcal{A}$,

$Z = \sum_{j=1}^{2m} \varphi_j \partial/\partial z_j$: vector field tangent to $\mathcal{A}$.

Along $\mathcal{F}_1$, that is to say regarding B as the space of parameters, set $J = \sum_{j=1}^{n} a_j(u)(\partial/\partial x_j)\otimes dx_j$, $H = \sum_{j,k=1}^{2m} a_{jk}(\partial/\partial z_j)\otimes dz_k$ where each $a_{jk} \in \mathbb{K}$, $\xi = \sum_{j=1}^{r} X_j \otimes \alpha_j$ where $X_1, ..., X_r \in \mathcal{A}$ and $\alpha_1, ..., \alpha_r$ are closed 1-forms functional combination of $dx_1, ..., dx_n$ [so their coefficients only depend on (x, u)] such that $\alpha_1 \wedge ... \wedge \alpha_r$ never vanishes, and $G = J + H + \xi$.

Let $\mathcal{F}$ be the foliation contained in $\mathcal{F}_1$ defined by $\alpha_1, ..., \alpha_r$. Suppose that ω, ω_1 are written with constant coefficients which respect to $dz_{1|\mathcal{A}}, ..., dz_{2m|\mathcal{A}}$, functions $a_1(u), ..., a_n(u)$ never vanish on B, H is nilpotent, $\omega_1 = \omega(G, \quad)$ and $(\mathcal{F}, G_{|\mathcal{F}})$ is a weak Veronese flag along $\mathcal{F}_1$ whose associated $G_{|\mathcal{F}}$-invariant foliation equals $\mathcal{A}$; therefore $\alpha_1, ..., \alpha_r, J$ defines a Veronese web along $\mathcal{F}_1/\mathcal{A}$ on $A \times B$

For any function φ one will denote X_φ its ω-hamiltonian.

Lemma 4.1. *Let X_f be the ω-hamiltonian of a function f. Then $(L_{X_f} G) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$ if and only if $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(df \circ G)$ is semi-basic for $\mathcal{A}$ (that is $i_U(\alpha_1 \wedge ... \wedge \alpha_r \wedge d(df \circ G)) = 0$ for any $U \in \mathcal{A}$).*

Proof. As $(L_{X_f}\xi) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$ and $d(df \circ \xi) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$ we may suppose $\xi = 0$ without loss of generality. Now consider new coordinates (z_j^i), $j = 1, ..., 2m_i$, $i = 1, ..., s$, on A' constant linear combination of $(z_1, ..., z_{2m})$ such that $\omega = (\sum_{i=1}^{s} \sum_{k=1}^{m_i} dz_{2k-1}^i \wedge dz_{2k}^i)_{|\mathcal{A}}$ and $\omega_1 = (\sum_{i=1}^{s} \sum_{k=1}^{m_i - 1} dz_{2k-1}^i \wedge dz_{2k+2}^i)_{|\mathcal{A}}$. Then $H = \sum_{i=1}^{s} \sum_{k=1}^{m_i - 1} [(\partial/\partial z_{2k+1}^i) \otimes dz_{2k-1}^i + (\partial/\partial z_{2k}^i) \otimes dz_{2k+2}^i]$ since $\omega_1 = \omega(G, \quad) = \omega(H, \quad)$, and $L_{X_f} G = S + T$ where

$$S = \sum_{i=1}^{s} \frac{\partial}{\partial z_1^i} \otimes \left(\sum_{j=1}^{n} a_j \frac{\partial^2 f}{\partial z_2^i \partial x_j} dx_j \right)$$

$$+ \sum_{i=1}^{s} \sum_{k=1}^{m_i - 1} \frac{\partial}{\partial z_{2k+1}^i} \otimes \left(\sum_{j=1}^{n} \left[a_j \frac{\partial^2 f}{\partial z_{2k+2}^i \partial x_j} - \frac{\partial^2 f}{\partial z_{2k}^i \partial x_j} \right] dx_j \right)$$

$$+\sum_{i=1}^{s}\sum_{k=1}^{m_i-1}\frac{\partial}{\partial z^i_{2k}}\otimes\left(\sum_{j=1}^{n}\left[\frac{\partial^2 f}{\partial z^i_{2k+1}\partial x_j}-a_j\frac{\partial^2 f}{\partial z^i_{2k-1}\partial x_j}\right]dx_j\right)$$

$$-\sum_{i=1}^{s}\frac{\partial}{\partial z^i_{2m_i}}\otimes\left(\sum_{j=1}^{n}a_j\frac{\partial^2 f}{\partial z^i_{2m_i-1}\partial x_j}dx_j\right)$$

and T does not involve any $\partial/\partial x_j$ nor dx_j, $j=1,...,n$.

Note that $T=0$ if and only if $(L_{X_f}G)_{|\mathcal{A}}=0$.

On the other hand $df\circ G=\sum_{j=1}^{n}a_j(\partial f/\partial x_j)dx_j$
$+\sum_{i=1}^{s}\sum_{k=1}^{m_i-1}[(\partial f/\partial z^i_{2k+1})dz^i_{2k-1}+(\partial f/\partial z^i_{2k})dz^i_{2k+2}]$.

Therefore $d(df\circ G)=\rho+\lambda+\mu$ where

$$\rho=\sum_{i=1}^{s}\sum_{k=1}^{m_i-1}\left(\sum_{j=1}^{n}\left[\frac{\partial^2 f}{\partial z^i_{2k+1}\partial x_j}-a_j\frac{\partial^2 f}{\partial z^i_{2k-1}\partial x_j}\right]dx_j\right)\wedge dz^i_{2k-1}$$

$$-\sum_{i=1}^{s}\left(\sum_{j=1}^{n}a_j\frac{\partial^2 f}{\partial z^i_{2m_i-1}\partial x_j}dx_j\right)\wedge dz^i_{2m_i-1}-\sum_{i=1}^{s}\left(\sum_{j=1}^{n}a_j\frac{\partial^2 f}{\partial z^i_{2}\partial x_j}dx_j\right)\wedge dz^i_{2}$$

$$+\sum_{i=1}^{s}\sum_{k=1}^{m_i-1}\left(\sum_{j=1}^{n}\left[\frac{\partial^2 f}{\partial z^i_{2k}\partial x_j}-a_j\frac{\partial^2 f}{\partial z^i_{2k+2}\partial x_j}\right]dx_j\right)\wedge dz^i_{2k+2}\,,$$

$\lambda=\sum_{jb}h_{jb}dx_j\wedge dx_b$ and $\mu=\sum_{iajb}\widetilde{h}_{iajb}dz^i_j\wedge dz^a_b$; thus $(d(df\circ G))_{|\mathcal{A}}=\mu_{|\mathcal{A}}$.

But $L_{X_f}\omega_1=\omega(L_{X_f}G,\quad)$ and at the same time $L_{X_f}\omega_1=D(\omega_1(X_f,\quad))=-d(df\circ G))_{|\mathcal{A}}$; therefore $T=0$ if and only if $\mu=0$ since ω is symplectic.

Finally, remark that the 1-forms functional combination of $dx_1,...,dx_n$ which are the coefficients of $\partial/\partial z^i_b$ or dz^i_b in the expressions of S and ρ, respectively, are the same up to sign and change of order, so $S\wedge\alpha_1\wedge...\wedge\alpha_r=0$ if and only if $\rho\wedge\alpha_1\wedge...\wedge\alpha_r=0$. As λ is semi-basic for $\mathcal{A}$, the lemma is proved. □

Remark. From lemma 4.1 immediately follows that $(\mathcal{F},G_{|\mathcal{F}},\omega,\omega_1)$ is a Veronese flag along $\mathcal{F}_1$.

Proposition 4.1. *Given an integer $k\geq 0$, $p\in A$, $q\in A'$, a compact set $K\subset B$, three scalars a,c,c' and a function $g:A\times A'\times B\to\mathbb{K}$ such that:*
(1) $(L_ZG-cH)\wedge\alpha_1\wedge...\wedge\alpha_r=0$ and $L_Z\omega=c'\omega$,
(2) Z is H-generic on $\{(p,q)\}\times K$,
(3) $(L_{X_g}G)\wedge\alpha_1\wedge...\wedge\alpha_r=0$ and $Dg(KerH^k)=0$,

then there exist a product open set $U \times U' \times V \subset A \times A' \times B$, which contains $\{(p,q)\} \times K$, and a function $f : U \times U' \times V \to \mathbb{K}$ such that:
(I) $Df(KerH^k) = 0$ and $D(Zf - af - g)(KerH^{k+1}) = 0$,
(II) $(L_{X_f} G) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$,
(III) $Df = 0$ on $\{(p,q)\} \times V$.

The next goal will be to prove proposition 4.1. Note that we may assume $\xi = 0$ since $(L_{X_g}\xi) \wedge \alpha_1 \wedge ... \wedge \alpha_r = (L_{X_f}\xi) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$. Set $\varphi_g(x,z,u) = g(x,q,u)$, then $D\varphi_g = 0$ and $d(d\varphi_g \circ G)$ is $\mathcal{A}$-basic; thus any solution of our problem for $g - \varphi_g$ is a solution for g too, which allows to suppose $g(A \times \{q\} \times B) = 0$ by considering $g - \varphi_g$ instead of g if necessary. On the other hand by shrinking A and modifying the order of variables $(x_1, ..., x_n)$ one may assume that $dx_1 \wedge ... \wedge dx_{n-r} \wedge \alpha_1 \wedge ... \wedge \alpha_r$ never vanishes. By lemma 4.1, the first statement (3) and part (II) of proposition 4.1 are respectively equivalent to suppose $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dg \circ G)$ and $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(df \circ G)$ semi-basic for $\mathcal{A}$; *throughout the proof one will use these statements instead of original ones.*

We start reducing the problem to the case $k = 0$. Consider H as a tensor field on A' and linearly rearrange coordinates z in such a way that $dz_1 = ... = dz_{2\widetilde{m}} = 0$ defines $KerH^k$. Let A'' be the quotient (close to q) of A' by $KerH^k$ endowed with coordinates $(z_1, ..., z_{2\widetilde{m}})$, and $\pi : A \times A' \times B \to A \times A'' \times B$ the canonical projection. As $Dg(KerH^k) = 0$ there is a function $\bar{g}$ on $A \times A'' \times B$ such that $g = \bar{g} \circ \pi$. Obviously $Z, H, G, \mathcal{F}, \mathcal{A}$ project in similar object $\bar{Z}, \bar{H}, \bar{G}, \bar{\mathcal{F}}, \bar{\mathcal{A}}$ defined on $A \times A'' \times B$.

On the other hand $\omega(H^k, \quad)$ and $\omega_1(H^k, \quad)$ project in a couple of 2-forms $\bar{\omega}, \bar{\omega}_1$ defined along $\bar{\mathcal{A}}$. It is easily checked that the hypothesis of proposition 4.1 still hold on $A \times A'' \times B$ for the scalars a, c and $c' + kc$. Therefore, if the result is proved for $k = 0$, there exists a solution $\bar{f}$ and it suffices to set $f = \bar{f} \circ \pi$.

In short $k = 0$ is the only case to deal with. We do that by induction on the order $\widetilde{k}$ of nilpotency of H. First consider the case $H = 0$, that is $\widetilde{k} = 1$ and $G = J$. Assume $m \geq 1$, otherwise it suffices setting $f = 0$. As Z has no zeros on the compact set $\{(p,q)\} \times K$, we may suppose that φ_1 does not vanish on $A \times A' \times B$ by shrinking these three factor and changing the order of variables $z = (z_1, ..., z_{2m})$ if necessary. From (1) of proposition 4.1 follows $\alpha_1 \wedge ... \wedge \alpha_r \wedge (d_x\varphi_j \circ J) = 0$, $j = 1, ..., 2m$, that is to say $(\alpha_1 \circ J^{-1}) \wedge ... \wedge$

$(\alpha_r \circ J^{-1}) \wedge d_x\varphi_j = 0$, $j = 1, ..., 2m$. Consider new coordinates $y = (y_1, ..., y_n)$ around p on A such that $dy_1 = ... = dy_r = 0$ defines the same foliation as $\alpha_1 \circ J^{-1}, ..., \alpha_r \circ J^{-1}$ (recall that $\alpha_1, ..., \alpha_r, J$ gives rise to a Veronese web). Since every coordinate y_i only depends on x one has that $d_x = d_y$ and each vector field $\partial/\partial z_j$, $j = 1, ..., 2m$, belongs to the dual basis of $\{dy_1, ..., dy_n, dz_1, ..., dz_{2m}\}$ as well. Thus $\varphi_j = \varphi_j(y_1, ..., y_r, z, u)$, $j = 1, ..., 2m$.

On the other hand given a function h then $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dh \circ G)$ is semi-basic for $\mathcal{A}$ if and only if $\alpha_1 \wedge ... \wedge \alpha_r \wedge (d_x(\partial h/\partial z_j) \circ J) = 0$, $j = 1, ..., 2m$, that is $(\alpha_1 \circ J^{-1}) \wedge ... \wedge (\alpha_r \circ J^{-1}) \wedge d_x(\partial h/\partial z_j) = 0$, $j = 1, ..., 2m$, or in coordinates (y, z, u) if and only if $\partial^2 h/\partial z_j \partial y_i = 0$, $i = r+1, ..., n$, $j = 1, ..., 2m$. In these last coordinates consider the open neighbourhoods of p and q, respectively, $U = U_1 \times U_2$ and U', where $U_1 \subset \mathbb{K}^r$, $U_2 \subset \mathbb{K}^{n-r}$ and U_1, U_2, U' are polycylinders (that is product of open intervals if $\mathbb{K} = \mathbb{R}$ or open disks if $\mathbb{K} = \mathbb{C}$). Then $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dh \circ G) = 0$ is semi-basic for $\mathcal{A}$ on $U \times U' \times B$ if and only if $h = h_1(y_1, ..., y_r, z, u) + h_2(y, u)$; moreover we may suppose $h_1(p_1, ..., p_r, q, u) = 0$, $u \in B$, by taking $h_1 - \widetilde{h}$ and $h_2 + \widetilde{h}$ where $\widetilde{h}(y, z, u) = h_1(y_1, ..., y_r, q, u)$ if necessary.

In particular as $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dg \circ G)$ is semi-basic for $\mathcal{A}$, on $U \times U' \times B$ one has $g = g_1(y_1, ..., y_r, z, u) + g_2(y, u)$ where $g_1(p_1, ..., p_r, q, u) = 0$, $u \in B$. But $g(A \times \{q\} \times B) = 0$ so $g_2 = 0$, that is $g = g(y_1, ..., y_r, z, u)$.

Let $f : U_1 \times U' \times B \to \mathbb{K}$ be the function defined by the ordinary differential equation $Zf = af + g$ and the initial condition $f(U_1 \times T \times B) = 0$ where $T = \{z \in U' \mid z_1 = q_1\}$ [again shrink U_1, U', B if necessary]. Then regarded on $U \times U' \times B$ in the obvious way $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(df \circ G)$ is semi-basic for $\mathcal{A}$ and $D(Zf - af - g) = 0$. By construction $(\partial f/\partial z_j)(\{(p, q)\} \times B) = 0$, $j = 2, ..., 2m$. But $(Zf)(\{(p, q)\} \times B) = af(\{(p, q)\} \times B) + g(\{(p, q)\} \times B) = 0$ so $(\partial f/\partial z_1)(\{(p, q)\} \times B) = 0$; in short $Df = 0$ on $\{(p, q)\} \times B$, *which proves proposition 4.1 when* $\widetilde{k} = 0$.

Remark. Observe that in this step proposition 4.1 was established without making use of ω or ω_1; therefore the result stated in terms of being semi-basic for $\mathcal{A}$ is true regardless the existence or not of ω, ω_1. This fact implies that proposition 4.1 also holds if $Dg \circ H = 0$; even more in this case there exists f satisfying (I), (II) and (III) such that $Df \circ H = 0$ and $D(Zf - af - g) =$

0. Indeed, regard H as a tensor field on A' and consider the quotient $A'' = A'/ImH$. Let $\pi : A \times A' \times B \to A \times A'' \times B$ be the canonical projection. Then $g = \bar{g} \circ \pi$ for some $\bar{g} : A \times A'' \times B \to \mathbb{K}$. Since all the relevant objects project on $A \times A'' \times B$ and H does in the zero tensor field, from the case $\widetilde{k} = 0$ follows the existence of a suitable function $\bar{f}$ for $\bar{g}$ and it suffices setting $f = \bar{f} \circ \pi$.

Now suppose true proposition 4.1 up to the order of nilpotency $\widetilde{k} - 1 \geq 1$ and for any scalars a, c, c'. One will need the following result.

Lemma 4.2. *Given a function $h : A \times A' \times B \to \mathbb{K}$ such that $Dh(KerH) = 0$ and $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dh \circ G)$ is semi-basic for $\mathcal{A}$, then there exist a product open set $U \times U' \times V \subset A \times A' \times B$, which contains $\{(p,q)\} \times K$, and a function $\varphi : U \times U' \times V \to \mathbb{K}$ such that:*
(I) $D\varphi \circ H = Dh$ and $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(d\varphi \circ G)$ is semi-basic for $\mathcal{A}$,
(II) $\varphi(\{(p,q)\} \times V) = 0$ and $D\varphi(p,q,u) = 0$ for every $u \in V$ such that $Dh(p,q,u) = 0$.

Proof. Consider coordinates (z^i_j), $j = 1, ..., m_i$, $i = 1, ..., s$, on A' as in the proof of lemma 4.1 and shrink this open set in such a way that, in these coordinates, A' is a polycylinder. Then $dz^i_{2k+1} \circ H = dz^i_{2k-1}$, $dz^i_{2k} \circ H = dz^i_{2k+2}$, $k = 1, ..., m_i - 1$, $dz^i_1 \circ H = dz^i_{2m_i} \circ H = 0$, $i = 1, ..., s$.

Since $Dh(KerH) = 0$ one has $\partial h/\partial z^i_{2m_i-1} = \partial h/\partial z^i_2 = 0$, $i = 1, ..., s$ and $\beta \circ H = Dh$ where

$$\beta = \sum_{i=1}^{s} \sum_{k=1}^{m_i-1} \left(\frac{\partial h}{\partial z^i_{2k-1}} dz^i_{2k+1} + \frac{\partial h}{\partial z^i_{2k+2}} dz^i_{2k} \right)_{|\mathcal{A}}.$$

Note that $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dh \circ G)$ semi-basic for $\mathcal{A}$ implies $D(Dh \circ H) = 0$. Now from lemma 1.1 follows that $D\beta(ImH, ImH) = 0$, so $\beta_{|ImH}$ is closed and there exists a function $\psi : A \times A' \times B \to \mathbb{K}$ such that $(D\psi - \beta)_{|ImH} = 0$; therefore $D\psi \circ H = Dh$. Hence $\partial\psi/\partial z^i_{2k+1} = \partial h/\partial z^i_{2k-1}$, $\partial\psi/\partial z^i_{2k} = \partial h/\partial z^i_{2k+2}$, $k = 1, ..., m_i - 1$.

In general $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(d\psi \circ G)$ is not semi-basic for $\mathcal{A}$ and we need to modify ψ.

If following the terminology of the proof of lemma 4.1 we set $d(dh \circ G) = \rho_h + \lambda_h + \mu_h$ and $d(\psi \circ G) = \rho_\psi + \lambda_\psi + \mu_\psi$ then $\alpha_1 \wedge ... \wedge \alpha_r \wedge \rho_h$ and $\mu_h = \mu_\psi = 0$ because $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dh \circ G)$ is semi-basic for $\mathcal{A}$ and $D(D\psi \circ H) = D(Dh) = 0$.

Therefore $\alpha_1\wedge...\wedge\alpha_r\wedge d(d\psi\circ G)$ is semi-basic for $\mathcal{A}$ if and only if $\alpha_1\wedge...\wedge\alpha_r\wedge\rho_\psi = 0$.

When $k = 1, ..., m_i - 1$, $i = 1, ..., s$, the 1-form coefficient of dz^i_{2k+1} in the expression of ρ_ψ equals that of dz^i_{2k-1} in the expression of ρ_h, and the coefficient of dz^i_{2k} that of dz^i_{2k+2} (recall that $\partial h/\partial z^i_{2m_i-1} = \partial h/\partial z^i_2 = 0$), so they vanish modulo $\alpha_1, ..., \alpha_r$. Thus we have only to study the coefficients of dz^i_1 and $dz^i_{2m_i}$, $i = 1, ..., s$, denoted by β_{2i-1}, β_{2i} hereafter, which are

$$\beta_{2i-1} = \sum_{j=1}^n \left(\frac{\partial^2\psi}{\partial z^i_3\partial x_j} - a_j\frac{\partial^2\psi}{\partial z^i_1\partial x_j}\right)dx_j = \sum_{j=1}^n \left(\frac{\partial^2 h}{\partial z^i_1\partial x_j} - a_j\frac{\partial^2\psi}{\partial z^i_1\partial x_j}\right)dx_j$$

and

$$\beta_{2i} = \sum_{j=1}^n \left(\frac{\partial^2\psi}{\partial z^i_{2m_i-2}\partial x_j} - a_j\frac{\partial^2\psi}{\partial z^i_{2m_i}\partial x_j}\right)dx_j$$

$$= \sum_{j=1}^n \left(\frac{\partial^2 h}{\partial z^i_{2m_i}\partial x_j} - a_j\frac{\partial^2\psi}{\partial z^i_{2m_i}\partial x_j}\right)dx_j \quad \text{respectively.}$$

For the sake of simplicity, set $z^i_1 = \bar z_{2i-1}$ and $z^i_{2m_i} = \bar z_{2i}$, $i = 1, ..., s$. From the expression of β_j and β_k immediately follows $\partial\beta_j/\partial\bar z_k = \partial\beta_k/\partial\bar z_j$, $j, k = 1, ..., 2s$. Moreover $\alpha_1\wedge...\wedge\alpha_r\wedge(\partial\beta_{\bar a}/\partial z^i_{\bar b}) = 0$, $\bar a = 1, ..., 2s$, $\bar b = 2, ..., 2m_i-1$, $i = 1, ..., s$. Indeed,

$$\frac{\partial\beta_{\bar a}}{\partial z^i_{\bar b}} = \sum_{j=1}^n \frac{\partial}{\partial\bar z_{\bar a}}\left(\frac{\partial^2 h}{\partial z^i_{\bar b}\partial x_j} - a_j\frac{\partial^2\psi}{\partial z^i_{\bar b}\partial x_j}\right)dx_j$$

which is the derivative with respect to $\bar z_{\bar a}$ of the coefficient of $dz^i_{\bar b-2}$, if $\bar b$ is odd, or $dz^i_{\bar b+2}$, if $\bar b$ is even, in the expression of ρ_h. Thus, if we set $\beta_{\bar a} = \bar\beta_{\bar a} + \beta'_{\bar a}$ where $\bar\beta_{\bar a}$ is a functional combination of $dx_1, ..., dx_{n-r}$ and $\beta'_{\bar a}$ a functional combination of $\alpha_1, ..., \alpha_r$ (recall that $dx_1, ..., dx_{n-r}, \alpha_1, ..., \alpha_r$ are linearly independent everywhere) one has $\partial\bar\beta_{\bar a}/\partial z^i_{\bar b} = 0$; that is every $\bar\beta_{\bar a}$, $\bar a = 1, ..., 2s$, only depends on x, $\bar z = (\bar z_1, ..., \bar z_{2s})$ and u.

Of course $\partial\bar\beta_j/\partial\bar z_k = \partial\bar\beta_k/\partial\bar z_j$, $j, k = 1, ..., 2s$.

The coefficient of $d\bar z_k$, $k = 1, ..., 2s$, in the expression of ρ_h equals

$$\sum_{j=1}^n \left(\frac{\partial^2 h}{\partial z^i_{\bar b}\partial x_j} - a_j\frac{\partial^2 h}{\partial\bar z_k\partial x_j}\right)dx_j = d_x\left(\frac{\partial h}{\partial z^i_{\bar b}}\right) - d_x\left(\frac{\partial h}{\partial\bar z_k}\right)\circ J$$

where i and $\bar b$ depend on k. This coefficient is a functional combination of $\alpha_1, ..., \alpha_r$ therefore, as $\alpha_1, ..., \alpha_r$ define a foliation, one has $\alpha_1\wedge...\wedge\alpha_r\wedge$

$d_x(d_x(\partial h/\partial \bar{z}_k)\circ J) = 0$. In turn from lemma 1.1 applied to $d_x(\partial h/\partial \bar{z}_k)\circ J^{-1}$ and J follows $(\alpha_1 \circ J^{-1}) \wedge ... \wedge (\alpha_r \circ J^{-1}) \wedge d_x(d_x(\partial h/\partial \bar{z}_k)\circ J^{-1}) = 0$, whence taking into account the expression of β_k given before results $(\alpha_1 \circ J^{-1}) \wedge ... \wedge (\alpha_r \circ J^{-1}) \wedge d_x(\beta_k \circ J^{-1}) = 0$, $k = 1, ..., 2s$. Finally, since $\alpha_1 \wedge ... \wedge \alpha_r \wedge (\beta_k - \bar{\beta}_k) = 0$ and $\alpha_1 \circ J^{-1}, ..., \alpha_r \circ J^{-1}$ define a foliation, one has $(\alpha_1 \circ J^{-1}) \wedge ... \wedge (\alpha_r \circ J^{-1}) \wedge d_x(\bar{\beta}_k \circ J^{-1}) = 0$, $k = 1, ..., 2s$.

After shrinking A and A' if necessary, we may suppose that A in coordinates $y = (y_1, ..., y_n)$ and A' in coordinates (z^i_j) are polycylinders. Set $A = A_1 \times A_2 \subset \mathbb{K}^r \times \mathbb{K}^{n-r}$ and $p = (p_1, ..., p_n)$ following coordinates $(y_1, ..., y_n)$. Then there exist functions $f_k : A \times A' \times B \to \mathbb{K}$, $k = 1, ..., 2s$, only depending on x, $\bar{z}$ and u such that $f_k(A_1 \times \{(p_{r+1}, ..., p_n)\} \times A' \times B) = 0$ and $(\alpha_1 \circ J^{-1}) \wedge ... \wedge (\alpha_r \circ J^{-1}) \wedge (d_x f_k - \bar{\beta}_k \circ J^{-1}) = 0$. Moreover $d_{\bar{z}}(\sum_{k=1}^{2s} f_k d\bar{z}_k) = 0$ where $d_{\bar{z}}$ is the exterior derivative with respect to $\bar{z} = (\bar{z}_1, ..., \bar{z}_{2s})$. Indeed, $d_x(\partial f_k/\partial \bar{z}_j - \partial f_j/\partial \bar{z}_k)$, $j, k = 1, ..., 2s$, equals $(\partial \bar{\beta}_k/\partial \bar{z}_j - \partial \bar{\beta}_j/\partial \bar{z}_k) \circ J^{-1} = 0$ modulo $\alpha_1 \circ J^{-1}, ..., \alpha_r \circ J^{-1}$, that is modulo $dy_1, ..., dy_r$; in other words $\partial f_k/\partial \bar{z}_j - \partial f_j/\partial \bar{z}_k$ does not depend on $(y_{r+1}, ..., y_n)$. But clearly $\partial f_k/\partial \bar{z}_j - \partial f_j/\partial \bar{z}_k$ vanishes on $A_1 \times \{(p_{r+1}, ..., p_n)\} \times A' \times B$ so $\partial f_k/\partial \bar{z}_j - \partial f_j/\partial \bar{z}_k = 0$, $j, k = 1, ..., 2s$.

Thus there is a function $\psi_1 : A \times A' \times B \to \mathbb{K}$ only depending on x, $\bar{z}$ and u such that $\partial \psi_1/\partial \bar{z}_k = f_k$, $k = 1, ..., 2s$. Therefore $\alpha_1 \wedge ... \wedge \alpha_r \wedge (d_x(\partial \psi_1/\partial \bar{z}_k) \circ J - \bar{\beta}_k) = 0$.

Now set $\widetilde{\varphi} = \psi + \psi_1$. Then $D\widetilde{\varphi} \circ H = D\psi \circ H = Dg$ and $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(d\widetilde{\varphi} \circ G)$ is semi-basic for $\mathcal{A}$, that is $\widetilde{\varphi}$ satisfies (I).

Finally, let $\widetilde{\varphi}_1$ be the function given by

$$\widetilde{\varphi}_1(x, z, u) = \sum_{k=1}^{2s} (\bar{z}_k - \bar{z}_k(q)) \frac{\partial \widetilde{\varphi}}{\partial \bar{z}_k}(p, q, u) + \widetilde{\varphi}(p, q, u);$$

then $\varphi = \widetilde{\varphi} - \widetilde{\varphi}_1$ satisfies (I) and (II). $\square$

A *box* [of coordinates (z^i_j)] will mean a block of coordinates $(z^i_1, ..., z^i_{2m_i})$ for any fixed i; so one has s boxes. A box will be named *short* if $m_i = 1$ and *long* otherwise.

Consider a function $h : A \times A' \times B \to \mathbb{K}$ such that $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dh \circ G)$ is semi-basic for $\mathcal{A}$. Then there exists a function $\widetilde{h}$, perhaps after shrinking A', such that $Dh \circ H = D\widetilde{h}$. From lemma 1.1, applied to Dh and H along $\mathcal{A}$, follows $D(D\widetilde{h} \circ H) = D(Dh \circ H^2) = 0$, that is $\mu_{\widetilde{h}} = 0$ in the terminology of the proof of lemma 4.1. Moreover $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(\widetilde{dh} \circ G)$ is semi-basic for $\mathcal{A}$ since the coefficient of each dz_j^i in the expression of $\rho_{\widetilde{h}}$ equals that of some $dz_{\bar{b}}^{\bar{a}}$ in the expression of ρ_h.

Observe that $\widetilde{h}$ does not depend on the short boxes. By lemma 4.2, applied to $\widetilde{h}$ but considering long boxes only, there exists a function $h_1 : A \times A' \times B \to \mathbb{K}$ [after shrinking we identify $A \times A' \times B$ and $U \times U' \times V$ for the sake of simplicity of the notation], which does not depend on the short boxes, such that $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dh_1 \circ G)$ is semi-basic for $\mathcal{A}$ and $Dh_1 \circ H = Dh \circ H$. Now set $h_2 = h - h_1$; then $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dh_2 \circ G)$ is semi-basic for $\mathcal{A}$ and $Dh_2 \circ H = 0$.

In other words, after shrinking A, A' and B if necessary, the function h decompose into a sum $h = h_1 + h_2$ in such a way that $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dh_1 \circ G)$ and $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dh_2 \circ G)$ are semi-basic for $\mathcal{A}$, h_1 only depend on x, u and the long boxes, and h_2 does on $(x, \bar{z}_1, ..., \bar{z}_{2s}, u)$ that is $Dh_2 \circ H = 0$.

Moreover $h_1(\{(p,q)\} \times B) = 0$ and $Dh_1(p,q,u) = 0$ whenever $(Dh \circ H)(p,q,u) = 0$.

Consider a function $\varphi : A \times A' \times B \to \mathbb{K}$ such that $D\varphi \circ H = 0$ and $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(d\varphi \circ G)$ is semi-basic for $\mathcal{A}$. If there is one long box at least, then *Z and $Z + X_\varphi$ are equivalent for the purpose of proposition 4.1.* Let us see that. On $A \times A' \times (B \times \mathbb{K})$ consider the vector field $Z_t = Z + tX_\varphi$ and the obvious extensions of G, ω, ω_1 and $\mathcal{A}$, where now the space of parameters is $B \times \mathbb{K}$ endowed with coordinates $(u_1, ..., u_{\bar{s}}, t)$. Note that the vector field Z_t is H-generic on $\{(p,q)\} \times (K \times \mathbb{K})$ since $X_\varphi \in KerH$ and $H \neq \{0\}$.

By the remark preceding lemma 4.2 applied to $a = c'$, c, c', Z_t, φ and the compact set $K' = K \times [0,1]$, as $D\varphi \circ H = 0$ there exists a function f such that $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(df \circ G)$ is semi-basic for $\mathcal{A}$, $Df \circ H = 0$, $D(Z_t f - c'f - \varphi) = 0$ and $Df = 0$ on $\{(p,q)\} \times K'$. Then $(L_{X_f} G) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$, $L_{X_f}\omega = L_{X_f}\omega_1 = 0$ since $\omega_1 = \omega(G, \quad)$, and $[X_f, Z_t] = -X_\varphi$ because $L_{Z_t}\omega = L_Z\omega = c'\omega$ and $i_{[X_f, Z_t]}\omega = i_{X_f} L_{Z_t}\omega - L_{Z_t}(i_{X_f}\omega) = D(Z_t f - c'f) = D\varphi$.

Let $\Psi_{\bar{t}}$ be the flow of the time depending vector field X_f. As $X_{f|\{(p,q)\} \times K'} = 0$, Ψ_1 is defined around $\{(p,q)\} \times K$, preserves $\mathcal{A}, \omega, \omega_1$, and transforms Z in $Z + X_\varphi$ and G in $G + \xi$ where $\xi = \sum_{j=1}^r X_j \otimes \alpha_j$ and $X_1, ..., X_r \in \mathcal{A}$. As ξ is irrelevant, that is the problem is the same for G and $G + \xi$, the equivalence is established.

Coming back to the main question, suppose $m_i \geq 2$ when $i = 1, ..., s'$ and $m_i = 1$ otherwise. Set $Z = Z_1 + Z_2$ where Z_1 corresponds to the long boxes and Z_2 to the sort ones. Remark that Z_1 is H-generic since $Z_2 \in KerH$. On the other hand set

$$\widetilde{Z} = \sum_{i=1}^{s} \left[c \sum_{k=1}^{m_i} \left(-kz^i_{2k-1} \frac{\partial}{\partial z^i_{2k-1}} + kz^i_{2k} \frac{\partial}{\partial z^i_{2k}} \right) + \frac{c'}{2} \sum_{k=1}^{2m_i} z^i_k \frac{\partial}{\partial z^i_k} \right].$$

Then $L_{\widetilde{Z}}\omega = c'\omega$ and $L_{\widetilde{Z}}G = cH$ so $(L_{\widetilde{Z}}G - cH) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$. Thus, after shrinking A, A' and B if necessary, there exists a function $h = h_1 + h_2$ such that $\widetilde{Z} = Z + X_h$, $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dh_1 \circ G)$ and $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dh_2 \circ G)$ are semi-basic for $\mathcal{A}$, h_1 does not depend on the short boxes and $Dh_2 \circ H = 0$ (indeed, $L_{\widetilde{Z}}\omega = c'\omega = L_Z\omega$ implies that $\widetilde{Z} = Z + X_h$ for some h; now decompose this function into a sum $h = h_1 + h_2$ as it was showed before).

Since the components of X_{h_1} in the short boxes vanish, $\widetilde{Z} = Z + X_{h_1} + X_{h_2}$ and the vector fields Z, $Z + X_{h_2}$ are equivalent, we may assume

$$Z_2 = \sum_{i=s'+1}^{s} \left[c \sum_{k=1}^{m_i} \left(-kz^i_{2k-1} \frac{\partial}{\partial z^i_{2k-1}} + kz^i_{2k} \frac{\partial}{\partial z^i_{2k}} \right) + \frac{c'}{2} \sum_{k=1}^{2m_i} z^i_k \frac{\partial}{\partial z^i_k} \right],$$

which implies that the coefficients functions of Z_1 do not depend on the short boxes, otherwise $L_Z\omega \neq c'\omega$.

Decompose g into a sum $g = g_1 + g_2$ in such a way that g_1 does not depend on the short boxes, $Dg_2 \circ H = 0$ and $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dg_1 \circ G)$, $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(dg_2 \circ G)$ are semi-basic for $\mathcal{A}$. As proposition 4.1 was already proved for g_2 since $Dg_2 \circ H = 0$, it suffices to show it for g_1. But Z_1 is H-generic and its components do not depend on the short boxes, therefore it is enough considering the problem on the long boxes only.

In other words, we may suppose that there is no short box. Note that in this case $KerH \subset ImH$, therefore if τ is a 1-form such that $(\tau \circ H)(KerH^2) = 0$ then $\tau(KerH) = 0$.

After shrinking A, A' and B if necessary, consider a function $\widetilde{g}$ such that $D\widetilde{g} = Dg \circ H$; then $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(d\widetilde{g} \circ G)$ is semi-basic for $\mathcal{A}$ (it suffices reasoning as in the case $D\widetilde{h} = Dh \circ H$). On the quotient $A \times (A'/KerH) \times B$ one may project $\widetilde{g}$, Z, G, $\mathcal{A}$, H and the 2-forms ω_1, which becomes symplectic, and $\omega_1(G, \quad)$. Then the order of nilpotency of the projection $\widetilde{H}$ of H equals $\widetilde{k} - 1$ and by the induction hypothesis there is a solution of the problem for the scalars $a+c$, c and $c+c'$ [now $L_Z\omega_1 = L_Z(\omega(G, \quad)) = (c+c')\omega_1$ and the same equality holds on the quotient $A \times (A'/KerH) \times B$] and the projection of $\widetilde{g}$. Pulling-back this solution yields a function $\widetilde{f} : \widetilde{U} \times \widetilde{U}' \times \widetilde{V} \to \mathbb{K}$ such that $D\widetilde{f}(KerH) = 0$, $D(Z\widetilde{f} - [a+c]\widetilde{f} - \widetilde{g})(KerH^2) = 0$ since the pull-back of $Ker\widetilde{H}$ is $KerH^2$, $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(d\widetilde{f} \circ G)$ is semi-basic for $\mathcal{A}$ and $D\widetilde{f} = 0$ on $\{(p,q)\} \times \widetilde{V}$.

Let $f : U \times U' \times V \to \mathbb{K}$ a function given by lemma 4.2 applied to $\widetilde{f}$. Then $Df = 0$ on $\{(p,q)\} \times V$ since $D\widetilde{f} = 0$ on this set; besides $D(Zf - af - g) \circ H = D(Z\widetilde{f} - [a+c]\widetilde{f} - \widetilde{g})$ because $(L_Z Df) \circ H = L_Z(Df \circ H) - cDf \circ H$. But $D(Z\widetilde{f} - [a+c]\widetilde{f} - \widetilde{g})(KerH^2) = 0$ therefore $D(Zf - af - g)(KerH) = 0$, *which finishes the proof of proposition 4.1.*

From now on and until the end of this section, consider an open set $A \subset \mathbb{K}^m$ endowed with coordinates $z = (z_1, ..., z_m)$, a manifold B whose points are denoted by u and on $A \times B$ the following objects:

$\mathcal{G}$: foliation defined by setting u constant,

d: exterior derivative along $\mathcal{G}$,

Z: vector field tangent to $\mathcal{G}$,

H: $(1,1)$-tensor field along $\mathcal{G}$.

Suppose that H is nilpotent and written with constant coefficients with respect to $(\partial/\partial z_j) \otimes dz_k$, $j,k = 1, ..., m$, where $(z_1, ..., z_m)$ are regarded as coordinates along $\mathcal{G}$.

Proposition 4.2. *Given an integer $k \geq 0$, a point $p \in A$, a compact set $K \subset B$, two scalars a, c and a function $g : A \times B \to \mathbb{K}$ such that:*

(1) $L_Z H = cH$,

(2) Z is H-generic on $\{p\} \times K$,

(3) $d(dg \circ H) = 0$, $dg(KerH^k) = 0$ and $g(\{p\} \times B) = 0$,

then there exist a product open set $U \times V \subset A \times B$, which contains $\{p\} \times K$, such that:

(I) $Zf = af + g$,

(II) $d(df \circ H) = 0$ and $df(KerH^k) = 0$,

(III) $df = 0$ on $\{p\} \times V$.

Let us proof proposition 4.2. Reasoning as in the proof of proposition 4.1

reduces the problem to the case $k = 0$. On the other hand, if $H = 0$ one has just a ordinary differential equation and it suffices considering a solution f that vanishes on a suitable transverse section of Z containing $\{p\} \times V$, where V is an open neighbourhood of K on B (note that $(Zf)(\{p\} \times B) = 0$ which implies $df = 0$ on $\{p\} \times V$).

Now suppose that proposition 4.2 holds up to the dimension $m - 1$ for any scalars a, c.

Lemma 4.3. *Given a function $h : A \times B \to \mathbb{K}$ such that $dh(KerH) = 0$ and $d(dh \circ H) = 0$, then there exist a product open set $U \times V \subset A \times B$, which contains $\{p\} \times K$, and a function $\varphi : U \times V \to \mathbb{K}$ such that:*
(I) $d\varphi \circ H = dh$ and $d(d\varphi \circ H) = 0$,
(II) $\varphi(\{p\} \times V) = 0$ and $d\varphi(p, u) = 0$ for every $u \in V$ such that $dh(p, u) = 0$.

Proof. Consider coordinates (z^i_j), $j = 1, ..., m_i$, $i = 1, ..., s$, on A, constant linear combination of $(z_1, ..., z_m)$, such that $H = \sum_{i=1}^{s} \sum_{k=1}^{m_i - 1} (\partial/\partial z^i_{k+1}) \otimes dz^i_k$; that is $dz^i_k \circ H = dz^i_{k-1}$ if $k \geq 2$ and $dz^i_1 \circ H = 0$. Therefore $\partial h/\partial z^i_{m_i} = 0$, $i = 1, ..., s$, and $\beta \circ H = dh$ where $\beta = \sum_{i=1}^{s} \sum_{k=1}^{m_i - 1} (\partial h/\partial z^i_k) dz^i_{k+1}$. On the other hand, after shrinking A, we may suppose that in these coordinates A is a polycylinder.

Now from lemma 1.1 follows that $\beta_{|ImH}$ is closed; therefore there exists a function $\psi : A \times B \to \mathbb{K}$ such that $(d\psi - \beta)_{|ImH} = 0$, whence $d\psi \circ H = dh$. Let $\widetilde{\psi}$ be the function given by $\widetilde{\psi}(z, u) = \sum_{i=1}^{s} (z^i_1 - z^i_1(p))(\partial\psi/\partial z^i_1)(p, u) + \psi(p, u)$; then $\varphi = \psi - \widetilde{\psi}$ satisfies (I) and (II). $\square$

Let us resume the proof of proposition 4.2. Since $d(dg \circ H) = 0$, after shrinking A if necessary, there is a function $\widetilde{g} : A \times B \to \mathbb{K}$ vanishing on $\{p\} \times B$ such that $d\widetilde{g} = dg \circ H$; moreover by lemma 1.1 $d(d\widetilde{g} \circ H) = 0$. The equation $Z\widetilde{f} = (a + c)\widetilde{f} + \widetilde{g}$ has some solution satisfying (II) and (III) and such that $d\widetilde{f}(KerH) = 0$. Indeed, project Z, H and $\widetilde{g}$ on the quotient $(A/KerH) \times B$, apply the induction hypothesis and then pull-back a suitable solution. Note that $\widetilde{f}$ is defined on a product open set containing $\{p\} \times K$ and which we will call $A \times B$ for simplifying.

Let $\varphi : U \times V \to \mathbb{K}$ be a function given by lemma 4.3 applied to $\widetilde{f}$. Set $g_0 = g + a\varphi - Z\varphi$; then $g_0(\{p\} \times V) = 0$ and $dg_0 \circ H = 0$ since $(L_Z d\varphi) \circ$

$H = L_Z(d\varphi \circ H) - cd\varphi \circ H$. In turn, the equation $Zf_0 = af_0 + g_0$ has some solution satisfying (II) and (III). Indeed, project Z, H and g_0 on the quotient $(U/ImH) \times V$ and reason as before. Now it suffices to set $f = f_0 + \varphi$ *for finishing the proof of proposition 4.2.*

5. The non-real eigenvalue case

In this section $\mathbb{K} = \mathbb{R}$ and the manifolds considered will be real unless another thing is stated. Let $(\mathcal{F}, \ell, \omega, \omega_1)$ be a Veronese flag on a manifold P or at some point of P, $\mathcal{A}$ the foliation of the largest ℓ-invariant vector subspace (see section 1) and $\pi : P \to N$ a local quotient of P by $\mathcal{A}$. Set $codim\mathcal{F} = r$, $dim\mathcal{A} = 2m$ and $dimN = n$. Recall that N is endowed with a r-codimensional Veronese web whose limit when $t \to \infty$ equals the quotient foliation $\mathcal{F}' = \mathcal{F}/\mathcal{A}$ and ℓ projects in the morphism ℓ' associated to this Veronese web.

Suppose that the characteristic polynomial φ of $\ell_{|\mathcal{A}}$ equals $(t^2 + ft + g)^m$ where $f^2 < 4g$, that is φ has no real roots. Set $g_k = trace((\ell_{|\mathcal{A}})^k)$; by lemma 1.2 one has $kdg_{k+1} = (k+1)dg_k \circ \ell$ on $\mathcal{F}$.

When $df_{|\mathcal{A}(p)} \neq 0$ and the algebraic type of $\ell_{|\mathcal{A}}$ is constant about p, one may construct the symplectic reduction of the Veronese flag as follows. First observe that each g_k is function of g_1, g_2 since f, g are the only significant coefficients of the elementary divisors, $g_1 = -mf$, $g_2 = m(f^2 - 2g)$ and $dg_2 = 2dg_1 \circ \ell$ on $\mathcal{F}$. Therefore $X_{g_2} = 2\ell X_{g_1}$ and $(dg_1 \wedge dg_2)_{|\mathcal{A}(p)} \neq 0$, otherwise $(X_{g_1} \wedge X_{g_2})(p) = 0$ and ℓ has an eigenvalue on $\mathcal{A}(p) - \{0\}$. Thus $X_{g_1}, X_{g_2}, X_{g_3}, ...$ give rise to a ℓ-invariant vector sub-bundle E of dimension two.

On the other hand $\omega(X_{g_1}, X_{g_2}) = 2\omega(X_{g_1}, \ell X_{g_1}) = \omega_1(X_{g_1}, X_{g_1}) = 0$ and $\omega_1(X_{g_1}, X_{g_2}) = 2\omega(\ell X_{g_1}, \ell X_{g_1}) = 0$. Hence $X_{g_1}g_1 = X_{g_1}g_2 = X_{g_2}g_1 = X_{g_2}g_2 = 0$ and $[X_{g_1}, X_{g_2}] = 0$; in particular E is a foliation. Besides $L_{X_{g_1}}\ell = L_{X_{g_2}}\ell = 0$ since $kdg_{k+1} = (k+1)dg_k \circ \ell$ on $\mathcal{F}$.

Denoted by $\bar{P}$ and $\bar{\pi} : P \to \bar{P}$, respectively, the local quotient of P by E and its canonical projection. Consider coordinates $(y, z) = (y_1, ..., y_n, z_1, ..., z_{2m})$ around p such that $dy_1 = ... = dy_r = 0$ defines $\mathcal{F}$, $dy_1 = ... = dy_n = 0$ the foliation $\mathcal{A}$, $g_1 = z_{2m-1}$, $g_2 = z_{2m}$, $X_{g_1} = -\partial/\partial z_{2m-3}$ and $X_{g_2} = -\partial/\partial z_{2m-2}$. Thus $(y_1, ..., y_n)$ can be regarded as coordinates on N, $(y_1, ..., y_n, z_1, ..., z_{2m-4}, z_{2m-1}, z_{2m})$ as coordinates on $\bar{P}$ and g_1, g_2 as functions on this last manifold. Now it is obvious that $Ker(dg_1 \wedge dg_2)$ and $\mathcal{F} \cap Ker(dg_1 \wedge dg_2)$ project in two foliations $\bar{\mathcal{F}}_1$

and $\bar{\mathcal{F}}$ on $\bar{P}$, respectively, and $\ell_{|\mathcal{F}\cap Ker(dg_1\wedge dg_2)}$ does in a morphism $\bar{\ell}:\bar{\mathcal{F}}\to\bar{\mathcal{F}}_1$; moreover $(\bar{\mathcal{F}},\bar{\ell})$ is a weak Veronese flag along $\bar{\mathcal{F}}_1$ (locally any extension of $\bar{\ell}$ can be lifted to an extension of ℓ), whose foliation $\bar{\mathcal{A}}$ of the largest $\bar{\ell}$-invariant vector subspaces equals the projection of $\mathcal{A}\cap Ker(dg_1\wedge dg_2)$, $\bar{P}/\bar{\mathcal{A}}$ is identified to $N\times B$, where B is an open neighbourhood of $(g_1(p),g_2(p))$ on $\mathbb{R}^2$, and $\bar{\mathcal{F}}_1$ projects in the foliation of $N\times B$ by the first factor. Besides, the Veronese web induced by $(\bar{\mathcal{F}},\bar{\ell})$ on each leaf $N\times\{b\}$ of this last foliation equals the pull-back, by the first projection $\pi_1:N\times B\to N$, of that induced by $(\mathcal{F},\ell)$.

On the other hand, since $i_{X_{g_1}}\omega$, $i_{X_{g_2}}\omega$, $i_{X_{g_1}}\omega_1$ and $i_{X_{g_2}}\omega_1$ are functional combination of $dg_{1|\mathcal{A}}$, $dg_{2|\mathcal{A}}$ (recall that every g_k is function of g_1,g_2 and $kdg_{k+1}=(k+1)dg_k\circ\ell$ on $\mathcal{F}$) one has $Ker(\omega_{|\mathcal{A}\cap Ker(dg_1\wedge dg_2)})$
$=Ker(\omega_{1|\mathcal{A}\cap Ker(dg_1\wedge dg_2)})=E$. Therefore $\omega_{|\mathcal{A}\cap Ker(dg_1\wedge dg_2)}$ and $\omega_{1|\mathcal{A}\cap Ker(dg_1\wedge dg_2)}$ project in two symplectic forms $\bar{\omega},\bar{\omega}_1$ along $\bar{\mathcal{A}}$; moreover $\bar{\omega}_1=\bar{\omega}(\bar{\ell},\quad)$. The family $(\bar{\mathcal{F}},\bar{\ell},\bar{\omega},\bar{\omega}_1)$ will be called *the symplectic reduction (near p) of* $(\mathcal{F},\ell,\omega,\omega_1)$. As in section 3, for proving that this family is a Veronese flag it suffices to check the third condition of its definition.

On N consider coordinates $(x_1,...,x_n)$ and a $(1,1)$-tensor field
$J=\sum_{j=1}^n a_j(\partial/\partial x_j)\otimes dx_j$ where $a_1,...,a_n$ are real numbers.

Theorem 5.1. *In the real analytic category consider a $(1,1)$-tensor field G, which extends ℓ and projects in J, defined around a point p of P such that $(\mathcal{F},\ell,\omega,\omega_1)$ is a Veronese flag at this point. Assume that:*
(a) the characteristic polynomial of $\ell_{|\mathcal{A}}$ equals $(t^2+ft+g)^m$ where $f^2<4g$,
(b) p is a regular point of $\ell_{|\mathcal{A}}$,
(c) if $df_{|\mathcal{A}(p)}=0$ then f is constant close to p,
(d) if $df_{|\mathcal{A}(p)}\neq 0$ then the symplectic reduction of $(\mathcal{F},\ell,\omega,\omega_1)$ is a Veronese flag at $\bar{\pi}(p)$,
then around p there exist a $(1,1)$-tensor field G' extending ℓ and projecting in J and functions $z_1,...,z_{2m}$ such that $(x,z)=(x_1,...,x_n,z_1,...,z_{2m})$ is a system of coordinates,

$$G'=\sum_{j=1}^n a_j(\partial/\partial x_j)\otimes dx_j+\sum_{j,k=1}^{2m}h_{jk}(z)(\partial/\partial z_j)\otimes dz_k$$

and ω,ω_1 are expressed relative to $dz_{1|\mathcal{A}},...,dz_{2m|\mathcal{A}}$ with coefficient functions only depending on z.

Before proving this result, let us recall a few facts on the relationship between complex and real manifolds. Let Q be a real manifold of dimension $2k$ endowed with a complex structure H, which allows us to regard Q as a complex manifold of dimension k. A real tangent vector field at $q \in Q$ is a linear derivation of the algebra of germs at this point of differentiable functions; therefore it acts too on the germs at q of holomorphic functions and it can be regarded as a complex tangent vector at this point. In other words, the real and the complex tangent vector space at the same point may be identified in a canonical way.

In turn, if X is a real vector field then $L_X H = 0$ if and only if the (infinitesimal) action of X sends holomorphic functions into holomorphic functions. In terms of complex coordinates $\mathbf{z}_1 = \mathbf{x}_1 + \imath \mathbf{y}_1, ..., \mathbf{z}_k = \mathbf{x}_k + \imath \mathbf{y}_k$ and the associated real coordinates $(\mathbf{x}_1, \mathbf{y}_1, ..., \mathbf{x}_k, \mathbf{y}_k)$, if $X = \sum_{j=1}^{k} (\varphi_j \partial/\partial \mathbf{x}_j + \psi_j \partial/\partial \mathbf{y}_j)$ then $L_X H = 0$ if and only if $\varphi_1 + \imath\psi_1, ..., \varphi_k + \imath\psi_k$ are holomorphic functions of $\mathbf{z} = (\mathbf{z}_1, ..., \mathbf{z}_k)$; in this case from the complex viewpoint $X = \sum_{j=1}^{k} (\varphi_j + \imath\psi_j)\partial/\partial \mathbf{z}_j$ (warning this identification only works for the action of X on holomorphic functions but not for any complex-valued function). This kind of vector fields are named *holomorphic*.

A complex $\widetilde{k}$-form [that is of type $(\widetilde{k}, 0)$] β decompose into a sum $\beta = \beta_1 + \imath\beta_2$, where β_1, β_2 are real $\widetilde{k}$-forms such that $\beta_1(H\quad, \quad, ..., \quad) = -\beta_2$ and $\beta_2(H\quad, \quad, ..., \quad) = \beta_1$ [which implies $\beta_j(H\quad, \quad, ..., \quad) = \beta_j(\quad, H\quad, ..., \quad)$ $= \beta_j(\quad, \quad, ..., H\quad)$, $j = 1, 2$], and conversely. Besides β is holomorphic if and only if regarded from the real viewpoint $\beta_1(X_1, ..., X_{\widetilde{k}}) + \imath\beta_2(X_1, ..., X_{\widetilde{k}})$ is a holomorphic function whatever $X_1, ..., X_{\widetilde{k}}$ are holomorphic vector fields. In particular if β is a complex $\widetilde{k}$-form and its real part is closed then β is holomorphic and closed.

Finally, a holomorphic $(1,1)$-tensor field regarded from the real point of view is just a real $(1,1)$-tensor field that commutes with H and transforms holomorphic vector fields into holomorphic vector fields.

Until the end of this section one works in the analytic category, in which theorem 5.1 will be deduced from the complex case of theorems 2.1 and 3.1. We start constructing a complex structure along $\mathcal{A}$. Shrinking P if necessary, one may suppose that the algebraic type of $\ell_{|\mathcal{A}}$ and that of $\ell_{|\mathcal{A}\cap Ker(df\wedge dg)}$ are constant. Set $H_0 = (4g - f^2)^{-1/2}(2\ell_{|\mathcal{A}} + fI)$; then $(H_0^2 + I)^m = 0$. Therefore

H_0 is 0-deformable. Let H be its semi-simple part; by construction $H^2 = -I$ and there is a polynomial $\psi(t)$ with real coefficients such that $H = \psi(H_0)$, so $H = \tilde{\psi}(\ell_{|\mathcal{A}})$ for some polynomial $\tilde{\psi} \in \mathbb{R}_P[t]$. From section 6 of [13] applied on each leaf of $\mathcal{A}$ follows $N_H = 0$; in other words H is a complex structure along $\mathcal{A}$. Moreover $\omega(H, \quad) = \omega(\quad, H)$ and $\omega_1(H, \quad) = \omega_1(\quad, H)$ since $H = \tilde{\psi}(\ell_{|\mathcal{A}})$.

Thus the 2-forms $\Omega = \omega + \imath\widetilde{\omega}$ and $\Omega_1 = \omega_1 + \imath\widetilde{\omega}_1$, where $\widetilde{\omega}(X,Y) = -\omega(HX,Y)$ and $\widetilde{\omega}_1(X,Y) = -\omega_1(HX,Y)$, are holomorphic and closed because $d\omega = d\omega_1 = 0$. Observe that Ω and Ω_1 are symplectic, $\ell_{|\mathcal{A}} \circ H = H \circ \ell_{|\mathcal{A}}$ and $\Omega_1 = \Omega(\ell_{|\mathcal{A}}, \quad)$, so m is even and $\ell_{|\mathcal{A}}$ is holomorphic along $\mathcal{A}$.

As H is the semi-simple part of H_0 the tensor field $H_0 - H$ is nilpotent and commutes with H; therefore $(H_0 - H)^m = 0$. Hence $(\ell_{|\mathcal{A}} + \frac{1}{2}[fI - (4g - f^2)^{\frac{1}{2}}H])^m = 0$. In other words the complex polynomial $(t+h)^m$ where $h = \frac{1}{2}[f - \imath(4g-f^2)^{\frac{1}{2}}]$ annuls $\ell_{|\mathcal{A}}$, which implies that $(t+h)^m$ is the complex characteristic polynomial of the holomorphic tensor $\ell_{|\mathcal{A}}$. In particular h is holomorphic along $\mathcal{A}$ and $Kerdh = Kerdf \cap Kerdg = Kerdg_1 \cap Kerdg_2$.

Shrinking P allows us to suppose that the elementary divisors of $\ell_{|\mathcal{A}}$ on this manifold are $(t^2+ft+g)^{a_1}$,..., $(t^2+ft+g)^{a_{\tilde{k}}}$. Consider any point $q \in P$ and a cyclic decomposition $\mathcal{A}(q) = \mathcal{B} \oplus ... \oplus \mathcal{B}_{\tilde{k}}$ associated to these elementary divisors. Then $H\mathcal{B}_j = \mathcal{B}_j$ since $\ell\mathcal{B}_j \subset \mathcal{B}_j$; that is each $\mathcal{B}_j$ is a cyclic complex vector subspace. Now reasoning as before on every $\mathcal{B}_j$ at each $q \in P$ shows that $(t+h)^{a_1}$,..., $(t+h)^{a_{\tilde{k}}}$ are the complex elementary divisors of $\ell_{|\mathcal{A}}$. By the same reason if $(t^2+ft+g)^{b_1}$,..., $(t^2+ft+g)^{b_{k'}}$ are the elementary divisors of $\ell_{|\mathcal{A}\cap Kerdh}$ then $(t+h)^{b_1}$,..., $(t+h)^{b_{k'}}$ are the complex elementary divisors of $\ell_{|\mathcal{A}\cap Kerdh}$.

The analytic complex Frobenius theorem yields functions $z_1, ..., z_{2m}$ such that $(x_1, ..., x_n, z_1, ..., z_{2m})$ is a system of coordinates around p and
$H = \sum_{k=1}^{2m}((\partial/\partial z_{2k}) \otimes dz_{2k-1} - (\partial/\partial z_{2k-1}) \otimes dz_{2k})_{|\mathcal{A}}$. Now we may consider the complex coordinates $v_1 = z_1 + \imath z_2$,..., $v_m = z_{2m-1} + \imath z_{2m}$ and, after shrinking, identify P to a product open set $A \times B \subset \mathbb{R}^n \times \mathbb{C}^m$. In complex notation H equals $\imath I$ on $\mathcal{A}$.

Moreover functions $z_1, ..., z_{2m}$ can be choose in such a way that $\Omega = (dv_1 \wedge dv_2 + ... + dv_{m-1} \wedge dv_m)_{|\mathcal{A}}$, and $h = v_m$ if $df_{|\mathcal{A}(p)} \neq 0$. Indeed, consider complex variables $u_1 = x_1 + \imath y_1$,..., $u_n = x_n + \imath y_n$ on an open set $A' \subset \mathbb{C}^n$ such that

$A' \cap \mathbb{R}^n = A$ and by means of the analyticity extend $\mathcal{A}$, Ω and h, in the obvious way, to an open neighbourhood of $(p, 0)$ on $A' \times B$. Then apply the Darboux theorem for obtaining suitable coordinates $(u_1, ..., u_n, \widetilde{v}_1, ..., \widetilde{v}_m)$ and, finally, restraint functions $\widetilde{v}_1, ..., \widetilde{v}_m$ to $P = A \times B$.

On the other hand $G = \sum_{j=1}^n a_j(\partial/\partial x_j) \otimes dx_j + \sum_{j,k=1}^{2m} g_{jk}(\partial/\partial z_j) \otimes dz_k + \sum_{j=1}^n X_j \otimes dx_j$ where $X_1, ..., X_n \in \mathcal{A}$.

After changing the other of variables $x_1, ..., x_n$ and shrinking A, we may assume that $dx_1 \wedge ... \wedge dx_{n-r} \wedge \alpha_1 \wedge ... \wedge \alpha_r$ has no zeros. Set $T = \sum_{j=1}^n X_j \otimes dx_j$. Since it is enough proving theorem 5.1 for some $G + \sum_{k=1}^r Y_k \otimes \alpha_k$ where $Y_1, ..., Y_r \in \mathcal{A}$, one can suppose $X_{n-r+1} = ... = X_n = 0$. Thus there exist vector fields $X'_1, ..., X'_n \in \mathcal{F}$ functional combinations of $\partial/\partial x_1, ..., \partial/\partial x_n$ whose coefficients do not depend on z such that $TX'_j = X_j$, $j = 1, ..., n$. Observe that $L_{X'_j} H = 0$.

But $N_G(\mathcal{F}, \mathcal{F}) = 0$ so $N_G(X'_j, \mathcal{A}) = 0$, whence $G \circ L_{X'_j}(G_{|\mathcal{A}}) - L_{(JX'_j + X_j)}(G_{|\mathcal{A}}) = 0$ that is $L_{X_j}(G_{|\mathcal{A}}) = G \circ L_{X'_j}(G_{|\mathcal{A}}) - L_{JX'_j}(G_{|\mathcal{A}})$. As JX'_j is a functional combination of $\partial/\partial x_1, ..., \partial/\partial x_n$ with coefficients only depending on x one has $L_{JX'_j} H = 0$. By construction $G_{|\mathcal{A}}$ and H commute, therefore $L_{X'_j}(G_{|\mathcal{A}})$ and $L_{JX'_j}(G_{|\mathcal{A}})$ commute with H and from the expression above follows that $L_{X_j}(G_{|\mathcal{A}})$ does too.

On the other hand $H = \widetilde{\psi}(G_{|\mathcal{A}})$ so $L_{X_j} H$ equals a polynomial in $G_{|\mathcal{A}}$ and $L_{X_j}(G_{|\mathcal{A}})$, which implies that $H \circ L_{X_j} H = (L_{X_j} H) \circ H$. In turn from $H^2 = -I$ follows $H \circ L_{X_j} H = -(L_{X_j} H) \circ H$, therefore $L_{X_j} H = 0$ since H is invertible. In short, we may assume $X_1, ..., X_n$ holomorphic without loss of generality. Now in complex notation one has $G = \sum_{j=1}^n a_j(\partial/\partial x_j) \otimes dx_j + \sum_{j,k=1}^m h_{jk}(\partial/\partial v_j) \otimes dv_k + \sum_{j=1}^n (\sum_{k=1}^m h'_{jk} \partial/\partial v_k) \otimes dx_j$ where h_{jk} and h'_{jk} are holomorphic along $\mathcal{A}$.

If as before we consider complex variables $u_1 = x_1 + \imath y_1$,..., $u_n = x_n + \imath y_n$ on an open set $A' \subset \mathbb{C}^n$ such that $A' \cap \mathbb{R}^n = A$, then through the analyticity $\alpha_1, ..., \alpha_r$, $\mathcal{F}$, ℓ, $\mathcal{A}$, Ω, Ω_1 and G may be extended to similar holomorphic objects $\widetilde{\alpha}_1, ..., \widetilde{\alpha}_r$, $\widetilde{\mathcal{F}}$, $\widetilde{\ell}$, $\widetilde{\mathcal{A}}$, $\widetilde{\Omega}$, $\widetilde{\Omega}_1$ and $\widetilde{G}$, which are defined on an open neighbourhood $\widetilde{P}$ of $\widetilde{p} = (p, 0)$ on $A' \times B$. In particular $\widetilde{\alpha}_1, ..., \widetilde{\alpha}_r$ defines $\widetilde{\mathcal{F}}$, $du_1 = ... = du_n = 0$ defines $\widetilde{\mathcal{A}}$ and $\widetilde{\Omega} = (dv_1 \wedge dv_2 + ... + dv_{m-1} \wedge dv_m)_{|\widetilde{\mathcal{A}}}$. After shrinking $\widetilde{P}$ if necessary, it is easily checked that $(\widetilde{\mathcal{F}}, \widetilde{\ell}, \widetilde{\Omega}, \widetilde{\Omega}_1)$ verifies the two first conditions of Veronese flag.

Observe that given a holomorphic function $\mu = \mu_1 + \imath\mu_2$ then its $\widetilde{\Omega}$-hamiltonian X_μ equals the $\omega_{\mathbb{R}}$-hamiltonian of μ_1 where $\omega_{\mathbb{R}}$ is the real part of $\widetilde{\Omega}$ (note that $\omega_{\mathbb{R}} = \omega$ on P). On the other hand if μ is defined around $\widetilde{p}$ and $\widetilde{\ell}^* d\mu$ is closed on $\widetilde{\mathcal{F}}$, that is $\widetilde{\alpha}_1 \wedge ... \wedge \widetilde{\alpha}_r \wedge d(d\mu \circ \widetilde{G}) = 0$, then $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(d\mu_1 \circ G) = 0$ on P which implies $\alpha_1 \wedge ... \wedge \alpha_r \wedge L_{X_\mu} G = 0$ around p on P; therefore $\widetilde{\alpha}_1 \wedge ... \wedge \widetilde{\alpha}_r \wedge L_{X_\mu} \widetilde{G} = 0$ since this last tensor field is the extension of $\alpha_1 \wedge ... \wedge \alpha_r \wedge L_{X_\mu} G$. Hence $L_{X_\mu} \widetilde{\ell} = 0$ near $\widetilde{p}$; in other words $(\widetilde{\mathcal{F}}, \widetilde{\ell}, \widetilde{\Omega}, \widetilde{\Omega}_1)$ is a Veronese flag at $\widetilde{p}$.

Set $\widetilde{G} = \sum_{j=1}^n a_j (\partial/\partial u_j) \otimes du_j + \sum_{j,k=1}^m \widetilde{h}_{jk} (\partial/\partial v_j) \otimes dv_k + \sum_{j=1}^n \widetilde{X}_j \otimes du_j$. Then $\widetilde{h}_{jk}$ is the prolongation to $\widetilde{P}$ of h_{jk} and $\widetilde{X}_j$ that of X_j. Consider the $m \times m$ matrix $\mathcal{M} = (h_{jk}) + hI$ and its prolongation $\widetilde{\mathcal{M}} = (\widetilde{h}_{jk}) + \widetilde{h}I$ where $\widetilde{h}$ is the prolongation of h, which equals v_m when h is not constant. Recall that $(t+h)^{a_1}$,..., $(t+h)^{a_{\tilde{k}}}$ are the complex elementary divisors of $\ell_{|\mathcal{A}}$ on P; since these elementary divisors up to change of order are determined by $dim Ker(\ell_{|\mathcal{A}} + hI)^a$, $a = 1, ..., m$, these dimensions have to be constant, that is to say each function $rank \mathcal{M}^a$, $a = 1, ..., m$, is constant on P. This fact implies that $rank \widetilde{\mathcal{M}}^a$, $a = 1, ..., m$, is constant near $\widetilde{p}$ on $\widetilde{P}$ and equals $rank \mathcal{M}^a$.

Indeed, as $\widetilde{\mathcal{M}}(\widetilde{p}) = \mathcal{M}(p)$ one has $rank \widetilde{\mathcal{M}}^a \geq rank \mathcal{M}^a$. Let $\widetilde{\rho}$ be a minor of $\widetilde{\mathcal{M}}^a$ and ρ the similar one of $\mathcal{M}^a$. Then $\widetilde{\rho}$ is the prolongation of ρ, so $\widetilde{\rho}$ vanishes on $\widetilde{P}$ if ρ does on P. Therefore $rank \widetilde{\mathcal{M}}^a \leq rank \mathcal{M}^a$. Thus $dim Ker(\widetilde{\ell}_{|\widetilde{\mathcal{A}}} + \widetilde{h}I)^a = dim Ker(\ell_{|\mathcal{A}} + hI)^a$, $a = 1, ..., m$, and consequently $(t + \widetilde{h})^{a_1}$,..., $(t + \widetilde{h})^{a_{\tilde{k}}}$ are the elementary divisors of $\widetilde{\ell}_{|\widetilde{\mathcal{A}}}$ closed to $\widetilde{p}$.

A similar argument shows that $(t + \widetilde{h})^{b_1}$,..., $(t + \widetilde{h})^{b_{k'}}$ are the elementary divisors of $\widetilde{\ell}_{|\widetilde{\mathcal{A}} \cap Ker dh}$ closed to $\widetilde{p}$. In short, the point $\widetilde{p}$ is regular for $\widetilde{\ell}_{|\widetilde{\mathcal{A}}}$.

When h is not constant, in coordinates (u, v) one has $\widetilde{h} = v_m$ and $\widetilde{\Omega} = (dv_1 \wedge dv_2 + ... + dv_{m-1} \wedge dv_m)_{|\widetilde{\mathcal{A}}}$; in particular $\omega = \sum_{j=1}^{m/2} (dz_{4j-3} \wedge dz_{4j-1} - dz_{4j-2} \wedge dz_{4j})_{|\mathcal{A}}$. Thus the symplectic reduction of $(\widetilde{\mathcal{F}}, \widetilde{\ell}, \widetilde{\Omega}, \widetilde{\Omega}_1)$ can be identified to the extension by means of complex variables $u_1 = x_1 + \imath y_1$,..., $u_n = x_n + \imath y_n$ of the symplectic reduction of $(\mathcal{F}, \ell, \omega, \omega_1)$. Indeed, g_1, g_2 are function of z_{2m-1}, z_{2m} only so E is spanned by $\partial/\partial z_{2m-3}, \partial/\partial z_{2m-2}$ and from the complex viewpoint it is the foliation spanned by $\partial/\partial v_{m-1}$. Since the symplectic reduction of $(\mathcal{F}, \ell, \omega, \omega_1)$ is a Veronese flag at $\bar{\pi}(p)$, the symplectic reduction of $(\widetilde{\mathcal{F}}, \widetilde{\ell}, \widetilde{\Omega}, \widetilde{\Omega}_1)$ is a Veronese flag at $(\bar{\pi}(p), 0)$, which is the image of $\widetilde{p} = (p, 0)$ by the canonical projection [just adapt the argument showing that $(\widetilde{\mathcal{F}}, \widetilde{\ell}, \widetilde{\Omega}, \widetilde{\Omega}_1)$ verifies condition 3') at $\widetilde{p}$].

Since $\widetilde{h}(\widetilde{p}) \notin \mathbb{R}$ from theorems 2.1 and 3.1 follow the existence around $\widetilde{p}$ of a $(1,1)$-tensor field $\widetilde{G}'$ extending $\widetilde{\ell}$ and projecting in $\sum_{j=1}^{n} a_j(\partial/\partial u_j) \otimes du_j$ and functions $w_1, ..., w_m$ such that $(u, v) = (u_1, ..., u_n, w_1, ..., w_m)$ is a system of coordinates,

$$\widetilde{G}' = \sum_{j=1}^{n} a_j(\partial/\partial u_j) \otimes du_j + \sum_{j,k=1}^{m} \theta_{jk}(w)(\partial/\partial w_j) \otimes dw_k$$

and $\widetilde{\Omega}, \widetilde{\Omega}_1$ are expressed with coefficient functions only depending on w (constant when h is constant).

Finally, set $w_1 = \widetilde{z}_1 + \imath\widetilde{z}_2$,..., $w_m = \widetilde{z}_{2m-1} + \imath\widetilde{z}_{2m}$ and observe that $\widetilde{G}'(TP) \subset TP$; therefore the restriction to P of $\widetilde{G}'$ defines a $(1,1)$-tensor field G' which projects in J and extends ℓ. Now for finishing the proof of theorem 5.1 it is enough considering G' and functions $\widetilde{z}_1, ..., \widetilde{z}_{2m}$.

6. The blocks of a Veronese flag

The aim of this section is to reduce the local study of Veronese flags to the case where their characteristic polynomial is a power of an irreducible one. Let $(\mathcal{F}, \ell, \omega, \omega_1)$ be a Veronese flag on a manifold P or at some point of P, $\mathcal{A}$ the foliation of the largest vector subspaces and $\pi : P \to N$ a local quotient of P by $\mathcal{A}$. Set $codim\mathcal{F} = r$, $dim\mathcal{A} = 2m$ and $dimN = n$. On N consider coordinates $(x_1, ..., x_n)$, closed 1-forms $\alpha_1, ..., \alpha_r$ and a $(1,1)$-tensor field $J = \sum_{j=1}^{n}(\partial/\partial x_j) \otimes dx_j$, where $a_1, ..., a_n \in \mathbb{K}$, such that the associated Veronese web is given by $J, \alpha_1, ..., \alpha_r$ and $dx_1 \wedge ... \wedge dx_{n-r} \wedge \alpha_1 \wedge ... \wedge \alpha_r$ never vanishes.

Assume that on an open neighbourhood of a regular point p of $\ell_{|\mathcal{A}}$ the characteristic polynomial φ of $\ell_{|\mathcal{A}}$ is the product of two monic relatively prime polynomials φ_1 and φ_2. Then $Im\varphi_1(\ell_{|\mathcal{A}}) = Ker\varphi_2(\ell_{|\mathcal{A}})$, $Im\varphi_2(\ell_{|\mathcal{A}}) = Ker\varphi_1(\ell_{|\mathcal{A}})$ and $\mathcal{A} = Im\varphi_1(\ell_{|\mathcal{A}}) \oplus Im\varphi_2(\ell_{|\mathcal{A}})$; moreover $Im\varphi_1(\ell_{|\mathcal{A}})$ and $Im\varphi_2(\ell_{|\mathcal{A}})$ are foliations because $N_\ell = 0$ (apply lemma 2 of [13]). Thus around p there exist coordinates $(x, z, \widetilde{z}) = (x_1, ..., x_n, z_1, ..., z_{m'}, \widetilde{z}_1, ..., \widetilde{z}_{\widetilde{m}})$ such that $Im\varphi_1(\ell_{|\mathcal{A}})$ is spanned by $\partial/\partial z_1, ..., \partial/\partial z_{m'}$ and $Im\varphi_2(\ell_{|\mathcal{A}})$ is spanned by $\partial/\partial\widetilde{z}_1, ..., \partial/\partial\widetilde{z}_{\widetilde{m}}$.

On the other hand $\omega_k(Im\varphi_1(\ell_{|\mathcal{A}}), Im\varphi_2(\ell_{|\mathcal{A}})) = \omega_k(Im\varphi(\ell_{|\mathcal{A}}), \quad) = 0$, $k = 0, 1$, where by definition $\omega_0 = \omega$. Hence

$$\omega_k = \sum_{1 \le i < j \le m'} f_{ijk}(x, z) dz_i \wedge dz_j + \sum_{1 \le i < j \le \widetilde{m}} \widetilde{f}_{ijk}(x, \widetilde{z}) d\widetilde{z}_i \wedge d\widetilde{z}_j$$

because $d\omega_k = 0$. In particular m' and $\widetilde{m}$ are even since ω_0 is symplectic.

Therefore if G is a $(1,1)$-tensor field extending ℓ and projecting in J one has $G = J + \sum_{j,k=1}^{m'} h_{jk}(x,z)(\partial/\partial z_j) \otimes dz_k + \sum_{j,k=1}^{\widetilde{m}} \widetilde{h}_{jk}(x,\widetilde{z})(\partial/\partial \widetilde{z}_j) \otimes d\widetilde{z}_k + \sum_{j=1}^{n} X_j \otimes dx_j$ where $X_1, ..., X_n \in \mathcal{A}$.

Let us see that G may be chosen in such a way that $X_1, ..., X_n$ are foliate both for $Im\varphi_1(G_{|\mathcal{A}})$ and $Im\varphi_2(G_{|\mathcal{A}})$. Set $T = \sum_{j=1}^{n} X_j \otimes dx_j$. By considering $G + \sum_{j=1}^{r} Y_j \otimes \alpha_j$ instead of G where $Y_1, ..., Y_r$ are suitable vector fields tangent to $\mathcal{A}$, we can suppose $X_{n-r+1} = ... = X_n = 0$ without loss of generality. Thus locally there exist $X'_1, ..., X'_n \in \mathcal{F}$ functional combinations of $\partial/\partial x_1, ..., \partial/\partial x_n$ with coefficients only depending on x such that $TX'_j = X_j$, $j = 1, ..., n$.

As $Im\varphi_1(G_{|\mathcal{A}})$ is spanned by $\partial/\partial z_1, ..., \partial/\partial z_{m'}$ and $Ker\varphi_1(G_{|\mathcal{A}})$ by $\partial/\partial \widetilde{z}_1$, ..., $\partial/\partial \widetilde{z}_{\widetilde{m}}$, the morphism $\varphi_1(G_{|\mathcal{A}}) : Im\varphi_1(G_{|\mathcal{A}}) \to Im\varphi_1(G_{|\mathcal{A}})$ is in fact an isomorphism whose inverse equals $\psi(\varphi_1(G_{|\mathcal{A}}))$ for some polynomial $\psi(t)$ [indeed, if $t^{m'} + \sum_{j=0}^{m'-1} g_j t^j$ is the characteristic polynomial of $\varphi_1(G_{|\mathcal{A}})$ restricted to $Im\varphi_1(G_{|\mathcal{A}})$ set $\psi(t) = -g_0^{-1}(t^{m'-1} + \sum_{j=1}^{m'-1} g_j t^{j-1})$]. Therefore $\sum_{j=1}^{m'} (\partial/\partial z_j) \otimes dz_{j\,|\mathcal{A}} = \rho(G_{|\mathcal{A}})$ where $\rho(t) = \varphi_1(t) \cdot \psi(\varphi_1(t))$.

Analogously there is a polynomial $\widetilde{\rho}(t)$ such that $\sum_{k=1}^{\widetilde{m}} (\partial/\partial \widetilde{z}_k) \otimes d\widetilde{z}_{k|\mathcal{A}} = \widetilde{\rho}(G_{|\mathcal{A}})$. Set $H = \sum_{j=1}^{m'} (\partial/\partial z_j) \otimes dz_{j\,|\mathcal{A}} - \sum_{k=1}^{\widetilde{m}} (\partial/\partial \widetilde{z}_k) \otimes d\widetilde{z}_{k|\mathcal{A}}$; then $H = \psi_1(G_{|\mathcal{A}})$ where $\psi_1(t) = \rho(t) - \widetilde{\rho}(t)$. Observe that $L_X H = 0$ for any vector field X such that $X = \sum_{j=1}^{n} f_j(x)\partial/\partial x_j$.

As in section 5, from $N_G(\mathcal{F}, \mathcal{F}) = 0$ follows $L_{X_j}(G_{|\mathcal{A}}) = G \circ L_{X'_j}(G_{|\mathcal{A}}) - L_{JX'_j}(G_{|\mathcal{A}})$; therefore $L_{X_j}(G_{|\mathcal{A}})$ and H commute since $L_{X'_j} H = L_{JX'_j} H = 0$ and $(G_{|\mathcal{A}}) \circ H = H \circ (G_{|\mathcal{A}})$ because $H = \psi_1(G_{|\mathcal{A}})$. In turn $L_{X_j} H = L_{X_j}(\psi_1(G_{|\mathcal{A}}))$ is a polynomial in $G_{|\mathcal{A}}$ and $L_{X_j}(G_{|\mathcal{A}})$, which implies that H and $L_{X_j} H$ commute. On the other hand since $H^2 = I$ one has $H \circ L_{X_j} H = -(L_{X_j} H) \circ H$, so $H \circ L_{X_j} H = 0$ and finally $L_{X_j} H = 0$. Therefore each X_j is foliate for $Im\varphi_1(G_{|\mathcal{A}}) = Im(H + I)$ and $Im\varphi_2(G_{|\mathcal{A}}) = Im(H - I)$, that is $X_j = \sum_{i=1}^{m'} f_{ji}(x,z)\partial/\partial z_i + \sum_{k=1}^{\widetilde{m}} \widetilde{f}_{jk}(x,\widetilde{z})\partial/\partial \widetilde{z}_k$.

Now in variables (x,z) we can consider the foliation $\mathcal{F}'$ defined by $\alpha_1, ..., \alpha_r$, the $(1,1)$-tensor field

$G' = J + \sum_{j,k=1}^{m'} h_{jk}(x,z)(\partial/\partial z_j) \otimes dz_k + \sum_{j=1}^{n} (\sum_{i=1}^{m'} f_{ji}(x,z)\partial/\partial z_i) \otimes dx_j$,

and its restriction ℓ' to $\mathcal{F}'$, the 2-forms $\omega' = \sum_{1 \leq i < j \leq m'} f_{ij0}(x,z) dz_i \wedge dz_j$, $\omega'_1 = \sum_{1 \leq i < j \leq m'} f_{ij1}(x,z) dz_i \wedge dz_j$ (more exactly their restriction to $Im\varphi_1(G_{|\mathcal{A}})$ but we omit it for simplifying the notation) and the point p' corresponding to

p. It is easily checked that $(\mathcal{F}', \ell', \omega', \omega'_1)$ is a Veronese flag, respectively a Veronese flag at p', if that was the case of $(\mathcal{F}, \ell, \omega, \omega_1)$. Similarly in variables $(x, \widetilde{z})$ one may consider the foliation $\widetilde{\mathcal{F}}$ defined by $\alpha_1, ..., \alpha_r$, the $(1,1)$-tensor field $\widetilde{G} = J + \sum_{j,k=1}^{\widetilde{m}} \widetilde{h}_{jk}(x, \widetilde{z})(\partial/\partial \widetilde{z}_j) \otimes d\widetilde{z}_k + \sum_{j=1}^{n} (\sum_{k=1}^{\widetilde{m}} f_{jk}(x, \widetilde{z})\partial/\partial \widetilde{z}_k) \otimes dx_j$, and its restriction $\widetilde{\ell}$ to $\widetilde{\mathcal{F}}$, the 2-forms $\widetilde{\omega} = \sum_{1 \leq i < j \leq \widetilde{m}} \widetilde{f}_{ij0}(x, \widetilde{z}) d\widetilde{z}_i \wedge d\widetilde{z}_j$, $\widetilde{\omega}_1 = \sum_{1 \leq i < j \leq \widetilde{m}} \widetilde{f}_{ij1}(x, \widetilde{z}) d\widetilde{z}_i \wedge d\widetilde{z}_j$ and the point $\widetilde{p}$ corresponding to p; then $(\widetilde{\mathcal{F}}, \widetilde{\ell}, \widetilde{\omega}, \widetilde{\omega}_1)$ is a Veronese flag or a Veronese flag at $\widetilde{p}$ if that is case of $(\mathcal{F}, \ell, \omega, \omega_1)$.

Moreover p' is regular for $\ell'_{|\mathcal{A}'}$ and $\widetilde{p}$ for $\widetilde{\ell}_{|\widetilde{\mathcal{A}}}$ since p was regular for $\ell_{|\mathcal{A}}$, φ_2 is the characteristic polynomial of $\ell'_{|\mathcal{A}'}$ and φ_1 that of $\widetilde{\ell}_{|\widetilde{\mathcal{A}}}$. In a more technical way we will say that, around p, $(\mathcal{F}, \ell, \omega, \omega_1)$ is the fibered product over N, around p' and $\widetilde{p}$, of $(\mathcal{F}', \ell', \omega', \omega'_1)$ and $(\widetilde{\mathcal{F}}, \widetilde{\ell}, \widetilde{\omega}, \widetilde{\omega}_1)$.

Obviously one may reiterate the process until the characteristic polynomial of each factor is power of an irreducible one, which thus becomes the only case to take into account.

7. The local product theorem

In this section is showed that, around every point of some dense open set, an analytic bihamiltonian structure decomposes into a product of a Kronecker bihamiltonian structure and a symplectic one if a necessary condition stated later on holds (see [18]).

Consider a bihamiltonian structure (Λ, Λ_1) on a real or complex manifold M of dimension m. The set of all $p \in M$ such that $rank(\Lambda, \Lambda_1)$ is constant about p is open (obvious) and dense. Indeed, first recall that at any $q \in M$ $rank((1-t)\Lambda + t\Lambda_1)(q) = rank(\Lambda, \Lambda_1)(q)$ except for a finite number of scalars t, which is $\leq m/2$ (see section 1.2 of [17]). Now choose non-equal scalars $b_1, ..., b_k$ with $k \geq (m/2) + 2$; then the set of all $p \in M$ such that the rank of each $rank((1-b_j)\Lambda + b_j\Lambda_1)$, $j = 1, ..., k$, is locally constant at p is dense, open and contained in the foregoing open set.

For simplicity sake suppose $r = corank(\Lambda, \Lambda_1)$ locally constant. Since our problem is local, by considering $rank((1-b_j)\Lambda + b_j\Lambda_1)$ and $rank((1-b_{\widetilde{j}})\Lambda + b_{\widetilde{j}}\Lambda_1)$ for suitable indices $j, \widetilde{j}$ instead of Λ, Λ_1, we may assume maximal (Λ, Λ_1), that is $r = corank\Lambda = corank\Lambda_1 = corank(\Lambda, \Lambda_1)$, without loss of generality.

As in sub-section 1.1, for each $p \in M$ let $\mathcal{A}_1(p)$ be the intersection of all vector subspaces $Im(\Lambda + t\Lambda_1)(p)$, $t \in \mathbb{K}$, such that $rank(\Lambda + t\Lambda_1)(p) = m - r$.

From the algebraic model follows that the dimension of the symplectic factor at p equals $2dim\mathcal{A}_1(p) + r - m$. But if $c_1, ..., c_m$ are different scalars such that $rank(\Lambda + c_j\Lambda_1)(p) = m - r$, $j = 1, ..., m$, then $\mathcal{A}_1(p) = \cap_{j=1}^{m} Im(\Lambda + c_j\Lambda_1)(p)$ [see section 1.2 of [17] again]. By continuity $\mathcal{A}_1(q) = \cap_{j=1}^{m} Im(\Lambda + c_j\Lambda_1)(q)$ when q is close to p therefore $dim\mathcal{A}_1$ is a locally decreasing function, which implies that the dimension of $\mathcal{A}_1$ and that of the symplectic factor are locally constant on a dense open set.

Observe that if (Λ, Λ_1) decomposes into a product near p, then the dimension of the symplectic factor has to be constant close to p.

In short, suppose that on an open set $M' \subset M$ the bihamiltonian structure is maximal and its rank and the dimension of the symplectic factor are constant. Then, following sub-section 1.1, set $m = 2m' + 2n - r$ where $2m'$ is the dimension of the symplectic factor and consider the Veronese flag $(\mathcal{F}, \ell, \omega, \omega_1)$ on the local quotient P of M' by the secondary axis $\mathcal{A}_2$.

Given a (linear) symplectic form τ and a 2-form τ_1 on an even dimensional vector space V, let K be the endomorphism of V defined by $\tau_1 = \tau(K, \quad)$. By definition the characteristic polynomial of (τ, τ_1) will be that of K. Let $\widetilde{\varphi} = t^{2m'} + \sum_{j=0}^{2m'-1} \widetilde{h}_j t^j$ be the characteristic polynomial of the symplectic factor of (Λ, Λ_1) on M', that is $t^{2m'} + \sum_{j=0}^{2m'-1} \widetilde{h}_j(p) t^j$, for each $p \in M'$, is the characteristic polynomial of the symplectic factor of $(\Lambda(p), \Lambda_1(p))$ when regarded as a couple of (linear) symplectic forms.

On the other hand let $\varphi = t^{2m'} + \sum_{j=0}^{2m'-1} h_j t^j$ be the characteristic polynomial of $\ell_{|\mathcal{A}}$. By means of the algebraic model of $(\Lambda(p), \Lambda_1(p))$ it is not hard to see that the symplectic factor of $(\Lambda(p), \Lambda_1(p))$ is isomorphic to $(\omega(\pi_P(p)), \omega_1(\pi_P(p)))$. Thus the characteristic polynomial of $(\ell_{|\mathcal{A}})(\pi_P(p))$ equals $\widetilde{\varphi}(p)$, that is locally $\widetilde{h}_j = h_j \circ \pi_P$, $j = 0, ..., 2m' - 1$, which in particular shows the differentiability of $\widetilde{h}_0, ..., \widetilde{h}_{2m'-1}$.

Proposition 7.1. *The functions $\widetilde{h}_0, ..., \widetilde{h}_{2m'-1}$ are in involution both for Λ and Λ_1. Moreover $\{\Lambda(d\widetilde{h}_j, \quad)(p)\}_{j=0,...,2m'-1}$ and $\{\Lambda_1(d\widetilde{h}_j, \quad)(p)\}_{j=0,...,2m'-1}$ span the same vector subspace of T_pM' for any $p \in M'$.*

Proof. Let $\{\quad,\quad\}_\omega$ be the Poisson structure on P defined by $(\mathcal{A}, \omega)$ and $\{\quad,\quad\}_{\omega_1}$ that defined by $(\mathcal{A}, \omega_1)$. Recall that $\{\quad,\quad\}_\omega$ is the projection of Λ

and $\{\ ,\ \}_{\omega_1}$ that of Λ_1. Thus for proving the involution of $\widetilde{h}_0, ..., \widetilde{h}_{2m'-1}$ it is enough showing that $h_0, ..., h_{2m'-1}$ are in involution with respect to $\{\ ,\ \}_\omega$ and $\{\ ,\ \}_{\omega_1}$.

On the other hand by lemma 1.2, $kdg_{k+1} = (k+1)dg_k \circ \ell$ on $\mathcal{F}$ where $g_k = trace(\ell_{|\mathcal{A}})^k$, $k \geq 0$, whence $(k+1)\ell X_{g_k} = kX_{g_{k+1}}$. Therefore if $1 \leq k \leq \widetilde{k}$ one has $\omega(X_{g_{\widetilde{k}}}, X_{g_k}) = C(k, \widetilde{k}) \cdot \omega(\ell^{\widetilde{k}-k} X_{g_k}, X_{g_k}) = 0$ since $\omega(\ell^{\widetilde{k}-k},\ \ )$ is a 2-form on $\mathcal{F}$; so $\{g_{\widetilde{k}}, g_k\}_\omega = 0$. But $h_0, ..., h_{2m'-1}$ are function of $g_1, ..., g_k, ...$ (see section 3) therefore $\{h_i, h_j\}_\omega = 0$.

As it was pointed out in sub-section 1.1, from the algebraic model follows that $\Lambda_1(\pi_P^* \ell^* \beta,\ \ ) = \Lambda(\pi_P^* \beta,\ \ )$ for any $\beta \in T^*_{\pi_P(q)}P$ and $q \in M'$. Thus in our case $kdg_{k+1} = (k+1)dg_k \circ \ell$ on $\mathcal{F}$ implies $\Lambda(d(g_k \circ \pi_P),\ \ ) = k(k+1)^{-1}\Lambda_1(d(g_{k+1} \circ \pi_P),\ \ )$. Since $h_0, ..., h_{2m'-1}$ are function of $g_1, ..., g_k, ...$ and the traces are function of $h_0, ..., h_{2m'-1}$ (see section 3 again), the same thing happens with $\widetilde{h}_0, ..., \widetilde{h}_{2m'-1}$ and $g_1 \circ \pi_P, ..., g_k \circ \pi_P, ...$ Therefore the vector subspace spanned by $\{\Lambda(d\widetilde{h}_j,\ \ )(p)\}_{j=0,...,2m'-1}$ is contained in that spanned by $\{\Lambda_1(d\widetilde{h}_j,\ \ )(p)\}_{j=0,...,2m'-1}$.

For finishing the proof it is enough inverting the roles of Λ and Λ_1 because the characteristic polynomial of the symplectic factor of (Λ_1, Λ) equals $t^{2m'} + \sum_{j=1}^{2m'-1} \widetilde{h}_{2m'-j} \widetilde{h}_0^{-1} t^j + \widetilde{h}_0^{-1}$. $\square$

Now assume that (M', Λ, Λ_1) is diffeomorphic to a product of a Kronecker bihamiltonian structure and a symplectic one $(M_1, \Lambda', \Lambda'_1) \times (M_2, \Lambda'', \Lambda''_1)$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the foliations given by the first and second factor respectively. Then $\mathcal{A}_1 \supset \mathcal{B}_2$ and $\widetilde{h}_0, ..., \widetilde{h}_{2m'-1}$ are $\mathcal{B}_1$-foliate functions; therefore the dimension of the vector subspace of $T^*_q M'$ spanned by $d\widetilde{h}_0(q), ..., d\widetilde{h}_{2m'-1}(q)$ equals the dimension of the vector subspace of $\mathcal{A}_1^*(q)$ spanned by $d\widetilde{h}_{0|\mathcal{A}_1(q)}, ..., d\widetilde{h}_{2m'-1|\mathcal{A}_1(q)}$ whenever $q \in M'$.

Thus *the foregoing property is necessary* for the existence of a local decomposition into a product of a Kronecker bihamiltonian structure and a symplectic one.

A point p of M is called *regular* for (Λ, Λ_1) if the three following conditions hold:

1) The rank (Λ, Λ_1) is constant on an open neighbourhood M' of this point.

Observe that this first condition allows assuming maximal (Λ, Λ_1) by replacing (Λ, Λ_1) by $(1-b)\Lambda + b\Lambda_1$ and $(1-b')\Lambda + b'\Lambda_1$, for suitable scalars b, b', and shrinking M'. Then:

2) The dimension of the symplectic factor is constant near p, that is on M' by shrinking this neighbourhood again if necessary.

3) The point $\pi_P(p)$ is regular for $\ell_{|\mathcal{A}}$.

Obviously there are many choices of scalars b, b' such that $((1-b)\Lambda+b\Lambda_1, (1-b')\Lambda + b'\Lambda_1)$ is maximal around p, but it is easily checked that conditions 2) and 3) do not depend on them.

Since the set of regular points of $\ell_{|\mathcal{A}}$ is open and dense and the projection π_P is a submersion, the set of regular points of (Λ, Λ_1) is dense and open on M; it will be named *the regular open set*.

Theorem 7.1. *Consider a real analytic or holomorphic bihamiltonian structure (Λ, Λ_1) on M and a regular point p. Let $\widetilde{\varphi} = t^{2m'} + \sum_{j=0}^{2m'-1} \widetilde{h}_j t^j$ be the characteristic polynomial of the symplectic factor of (Λ, Λ_1) near p. Assume that when q is close to p the vector subspace spanned by $d\widetilde{h}_0(q), ..., d\widetilde{h}_{2m'-1}(q)$ and that spanned by $d\widetilde{h}_{0|\mathcal{A}_1(q)}, ..., d\widetilde{h}_{2m'-1|\mathcal{A}_1(q)}$ have the same dimension. Then, around p, (Λ, Λ_1) decomposes into a product of a Kronecker bihamiltonian structure and a symplectic one.*

Moreover, if $\varphi(p)$ only has real roots then in the C^∞ category (Λ, Λ_1) locally decomposes into a product Kronecker-symplectic.

Let us prove theorem 7.1. Shrinking M we may assume that the hypothesis of theorem hold for every point of this manifold. Around $\pi_N(p)$ consider coordinates $(x_1, ..., x_n)$, scalars $a_1, ..., a_n$, the tensor field $J = \sum_{j=1}^n a_j(\partial/\partial x_j) \otimes dx_j$ and closed 1-forms $\alpha_1, ..., \alpha_n$ such that $a_1, ..., a_n$ are not roots of the characteristic polynomial $\varphi(\pi_P(p))$ of $(\ell_{|\mathcal{A}})(\pi_P(p))$ and $\alpha_1, ..., \alpha_n, J$ define the Veronese web associated to the Veronese flag $(\mathcal{F}, \ell, \omega, \omega_1)$ on P induced in turn by (Λ, Λ_1). Now shrinking P allows supposing that $a_1, ..., a_n$ never are roots of the characteristic polynomial φ of $\ell_{|\mathcal{A}}$.

The next aim will be to show the existence near $\pi_P(p)$ of functions $z_1, ..., z_{2m'}$ and a $(1,1)$-tensor field G extending ℓ such that $(x, z) = (x_1, ..., x_n, z_1, ..., z_{2m'})$ is a system of coordinates, $G = \sum_{j=1}^n a_j(\partial/\partial x_j) \otimes dx_j + \sum_{j,k=1}^n h_{jk}(z)(\partial/\partial z_j) \otimes dz_k$ and ω, ω_1 are expressed relative to $dz_{1|\mathcal{A}}, ..., dz_{2m'|\mathcal{A}}$ with coefficient functions only depending on z.

First, around $\pi_P(p)$, consider a $(1,1)$-tensor field G_0 extending ℓ and projecting in J. Taking into account section 6 one may suppose G_0 adapted to the blocks of $(\mathcal{F}, \ell, \omega, \omega_1)$, where each of them has a characteristic polynomial power of an irreducible one. Therefore it suffices dealing with the problem in every block $(\mathcal{F}', \ell', \omega', \omega'_1)$. Observe that the corresponding point p' is regular for $\ell'_{|\mathcal{A}'}$.

On the other hand each $\widetilde{h}_j = h_j \circ \pi_P$ and the foliation $\mathcal{A}_1$ projects in $\mathcal{A}$, therefore at every point the vector subspace spanned by $dh_0, ..., dh_{2m'-1}$ and that spanned by $dh_{0|\mathcal{A}}, ..., dh_{2m'-1|\mathcal{A}}$ have the same dimension. Thus since φ is the product of the characteristic polynomial of the blocks one has the following two cases:

(I) If $(t-f)^{2m''}$ is the characteristic polynomial of $\ell'_{|\mathcal{A}'}$ then f is either constant or $(df_{|\mathcal{A}'})(p') \neq 0$; besides f never takes the values $a_1, ..., a_n$.

(II) If $(t^2 + ft + g)^{m''}$, where $\mathbb{K} = \mathbb{R}$ and $f^2 < 4g$, is the the characteristic polynomial of $\ell'_{|\mathcal{A}'}$ then f is either constant or $(df_{|\mathcal{A}'})(p') \neq 0$.

When f is constant theorems 2.1 and 5.1 give us the required coordinates and the $(1,1)$-tensor field. If $(df_{|\mathcal{A}'})(p') \neq 0$ these objects are given by theorems 3.1 and 5.1, provided that we are able to show that the symplectic reduction is a Veronese flag or, more exactly, to check the third condition of this notion. Let $(\bar{\mathcal{F}}', \bar{\ell}', \bar{\omega}', \bar{\omega}'_1)$ be the symplectic reduction of $(\mathcal{F}', \ell', \omega', \omega'_1)$ and π' its corresponding canonical projection. Consider a function h on an open set of the symplectic reduction such that $(\bar{\ell}')^* dh$ is closed along $\bar{\mathcal{F}}'$. Then regarded as a function on an open set of P in the obvious way (that is first compose with π' and then extend from the block to P) $\ell^* dh$ is closed along the foliation $\mathcal{F} \cap Kerdf$ or $\mathcal{F} \cap Kerdf \cap Kerdg$. By lemma 1.6, at each point $L_{X_f}\ell$ sends $\mathcal{F} \cap Kerdf$, respectively $\mathcal{F} \cap Kerdf \cap Kerdg$, into the vector space spanned by X_f, respectively X_f, X_g.

But X_h is tangent to the block corresponding to $(\mathcal{F}', \ell', \omega', \omega'_1)$ since h only depends on the variables of this block. Therefore $L_{X'_f}\ell'$ sends $\mathcal{F}' \cap Kerdf$ or $\mathcal{F}' \cap Kerdf \cap Kerdg$ into the vector space spanned by X'_f or X'_f, X'_g, where X'_h, X'_f, X'_g are the ω'-hamiltonians of h, f, g respectively.

On the other hand $X'_f h = X'_g h = 0$, therefore $X'_h f = X'_h g = 0$; that is to say X'_h is tangent to $\mathcal{A}' \cap Kerdf$ or to $\mathcal{A}' \cap Kerdf \cap Kerdg$. Moreover by π' the vector field X'_h projects in the $\bar{\omega}'$-hamiltonian $\bar{X}'_h$ of h, whereas $\ell'_{|\mathcal{F}' \cap Kerdf}$ or $\ell'_{|\mathcal{F}' \cap Kerdf \cap Kerdg}$ do in $\bar{\ell}'$. Thus $L_{X'_h}\ell'$ restricted to $\mathcal{F}' \cap Kerdf$ or to $\mathcal{F}' \cap Kerdf \cap Kerdg$ projects in $L_{\bar{X}'_h}\bar{\ell}'$, whence $L_{\bar{X}'_h}\bar{\ell}' = 0$. *In short, the symplectic reduction is a Veronese flag.*

We need the following lemma whose proof is an exercise on Poisson structures (see [20]).

Lemma 7.1. *On a manifold $\widetilde{M}$ consider a Poisson structure $\widetilde{\Lambda}$ and a $2\widetilde{m}$-codimensional foliation $\mathcal{G}$. Assume that:*
(a) The bracket of any two foliate functions is a foliate function.
(b) The hamiltonians of the foliate functions give rise to a $2\widetilde{m}$-dimensional vector sub-bundle $\widetilde{\mathcal{G}}$ of $T\widetilde{M}$.

Then $\widetilde{\mathcal{G}}$ is a foliation, $TM = \mathcal{G} \oplus \widetilde{\mathcal{G}}$ and, in coordinates $(u, v) = (u_1, ..., u_k, v_1, ..., v_{2\widetilde{m}})$ such that $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are defined by $dv_1 = ... = dv_{2\widetilde{m}} = 0$ and $du_1 = ... = du_k = 0$ respectively, one has
$\widetilde{\Lambda} = \sum_{1 \le i < j \le k} \theta_{ij}(u)(\partial/\partial u_i) \wedge (\partial/\partial u_j) + \sum_{1 \le i < j \le 2\widetilde{m}} \widetilde{\theta}_{ij}(v)(\partial/\partial v_i) \wedge (\partial/\partial v_j)$.

Moreover $\sum_{1 \le i < j \le 2\widetilde{m}} \widetilde{\theta}_{ij}(v)(\partial/\partial v_i) \wedge (\partial/\partial v_j)$ is a symplectic Poisson structure in variables $(v_1, ..., v_{2\widetilde{m}})$.

By means of π_P functions $z_1, ..., z_{2m'}$ may be regarded as functions defined around p on M; since π_P is a submersion $\mathcal{G}_0 = Ker(dz_1 \wedge ... \wedge dz_{2m'})$ is a $2m'$-codimensional foliation about p. On the other hand, since $\{z_i, z_j\}_\omega$ and $\{z_i, z_j\}_{\omega_1}$ are only function of z and Λ, Λ_1 project in the bivectors associated to $(\mathcal{A}, \omega)$ and $(\mathcal{A}, \omega_1)$ respectively, the functions $\Lambda(dh_1, dh_2)$ and $\Lambda_1(dh_1, dh_2)$ are $\mathcal{G}_0$-foliate whenever h_1, h_2 are $\mathcal{G}_0$-foliate. Besides, near p, the Λ-hamiltonians of the $\mathcal{G}_0$-foliate functions give rise to a vector sub-bundle $\mathcal{G}_1$ of dimension $2m'$ because ω is symplectic on $\mathcal{A}$. In the same way, the Λ_1-hamiltonians of the $\mathcal{G}_0$-foliate functions give rise to a vector sub-bundle ${\mathcal{G}'}_1$ of dimension $2m'$. But $\Lambda_1(\pi_P^* \ell^* \beta, \quad) = \Lambda(\pi_P^* \beta, \quad)$, $dz_j \circ G = \sum_{k=1}^{2m'} h_{jk}(z) dz_k$, $j = 1, ..., 2m'$, and $\ell_{|\mathcal{A}} = G_{|\mathcal{A}}$ is invertible; therefore ${\mathcal{G}'}_1 = \mathcal{G}_1$.

By lemma 7.1 applied to Λ, Λ_1 and $\mathcal{G}_0$, the vector sub-bundle $\mathcal{G}_1$ is a foliation and locally $TM = \mathcal{G}_0 \oplus \mathcal{G}_1$. Thus around p there exist functions $u_1, ..., u_{m-2m'}$ such that $(u, z) = (u_1, ..., u_{m-2m'}, z_1, ..., z_{2m'})$ is a system of coordinates, $\mathcal{G}_0$ is defined by $dz_1 = ... = dz_{2m'} = 0$ and $\mathcal{G}_1$ by $du_1 = ... = du_{m-2m'} = 0$. Now from lemma 7.1 follows that

$$\Lambda = \sum_{1\leq i<j\leq m-2m'} \theta_{ij}(u)(\partial/\partial u_i) \wedge (\partial/\partial u_j) + \sum_{1\leq i<j\leq 2m'} \widetilde{\theta}_{ij}(z)(\partial/\partial z_i) \wedge (\partial/\partial z_j)$$

$$\Lambda_1 = \sum_{1\leq i<j\leq m-2m'} \theta_{1ij}(u)(\partial/\partial u_i) \wedge (\partial/\partial u_j) + \sum_{1\leq i<j\leq 2m'} \widetilde{\theta}_{1ij}(z)(\partial/\partial z_i) \wedge (\partial/\partial z_j)$$

which decomposes (Λ, Λ_1) into a product of a Kronecker bihamiltonian structure [variables $(u_1, ..., u_{m-2m'})$] and a symplectic one [variables $(z_1, ..., z_{2m'})$] and *finishes the proof of theorem 7.1.*

8. A counter-example

In this section one will give an example, in the C^∞ category, of a bihamiltonian structure for which theorem 7.1 fails (see [19]); more exactly one will show that the partial tensor field ℓ, of the associated Veronese web, cannot be extended to a $(1,1)$-tensor field with no Nijenhuis torsion. In our example the bihamiltonian structured considered defines a G-structure and the Lewy's result [7] prevents us to find an extension of ℓ with vanishing Nijenhuis torsion, which clearly contradicts theorem 7.1 (the reader interested in a classic example of non-equivalent G-structures may see [6]).

First let us establish some auxiliary results. Consider on a manifold P endowed with coordinates $(x, y) = (x_1, ..., x_n, y_1, ..., y_m)$, for example on an open set of $\mathbb{K}^{n+m}$, the foliation $\mathcal{A}$ given by $dx_1 = ... = dx_n = 0$, and in coordinates $x = (x_1, ..., x_n)$, that is on the quotient of P by $\mathcal{A}$, a Veronese web defined by $J = \sum_{j=1}^n a_j(\partial/\partial x_j) \otimes dx_j$ where $a_1, ..., a_n \in \mathbb{K} - \{0\}$ and the closed 1-forms $\alpha_1, ..., \alpha_r$. Recall that in this case $\alpha_1 \wedge ... \wedge \alpha_r \wedge d(\alpha_j \circ J) = 0$, $j = 1, ..., r$, and $\alpha_1, ..., \alpha_r, J^*$ span, at each point, the same vector space that $dx_1, ..., dx_n$. In the obvious way $J, \alpha_1, ..., \alpha_r$ will be regarded as objects on P too. On the other hand, assume that the n-form $dx_1 \wedge ... \wedge dx_{n-r} \wedge \alpha_1 \wedge ... \wedge \alpha_r$ never vanishes.

On P let $G = J + H + \sum_{j=1}^{n-r} X_j \otimes dx_j$ where $X_1, ..., X_{n-r} \in \mathcal{A}$, $H = \sum_{j,k=1}^m a_{jk}(y)(\partial/\partial y_j) \otimes dy_k$ and the Nijenhuis torsion of $H_{|\mathcal{A}}$ vanishes.

Lemma 8.1. *One has:*

(a) If $N_G \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$ *then* $(L_{X_j} H)_{|\mathcal{A}} = 0$, $j = 1, ..., n-r$.

(b) If $(L_{X_j} H)_{|\mathcal{A}} = 0$, $j = 1, ..., n-r$, *then* $N_G(\mathcal{A}, \quad) = 0$.

Proof. (a) From the formula $N_G(X, \quad) = L_{GX}G - GL_XG$ straightforward follows $N_G(\partial/\partial x_i, \mathcal{A}) = 0$, $i = n-r+1, ..., n$. But $N_G(\partial/\partial y_k, \quad) \wedge \alpha_1 \wedge ... \wedge \alpha_r = 0$, $k = 1, ..., m$, so $N_G(\partial/\partial y_k, \quad) = 0$ since the 1-forms $\alpha_1, ..., \alpha_r$ restricted to $dx_1 = ... = dx_{n-r} = 0$ are linearly independent everywhere. Thus $N_G(\partial/\partial x_j, \mathcal{A}) = 0$, $j = 1, ..., n-r$, which implies $(L_{X_j}H)_{|\mathcal{A}} = 0$.

(b) Clearly $N_G(\mathcal{A}, \mathcal{A}) = 0$ and $N_G(\partial/\partial x_i, \mathcal{A}) = 0$ when i runs from $n-r+1$ to n. On the other hand $N_G(\partial/\partial x_j, \mathcal{A}) = (L_{X_j}H)(\mathcal{A})$ if $j = 1, ..., n-r$. □

Lemma 8.2. *Consider a tensor field* $G'' = J + H + \sum_{j=1}^{n} X_j \otimes dx_j$ *where* $X_1, ..., X_n \in \mathcal{A}$. *If* $N_{G''} = 0$ *then* $(L_{X_j}H)_{|\mathcal{A}} = 0$, $j = 1, ..., n$.

Proof. Now $(L_{\partial/\partial x_j}G'')(\mathcal{A}) = 0$ whereas $(L_{X_j}H)(\mathcal{A}) = (L_{G''(\partial/\partial x_j)}G'')(\mathcal{A}) = 0$. □

Hereafter $n = 3$ and $J = \sum_{j=1}^{3} a_j(\partial/\partial x_j) \otimes dx_j$ where a_1, a_2, a_3 are non-equal and non-vanishing real numbers. Besides one will replace m by $4m$, that is we will consider coordinates $(x_1, x_2, x_3, y_1, ..., y_{4m})$, and P will be an open set of $\mathbb{R}^{4m+3}$. On the other hand one will set $r = 2$, $\alpha_1 = dx_1 - dx_2$ and $\alpha_2 = x_2 dx_2 - dx_3$. Then α_1, α_2, $\alpha_1 \circ J$ and $\alpha_2 \circ J$ are closed. It is easily checked that α_1, α_2, J define a Veronese web of codimension two in variables x and $dx_1 \wedge \alpha_1 \wedge \alpha_2 = dx_1 \wedge dx_2 \wedge dx_3$.

For making calculations easy, we introduce a complex structure along $\mathcal{A}$ by means of the complex variables $(z, u) = (z_1, ..., z_m, u_1, ..., u_m)$, where $z_1 = y_1 + \imath y_2$, $u_1 = y_3 + \imath y_4$,..., $z_m = y_{4m-3} + \imath y_{4m-2}$, $u_m = y_{4m-1} + \imath y_{4m}$. Set $H = \imath I_{(z,u)} + \sum_{j=1}^{m}(\partial/\partial z_j) \otimes du_j$ where $I_{(z,u)} = \sum_{j=1}^{m}[(\partial/\partial z_j) \otimes dz_j + (\partial/\partial u_j) \otimes du_j]$.

From the real viewpoint H is a $(1,1)$-tensor field with constant coefficients and minimal polynomial $t(t^2+1)^2$, whose semi-simple and nilpotent parts equal $\imath I_{(z,u)}$ and $\sum_{j=1}^{m}(\partial/\partial z_j) \otimes du_j$ respectively.

Lemma 8.3. *Consider the* $(1,1)$*-tensor field* $G' = J + H$ *and a complex valued function* $f(x, u)$ *holomorphic along* $\mathcal{A}$. *If* $d(df \circ G') \wedge \alpha_1 \wedge \alpha_2 = 0$ *then, locally, there exists a complex valued function* $g(x, z, u)$ *holomorphic along* $\mathcal{A}$ *such that* $d(dg \circ G') \wedge \alpha_1 \wedge \alpha_2 = 0$ *and* $(dg \circ (H - \imath I_{(z,u)}))_{|\mathcal{A}} = df_{|\mathcal{A}}$.

Let us prove this result. First consider the basis of the cotangent bundle, with respect to variables x, $\{dx_1, \alpha_1, \alpha_2\}$ and its dual basis $X = \partial/\partial x_1 +$

$\partial/\partial x_2 + x_2\partial/\partial x_3$, $X_1 = -\partial/\partial x_2 - x_2\partial/\partial x_3$, $X_2 = -\partial/\partial x_3$. Taking into account that $dh = X(h)dx_1 + X_1(h)\alpha_1 + X_2(h)\alpha_2 + \sum_{j=1}^{m}[(\partial h/\partial z_j)dz_j + (\partial h/\partial u_j)du_j]$ when h is holomorphic along $\mathcal{A}$, a calculation shows that $d(df \circ G') \wedge \alpha_1 \wedge \alpha_2 = 0$ if and only if $(JX - \imath X) \cdot (\partial f/\partial u_j) = 0$, $j = 1, ..., m$. On the other hand $(dg \circ (H - \imath I_{(z,u)}))_{|\mathcal{A}} = df_{|\mathcal{A}}$ means that $g = \sum_{j=1}^{m} z_j \partial f/\partial u_j + \varphi(x, u)$. Therefore we have to find a function $\varphi(x, u)$ holomorphic along $\mathcal{A}$ in such a way that $d(dg \circ G') \wedge \alpha_1 \wedge \alpha_2 = 0$. But again a calculation shows that this last condition is equivalent to the equation $(JX - \imath X) \cdot (\partial\varphi/\partial u_j) = X \cdot (\partial f/\partial u_j)$, $j = 1, ..., m$ [observe that $d(d(\partial f/\partial u_k) \circ G') \wedge \alpha_1 \wedge \alpha_2 = 0$, $k = 1, ..., m$, since $d(df \circ G') \wedge \alpha_1 \wedge \alpha_2 = 0$].

Now consider a function $\psi(x, u)$ such that $(JX - \imath X) \cdot \psi = 0$ and set $Y = (a_1 - \imath)^{-1} x_1 \partial/\partial x_1 + (a_2 - \imath)^{-1} x_2 \partial/\partial x_2 + ((a_2 - \imath)^{-1} + (a_3 - \imath)^{-1}) x_3 \partial/\partial x_3$. Then $[JX - \imath X, Y] = X$, which implies $(JX - \imath X) \cdot h = X \cdot \psi$ where $h = Y \cdot \psi$.

Since Y commutes with $\partial/\partial u_j, \partial/\partial \bar{z}_j, \partial/\partial \bar{u}_j$, $j = 1, ..., m$, it suffices to set $\varphi = Y \cdot f$ *for finishing the proof of lemma 8.3.*

Now let $G = J + H + Z \otimes dx_1$ where $Z = \sum_{j=1}^{m} f_j(x, u)\partial/\partial z_j$ and each f_j is holomorphic along $\mathcal{A}$. Then $(L_Z H)_{|\mathcal{A}} = 0$ and, by lemma 8.1, one has $N_G(\mathcal{A}, \quad) = 0$; thus $N_G \wedge \alpha_1 \wedge \alpha_2 = 0$ since there are only three variables x.

Theorem 8.1. *There exists a complex valued function $f(x)$, $x \in \mathbb{R}^3$, such that, if one sets $f_1 = u_1 f$, then the Nijenhuis torsion of the $(1,1)$-tensor field $\widetilde{G} = G + Z_1 \otimes \alpha_1 + Z_2 \otimes \alpha_2$ never vanishes around any point whatever $Z_1, Z_2 \in \mathcal{A}$.*

Proof. The real characteristic polynomial ψ of $\widetilde{G}$ equals $\psi_1 \cdot \psi_2$ where $\psi_1 = (t - a_1)(t - a_2)(t - a_3)$ and $\psi_2 = (t^2 + 1)^{2m}$. Assume $N_{\widetilde{G}} = 0$ for some $Z_1, Z_2 \in \mathcal{A}$. Then locally $\widetilde{G}$ decompose into a product of two manifolds endowed each of them with a $(1,1)$-tensor field whose real characteristic polynomials are ψ_1 and ψ_2 respectively. Both factor tensor fields are flat because the first one can be identified to J and the second one to the restriction of $\widetilde{G}$ to a leaf of the foliation $\mathcal{A}$ and, obviously, this restriction is flat (in fact the foliation by the second factor equals $\mathcal{A}$).

Note that the complex structure along $\mathcal{A}$ is given by the semi-simple part of $\widetilde{G}_{|\mathcal{A}}$. So there exist coordinates $(\widetilde{x}, \widetilde{z}, \widetilde{u}) = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{z}_1, ..., \widetilde{z}_m, \widetilde{u}_1, ..., \widetilde{u}_m)$, where $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2$, $\tilde{x}_3 = x_3$, such that $d\widetilde{x}_1 = d\widetilde{x}_2 = d\widetilde{x}_3 = 0$ defines $\mathcal{A}$, $\alpha_1 = d\tilde{x}_1 - d\tilde{x}_2$, $\alpha_2 = \tilde{x}_2 d\tilde{x}_2 - d\tilde{x}_3$, $\widetilde{z}_1, ..., \widetilde{z}_m, \widetilde{u}_1, ..., \widetilde{u}_m$ are holomorphic along $\mathcal{A}$ and $\widetilde{G} = \widetilde{J} + \widetilde{H}$ where $\widetilde{J} = \sum_{j=1}^{3} a_j(\partial/\partial\widetilde{x}_j) \otimes d\widetilde{x}_j$ and $\widetilde{H} = \imath I_{(\widetilde{z},\widetilde{u})} + \sum_{j=1}^{m}(\partial/\partial\widetilde{z}_j) \otimes d\widetilde{u}_j$.

Clearly $d(du_1 \circ \widetilde{G}) \wedge \alpha_1 \wedge \alpha_2 = 0$ [calculate it in coordinates (x, z, u)]. Besides in coordinates $(\widetilde{x}, \widetilde{z}, \widetilde{u})$ function u_1 does not depend on $\widetilde{z}$ since it is foliate with respect to the foliation $Ker((\widetilde{G} - \imath I)_{|\mathcal{A}})$. Therefore from lemma 8.3 applied, in coordinates $(\widetilde{x}, \widetilde{z}, \widetilde{u})$, to u_1 and $\widetilde{G}$ follows the local existence of a function g holomorphic along $\mathcal{A}$ such that $d(dg \circ \widetilde{G}) \wedge \alpha_1 \wedge \alpha_2 = 0$ and $du_{1|\mathcal{A}} = (dg \circ (\widetilde{H} - \imath I_{(\widetilde{z},\widetilde{u})}))_{|\mathcal{A}}$. But $(\widetilde{H} - \imath I_{(\widetilde{z},\widetilde{u})})_{|\mathcal{A}} = (H - \imath I_{(z,u)})_{|\mathcal{A}}$ since this object is the nilpotent part of $\widetilde{G}_{|\mathcal{A}}$, so $du_{1|\mathcal{A}} = (dg \circ (H - \imath I_{(z,u)}))_{|\mathcal{A}}$; moreover $d(dg \circ G) \wedge \alpha_1 \wedge \alpha_2 = 0$ because $(\widetilde{G} - G) \wedge \alpha_1 \wedge \alpha_2 = 0$. The first condition implies that $g = z_1 + \rho(x, u)$ where ρ is holomorphic along $\mathcal{A}$.

Now take $f_1 = u_1 f(x)$; then $0 = (d(dg \circ G) \wedge \alpha_1 \wedge \alpha_2)(X, X_1, X_2, \partial/\partial u_1) = d(dg \circ G)(X, \partial/\partial u_1) = X(dg(G\partial/\partial u_1)) - \partial/\partial u_1(dg(GX))$, which yields the equation

$$(*) \qquad (JX - \imath X) \cdot (\partial\rho/\partial u_1) + f = 0.$$

Let $\widetilde{X} = [JX, -X]$. Then $\widetilde{X} = (a_3 - a_2)\partial/\partial x_3 \neq 0$ since $a_2 \neq a_3$. Regarded on $\mathbb{R}^3$, the vector fields $JX, -X, \widetilde{X}$, which are linearly independent everywhere, define a 3-dimensional Lie algebra whose center is spanned by $\widetilde{X}$. Moreover $b_1 JX - b_2 X + b_3 \widetilde{X}$ is complete for any $b_1, b_2, b_3 \in \mathbb{R}$.

On $\mathbb{R}^3$ endowed with coordinates $y = (y_1, y_2, y_3)$ set $Y_1 = -\partial/\partial y_1 - 2y_2\partial/\partial y_3$, $Y_2 = -\partial/\partial y_2 + 2y_1\partial/\partial y_3$ and $\widetilde{Y} = [Y_1, Y_2] = -4\partial/\partial y_3$; note that $Y_1, Y_2, \widetilde{Y}$ are linearly independent everywhere and define a 3-dimensional Lie algebra whose center is spanned by $\widetilde{Y}$; moreover $b_1 Y_1 + b_2 Y_2 + b_3 \widetilde{Y}$ is complete for any $b_1, b_2, b_3 \in \mathbb{R}$. As $\mathbb{R}^3$ is simply connected there is a diffeomorphism of this space transforming $JX, -X, \widetilde{X}$ in $Y_1, Y_2, \widetilde{Y}$ respectively.

From the Lewy's example (see (5) of page 156 of [7]) follows the existence of a C^∞ function $F : \mathbb{R}^3 \to \mathbb{C}$ such that the equation $(Y_1 + \imath Y_2)\widetilde{F} = F$ has no solution in any neighbourhood of any point of $\mathbb{R}^3$. Pulling-back $-F$ gives a function f for which equation (*) has no solution at all (regard $\partial\rho/\partial u_1$ as a function of x and $u_1, ..., u_m$ as parameters); in other words if one takes $f_1 = u_1 f$ then $N_{\widetilde{G}}$ never vanishes around any point. $\square$

The next step will be to apply the construction of sub-section 1.2 to a foliation and a particular $(1,1)$-tensor field on $\mathbb{R}^7$. More exactly, set $m=1$ and $S = J + H + u_1 f(x)(\partial/\partial z_1)\otimes dx_1$ or in real notation $S = \sum_{j=1}^{3} a_j(\partial/\partial x_j)\otimes dx_j + \sum_{j=1}^{2}[(\partial/\partial y_{2j})\otimes dy_{2j-1} - (\partial/\partial y_{2j-1})\otimes dy_{2j}] + (\partial/\partial y_1)\otimes dy_3 + (\partial/\partial y_2)\otimes dy_4 + [(y_3 g_1 - y_4 g_2)(\partial/\partial y_1) + (y_3 g_2 + y_4 g_1)(\partial/\partial y_2)]\otimes dx_1$ where $f = g_1 + \imath g_2$.

Note that, as it was pointed out before, $N_S \wedge \alpha_1 \wedge \alpha_2 = 0$. Moreover, if α is a closed 1-form such that $Ker\alpha \supset Ker(\alpha_1\wedge\alpha_2)$ then $\alpha_1\wedge\alpha_2\wedge d(\alpha\circ S) = 0$ since $\alpha = \sum_{j=1}^{3} h_k(x)dx_k$. In other words the construction of sub-section 1.2 applies to S and $\mathcal{G} = Ker(\alpha_1\wedge\alpha_2)$.

Let $(x,y,\tilde{x},\tilde{y}) = (x_1,x_2,x_3,y_1,...,y_4,\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{y}_1,...,\tilde{y}_4)$ be the coordinates of $T^*\mathbb{R}^7$ associated to (x,y). Then $\omega = \sum_{j=1}^{3} d\tilde{x}_j\wedge dx_j + \sum_{j=1}^{4} d\tilde{y}_j\wedge dy_j$, $\omega_1 = \omega(S^*,\quad)$ and

$$
\begin{aligned}
S^* = & \sum_{j=1}^{3} a_j[(\partial/\partial x_j)\otimes dx_j + (\partial/\partial\tilde{x}_j)\otimes d\tilde{x}_j] \\
& + \sum_{j=1}^{2}[(\partial/\partial y_{2j})\otimes dy_{2j-1} - (\partial/\partial y_{2j-1})\otimes dy_{2j} \\
& + (\partial/\partial\tilde{y}_{2j-1})\otimes d\tilde{y}_{2j} - (\partial/\partial\tilde{y}_{2j})\otimes d\tilde{y}_{2j-1}] \\
& + (\partial/\partial y_1)\otimes dy_3 + (\partial/\partial y_2)\otimes dy_4 + (\partial/\partial\tilde{y}_3)\otimes d\tilde{y}_1 + (\partial/\partial\tilde{y}_4)\otimes d\tilde{y}_2 \\
& + [(y_3 g_1 - y_4 g_2)(\partial/\partial y_1) + (y_3 g_2 + y_4 g_1)(\partial/\partial y_2) \\
& - (\tilde{y}_1 g_1 + \tilde{y}_2 g_2)(\partial/\partial\tilde{y}_3) + (\tilde{y}_1 g_2 - \tilde{y}_2 g_1)(\partial/\partial\tilde{y}_4)]\otimes dx_1 \\
& + \sum_{j=1}^{3}(\partial/\partial\tilde{x}_j)\otimes\beta_j
\end{aligned}
$$

where β_1,β_2,β_3 are functional combinations of $dx_1, dx_2, dx_3,\ dy_3, dy_4,\ d\tilde{y}_1, d\tilde{y}_2$.

Recall that in our case $\mathcal{G}_0$ is a 2-dimensional foliation, isotropic and symplecticly complete for ω and ω_1, spanned by the ω-hamiltonians of $\alpha_1\circ J^{-1}, \alpha_2\circ J^{-1}$ or by the ω_1-hamiltonians of α_1,α_2, when $\alpha_1,\alpha_2,\alpha_1\circ J^{-1},\alpha_2\circ J^{-1}$ are regarded as 1-forms on $T^*\mathbb{R}^7$ in the obvious way.

Therefore by projection the Poisson structures Λ_ω and Λ_{ω_1} give rise to a bihamiltonian structure (Λ,Λ_1) on the global quotient $M = T^*\mathbb{R}^7/\mathcal{G}_0$ (proposition 1.4).

Since the ω-hamiltonians of $\alpha_1\circ J^{-1},\alpha_2\circ J^{-1}$ equal $-a_1^{-1}\partial/\partial\tilde{x}_1 + a_2^{-1}\partial/\partial\tilde{x}_2$, $-a_2^{-1}x_2\partial/\partial\tilde{x}_2 + a_3^{-1}\partial/\partial\tilde{x}_3$, the submanifold of $T^*\mathbb{R}^7$ defined by $\tilde{x}_2 = \tilde{x}_3 = 0$ is transverse to $\mathcal{G}_0$, which allows us to identify it with M endowed with coordinates $(x_1,x_2,x_3,y_1,...,y_4,\tilde{x}_1,\tilde{y}_1,...,\tilde{y}_4)$, whereas Λ,Λ_1 are given by the restriction to M of $\alpha_1\circ J^{-1},\alpha_2\circ J^{-1},\omega$ and $\alpha_1,\alpha_2,\omega_1$ respectively.

In general (see the proof of proposition 1.4) $\Lambda + t\Lambda_1$ is defined by the restriction to M of $\alpha_1 \circ (S^* + tI)^{-1} = \alpha_1 \circ (J + tI)^{-1}$, $\alpha_2 \circ (S^* + tI)^{-1} = \alpha_1 \circ (J + tI)^{-1}$ and $\omega((I + t(S^*)^{-1})^{-1}, \quad)$. Therefore the rank of (Λ, Λ_1) equals 10, the primary axis of (Λ, Λ_1) is the foliation $dx_1 = dx_2 = dx_3 = 0$ and the secondary one the foliation spanned by $\partial/\partial \tilde{x}_1$; in particular the dimension of the symplectic factor is 8 everywhere and 4 that of the Kronecker factor. Thus the global quotient of M by the secondary axis is identified, in a natural way, to the submanifold P' of $T^*\mathbb{R}^7$ defined by the equations $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = 0$ endowed with coordinates $(x, y, \tilde{y})$, while the foliation $\mathcal{A}$ of the Veronese flag on P' induced by (Λ, Λ_1) is given by $dx_1 = dx_2 = dx_3 = 0$, and the Veronese web is defined in variables $x = (x_1, x_2, x_3)$ by J, α_1, α_2.

On the other hand, as $\omega_1 = \omega(S^*, \quad)$ and S^* projects on P' in the $(1,1)$-tensor field

$$
\begin{aligned}
G = {} & \textstyle\sum_{j=1}^{3} a_j (\partial/\partial x_j) \otimes dx_j \\
& + \textstyle\sum_{j=1}^{2} [(\partial/\partial y_{2j}) \otimes dy_{2j-1} - (\partial/\partial y_{2j-1}) \otimes dy_{2j} \\
& + (\partial/\partial \tilde{y}_{2j-1}) \otimes d\tilde{y}_{2j} - (\partial/\partial \tilde{y}_{2j}) \otimes d\tilde{y}_{2j-1}] \\
& + (\partial/\partial y_1) \otimes dy_3 + (\partial/\partial y_2) \otimes dy_4 + (\partial/\partial \tilde{y}_3) \otimes d\tilde{y}_1 + (\partial/\partial \tilde{y}_4) \otimes d\tilde{y}_2 \\
& + [(y_3 g_1 - y_4 g_2)(\partial/\partial y_1) + (y_3 g_2 + y_4 g_1)(\partial/\partial y_2) \\
& - (\tilde{y}_1 g_1 + \tilde{y}_2 g_2)(\partial/\partial \tilde{y}_3) + (\tilde{y}_1 g_2 - \tilde{y}_2 g_1)(\partial/\partial \tilde{y}_4)] \otimes dx_1,
\end{aligned}
$$

this last one is a prolongation of the partial $(1,1)$-tensor field $\ell : \mathcal{F} \to TP'$, which projects in J.

Since $\ell_{|\mathcal{A}} = G_{|\mathcal{A}}$ is 0-deformable because it is written with constant coefficients, the algebraic model of the symplectic factor of (Λ, Λ_1), which is completely determined by $\ell_{|\mathcal{A}}$, does not depend on the point considered. In particular its characteristic polynomial equals $(t^2 + 1)^4$ and the hypothesis of theorem 7.1 on the coefficients of this polynomial automatically holds.

Note that the algebraic model of the Veronese web in variables x does not depend on the point as in dimension three and codimension two there is only one model. Thus the algebraic model of the Kronecker factor is independent of the point, (Λ, Λ_1) defines a G-structure and M is the regular open set of (Λ, Λ_1).

Assume that, around some point q of M, the bihamiltonian structure (Λ, Λ_1) decomposes into a product Kronecker-symplectic. Then considering the local quotient by the secondary axis on each factor separately implies the existence about of some point $p \in P'$ of a $(1,1)$-tensor field $\widetilde{G}$, which prolongs ℓ and projects in J, whose Nijenhuis torsion vanishes. In other words, around p there exist two vector fields $Z_1, Z_2 \in \mathcal{A}$ such that $N_{\widetilde{G}} = 0$ where $\widetilde{G} = G + Z_1 \otimes \alpha_1 + Z_2 \otimes \alpha_2$.

Now set $y_5 = \tilde{y}_3$, $y_6 = -\tilde{y}_4$, $y_7 = \tilde{y}_1$, $y_8 = -\tilde{y}_2$ and consider complex variables $z_1 = y_1 + \imath y_2$, $u_1 = y_3 + \imath y_4$, $z_2 = y_5 + \imath y_6$ and $u_2 = y_7 + \imath y_8$. Then $G = J + H + Z \otimes dx_1$ where $Z = u_1 f\partial/\partial z_1 - u_2 f\partial/\partial z_2$ and $f = g_1 + \imath g_2$.

By theorem 8.1 one may choose function f in such a way that the Nijenhuis torsion of $\widetilde{G}$ never vanishes about any point, which implies that (Λ, Λ_1) does not decompose into a product Kronecker-symplectic around any point.

In short, *one has constructed a counter-example to theorem 7.1 in the C^∞ case.*

Appendix: A splitting property for $(1,1)$-tensor fields

Nowadays it is well known, and belongs to the mathematical folklore, that a $(1,1)$-tensor fields whose Nijenhuis torsion vanishes locally follows the decomposition of its characteristic polynomial (see [2]). Nevertheless, and for making our text more self-contained, we will prove this result here. More exactly:

Proposition A.1. *Consider a $(1,1)$-tensor fields G on a n-manifold M. Let φ be its characteristic polynomial. Assume that:*
(1) $N_G = 0$,
(2) $\varphi = \varphi_1 \cdot \varphi_2$ where φ_1, φ_2 are monic polynomials, of respective degrees n_1 and n_2, relatively prime at each point.

Then, around every point, (M, G) decomposes into a product $(M_1, G_1) \times (M_2, G_2)$, where $dimM_1 = n_1$, $dimM_2 = n_2$, $N_{G_1} = N_{G_2} = 0$, φ_1 is the characteristic polynomial of G_1 (more exactly φ_1 is the pull-back of the characteristic polynomial of G_1 by the first projection) and φ_2 that of G_2.

Let us prove proposition A.1. Set $H_1 = \varphi_2(G)$ and $H_2 = \varphi_1(G)$. By algebraic reasons $KerH_1 = ImH_2$, $KerH_2 = ImH_1$, ImH_1 and ImH_2 are vector sub-bundles of dimension n_1 and n_2 respectively and $TM = ImH_1 \oplus ImH_2$. Moreover ImH_1 and ImH_2 are G-invariant, $\varphi_1(H_1) = \varphi_2(H_2) = 0$, and $H_1, \varphi_2(H_1) : ImH_1 \to ImH_1$, $H_2, \varphi_1(H_2) : ImH_2 \to ImH_2$ are isomorphisms.

Since $N_G = 0$ one has $(L_{G^k X}(G^r))Y = (G^k L_X(G^r))Y$ for any vector fields X, Y and natural numbers k, r. Recall that if $\widetilde{H}$ is a $(1,1)$-tensor field then

$L_{fZ}\widetilde{H} = fL_Z\widetilde{H} + (\widetilde{H}Z)\otimes df - Z\otimes(df\circ\widetilde{H})$. Now a straightforward calculation shows:

Lemma A.1. *Consider functions $h_0,...,h_s$ and set $H=\sum_{k=0}^{s} h_k G^k$. Then $N_H(X,Y)=\sum_{j=0}^{n-1}[\alpha_j(X)G^jY-\alpha_j(Y)G^jX]$ where each α_j is a 1-form functional combination of $dh_k\circ G^r$, $k=0,...,s$, $r=0,...,n-1$.*

In particular $N_H=0$ if $h_0,...,h_s$ are constant.

By definition of Nijenhuis torsion $[H_1X,H_1Y]-N_{H_1}(X,Y)$ is a section of ImH_1. Therefore given vector fields $X,Y\in ImH_1$, since $G^k(ImH_1)\subset ImH_1$, from lemma A.1 follows that $[H_1X,H_1Y]\in ImH_1$. But the vector fields H_1Z such that $Z\in ImH_1$ span ImH_1, so ImH_1 is involutive; in turn and by a similar reason ImH_2 is involutive too.

In other words, locally, M can be regarded as a product $M_1\times M_2$ associated to the decomposition of the tangent bundle $TM=ImH_1\oplus ImH_2$; moreover $G(TM_1\times\{0\})\subset TM_1\times\{0\}$ and $G(\{0\}\times TM_2)\subset\{0\}\times TM_2$. Thus there exist two $(1,1)$-tensor field $G_1:TM_1\to TM_1$, perhaps depending on M_2, and $G_2:TM_2\to TM_2$, perhaps depending on M_1, such that $G=G_1+G_2$ when G_1,G_2 are considered on TM in the natural way (that is $G_1(\{0\}\times TM_2)=0$ and $G_2(TM_1\times\{0\})=0$). The proof will be finished if we are able to show that G_1,G_2 do not depend on M_2 and M_1 respectively, since in this case $N_G=0$ obviously implies $N_{G_1}=N_{G_2}=0$.

We start dealing with the case where there exist a symplectic form ω and a closed 2-form ω_1 such that $\omega_1=\omega(G,\quad)$; recall that $\omega(G,\quad)=\omega(\quad,G)$. Then $\omega(ImH_1,ImH_2)=\omega(Im(\varphi_2(G)),Im(\varphi_1(G)))=\omega(Im(\varphi_1(G)\circ\varphi_2(G)),TM)=0$; in a analogous way one has $\omega_1(ImH_1,ImH_2)=0$. Now consider coordinates $(x,y)=(x_1,...,x_{n_1},y_1,...,y_{n_2})$ on M such that $\partial/\partial x_1,...,\partial/\partial x_{n_1}$ span ImH_1 and $\partial/\partial y_1,...,\partial/\partial y_{n_2}$ span ImH_2. Then $\omega=\omega'+\omega''$ and $\omega_1=\omega_1'+\omega_1''$ where $\omega'=\sum_{1\leq i<j\leq n_1} f_{ij}(x)dx_i\wedge dx_j$, $\omega''=\sum_{1\leq i<j\leq n_2} g_{ij}(y)dy_i\wedge dy_j$, $\omega_1'=\sum_{1\leq i<j\leq n_1}\tilde{f}_{ij}(x)dx_i\wedge dx_j$ and $\omega_1''=\sum_{1\leq i<j\leq n_2}\tilde{g}_{ij}(y)dy_i\wedge dy_j$, because $d\omega=d\omega_1=0$ and $\omega(\partial/\partial x_k,\partial/\partial y_r)=\omega_1(\partial/\partial x_k,\partial/\partial y_r)=0$, $k=1,...,n_1$, $r=1,...,n_2$. Thus $\omega_1'=\omega'(G_1,\quad)$ in coordinates $(x_1,...,x_{n_1})$ regarded on M_1 and $\omega_1''=\omega''(G_2,\quad)$ in coordinates $(y_1,...,y_{n_2})$ on M_2; whereby G_1 only depends on $(x_1,...,x_{n_1})$ and G_2 on $(y_1,...,y_{n_2})$, which proves proposition A.1 in this case.

In the general case consider the prolongation G^* of G to T^*M (see sub-section 1.2) whose characteristic polynomial equals φ^2, or more exactly the pull-back of φ^2 by the canonical projection $\pi : T^*M \to M$. Now $N_{G^*} = 0$, $\varphi^2 = \varphi_1^2 \cdot \varphi_2^2$ and, since on T^*M there exist ω and ω_1 as before, G^* decomposes into a sum $G^* = G_1^* + G_2^*$ in such a way that $ImG_1^* = Im\varphi_2^2(G^*)$, $KerG_1^* = Im\varphi_1^2(G^*)$, $ImG_2^* = Im\varphi_1^2(G^*)$ and $KerG_2^* = Im\varphi_2^2(G^*)$. Moreover $N_{G_1^*} = N_{G_2^*} = 0$ as $N_{G^*} = 0$.

Again, consider coordinates $(x, y) = (x_1, ..., x_{n_1}, y_1, ..., y_{n_2})$ on M such that $\partial/\partial x_1, ..., \partial/\partial x_{n_1}$ span ImH_1 and $\partial/\partial y_1, ..., \partial/\partial y_{n_2}$ span ImH_2. Identify M to the zero section S_0 of T^*M. If $(x, y, \tilde{x}, \tilde{y})$ are the associated coordinates on T^*M, in which the zero section is given by $\tilde{x} = 0$, $\tilde{y} = 0$, from the formula of the prolongation given in sub-section 1.2 easily follows that $G^*(TS_0) \subset TS_0$, $G_1^*(TS_0) \subset TS_0$ and $G_2^*(TS_0) \subset TS_0$. Besides $(Im\varphi_2^2(G^*)) \cap TS_0 = Im\varphi_2(G)$, $(Im\varphi_1^2(G^*)) \cap TS_0 = Im\varphi_1(G)$, $G^*_{|S_0} = G$, $G^*_{1|S_0} = G_1$ and $G^*_{2|S_0} = G_2$ [it is just an algebraic verification at each point of S_0]. Thus $N_{G_1} = N_{G_2} = 0$ on M. In particular from $N_{G_1}(\partial/\partial y_r, \quad) = 0$ follows $L_{(\partial/\partial y_r)}G_1 = 0$, $r = 1, ..., n_2$, that is G_1 does not depend on M_2. Analogously one shows that G_2 does not depend on M_1. Therefore *the proof of proposition A.1 is finished.*

References

[1] Bouetou, B.T. and Dufour, J.P., *Veronese curves and bihamiltonian systems*, Int. J. Math. Sci. **19** 93142 (2006) 11p.

[2] Frölicher A. and Nijenhuis A., *Theory of vector-valued differential forms, Part I*, Indag. Math. **18** (1956), 338-359.

[3] Gelfand I.M. and Zakharevich I., *Webs, Veronese Curves and Bihamiltonian Systems*, J. Funct. An., **99** (1991), 150-178.

[4] Gelfand I.M. and Zakharevich I., *On the local geometry of a bihamiltonian structure*, Corwin, L. (ed) et al., The Gelfand Seminars, 1990-92, Birkhäuser, Basel, 51-112 (1993).

[5] Gelfand I.M. and Zakharevich I., *Webs, Lenard schemes, and the local geometry of bihamiltonian Toda and Lax structures*, Selecta Math. (N.S.) **6** (2000) 131-183.

[6] Guillemin V. W. and Sternberg S., *The Lewy counterexample and the local equivalence problem for G-structures*, J. of Diff. Geometry **1** (1967) 127-131.

[7] Lewy H., *An example of a smooth linear partial differential equation without solution*, Ann. of Math. **66** (1957) 155-158.

[8] Libermann P., *Problmes d'quivalence et géométrie symplectique*, Astérisque **107-108** (1983) 43-68.

[9] Magri, F., *A simple model of integrable hamiltonian equations*, J. Math. Phys. **19** (1978) 1156-1162.

[10] Panasyuk A., *Veronese Webs for Bihamiltonian Structures of Higher Corank*, Banach Center Publ. **51** (2000), 251-261.

[11] Panasyuk A., *On integrability of generalized Veronese curves of distributions*, Rep. Math. Phys. **50** (2002) 291-297.

[12] Turiel F.J., *Classification locale d'un couple de formes symplectiques Poisson compatibles*, C.R. Acad. Sci., Paris, Série I **308** (1989) 575-578.

[13] Turiel F.J., *Classification locale simultanée de deux formes symplectiques compatibles*, Manuscripta Math. **82** (1994), 349-362.

[14] Turiel F.J., *C^∞-équivalence entre tissus de Veronese et structures bihamiltoniennes*, C.R. Acad. Sci. Paris Série I **328** (1999) 891-894.

[15] Turiel F.J., *C^∞-classification des germes de tissus de Veronese*, C.R. Acad. Sci. Paris Série I **329** (1999) 425-428.

[16] Turiel F.J., *Tissus de Veronese analytiques de codimension supérieure et structures bihamiltoniennes*, C.R. Acad. Sci., Paris, Série I **331** (2000) 61-64.

[17] Turiel F.J., *On the local theory of Veronese webs*, arXiv:1001.3098v1 [math.DG]

[18] Turiel F.J., *Décomposition locale d'une structure bihamiltonienne en produit Kronecker-symplectique*, C.R. Acad. Sci. Paris Série I **349** (2011) 85-87.

[19] Turiel F.J., *Un exemple en classe C^∞ de structure bihamiltonienne non décomposable localement en produit Kronecker-symplectique*, C.R. Acad. Sci. Paris Série I **349** (2011) 451-454.

[20] Weinstein A., *The local structure of Poisson manifolds*, J. of Diff. Geometry **18** (1983) 523-557.

[21] Zakharevich, I., *Kronecker webs, bihamiltonian structures, and the method of argument translation*, Transform. Groups **6** (2001) 267-300.