THE HIGHER ORDER TERMS IN ASYMPTOTIC EXPANSION OF COLOR JONES POLYNOMIALS

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Abstract. Color Jones polynomial is one of the most important quantum invariants in knot theory. Finding the geometric information from the color Jones polynomial is an interesting topic. In this paper, we study the general expansion of color Jones polynomial which includes the volume conjecture expansion and the Melvin-Morton-Rozansky (MMR) expansion as two special cases. Following the recent works on $SL(2,\mathbb{C})$ Chern-Simons theory, we present an algorithm to calculate the higher order terms in general asymptotic expansion of color Jones polynomial from the view of A-polynomial and noncommutative A-polynomial. Moreover, we conjecture that the MMR expansion corresponding to the abelian branch of A-polynomial. Lastly, we give some examples to illustrate how to calculate the higher order terms. These results support our conjecture.

Contents

1. Introduction

Let $J_N(\mathcal{K};q)$ be the *normalized colored Jones polynomial* of a knot $\mathcal{K}$ colored by the N-dimensional irreducible representation of $SU(2)$. Thus, $J_N(\text{unknot};q)=1$, $J_1(\mathcal{K};q)=1$ for all $\mathcal{K}$ and $J_2(\mathcal{K};q)$ is the Jones polynomial of $\mathcal{K}$. $J_N(\mathcal{K};q)$ is an important quantum invariant in knot theory. People want to find the geometric information from $J_N(\mathcal{K};q)$. More precisely, let $q=e^{\frac{2\pi i}{k}}$ and consider the following limit,

$$k,\ N\to\infty,\quad u=\pi i\frac{N}{k}\quad \text{fixed.}$$

Now the question is what's the behavior of the limit

$$\lim_{N\to\infty} J_N(\mathcal{K};e^{\frac{2u}{N}}). \tag{1}$$

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The first progress in this direction is the volume conjecture. Let us briefly review it. R.M. Kashaev defined a knot invariant associated with the quantum dilogarithm and integer N, denoted by $\langle\mathcal{K}\rangle_N$. He conjectured that for any hyperbolic knot $\mathcal{K}$ [23], when $N\to\infty$,

$$|\langle\mathcal{K}\rangle|_N \sim_{N\to\infty} \exp\left(\frac{N}{2\pi}Vol(M_{\mathcal{K}})\right) \tag{2}$$

where $M_{\mathcal{K}}$ is equal to the knot complement $S^3\setminus\mathcal{K}$. $Vol(M_{\mathcal{K}})$ is the hyperbolic volume of $M_{\mathcal{K}}$. Then, in [26], H. Murakami and J. Murakami proved that for any knot $\mathcal{K}$,

$$\langle\mathcal{K}\rangle_N = J_N(\mathcal{K}; e^{\frac{2\pi i}{N}}).$$

Moreover, they generalized the volume definition at right hand side of (2) to simplicial volume of any knot complement $M_{\mathcal{K}}$. Now, the volume conjecture is formulated as follow [26],

Conjecture 1.1 (Volume conjecture)**.** *For a knot $\mathcal{K}$,*

$$|J_N(\mathcal{K}; e^{\frac{2\pi i}{N}})| \sim_{N\to\infty} \exp\left(\frac{N}{2\pi}Vol(M_{\mathcal{K}})\right)$$

where $Vol(M_{\mathcal{K}})$ is the simplicial volume of knot complement $M_{\mathcal{K}}=S^3\setminus\mathcal{K}$. In particular, when $\mathcal{K}$ is hyperbolic, $Vol(M_{\mathcal{K}})$ is the hyperbolic volume of $M_{\mathcal{K}}$.

We remark that the original volume conjecture was proposed for link $\mathcal{L}$ [23, 26], but in this paper, we only consider the case of knot $\mathcal{K}$. It is also possible to remove the absolute value to consider the complexified volume conjecture [27]: For any hyperbolic knot $\mathcal{K}$,

$$J_N(\mathcal{K}; e^{\frac{2\pi i}{N}}) \sim_{N\to\infty} \exp\left(\frac{N}{2\pi}\left(Vol(M_{\mathcal{K}})+iCS(M_{\mathcal{K}})\right)\right) \tag{3}$$

where $CS(M_{\mathcal{K}})$ is the Chern-Simons invariant of $M_{\mathcal{K}}$ [7]. Furthermore , S. Gukov proposed a u-parameterized version of complexified volume conjecture for any hyperbolic knot $\mathcal{K}$ [19],

$$J_N(\mathcal{K}; e^{\frac{2u}{N}}) \sim_{N\to\infty} \exp\left(\frac{k}{\pi i}S_0(u)\right) \tag{4}$$

where $S_0(u)$ is a geometric invariant related the u-deformation volume of $M_{\mathcal{K}}$ [39]. In fact, formula (4) is a generalization of (3) for u near the point πi in $\mathbb{C}$. Moreover, the expansion form of (4) has been extend to the higher order terms by S. Gukov and H. Murakami [20].

It is also interesting to consider the situation when u near the point 0. Another expansion form of color Jones polynomial called the Melvin-Morton-Rozansky (MMR) conjecture was proposed in [25] and generalized by [35]. The MMR conjecture has been proved by D. Bar-Natan and S. Garoufalidis in [4]. Recently, S. Garoufalidis and T. T. Q. Le obtained the following analytic version of MMR expansion.

Theorem 1.2 ([17])**.** *For every knot $\mathcal{K}$, there exist a neighborhood $\mathcal{O}\subset\mathbb{C}$ at $u=0$, such that for any $u\in\mathcal{O}$, we have*

$$J_N(\mathcal{K}; e^{\frac{2u}{N}}) \sim_{N\to\infty} \sum_{d=0}^{\infty}\frac{P_{\mathcal{K},d}(e^{2u})}{\Delta_{\mathcal{K}}(e^{2u})^{2d+1}}\left(\frac{2u}{N}\right)^d, \tag{5}$$

where $\Delta_{\mathcal{K}}(t)$ is the Alexander polynomial of $\mathcal{K}$, $\{P_{\mathcal{K},d}(t), d\geq 0\}$ is a sequence of Laurent polynomials with $P_{\mathcal{K},0}(t)=1$.

Now we focus on the general expansion form of the limit (1), it was conjectured in [19] that the perturbative expansion of $J_N(\mathcal{K};q)$ at the limit $N\to\infty,\ q\to 1$ was equal to the $SL(2,\mathbb{C})$ Chern-Simons partition $Z(M_{\mathcal{K}})$ function up to a certain normalization. Based on the standard perturbative Chern-Simons theory [1, 5, 2], the general perturbative computations of $Z(M_{\mathcal{K}})$

were explored in [9, 10]. Therefore, motivated by the conjectured intimate relation between the color Jones polynomial and Chern-Simons partition, it is rational to consider the higher order expansion of color Jones polynomial [20]. If we introduce the quantum parameter $\hbar$ as $\hbar = \frac{i\pi}{k}$. The two parameters (k, N) in color Jones polynomial are changed to two parameters $(\hbar, u)$. Then the general asymptotic expansion of color Jones polynomial takes the following form [9, 8],

$$J_N(\mathcal{K}; e^{\frac{2u}{N}}) \sim_{N\to\infty} \exp\left(\frac{S_0(u)}{\hbar} - \frac{\delta_{\mathcal{K}}(u)}{2}\log\hbar + \sum_{n=1}^{\infty} S_n(u)\hbar^{n-1}\right). \tag{6}$$

In this paper, we study the calculation of general terms $S_n(u)$ appearing above expansion (5). We propose the two expansion formulas (4) and (5) can be unified from the view of A-polynomial and noncommutative A-polynomial of a knot $\mathcal{K}$. In order to determine every terms $S_n(u)$ appearing at the right side of (6), one need to solve the following equation with initial value $S_{\text{Initial}}(u)$:

$$\begin{cases} \hat{A}_{\mathcal{K}}(\hat{l}, \hat{m}; q) J_N(\mathcal{K}; e^{\frac{2u}{N}}) = 0 \\ S_0(u) = S_{\text{Initial}}(u) \end{cases} \tag{7}$$

where the initial value $S_{\text{Initial}}(u)$ is determined by the solution of the equation $A_{\mathcal{K}}(e^v, e^u) = 0$ up to a constant, where $A_{\mathcal{K}}(l, m)$ is the A-polynomial of $\mathcal{K}$ and $\hat{A}_{\mathcal{K}}(\hat{l}, \hat{m}; q)$ is an operator defined from the noncommutative A-polynomial $\mathcal{K}$ which will be defined in section 2. More precisely, we propose the following conjecture based on the work [9].

Conjecture 1.3. *i) There exists a solution of equation $A_{\mathcal{K}}(e^v, e^u) = 0$ called* **geometric branch** *of A-polynomial: $v = v^G(u)$. In this branch, we have a neighborhood $\mathcal{O}^G \subset \mathbb{C}$ at $u = \pi i$, such that for any $u \in \mathcal{O}^G$,*

$$J_N(\mathcal{K}; e^{\frac{2u}{N}}) \sim_{N\to\infty} \exp\left(\frac{S_0^G(u)}{\hbar} - \frac{3}{2}\log\hbar + \sum_{n=1}^{\infty} S_n^G(u)\hbar^{n-1}\right), \tag{8}$$

with $\frac{dS_0^G(u)}{du} = v^G(u)$, $S_0^G(u)$ is related to the u-deformed volume of $M_{\mathcal{K}}$ by Gukov's conjecture formula (4)*, and $S_1^G(u) = \frac{1}{2}\log\frac{iT_{\mathcal{K}}(u)}{4\pi}$* [20]*. Moreover, every $S_n^G(u)$ for $n \geq 2$ can be obtained by the algorithm introduced in section 2.*

ii) By the properties of A-polynomial, we know that there exist an **abelian branch** *which corresponding to the branch $l = 1$ of $A_{\mathcal{K}}(l, m) = 0$. In this branch, we have a neighborhood $\mathcal{O}^A \subset \mathbb{C}$ of 0, such that for any $u \in \mathcal{O}^A$,*

$$J_N(\mathcal{K}; e^{\frac{2u}{N}}) \sim_{N\to\infty} \exp\left(\frac{S_0^A(u)}{\hbar} + \sum_{n=1}^{\infty} S_n^A(u)\hbar^{n-1}\right), \tag{9}$$

with $S_0^A(u) = 0$ and $S_1^A(u) = \log\frac{1}{\Delta_{\mathcal{K}}(2u)}$, where $\Delta_{\mathcal{K}}(t)$ is the Alexander polynomial of knot $\mathcal{K}$. Moreover, every $S_n^A(u)$ for $n \geq 2$ can also be obtained by the same algorithm.

Furthermore, we promote that

Conjecture 1.4. *The expansion formula* (9) *is consistent with the analytic version of MMR expansion* (5)*.*

Remark 1.5. In fact, there exists a sequence of Laurent polynomials $\{Q_{\mathcal{K},n}(t)\}$ such that for $n \geq 2$, we have

$$S_n^A(u) = \frac{Q_{\mathcal{K}}(e^{2u})}{\Delta_{\mathcal{K}}(e^{2u})}.$$

By the consistence of (9) and (5), if we let $C_{\mathcal{K},d}(u)=\frac{2^dP_{\mathcal{K},d}(e^{2u})}{\Delta_{\mathcal{K}}(e^{2u})^{2d+1}}$, then

$$C_{\mathcal{K},d}(u)=\exp(S_1^A(u))\sum_{\mu\mapsto d}\frac{\prod_{i=1}^{l(\mu)}S^A_{\mu_i+1}(u)}{|Aut(\mu)|},$$

where μ is the partition of d with length $l(\mu)$. In other words, conjecture 1.3 provides a method to compute every $P_{\mathcal{K},d}(e^{2u})$ appears in the analytic version of MMR expansion.

Remark 1.6. The algorithm mentioned above to calculate the higher order terms $S_n(u)$ is extracted from the recent works on perturbative computation of $SL(2,\mathbb{C})$ Chern-Simons theory [9, 10]. We note that, by the definitions in their works, the color Jones polynomial $J_N(\mathcal{K};e^{\frac{2\pi i}{k}})$ and $Z(M_{\mathcal{K}};u,\hbar)$ are only difference with a normalization $\frac{q^{\frac{N}{2}}-q^{-\frac{N}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$. Thus, we have the similar calculations for $J_N(\mathcal{K};e^{\frac{2\pi i}{k}})$. Besides the geometric and abelian branches, the authors also introduced the conjugate branch of A-polynomial. See [9] for more details.

The rest of this paper is organized as follows: In section 2, we review the definitions of A-polynomial, non-commutative A-polynomial, AJ conjecture for color Jones polynomial and their recent progresses. Then, we illustrate the quantization algorithm to compute $S_n(u)$ which was introduced in [9] to study the pertubative computation of $SL(2,\mathbb{C})$ Chern-Simons theory. In section 3, we give some examples to illustrate the calculations of the higher order terms with the quantization algorithm. More precisely, we have calculated the following examples:

$i)$ Figure-8 knot 4_1 in both geometric and abelian branches which has been computed in [9, 10] with three different methods under the context of $SL(2,\mathbb{C})$ Chern-Simons theory.

$ii)$ Twist knots 5_2 and 6_1 in abelian branch.

Lastly, the results on the abelian branch support our Conjecture 1.4.

2. A-polynomial, noncommutative A-polynomial and the quantization algorithm

2.1. **A-polynomial $A_{\mathcal{K}}(l,m)$ of a knot $\mathcal{K}$.** Let us start with the review of definition of A-polynomial of a knot $\mathcal{K}$ in S^3 [6]. Denoted by $R(M)=Hom(\pi_1(M),SL(2,\mathbb{C}))$ the set of all homomorphisms ρ from $\pi_1(M)$ to $SL(2,\mathbb{C})$ where $M=S^3\backslash\mathcal{K}$. Let $R_U(M)$ be the subset of $R(M)$ consisting of a representation ρ such that $\rho(\mu)$ and $\rho(\lambda)$ are upper triangular matrices for a fixed meridian μ and longitude λ of $\mathcal{K}$. Then one can define a projection $\xi=(\xi_\lambda,\xi_\mu):R_U(M)\to\mathbb{C}^2$ by $\xi(\rho)=(l,m)$ for $\rho\in R_U(M)$ with

$$\rho(\lambda)=\begin{pmatrix} l & *\\ 0 & l^{-1}\end{pmatrix},\ \rho(\mu)=\begin{pmatrix} m & *\\ 0 & m^{-1}\end{pmatrix}.$$

The Zariski closure of $\xi(R_U(M))$ is an algebraic variety in $\mathbb{C}^2$ and each of its irreducible components is a curve, which is defined by zeros of polynomial with integer coefficients in l and m. Then the product of those defining polynomials is defined as the **A-polynomial** of knot $\mathcal{K}$. Note that the A-polynomial of $\mathcal{K}$ has a factor $l-1$, which corresponds to to abelian representations which related to the Alexander polynomial of $\mathcal{K}$. Thus someone define the A-polynomial $A_{\mathcal{K}}(l,m)$ as the original A-polynomial divided by $l-1$. The A-polynomial reflects the geometric properties of the knot $\mathcal{K}$. More algebraic properties of A-polynomial are listed in [19].

Many A-polynomial of knot has been computed by now. Here we give the A-polynomial of two types of knots. For a (p,q)-torus knot $\mathcal{K}_{p,q}$, the A-polynomial is given by [6]:

$$A_{\mathcal{K}_{p,q}}(l,m)=1+lm^{pq}.$$

Denote by $\mathcal{K}_p$ $p\in\mathbb{Z}$ the p-twist knot, its A-polynomial was computed in [22].

When $p\neq-1,0,1,2$, $A_{\mathcal{K}_p}(l,m)$ is given recursively by

$$A_{\mathcal{K}_p}(l,m) = \begin{cases} cA_{\mathcal{K}_{p-1}}(l,m) - dA_{\mathcal{K}_{p-2}}(l,m), & p > 0, \\ cA_{\mathcal{K}_{p+1}}(l,m) - dA_{\mathcal{K}_{p+2}}(l,m), & p < 0. \end{cases} \tag{10}$$

where

$$\begin{aligned} c &= -l + l^2 + 2lm^2 + m^4 + 2lm^4 + l^2m^4 + 2lm^6 + m^8 - lm^8, \\ d &= m^4(l+m^2)^4, \end{aligned}$$

and with the initial conditions

$$\begin{aligned} A_{\mathcal{K}_2}(l,m) &= -l^2 + l^3 + 2l^2m^2 + lm^4 + 2l^2m^4 - lm^6 - l^2m^8 \\ &\quad + 2lm^{10} + l^2m^{10} + 2lm^{12} + m^{14} - lm^{14}, \\ A_{\mathcal{K}_1}(l,m) &= l + m^6, \\ A_{\mathcal{K}_0}(l,m) &= 1, \\ A_{\mathcal{K}_{-1}}(l,m) &= -l + lm^2 + m^4 + 2lm^4 + l^2m^4 + lm^6 - lm^8. \end{aligned}$$

For example, by the recursion (10),

$$\begin{aligned} A_{\mathcal{K}_{-2}}(l,m) &= l^2 - l^3 - 3l^2m^2 + l^3m^2 - 2lm^4 - l^2m^4 + 3lm^6 + 3l^2m^6 \\ &\quad + m^8 + 3lm^8 + 6l^2m^8 + 3l^3m^8 + l^4m^8 + 3l^2m^10 + 3l^3m^{10} \\ &\quad - l^2m^{12} - 2l^3m^{12} + lm^{14} - 3l^2m^{14} - lm^{16} + l^2m^{16}. \end{aligned}$$

Remark 2.1. The twist knots $\mathcal{K}_p$ for $p \in \mathbb{Z}$ include some basic knots from Rolfsen's table.

$$\begin{aligned} &\mathcal{K}_1 = 3_1, \mathcal{K}_2 = 5_2, \mathcal{K}_3 = 7_2, \mathcal{K}_4 = 9_2, \\ &\mathcal{K}_{-1} = 4_1, \mathcal{K}_{-2} = 6_1, \mathcal{K}_{-3} = 8_1, \mathcal{K}_{-4} = 10_1. \end{aligned}$$

Recently, S. Garoufalidis and T. Mattman [13] give a recursion formula for the A-polynomial of the $(-2,3,n)$ Pretzel knots.

2.2. **Noncommutative A-polynomial $\hat{A}_{\mathcal{K}}(E,Q;q)$.** The colored Jones polynomial $J_N(\mathcal{K};q)$ has many beautiful structures. It was shown by S. Garoufalidis and TTQ Le [11, 16] that the colored Jones function is q-holonomic, i.e. it satisfies a nontrivial linear recursion relation with appropriate coefficients. With such holonomicity, they introduce a geometric invariant of a knot: the characteristic variety which is an affine 1-dimensional variety in $\mathbb{C}^2$. By comparing the character variety of $SL(2,\mathbb{C})$ representations in the case of the trefoil and figure-eight knots, they stated a conjecture that these two varieties must be equal [11, 16]. They also define the noncommutative A-polynomial $\hat{A}_{\mathcal{K}}(E,Q;q)$ for a knot $\mathcal{K}$ which is the unique monic, linear, minimal order q-difference equation satisfied by the sequence of color Jones polynomials $\{J_N(\mathcal{K};q)\}$. Considering two operators E and Q acting on the Jones polynomial $J_N(\mathcal{K};q)$ by

$$(QJ_{\mathcal{K}})(N) = q^N J_N(\mathcal{K};q), (EJ_{\mathcal{K}})(N) = J_{N+1}(\mathcal{K};q). \tag{11}$$

It is easy to see that $EQ = qQE$.

Then $\hat{A}_{\mathcal{K}}(E,Q;q)$ controls the recursion structure of color Jones polynomial

$$\hat{A}_{\mathcal{K}}(E,Q;q)J_N(\mathcal{K};q) = 0. \tag{12}$$

Note that $\hat{A}_{\mathcal{K}}(E,Q;q)$ can be written as the form

$$\hat{A}_{\mathcal{K}}(E,Q;q) = \sum_{k\geq 0} a_k(Q;q)E^k \tag{13}$$

with $a_k(Q;q) \in \mathbb{Z}[q,Q]$. Then S. Garoufalidis conjectured that

Conjecture 2.2 (AJ Conjecture)**.** *For every knot $\mathcal{K}$ in S^3, $A_{\mathcal{K}}(l,m) = \epsilon \hat{A}_{\mathcal{K}}(l,m^2;q)$, where ϵ is the evaluation map at $q = 1$.*

In order to prove the AJ conjecture, a natural way is to compute the non-commutative A-polynomial. So far, we have known an explicit formula for torus knot in [18], figure-eight knot 4_1 in [11], and 2-bridge knots [24]. Moreover, Takata found out an explicit inhomogeneous q-difference equations for knots 5_2 and 6_1 with degree 5 and 6 respectively [37]. But it is not the really non-commutative A-polynomial in the sense of our definition. Then S. Garoufalidis and X. Sun [14, 15] gave an explicit formula for non-commutative A-polynomial of twist knots $\mathcal{K}_p$ for $p = -8, .., 11$. Recently, S. Garoufalidis and C. Koutschan [12] obtained the non-commutative A-polynomial the for Pretzel knot $(-2, 3, 3+2p)$ for $p = -5, .., 5$ using the method of guessing.

Let us briefly describe the philosophy to calculate the noncommutative A-polynomial $\hat{A}_{\mathcal{K}}(E, Q; q)$ of knot $\mathcal{K}$.

For a generic planar projection of a knot $\mathcal{K}$, S. Garoufalidis and T.T.Q. Le proved that the colored Jones polynomial of a knot $\mathcal{K}$ can be written as a multisum [16]

$$J_N(\mathcal{K}; q) = \sum_{k_1,..,k_r}^{\infty} F(N, k_1, .., k_r), \tag{14}$$

where $F(N, k_1, .., k_r)$ is a proper q-hypergeometric function and for a fixed $N \in \mathbb{Z}^+$, only finitely many terms are nonzero. Because $F(N, k_1, .., k_r)$ is a proper q-hypergeometric function, one can use the algorithm invented by Wilf-Zeilberger [34, 42](the WZ algorithm), also called creative telescoping method, to produce the noncommutative operator eliminate $J_N(\mathcal{K}; q)$. See [32, 33] for a mathematica implementation of WZ-algorithm. We will give some examples to demonstrate how to use this computer program to derive the noncommutative A-polynomial in next section.

2.3. **The algorithm to compute the asymptotic expansion of** $J_N(\mathcal{K}; q)$**.** Let $A_{\mathcal{K}}(l,m)$ be the A-polynomial of a knot $\mathcal{K}$. Define the operator $\hat{l}$ and $\hat{m}$ such that

$$\hat{l} = E, \ \hat{m}^2 = Q. \tag{15}$$

Then by (12), we known that $\hat{A}_{\mathcal{K}}(\hat{l}, \hat{m}^2; q) J_N(\mathcal{K}; q) = 0$.

Recall the parameters we have described in the introduction section

$$\hbar = \frac{\pi i}{k}, \ u = \frac{\pi i N}{k}, \ q = e^{\frac{2\pi i}{k}}.$$

Then $q = e^{2\hbar}$, the operator $\hat{A}_{\mathcal{K}}(\hat{l}, \hat{m}^2; q)$ annihilates

$$J(\mathcal{K}; \hbar, u) := J_N(\mathcal{K}; e^{2\hbar})$$

i.e. we have the equation

$$\hat{A}_{\mathcal{K}}(\hat{l}, \hat{m}^2; q) J(\mathcal{K}; \hbar, u) = 0, \tag{16}$$

and by (11) and (15), the action of the operators $\hat{l}, \hat{m}$ is

$$\hat{m} J(\mathcal{K}; \hbar, u) = e^u J(\mathcal{K}; \hbar, u), \quad \hat{l} J(\mathcal{K}; \hbar, u) = J(\mathcal{K}; \hbar, u + \hbar). \tag{17}$$

It is clear that $\hat{l}\hat{m} = q^{\frac{1}{2}} \hat{m}\hat{l}$. As in (13), we expand $\hat{A}_{\mathcal{K}}(\hat{l}, \hat{m}^2; q)$ as,

$$\hat{A}_{\mathcal{K}}(\hat{l}, \hat{m}^2; q) = \sum_{j=0}^{d} a_j(\hat{m}, \hbar) \hat{l}^j. \tag{18}$$

Then we obtain

$$\sum_{j=0}^{d} a_j(\hat{m},\hbar) J(\mathcal{K};\hbar,u+j\hbar)=0. \tag{19}$$

With the formula (6), one can assume that at large N,

$$J(\mathcal{K};\hbar,u)=\exp\left(\frac{S_0(u)}{\hbar}-\frac{\delta_{\mathcal{K}}(u)}{2}\log\hbar+\sum_{n=1}^{\infty} S_n(u)\hbar^{n-1}\right).$$

Therefore, from the above restriction equation for $J(\mathcal{K};q,u)$, one can obtain the sequence of expansion coefficients $\{S_n(u)\}$ recursively by solving the equation (16) for a given initial value $S_0(u)$ [9]. In following, we will show how to get the recursion formula for $S_n(u)$ step by step.

Equation (19) is equivalent to

$$\begin{aligned} 0&=\sum_{j=0}^{d} a_j(\hat{m},\hbar)\exp\left(\frac{1}{\hbar}S_0(u+j\hbar)-\frac{3}{2}\cdot\log\hbar+\sum_{n\geq 0}\hbar^n S_{n+1}(u+j\hbar)\right) \\ &=\exp\left(-\frac{\delta_{\mathcal{K}}(u)}{2}\cdot\log\hbar\right)\sum_{j=0}^{d} a_j(\hat{m},\hbar)\exp\left(\sum_{n\geq -1}\hbar^n S_{n+1}(u+j\hbar)\right). \end{aligned} \tag{20}$$

And by Taylor expansion

$$\begin{aligned} \sum_{n\geq -1}\hbar^n S_{n+1}(u+j\hbar) &= \sum_{n\geq -1}\sum_{k\geq 0}\frac{S_{n+1}^{(k)}(u)}{k!}j^k\hbar^{k+n} \\ &= \sum_{t\geq -1}\sum_{r=-1}^{t}\frac{S_{r+1}^{(t-r)}(u)}{(t-r)!}j^{t-r}\hbar^t \\ &= \sum_{t\geq -1} S_{t+1}(u)\hbar^t+\sum_{t\geq 0}\sum_{r=-1}^{t-1}\frac{S_{r+1}^{(t-r)}(u)}{(t-r)!}j^{t-r}\hbar^t \\ &= \sum_{t\geq -1} S_{t+1}(u)\hbar^t + jS_0'(u)+\sum_{t\geq 1}\sum_{r=-1}^{t-1}\frac{S_{r+1}^{(t-r)}(u)}{(t-r)!}j^{t-r}\hbar^t. \end{aligned}$$

It follows that

$$\sum_{j=0}^{d} a_j(\hat{m},\hbar)\exp\left(jS_0'(u)+\sum_{t\geq 1}B_t(u,j)\hbar^t\right)=0,$$

where we have defined

$$B_t(u,j)=\sum_{r=-1}^{t-1}\frac{S_{r+1}^{(t-r)}(u)}{(t-r)!}j^{t-r}.$$

Furthermore, one can expand $a_j(\hat{m},\hbar)$ and $\exp\left(\sum_{t\geq 1}B_t(u,j)\hbar^t\right)$ as

$$a_j(\hat{m},\hbar)=\sum_{p\geq 0}a_{i,p}(\hat{m})\hbar^p$$

and

$$\exp\left(\sum_{t\geq 1} B_t(u,j)\hbar^t\right) = 1 + \sum_{n\geq 1} \frac{\left(\sum_{t\geq 1} B_t(u,j)\hbar^t\right)^n}{n!}$$
$$= \sum_{\mu\in\mathcal{P}} \frac{B_\mu(u,j)}{|Aut(\mu)|}\hbar^{|\mu|}$$

where, $\mathcal{P} = \cup_{n\geq 1}\mathcal{P}_n \cup \{\emptyset\}$, and $\mathcal{P}_n$ is the set of all partitions of integer $n \in \mathbb{Z}^+$, and we denote $B_\mu(u,j) = B_{\mu_1}(u,j)\cdots B_{\mu_{l(\mu)}}(u,j)$ and $B_\emptyset(u,j) = 1$.

Then formula (20) is equal to

$$\sum_{j=0}^{d} e^{jS_0'(u)} \left(\sum_{p\geq 0}\sum_{\mu\in\mathcal{P}} a_{j,p}(\hat{m})\frac{B_\mu(u,j)}{|Aut(\mu)|}\hbar^{|\mu|+p}\right) = 0. \tag{21}$$

By the action of $\hat{m}$ defined in (17), one can replace the $\hat{m}$ with e^u in $a_{j,p}(\hat{m})$. As a series of $\hbar$, all the coefficients of left hand side of (21) must be zero. The constant term gives

$$\sum_{j=0}^{d} e^{jS_0'(u)} a_{j,0}(e^u) = 0. \tag{22}$$

which in fact is the A-polynomial.

When $n = |\mu| + p > 0$, one can solve the n-th equation obtained from the coefficient of $\hbar^n$ in equation (21) and get

$$\begin{aligned} S_n'(u) = &-\frac{1}{\sum_{j=0}^{d} e^{jS_0'(u)} a_{j,0}(e^u) j} \sum_{j=0}^{d} e^{jS_0'(u)} \left(\sum_{p=1}^{n} a_{j,p}(e^u) \sum_{\mu\in\mathcal{P}_{n-p}\cup\{\emptyset\}} \frac{B_\mu(u,j)}{|Aut(\mu)|}\right. \\ &\left. + a_{j,0}(e^u) \sum_{\mu\in\mathcal{P}\backslash\{(n)\}} \frac{B_\mu(u,j)}{|Aut(\mu)|} + a_{j,0}(e^u) \sum_{r=-1}^{n-2} \frac{S_{r+1}^{(n-r)}(u)}{(n-r)!} j^{n-r}\right). \end{aligned} \tag{23}$$

Example 2.3. When $n = 1$ and 2, we have

$$S_1'(u) = -\frac{1}{\sum_{j=0}^{d} e^{jS_0'(u)} a_{j,0}(e^u) j} \sum_{j=0}^{d} e^{jS_0'(u)} \left(a_{j,1}(e^u) + a_{j,0}(e^u)\frac{S_0''(u)}{2} j^2\right)$$

$$\begin{aligned} S_2'(u) = &-\frac{1}{\sum_{j=0}^{d} e^{jS_0'(u)} a_{j,0}(e^u) j} \sum_{j=0}^{d} e^{jS_0'(u)} \left[a_{j,1}(e^u)\left(\frac{S_0''(u)}{2} j^2 + S_1'(u) j\right)\right. \\ &\left. + a_{j,2}(e^u) + a_{j,0}(e^u)\left(\frac{1}{2}\left(\frac{S_0''(u)}{2} j^2 + S_1'(u) j\right)^2 + \frac{S_0'''(u)}{6} j^3 + \frac{S_1''(u)}{2} j^2\right)\right] \end{aligned}$$

The above formula (23) gives a recursion relation for $S_n(u)$. In other words, if one knows the initial value $S_0(u)$, then all the coefficients $S_n(u)$ are determined uniquely. How to choose $S_0(u)$ depends on the choice of the branch of A-polynomial as described in the introduction section, i.e. in the geometric branch: choosing $S_0(u) = S_0^G(u)$; and in abelian branch: choosing $S_0(u) = S_0^A(u) = 0$.

Remark 2.4. By AJ-conjecture, the classical limit $q \to 1$ of noncommutative A-polynomial is the A-polynomial. Thus, the noncommutative A-polynomial can be considered as the quantization of A-polynomial. So the above method to compute $S_n(u)$ can be called quantization algorithm. All the information of color Jones polynomial are implied in a hierarchy of equations (22), (23). The first one equation (22) is the A-polynomial if we let $l = e^{S_0'(u)}$. A-polynomial reflects some geometric information of the knot complement $M_{\mathcal{K}}$. Finding the geometric meaning of the generic equations (23) will be interesting.

Remark 2.5. The above quantization algorithm was introduce in [9] to study the $SL(2,\mathbb{C})$ Chern-Simons partition function of $M_{\mathcal{K}}$. They assumed that the color Jones polynomial $J_N(\mathcal{K}; e^{\frac{2\pi i}{k}})$ and Chern-Simons partition $Z(M_{\mathcal{K}}; u, \hbar)$ are only difference with a normalization $\frac{q^{\frac{N}{2}}-q^{-\frac{N}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$. So in order to get the quantization operator $\tilde{A}(\tilde{l},\tilde{m})$ of $Z(M_{\mathcal{K}}; u; \hbar)$ such that $\tilde{A}(\tilde{l},\tilde{m})Z(M_{\mathcal{K}}; u; \hbar) = 0$, one only needs to do some modifications on operator $\hat{A}_{\mathcal{K}}(\hat{l},\hat{m}^2; q)$. See [9, 10] for detail discussion.

3. Examples

Example 3.1. When $p = -1$, $\mathcal{K}_{-1} = 4_1$, this example has been calculated in [9]. But we still recalculate here as an basic example to illustrate the application of above algorithm.

Step 1. Finding the noncommutative A-polynomial of 4_1.

We download Paule and Riese's qZeil.m and qMultiSum.m package [32] and run them in Mathematica 7.0.

In[1]:= << c:/qZeil.m

q-Zeilberger Package by Axel Riese–@RISC Linz -V 2.42 (02/18/05)

In[2]:= << c:/qMultiSum.m

qMultiSum Package by Axel Riese–@RISC Linz -V 2.52 (30-Jul-2010)

In[3]:=summandfigure8 = $q^{nk}qfac[q^{-n-1}, q^{-1}, k]qfac[q^{-n+1}, q, k]$

Out[3]:=$q^{kn}qPochhammer[q^{-1-n}, 1/q, k]qPochhammer[q^{1-n}, q, k]$

In[4]:=qZeil[summandfigure8, k, 0, Infinity, n, 2]

Out[4]:=

$$\text{SUM[n]} == \frac{q^{-1-n}(q+q^n)(-q+q^{2n})}{-1+q^n} - \frac{(1-q^{-2+n})(1-q^{-1+2n})\text{SUM[-2 + n]}}{(1-q^n)(1-q^{-3+2n})} + \frac{q^{-2-2n}(1-q^{-1+n})^2(1+q^{-1+n})(q^4+q^{4n}-q^{3+n}-q^{1+2n}-q^{3+2n}-q^{1+3n})\text{SUM[-1 + n]}}{(1-q^n)(1-q^{-3+2n})}$$

This is a second-order inhomogeneous recursion relation, we convert it into a third-order homogeneous recursion relation:

In[5]:=MakeHomRec [%, SUM[n]];

Converting to forward shifts:

In[6]:=Rec41=ForwardShifts[%]

$$\begin{aligned}\text{Out[6]} :=& q^{5+n}(q-q^{3+n})(q^3-q^{3+n})(q+q^{3+n})(q-q^{6+2n})(q^3-q^{6+2n})\text{SUM[n]}\\ &- q^{-5-n}(q-q^{3+n})(q^2-q^{3+n})(q^2+q^{3+n})(q-q^{6+2n})(q^3-q^{6+2n})\\ &\times(q^8-2q^{9+n}+q^{10+n}-q^{9+2n}+q^{10+2n}-q^{11+2n}+q^{10+3n}-2q^{11+3n}+q^{12+4n})\\ &\times \text{SUM[1 + n]} + q^{-4-n}(q-q^{3+n})^2(q+q^{3+n})(q^3-q^{6+2n})(q^5-q^{6+2n})(q^4+q^{5+n}\\ &-2q^{6+n}-q^{7+2n}+q^{8+2n}-q^{9+2n}-2q^{10+3n}+q^{11+3n}+q^{12+4n})\ \text{SUM[2 + n]} + q^{4+n}\\ &\times(q-q^{3+n})(-1+q^{3+n})(q^2+q^{3+n})(q^3-q^{6+2n})(q^5-q^{6+2n})\text{SUM[3 + n]} == 0\end{aligned}$$

Converting it to operator form:

In[7]:=F=ToqHyper[Rec41[[1]] - rec41[[2]]]/.{SUM[N] $\to 1$, SUM[N $q^c_.$] :> X^c}/.N $\to Q$

Out[7] :=$q^5Q(q-q^3Q)(q^3-q^3Q)(q+q^3Q)(q-q^6Q^2)(q^3-q^6Q^2)$

$$
\begin{aligned}
&-\frac{1}{q^5Q}(q-q^3Q)(q^2-q^3Q)(q^2+q^3Q)(q-q^6Q^2)(q^3-q^6Q^2)\\
&\times(q^8-2q^9Q+q^{10}Q-q^9Q^2+q^{10}Q^2-q^{11}Q^2+q^{10}Q^3-2q^{11}Q^3+q^{12}Q^4)X\\
&+\frac{1}{q^4Q}(q-q^3Q)^2(q+q^3Q)(q^3-q^6Q^2)(q^5-q^6Q^2)(q^4+q^5Q-2q^6Q\\
&-q^7Q^2+q^8Q^2-q^9Q^2-2q^{10}Q^3+q^{11}Q^3+q^{12}Q^4)X^2\\
&+q^4Q(q-q^3Q)(-1+q^3Q)(q^2+q^3Q)(q^3-q^6Q^2)(q^5-q^6Q^2)X^3
\end{aligned}
$$

Then F is the non-commutative A-polynomial of 4_1 if we replace X by E.

Step 2. Finding the operator $\hat{A}_{4_1}(\hat{l},\hat{m};q)=\sum_{j=0}^{d}a_j(\hat{m},\hbar)\hat{l}^j$.

Substituting Q and X by m^2 and l respectively in F, we get

$$\hat{A}_{4_1}(\hat{l},\hat{m})=\sum_{j=0}^{3}a_j(\hat{m},\hbar)\hat{l}^j$$

where

$$
\begin{aligned}
\hat{a}_0(\hat{m},q)&=\hat{m}^2q^5(q-\hat{m}^2q^3)(q^3-\hat{m}^2q^3)(q+\hat{m}^2q^3)(q-\hat{m}^4q^6)(q^3-\hat{m}^4q^6)\\
\hat{a}_1(\hat{m},q)&=\frac{1}{\hat{m}^2q^5}(q-\hat{m}^2q^3)(q^2-\hat{m}^2q^3)(q^2+\hat{m}^2q^3)(q-\hat{m}^4q^6)(q^3-\hat{m}^4q^6)\\
&\quad\times(q^8-2\hat{m}^2q^9-\hat{m}^4q^9+\hat{m}^2q^{10}+\hat{m}^4q^{10}+\hat{m}^6q^{10}-\hat{m}^4q^{11}-2\hat{m}^6q^{11}+\hat{m}^8q^{12})\\
\hat{a}_2(\hat{m},q)&=\frac{1}{\hat{m}^2q^4}(q-\hat{m}^2q^3)^2(q+\hat{m}^2q^3)(q^3-\hat{m}^4q^6)(q^5-\hat{m}^4q^6)\\
&\quad\times(q^4+\hat{m}^2q^5-2\hat{m}^2q^6-\hat{m}^4q^7+\hat{m}^4q^8-\hat{m}^4q^9-2\hat{m}^6q^{10}+\hat{m}^6q^{11}+\hat{m}^8q^{12})\\
\hat{a}_3(\hat{m},q)&=\hat{m}^2q^4(q-\hat{m}^2q^3)(-1+\hat{m}^2q^3)(q^2+\hat{m}^2q^3)(q^3-\hat{m}^4q^6)(q^5-\hat{m}^4q^6)
\end{aligned}
$$

Step 3 Choosing the different branches.

The A-polynomial of 4_1 is

$$A_{4_1}(l,m)=(-1+l)(l-lm^2-m^4-2lm^4-l^2m^4-lm^6+lm^8)$$

Solving this equation, we obtain the three branches: $l_A=1$ is called the abelian branch and $l_G=-\frac{-1+m^2+2m^4+m^6-m^8+(-1+m^4)\sqrt{1-2m^2-m^4-2m^6+m^8}}{2m^4}$ is the geometric branch. The third one is the conjugate of l_G called conjugate branch which have the intimate relation with geometric branch discussed in [9].

Step 4 Calculating the expansion coefficients $S_n(u)$ in different branches by formula (23).

Abelian branch expansion: taking the initial value $S_0^A(u)=\log l_A=0$, then

$$
\begin{aligned}
&S_1^A(u)=\log\frac{1}{\Delta_{4_1}(m^2)};\\
&S_2^A(u)=\text{constant};\\
&S_3^A(u)=\frac{4(m^{-2}-1+m^2)}{\Delta_{4_1}(m^2)^3};\\
&S_4^A(u)=\text{constant};\\
&\ldots
\end{aligned}
$$

where $\Delta_{4_1}(t)=\frac{1}{t}+t-3$ is the Alexander polynomial of 4_1. The above results match the Conjecture 1.4.

Geometric branch expansion: the initial value $S_0^G(u) = \frac{i}{2}Vol(4_1) + \int_{i\pi}^{u} v_G(u)du - 2\pi^2$ [9].

$$S_1^G(u) = 2\log(m) - \log(-1+m^2) - \frac{1}{4}\log(1-2m^2-m^4-2m^6+m^8);$$
$$S_2^G(u) = \frac{1-m^2-2m^4+15m^6-2m^8-m^{10}+m^{12}}{12(-1+2m^2+m^4+2m^6-m^8)^{\frac{3}{2}}};$$
$$S_3^G(u) = -\frac{2m^6(-1+m^2+2m^4-5m^6+2m^8+m^{10}-m^{12})}{(1-2m^2-m^4-2m^6+m^8)^3};$$
$$S_4^G(u) = \frac{m^2}{90(1-2m^2-m^4-2m^6+m^8)^{\frac{9}{2}}}(1-4m^2-128m^4+36m^6$$
$$+1074m^8-5630m^{10}+5782m^{12}+7484m^{14}-18311m^{16}+7484m^{18}$$
$$+5782m^{20}-5630m^{22}+1074m^{24}+36m^{26}-128m^{28}-4m^{30}+m^{32});$$
$$\cdots$$

If we use the Ray-Singer torsion of 4_1 [20, 8]

$$T_{4_1}(u) = \frac{4\pi^2 m^2}{\sqrt{-1+2m^2+m^4+2m^6-m^8}},$$

we may conjecture that $S_n(u)$ has the form

$$S_n(u) = \left(\frac{T_{4_1}(u)}{4\pi^2}\right)^{3n-3} G_n(m) \text{ for } n \geq 2,$$

where $\{G_n(m)\}$ is a sequence of Laurent polynomial of m.

Remark 3.2. In [9], they have calculated the perturbative expansion for $Z(M_{4_1}; u; \hbar)$ assume that $Z(M_{4_1}; u; \hbar) = \frac{q^{N/2}-q^{-N/2}}{q^{1/2}-q^{-1/2}} J(\mathcal{K}; u, \hbar)$. By this relation, we should make a modification for $\hat{a}_j(\hat{m}, q)$,

$$\hat{a}_j(\hat{m}, q) \to \frac{\hat{a}_j(\hat{m}, q)}{m^2 q^{j/2} - q^{-j/2}}.$$

With these new $\hat{a}(\hat{m}, q)$, they calculated the $S_n(u)$ for $Z(M_{4_1}; u, \hbar)$ up to $n = 8$.

We give more examples in abelian branch expansion.

Example 3.3. When $p = 2$, the twist knot $\mathcal{K}_2 = 5_2$. Setting the initial value $S_0^A(u) = 0$, we get

$$S_1^A(u) = \log\left(\frac{1}{\Delta_{5_2}(m^2)}\right);$$
$$S_2^A(u) = \frac{-4(m^{-2}+m^2)+13}{2\Delta_{5_2}(m^2)^2};$$
$$S_3^A(u) = -\frac{-32+104m^2+200m^4-607m^6+200m^8+104m^{10}-32m^{12}}{8m^6\Delta_{5_2}(m^2)^4};$$
$$S_4(u) = -\frac{1}{24m^{10}\Delta_{5_2}(m^2)^6}(320-752m^2-3808m^4+3052m^6+39692m^8$$
$$-78163m^{10}+39692m^{12}+3052m^{14}-3808m^{16}-752m^{18}+320m^{20})$$
$$\cdots$$

where $\Delta_{5_2}(t) = 2(t^{-1}+t) - 3$ is the Alexander polynomial of 5_2. It is easy to see these results match the Conjecture 1.4.

Example 3.4. When $p=-2$, the twist knot $\mathcal{K}_{-2}=6_1$. Setting the initial value $S_0^A(u)=0$, we obtain

$$\begin{aligned}
&S_1^A(u)=\log\left(\frac{1}{\Delta_{6_1}(m^2)}\right);\\
&S_2^A(u)=\frac{-4m^2+7m^4-4m^6}{2m^4\Delta_{6_1}(m^2)^2};\\
&S_3^A(u)=-\frac{32-504m^2+1656m^4-2303m^6+1656m^8-504m^{10}+32m^{12}}{8m^6\Delta_{6_1}(m^2)^4};\\
&S_4(u)=-\frac{1}{24m^{10}\Delta_{6_1}(m^2)^6}(320-2512m^2+23968m^4-103404m^6+225900m^8\\
&-288925m^{10}+225900m^{12}-103404m^{14}+23968m^{16}-2512m^{18}+320m^{20})\\
&\cdots
\end{aligned}$$

where $\Delta_{6_1}(t)=2(t^{-1}+t)-5$ is the Alexander polynomial of 6_1. These results match the Conjecture 1.4.

Remark 3.5. In the above two examples, we have used the non-commutative A-polynomial for twist knot $\mathcal{K}_p$ obtained by S. Garoufalidis and X. Sun [14, 15]. In fact, they have calculated all the non-commutative A-polynomial of $\mathcal{K}_p$ for $p=-8,..,11$. So we can get more examples by using their results.

4. Conclusion and discussion

In this paper, we present an algorithm to calculate the higher order terms in general expansion of color Jones polynomial from the view of A-polynomial and noncommutative A-polynomial. In the large N limit, the color Jones polynomial $J_N(\mathcal{K};e^{2u})$ has the following expansion form

$$J(\mathcal{K};\hbar,u)=\exp\left(\frac{S_0(u)}{\hbar}-\frac{3}{2}\log\hbar+\sum_{n=1}^{\infty}S_n(u)\hbar^{n-1}\right). \tag{24}$$

In order to determine every terms $S_n(u)$ appearing at left side of (24), we need to solve the following equation with initial value $S_{\text{Initial}}(u)$:

$$\begin{cases}\hat{A}_{\mathcal{K}}(\hat{l},\hat{m};q)J(\mathcal{K};\hbar,u)=0\\ S_0(u)=S_{\text{Initial}}(u)\end{cases} \tag{25}$$

Up to a constant, the initial value $S_{\text{Initial}}(u)$ is determined by the solution of the equation $A_{\mathcal{K}}(e^v,e^u)=0$, where $A_{\mathcal{K}}(l,m)$ is the A-polynomial of knot $\mathcal{K}$. More precisely, if we assume $A_{\mathcal{K}}(l,m)=(l-1)f_d(l,m)$, where $f_d(l,m)=\sum_{i=1}^d a_i(m)l^i$. For a given m, the equation $f_d(l,m)=0$ has d solutions in $\mathbb{C}$ denoted by $l=l^\alpha(m)$, $\alpha=1,..,d$. Thus $A_{\mathcal{K}}(l,m)=0$ has $d+1$ branches: abelian branch $l^A=1$, and $l^\alpha(m)$, for $\alpha=1,..,d$. There are some symmetries between these different branches $\alpha=1,..,d$. See [9] for discussions from Chern-Simon theory.

One of the most interesting branch is called geometric branch denoted by $l^G(m)$ which is relevant with the hyperbolic volume of knot complement $M_{\mathcal{K}}$. In this geometric branch, the initial value is $\frac{dS_{\text{Initial}}(u)}{du}=v^G(u)$ and $S_{\text{Initial}}(u)$ is the complexified volume of $M_{\mathcal{K}}$ parameterized by u. Then all the terms $S_n^G(u)$ can be solved from the recursive relation (23). Moreover, $S_1^G(u)$ has the geometric interpretation $S_1^G(u)=\frac{1}{2}\log\frac{iT_{\mathcal{K}}(u)}{4\pi}$ [20], where $T_{\mathcal{K}}(u)$ u-deformed torsion of $M_{\mathcal{K}}$. But what's the geometric mean of $S_n^G(u)$ for $n\geq 2$ is still unknown.

In the abelian branch $l_A=1$, the initial value is $\frac{dS_{\text{Initial}}(u)}{du}=v^A=0$. So $S_{\text{Initial}}(u)$ is a constant. One can get every $S_n^A(u)$ by formula (23). Moreover, the first term $S_1^A(u)=$

$\log \frac{1}{\Delta_{\mathcal{K}}(2u)}$, where $\Delta_{\mathcal{K}}(t)$ is the Alexander polynomial of $\mathcal{K}$ which has the geometric meaning, but the geometric interpretation is still unclear for $S_n^A(u), n \geq 2$. We found that expansion terms $S_n^A(u)$ is in consistence with the Melvin-Morton-Rozansky expansion for color Jones polynomial [25, 8, 35, 36, 16], so we proposed the Conjecture 1.4.

We have calculated the following examples in this paper: i) figure-8 knot 4_1 in both geometric and abelian branches; ii) Twist knots 5_2 and 6_1 in abelian branch. Both of them match the conjecture. For the cases in ii) , we only compute the abelian branch, because the geometric branch in this situation is more complicated which consumes much computer time.

The recursive algorithm to get $S_n(u)$ was found in [9] to study the $SL(2,\mathbb{C})$ Chern-Simons partition function $Z(M_{\mathcal{K}}; u, \hbar)$ of $M_{\mathcal{K}}$. As to quantum invariant $Z(M_{\mathcal{K}}; u, \hbar)$, there are also two other approaches to compute the expansion coefficients $S_n(u)$: the state integral model introduced in [9] and the topological recursion method which is from string theory [10]. These two methods may also be used to study the geometric branch of cases in ii) which will be further studied in [45].

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