One-sided Lévy stable distributions

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Abstract

In this paper, we show new representations of one-sided Lévy stable distributions for irrational Lévy indices of the type $\left(\frac{p}{q}\right)^{\frac{l_2}{l_1}}$ which are not covered in [8] : for rational Lévy indices. Furthermore, other equivalent representations for a distribution of a rational Lévy index is described. We also give a simplest proof for the formulae which cover the cases for rational Lévy indices. Finally we introduce the concepts of Lévy smashing and Lévy-smashed gamma stochastic processes.

1 Introduction and preliminary results

In [8], Penson and Górska obtained a general form of one-sided Lévy stable distributions expressed as a Meijer G-function for rational Lévy indices by putting $v = 0$ and $a = 1$ in the formula 2.2.1.19 in vol. 5 of [10]. In [2], the role of Mathai transformation in the theory of fractional calculus, which connects ordinary integral to fractional integral through their kernels, is described and α-level space is defined. Furthermore it is insisted that fractional integral and derivative preserve the Lévy structure defined in [2]. The Lévy structure is nothing but the integrand $\frac{\Gamma(\frac{1}{\alpha}-\frac{s}{\alpha})}{\alpha\Gamma(1-s)}$ of the H-function representation of the Lévy distribution with α known as the Lévy index which lies between 0 and 1. Hence the case of simple irrational Lévy indices of the type $\left(\frac{p}{q}\right)^{\frac{l_2}{l_1}}$ can be handled.

In this paper, we provide formulae for one-sided Lévy distribution of irrational Lévy index for $0 < \alpha < 1$ and some techniques to generate infinitely many new representations of one-sided Lévy distribution. Furthermore, we introduce the concept of *Lévy smashing* as a consequence of Lévy effect on the family of gamma density functions and Lévy-smashed stochastic processes.

We will use the following integral representation of the gamma function:

$$\begin{aligned}\Gamma(z) &= p^z \int_0^\infty t^{z-1} e^{-pt} dt, \quad \Re(p) > 0, \Re(z) > 0 \\ &= \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)}, \quad z \neq 0, 1, 2, 3, \cdots .\end{aligned}$$

Pochammer symbol is defined as

$$\begin{aligned}(b)_k &= b(b+1)\cdots(b+k-1),\ (b)_0 = 1,\ b \neq 0 \\ &= (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \text{ whenever the gammas exist.}\end{aligned}$$

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For H-function representations and their convergence conditions, consult with [6], [7]:

$$H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),((b_2,B_2),\cdots,((b_q,B_q)}^{(a_1,A_1),(a_2,A_2),\cdots,(a_p,A_p)}\right]=\frac{1}{2\pi i}\oint_L\frac{\{\prod_{j=1}^m\Gamma(b_j+B_js)\}\{\prod_{j=1}^n\Gamma\left(1-a_j-A_js\right)\}}{\{\prod_{j=m+1}^q\Gamma\left(1-b_j-B_js\right)\}\{\prod_{j=n+1}^p\Gamma\left(a_j+A_js\right)\}}z^{-s}ds.$$

Generalized Wright function, which is well explained in [7] and which is a particular case of the H-function, is

$${}_p\Psi_q\left[z\Big|_{(b_1,B_1),(b_2,B_2),\cdots,(b_q,B_q)}^{(a_1,A_1),(a_2,A_2),\cdots,(a_p,A_p)}\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^p\Gamma\left(a_i+A_in\right)}{\prod_{j=1}^q\Gamma(b_j+B_jn)\ n!}z^n$$

where $a_i,b_j\in\mathbb{C}$ and $A_i,B_j\in\mathbb{R}$: $A_i\neq 0, B_j\neq 0, i=1,\cdots,p, j=1,\cdots,q$; $\sum_{j=1}^q B_j-\sum_{i=1}^p A_i>-1$ for absolute convergence.

Hypergeometric series, which is well explained in [6] and which is a particular case of the H-function, is

$${}_pF_q\left(a_1,\cdots,a_p;b_1,\cdots,b_q;z\right)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^p\left(a_i\right)}{\prod_{j=1}^q(b_j)\ n!}z^n$$

which is absolutely convergent for all z in $\mathbb{C}$ if $p\leq q$.

Definition 1.1 *Lévy jump function is defined as follows*

$$\left[\frac{j}{q}\right]_n=\begin{cases}\frac{j}{q}, & j<n,\ j,n\in\mathbb{Z}^+\\ \frac{j+1}{q}, & j\geq n,\ j,n\in\mathbb{Z}^+.\end{cases}$$

Lemma 1.1 *[Stirling asymptotic formula] [6]*
For $|z|\to\infty$ and α a bounded quantity,

$$\Gamma(z+\alpha)\approx(2\pi)^{1/2}z^{z+\alpha-1/2}e^{-z}. \tag{1.1}$$

Lemma 1.2

$$\Gamma\left(-v+\frac{i}{q}-\left[\frac{j}{q}\right]_i\right)=\frac{(-1)^v\Gamma\left(\frac{i}{q}-\left[\frac{j}{q}\right]_i\right)}{\left(1-\frac{i}{q}+\left[\frac{j}{q}\right]_i\right)_v}$$

$$\Gamma\left(-v+\frac{iq-jp}{pq}\right)=\frac{(-1)^v\Gamma\left(\frac{iq-jp}{pq}\right)}{\left(1-\frac{iq-jp}{pq}\right)_v}$$

and

$$H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]=k\ H_{p,q}^{m,n}\left[z^k\Big|_{(b_q,kB_q)}^{(a_p,kA_p)}\right]$$

where $v,n,p,q,i,j\in\mathbb{Z}^+$ and $k>0$.

Proof. Use the formula in [6], p44, $\Gamma(\beta+1-v)=\frac{(-1)^v\Gamma(\beta+1)}{(-\beta)_v}$ and the properties in [7], which complete the proof.

2 Lévy stable distribution of Lévy index $\left(\frac{p}{q}\right)^{\frac{l_2}{l_1}}$

Let

$$f_\alpha(x)=\frac{1}{2\pi i}\oint_L \frac{\Gamma(\frac{1}{\alpha}-\frac{s_1}{\alpha})}{\alpha\Gamma(1-s_1)}x^{-s_1}ds_1,\quad 1>\alpha>Re(s_1)>0. \tag{2.1}$$

$\frac{\Gamma(\frac{1}{\alpha}-\frac{s}{\alpha})}{\alpha\Gamma(1-s)}$ appears firstly in [5] in the literature. Then $f_\alpha(x)$ is the well known Levy density function having the Laplace transform e^{-t^α}.

Theorem 2.1 *Let* $0<\alpha<1$ *and* p,q,l_1,l_2 *be arbitrary positive integers such that* $0<\alpha=\left(\frac{p}{q}\right)^{\frac{l_2}{l_1}}<1,\quad p<q$ *and* $l=\left(\frac{p}{q}\right)^{\frac{l_2}{l_1}-1}$.

If l *is not a positive integer and* $l_1\neq l_2$, *then*

$$\begin{aligned}f_{(\alpha,l)}(x)&=\frac{l\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\;H^{q,0}_{p,q}\left[\frac{p^{pl}}{x^{pl}q^q}\Big|^{(1,l),(\frac{1}{p},l),\cdots,(\frac{p-1}{p},l)}_{(1,1),(\frac{1}{q},1),\cdots,(\frac{q-1}{q},1)}\right]\\&=\frac{l\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\left(\frac{p^{lp}}{x^{lp}q^q}\right)\;{}_{q-1}\Psi_p\left[-\frac{p^{pl}}{x^{pl}q^q}\Big|^{(-\frac{q-1}{q},-1)\cdots(-\frac{2}{q},-1)(-\frac{1}{q},-1)}_{(1-l,-l)(\frac{q-plq}{pq},-l)\cdots(\frac{(p-1)q-plq}{pq},-l)}\right]+\frac{l\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\sum_{j=1}^{q-1}\left(\frac{p^{jl\frac{p}{q}}}{x^{jl\frac{p}{q}}q^j}\right)\\&\times\;{}_{q-1}\Psi_p\left[-\frac{p^{pl}}{x^{pl}q^q}\Big|^{\left(1-\left[\frac{j}{q}\right]_q,-1\right)\left(\frac{2}{q}-\left[\frac{j}{q}\right]_2,-1\right)\left(\frac{3}{q}-\left[\frac{j}{q}\right]_3,-1\right)\cdots\left(\frac{q-1}{q}-\left[\frac{j}{q}\right]_{q-1},-1\right)}_{\left(\frac{pq-jpl}{pq},-l\right)\left(\frac{q-jpl}{pq},-l\right)\cdots\left(\frac{(p-1)q-jpl}{pq},-l\right)}\right].\end{aligned}\tag{2.2}$$

If $l=1$ *and* $l_1=l_2$, *then*

$$\begin{aligned}f_{(\alpha,1)}(x)&=\frac{\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\;H^{q-1,0}_{p-1,q-1}\left[\frac{p^p}{x^pq^q}\Big|^{(\frac{1}{p},1),\cdots,(\frac{p-1}{p},1)}_{(\frac{1}{q},1),\cdots,(\frac{q-1}{q},1)}\right]\\&=\frac{\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\sum_{j=1}^{q-1}\left(\frac{p^{j\frac{p}{q}}}{x^{j\frac{p}{q}}q^j}\right)\frac{\prod_{i_1=2}^{q-1}\Gamma\left(\frac{i_1}{q}-\left[\frac{j}{q}\right]_{i_1}\right)}{\prod_{i_2=1}^{p-1}\Gamma\left(\frac{i_2q-jp}{pq}\right)}\\&\times\;{}_{p-1}F_{q-2}\left[(-1)^{q-p}\frac{p^p}{x^pq^q}\Big|^{\left(1-\frac{q-jp}{pq}\right)\cdots\left(1-\frac{(p-1)q-jp}{pq}\right)}_{\left(1-\frac{2}{q}+\left[\frac{j}{q}\right]_2\right)\left(1-\frac{3}{q}+\left[\frac{j}{q}\right]_3\right)\cdots\left(1-\frac{q-1}{q}+\left[\frac{j}{q}\right]_{q-1}\right)}\right].\end{aligned}\tag{2.3}$$

If l *belongs to the set* $\{2,3,4,\cdots\}$ *and* $l_1\neq l_2$, *then*

$$\begin{aligned}f_{(\alpha,l)}(x)&=\frac{\sqrt{pql}}{x(2\pi)^{\frac{q+1-p-l}{2}}}\;H^{q-1,0}_{p+l-2,q-1}\left[\frac{p^{pl}l^l}{x^{pl}q^q}\Big|^{(\frac{1}{l},1),\cdots,(\frac{l-1}{l},1),(\frac{1}{p},l),\cdots,(\frac{p-1}{p},l)}_{(\frac{1}{q},1),\cdots,(\frac{q-1}{q},1)}\right]\\&=\frac{\sqrt{pql}}{x(2\pi)^{\frac{q+1-l-p}{2}}}\sum_{j=1}^{q-1}\left(\frac{p^{jl\frac{p}{q}}l^{j\frac{l}{q}}}{x^{jl\frac{p}{q}}q^j}\right)\\&\times\;{}_{q-2}\Psi_{p-1+l-1}\left[-\frac{p^{pl}l^l}{x^{pl}q^q}\Big|^{\left(\frac{2}{q}-\left[\frac{j}{q}\right]_2,-1\right)\left(\frac{3}{q}-\left[\frac{j}{q}\right]_3,-1\right)\cdots\left(\frac{q-1}{q}-\left[\frac{j}{q}\right]_{q-1},-1\right)}_{\left(\frac{q-jl}{lq},-1\right)\cdots\left(\frac{(l-1)q-jl}{lq},-1\right)\left(\frac{q-jpl}{pq},-l\right)\cdots\left(\frac{(p-1)q-jpl}{pq},-l\right)}\right].\end{aligned}\tag{2.4}$$

Proof. Here we use a transformation $1 - s_1 = \alpha s$, then

$$f_\alpha(x) = \frac{1}{2\pi i}\oint_L \frac{\Gamma(s)}{\Gamma(\alpha s)} x^{\alpha s - 1} ds, \quad \frac{1}{\alpha} > Re(s) > \frac{1-\alpha}{\alpha} > 0. \tag{2.5}$$

Now, by using the Gauss-Legendre formula (Multiplicative formula) for gamma function, namely

$$\Gamma(mz) \;=\; (2\pi)^{\frac{1-m}{2}} m^{mz-\frac{1}{2}} \Gamma(z)\Gamma(z+\frac{1}{m})\cdots\Gamma(z+\frac{m-1}{m}),\ m = 1, 2, \cdots,$$

we have

$$\Gamma(\alpha s) \;=\; \Gamma\left(\left(\frac{p}{q}\right)^{\frac{l_2}{l_1}} s\right) = \Gamma\left(p\frac{ls}{q}\right) = (2\pi)^{\frac{1-p}{2}} p^{p\frac{ls}{q}-\frac{1}{2}} \Gamma\left(\frac{ls}{q}\right)\Gamma\left(\frac{ls}{q}+\frac{1}{p}\right)\cdots\Gamma\left(\frac{ls}{q}+\frac{p-1}{p}\right).$$

Apply to the integrand in (2.5), then

$$f_{(\alpha,l)}(x) = \frac{1}{x}\frac{1}{2\pi i}\oint_L \frac{\Gamma(s)\ x^{\frac{p}{q}ls}}{(2\pi)^{\frac{1-p}{2}} p^{p\frac{ls}{q}-\frac{1}{2}} \Gamma(\frac{ls}{q})\Gamma(\frac{ls}{q}+\frac{1}{p})\cdots\Gamma(\frac{ls}{q}+\frac{p-1}{p})} ds$$

where $\alpha = \frac{p}{q}l = \frac{p}{q}\left(\frac{p}{q}\right)^{\frac{l_2}{l_1}-1} = \left(\frac{p}{q}\right)^{\frac{l_2}{l_1}}$.
We use

$$H^{m,n}_{p,q}\left[z\Big|^{(a_p,A_p)}_{(b_q,B_q)}\right] = k\ H^{m,n}_{p,q}\left[z^k\Big|^{(a_p,kA_p)}_{(b_q,kB_q)}\right]$$

in Lemma 1.2. Then

$$\begin{aligned}
f_{(\alpha,l)}(x) &= \frac{q}{2\pi i x}\oint_L \frac{\Gamma(qs)\ x^{pls}}{(2\pi)^{\frac{1-p}{2}} p^{pls-\frac{1}{2}}\Gamma(ls)\Gamma(ls+\frac{1}{p})\cdots\Gamma(ls+\frac{p-1}{p})} ds \\
&= \frac{q}{2\pi i x}\oint_L \frac{(2\pi)^{\frac{1-q}{2}} q^{qs-\frac{1}{2}}\Gamma(s)\Gamma(s+\frac{1}{q})\cdots\Gamma(s+\frac{q-1}{q})\ x^{pls}}{(2\pi)^{\frac{1-p}{2}} p^{pls-\frac{1}{2}}\Gamma(ls)\Gamma(ls+\frac{1}{p})\cdots\Gamma(ls+\frac{p-1}{p})} ds \\
&= \frac{1}{2\pi i}\oint_L \frac{\frac{\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\ ls\Gamma(s)\Gamma\left(s+\frac{1}{q}\right)\cdots\Gamma\left(s+\frac{q-1}{q}\right)}{ls\Gamma\left(ls\right)\Gamma\left(ls+\frac{1}{p}\right)\cdots\Gamma\left(ls+\frac{p-1}{p}\right)}\left[\frac{p^{pl}}{x^{pl}q^q}\right]^{-s} ds \\
&= \frac{l\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\ \frac{1}{2\pi i}\oint_L \frac{\Gamma(s+1)\Gamma\left(s+\frac{1}{q}\right)\cdots\Gamma\left(s+\frac{q-1}{q}\right)}{\Gamma\left(ls+1\right)\Gamma\left(ls+\frac{1}{p}\right)\cdots\Gamma\left(ls+\frac{p-1}{p}\right)}\left[\frac{p^{pl}}{x^{pl}q^q}\right]^{-s} ds \\
&= \frac{l\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}} H^{q,0}_{p,q}\left[\frac{p^{pl}}{x^{pl}q^q}\Big|^{(1,l),(\frac{1}{p},l),\cdots,(\frac{p-1}{p},l)}_{(1,1),(\frac{1}{q},1),\cdots,(\frac{q-1}{q},1)}\right]
\end{aligned}$$

Since we have the following

$$\begin{aligned}
&H^{q,0}_{p,q}\left[\frac{p^{pl}}{x^{pl}q^q}\Big|^{(1,l),(\frac{1}{p},l),\cdots,(\frac{p-1}{p},l)}_{(1,1),(\frac{1}{q},1),\cdots,(\frac{q-1}{q},1)}\right] \\
&= \sum_{k_0=1}^{\infty} \frac{(-1)^{k_0}\Gamma(-k_0-\frac{q-1}{q})\cdots\Gamma(-k_0-\frac{2}{q})\Gamma(-k_0-\frac{1}{q})}{k_0!\ \Gamma(-lk_0+\frac{pq-plq}{pq})\Gamma(-lk_0+\frac{q-plq}{pq})\cdots\Gamma(-lk_0+\frac{(p-1)q-plq}{pq})}\left(\frac{p^{pl}}{x^{pl}q^q}\right)^{k_0+1}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k_1=0}^{\infty}\frac{(-1)^{k_1}\Gamma(-k_1+\frac{q-1}{q})\Gamma(-k_1+\frac{1}{q})\cdots\Gamma(-k_1+\frac{q-2}{q})}{k_1!\ \Gamma(-lk_1+\frac{pq-pl}{pq})\Gamma(-lk_1+\frac{q-pl}{pq})\cdots\Gamma(-lk_1+\frac{(p-1)q-pl}{pq})}\left(\frac{p^{pl}}{x^{pl}q^q}\right)^{k_1+\frac{1}{q}}\\
&+\sum_{k_2=0}^{\infty}\frac{(-1)^{k_2}\Gamma(-k_2+\frac{q-2}{q})\Gamma(-k_2-\frac{1}{q})\Gamma(-k_2+\frac{1}{q})\cdots\Gamma(-k_2+\frac{q-3}{q})}{k_2!\ \Gamma(-lk_2+\frac{pq-2pl}{pq})\Gamma(-lk_2+\frac{q-2pl}{pq})\cdots\Gamma(-lk_2+\frac{(p-1)q-2pl}{pq})}\left(\frac{p^{pl}}{x^{pl}q^q}\right)^{k_2+\frac{2}{q}}\\
&+\cdots\\
&+\sum_{k_{q-1}=0}^{\infty}\frac{(-1)^{k_{q-1}}\Gamma(-k_{q-1}+\frac{1}{q})\Gamma(-k_{q-1}-\frac{q-2}{q})\cdots\Gamma(-k_{q-1}-\frac{1}{q})}{k_{q-1}!\ \Gamma(-lk_{q-1}+\frac{pq-(q-1)pl}{pq})\Gamma(-lk_{q-1}+\frac{q-(q-1)pl}{pq})\cdots\Gamma(-lk_{q-1}+\frac{(p-1)q-(q-1)pl}{pq})}\\
&\times\left(\frac{p^{pl}}{x^{pl}q^q}\right)^{k_{q-1}+\frac{q-1}{q}}=\left(\frac{p^{lp}}{x^{lp}q^q}\right)\ {}_{q-1}\Psi_p\left[-\frac{p^{pl}}{x^{pl}q^q}\Big|^{(-\frac{q-1}{q},-1)\cdots(-\frac{2}{q},-1)(-\frac{1}{q},-1)}_{(1-l,-l)(\frac{q-plq}{pq},-l)\cdots(\frac{(p-1)q-plq}{pq},-l)}\right]\\
&+\left(\frac{p^{l\frac{p}{q}}}{x^{l\frac{p}{q}}q}\right)\ {}_{q-1}\Psi_p\left[-\frac{p^{pl}}{x^{pl}q^q}\Big|^{(\frac{q-1}{q},-1)(\frac{1}{q},-1)\cdots(\frac{q-2}{q},-1)}_{(\frac{pq-pl}{p},-l)(\frac{q-pl}{pq},-l)\cdots(\frac{(p-1)q-pl}{p},-l)}\right]\\
&+\left(\frac{p^{2l\frac{p}{q}}}{x^{2l\frac{p}{q}}q^2}\right)\ {}_{q-1}\Psi_p\left[-\frac{p^{pl}}{x^{pl}q^q}\Big|^{(-\frac{1}{q},-1)(\frac{1}{q},-1)\cdots(\frac{q-3}{q},-1)(\frac{q-2}{q},-1)}_{(\frac{q-2pl}{pq},-l)\cdots(\frac{(p-1)q-2pl}{p},-l)(\frac{pq-2pl}{p},-l)}\right]\\
&+\cdots\\
&+\left(\frac{p^{(q-1)l\frac{p}{q}}}{x^{(q-1)l\frac{p}{q}}q^{(q-1)}}\right)\ {}_{q-1}\Psi_p\left[-\frac{p^{pl}}{x^{pl}q^q}\Big|^{(\frac{1}{q},-1)(-\frac{(q-2)}{q},-1)(\frac{(q-3)}{q},-1)\cdots(-\frac{1}{q},-1)}_{(\frac{pq-(q-1)pl}{p},-l)(\frac{q-(q-1)pl}{pq},-l)\cdots(\frac{(p-1)q-(q-1)pl}{p},-l)}\right],
\end{aligned}
$$

we get

$$
\begin{aligned}
&f_{(\alpha,l)}(x)=\frac{l\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\ H^{q,0}_{p,q}\left[\frac{p^{pl}}{x^{pl}q^q}\Big|^{(1,l),(\frac{1}{p},l),\cdots,(\frac{p-1}{p},l)}_{(1,1),(\frac{1}{q},1),\cdots,(\frac{q-1}{q},1)}\right]\\
&=\frac{l\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\left(\frac{p^{lp}}{x^{lp}q^q}\right)\ {}_{q-1}\Psi_p\left[-\frac{p^{pl}}{x^{pl}q^q}\Big|^{(-\frac{q-1}{q},-1)\cdots(-\frac{2}{q},-1)(-\frac{1}{q},-1)}_{(1-l,-l)(\frac{q-plq}{pq},-l)\cdots(\frac{(p-1)q-plq}{pq},-l)}\right]\\
&+\frac{l\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\sum_{j=1}^{q-1}\left(\frac{p^{jl\frac{p}{q}}}{x^{jl\frac{p}{q}}q^j}\right)\ {}_{q-1}\Psi_p\left[-\frac{p^{pl}}{x^{pl}q^q}\Big|^{\left(1-\left[\frac{j}{q}\right]_q,-1\right)\left(\frac{2}{q}-\left[\frac{j}{q}\right]_2,-1\right)\left(\frac{3}{q}-\left[\frac{j}{q}\right]_3,-1\right)\cdots\left(\frac{q-1}{q}-\left[\frac{j}{q}\right]_{q-1},-1\right)}_{\left(\frac{pq-jpl}{pq},-l\right)\left(\frac{q-jpl}{pq},-l\right)\cdots\left(\frac{(p-1)q-jpl}{pq},-l\right)}\right]
\end{aligned}
$$

where $\left[\frac{j}{q}\right]_n$ is the Lévy jump function.
Note that the series are absolutely convergent satisfying the condition $\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i>-1\Rightarrow -lp+q-1>-1\Rightarrow q>lp\Rightarrow \frac{q}{p}>l\Rightarrow \frac{q}{p}>\frac{p}{q}\left(\frac{p}{q}\right)^{\frac{l_2}{l_1}-1}\Rightarrow \frac{q}{p}>1>\left(\frac{p}{q}\right)^{\frac{l_2}{l_1}}=\alpha$ since $\alpha<1$.

If l belongs to the set $\{2,3,4,\cdots\}$, then we have

$$
\begin{aligned}
f_{(\alpha,l)}(x)&=\frac{\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}H^{q,0}_{p,q}\left[\frac{p^{pl}}{x^{pl}q^q}\Big|^{(0,l),(\frac{1}{p},l),\cdots,(\frac{p-1}{p},l)}_{(0,1),(\frac{1}{q},1),\cdots,(\frac{q-1}{q},1)}\right] \qquad (2.6)\\
&=\frac{\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\frac{1}{2\pi i}\oint_L\frac{\Gamma(s)\Gamma\left(s+\frac{1}{q}\right)\cdots\Gamma\left(s+\frac{q-1}{q}\right)}{\Gamma\left(ls\right)\Gamma\left(ls+\frac{1}{p}\right)\cdots\Gamma\left(ls+\frac{p-1}{p}\right)}\left[\frac{p^{pl}}{x^{pl}q^q}\right]^{-s}ds
\end{aligned}
$$

We use

$$
\Gamma(ls)=(2\pi)^{\frac{1-l}{2}}l^{ls-\frac{1}{2}}\Gamma(s)\Gamma\left(s+\frac{1}{l}\right)\cdots\Gamma\left(s+\frac{l-1}{l}\right).
$$

Hence we have

$$= \frac{\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\frac{1}{2\pi i}\oint_L \frac{\Gamma(s)\Gamma\left(s+\frac{1}{q}\right)\cdots\Gamma\left(s+\frac{q-1}{q}\right)}{(2\pi)^{\frac{1-l}{2}}l^{ls-\frac{1}{2}}\Gamma(s)\Gamma\left(s+\frac{1}{l}\right)\cdots\Gamma\left(s+\frac{l-1}{l}\right)\Gamma\left(ls+\frac{1}{p}\right)\cdots\Gamma\left(ls+\frac{p-1}{p}\right)}\left[\frac{p^{pl}}{x^{pl}q^q}\right]^{-s} ds$$

$$= \frac{\sqrt{pql}}{x(2\pi)^{\frac{q+1-p-l}{2}}}\frac{1}{2\pi i}\oint_L \frac{\Gamma\left(s+\frac{1}{q}\right)\cdots\Gamma\left(s+\frac{q-1}{q}\right)}{\Gamma\left(s+\frac{1}{l}\right)\cdots\Gamma\left(s+\frac{l-1}{l}\right)\Gamma\left(ls+\frac{1}{p}\right)\cdots\Gamma\left(ls+\frac{p-1}{p}\right)}\left[\frac{p^{pl}l^l}{x^{pl}q^q}\right]^{-s} ds$$

$$= \frac{\sqrt{pql}}{x(2\pi)^{\frac{q+1-p-l}{2}}} H^{q-1,0}_{p+l-2,q-1}\left[\frac{p^{pl}l^l}{x^{pl}q^q}\Big|^{(\frac{1}{l},1),\cdots,(\frac{l-1}{l},1),(\frac{1}{p},l),\cdots,(\frac{p-1}{p},l)}_{(\frac{1}{q},1),\cdots,(\frac{q-1}{q},1)}\right]$$

Note that for $l \in \{2,3,4,\cdots\}$, q should be of the form kp, $k=4,5,6,\cdots$ and $l_2 < l_1$. $\sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > 0$ means $q-1-(l-1)-l(p-1) > 0 \Rightarrow q > lp \Rightarrow kp > lp \Rightarrow k > l$. But this condition is always satisfied since $l = (k)^{1-\frac{l_2}{l_1}}$.

If we put $l = 1$ in (2.6), then

$$f_{(\alpha,1)}(x) = \frac{\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\; H^{q,0}_{p,q}\left[\frac{p^p}{x^p q^q}\Big|^{(0,1),(\frac{1}{p},1),\cdots,(\frac{p-1}{p},1)}_{(0,1),(\frac{1}{q},1),\cdots,(\frac{q-1}{q},1)}\right] = \frac{\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\; H^{q-1,0}_{p-1,q-1}\left[\frac{p^p}{x^p q^q}\Big|^{(\frac{1}{p},1),\cdots,(\frac{p-1}{p},1)}_{(\frac{1}{q},1),\cdots,(\frac{q-1}{q},1)}\right]$$

$$= \frac{\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\sum_{j=1}^{q-1}\left(\frac{p^{j\frac{p}{q}}}{x^{j\frac{p}{q}}q^j}\right)\; {}_{q-2}\Psi_{p-1}\left[-\frac{p^p}{x^p q^q}\Big|^{\left(\frac{2}{q}-\left[\frac{j}{q}\right]_2,-1\right)\left(\frac{3}{q}-\left[\frac{j}{q}\right]_3,-1\right)\cdots\left(\frac{q-1}{q}-\left[\frac{j}{q}\right]_{q-1},-1\right)}_{\left(\frac{q-jp}{pq},-1\right)\cdots\left(\frac{(p-1)q-jp}{pq},-1\right)}\right]$$

by applying the formula $\Gamma(\beta+1-v) = \frac{(-1)^v\Gamma(\beta+1)}{(1-(\beta+1))_v}$ in Lemma 1.1

$$= \frac{\sqrt{pq}}{x(2\pi)^{\frac{q-p}{2}}}\sum_{j=1}^{q-1}\left(\frac{p^{j\frac{p}{q}}}{x^{j\frac{p}{q}}q^j}\right)\frac{\prod_{i_1=2}^{q-1}\Gamma\left(\frac{i_1}{q}-\left[\frac{j}{q}\right]_{i_1}\right)}{\prod_{i_2=1}^{p-1}\Gamma\left(\frac{i_2 q-jp}{pq}\right)}$$

$$\times\; {}_{p-1}F_{q-2}\left[(-1)^{q-p}\frac{p^p}{x^p q^q}\Big|^{\left(1-\frac{q-jp}{pq}\right)\cdots\left(1-\frac{(p-1)q-jp}{pq}\right)}_{\left(1-\frac{2}{q}+\left[\frac{j}{q}\right]_2\right)\left(1-\frac{3}{q}+\left[\frac{j}{q}\right]_3\right)\cdots\left(1-\frac{q-1}{q}+\left[\frac{j}{q}\right]_{q-1}\right)}\right]$$

For the case of $l = 1$, the condition for their convergence is trivial since $p < q$.

Some special cases will be given. We will use (2.3) to show some known results.

Example 2.1 *When $p = 1,\ q = 4,\ \alpha = \frac{1}{4}$, we have*

$$f_{(\frac{1}{4},1)}(x) = \frac{2}{x(2\pi)^{\frac{3}{2}}}\; H^{3,0}_{0,3}\left[\frac{1}{4^4 x}\Big|^{-}_{(\frac{1}{4},1),(\frac{2}{4},1),(\frac{3}{4},1)}\right]$$

$$= \frac{2}{x(2\pi)^{\frac{3}{2}}}\sum_{j=1}^{3}\left(\frac{1}{x^{j\frac{1}{4}}4^j}\right)\frac{\prod_{i_1=2}^{3}\Gamma\left(\frac{i_1}{4}-\left[\frac{j}{4}\right]_{i_1}\right)}{\prod_{i_2=1}^{0}\Gamma\left(\frac{i_2 4-j}{4}\right)}\; {}_0F_2\left[(-1)^3\frac{1}{4^4 x}\Big|^{-}_{\left(1-\frac{2}{4}+\left[\frac{j}{4}\right]_2\right)\left(1-\frac{3}{4}+\left[\frac{j}{4}\right]_3\right)}\right]$$

$$= \frac{1}{2x^{\frac{5}{4}}(2\pi)^{\frac{3}{2}}}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)\; {}_0F_2\left[\frac{-1}{256x}\Big|^{-}_{(\frac{3}{4})(\frac{1}{2})}\right] + \frac{1}{8x^{\frac{3}{2}}(2\pi)^{\frac{3}{2}}}\Gamma\left(-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)\; {}_0F_2\left[\frac{-1}{256x}\Big|^{-}_{(\frac{3}{4})(\frac{5}{4})}\right]$$

$$+\frac{1}{32x^{\frac{7}{4}}(2\pi)^{\frac{3}{2}}}\Gamma\left(-\frac{1}{4}\right)\Gamma\left(-\frac{1}{2}\right)\ {}_0F_2\left[\frac{-1}{256x}|^{-}_{(\frac{5}{4})(\frac{3}{2})}\right]$$

$$=\frac{2}{(2\pi)^{\frac{3}{2}}x}\{\frac{2}{x^{\frac{1}{4}}}\Gamma\left(1+\frac{1}{4}\right)\Gamma\left(1+\frac{2}{4}\right)\ {}_0F_2\left[\frac{-1}{256x}|^{-}_{(\frac{3}{4})(\frac{2}{4})}\right]$$

$$-\frac{1}{x^{\frac{1}{2}}}\Gamma\left(\frac{3}{4}\right)\Gamma\left(1+\frac{1}{4}\right)\ {}_0F_2\left[\frac{-1}{256x}|^{-}_{(\frac{3}{4})(1+\frac{1}{4})}\right]+\frac{1}{8x^{\frac{3}{4}}}\Gamma\left(\frac{2}{4}\right)\Gamma\left(\frac{3}{4}\right)\ {}_0F_2\left[\frac{-1}{256x}|^{-}_{(1+\frac{1}{4})(1+\frac{2}{4})}\right]\}.$$

We will use (2.2) to show some new results.

Example 2.2 *Let* $p=1,\ q=2,\ l_1=2,\ l_2=1,\ \alpha=\frac{1}{\sqrt{2}},\ l=\sqrt{2}$, *then*

$$f_{(\frac{1}{\sqrt{2}},\sqrt{2})}(x)=\frac{\sqrt{2}}{x\sqrt{\pi}}\ H^{2,0}_{1,2}\left[\frac{1}{4x^{\sqrt{2}}}|^{(1,\sqrt{2})}_{(1,1),(\frac{1}{2},1)}\right]$$

$$=\frac{\sqrt{2}}{x\sqrt{\pi}}\left(\frac{1}{4x^{\sqrt{2}}}\ {}_1\Psi_1\left[-\frac{1}{4x^{\sqrt{2}}}|^{(-\frac{1}{2},-1)}_{(1-\sqrt{2},-\sqrt{2})}\right]+\frac{1}{2x^{\frac{1}{\sqrt{2}}}}\ {}_1\Psi_1\left[-\frac{1}{4x^{\sqrt{2}}}|^{(\frac{1}{2},-1)}_{(1-\frac{1}{\sqrt{2}},-\sqrt{2})}\right]\right).$$

3 A process to generate more representations

For the same rational α, more representations of one distribution can be generated by using (2.3) and (2.4).

Example 3.1 *When* $p=1,\ q=2,\ \alpha=\frac{1}{2}$ *in* (2.3), *we have*

$$f_{(\frac{1}{2},1)}(x)=\frac{1}{x\sqrt{\pi}}\ H^{1,0}_{0,1}\left[\frac{1}{4x}|^{-}_{(\frac{1}{2},1)}\right]=\frac{1}{2x^{\frac{3}{2}}\sqrt{\pi}}\ {}_0F_0\left[\frac{-1}{4x}|^{-}_{-}\right]=\frac{1}{2\sqrt{\pi}x^{\frac{3}{2}}}exp\left(\frac{-1}{4x}\right)$$

Example 3.2 *Let* $p=1,\ q=4,\ l_1=2,\ l_2=1,\ \alpha=\frac{1}{4}^{\frac{1}{2}}=\frac{1}{2},\ l=2$ *in* (2.4), *then we have*

$$f_{(\frac{1}{2},2)}(x)=\frac{\sqrt{8}}{2\pi x}H^{3,0}_{1,3}\left[\frac{4}{x^2 4^4}|^{(\frac{1}{2},1)}_{(\frac{1}{4},1),(\frac{2}{4},1),(\frac{3}{4},1)}\right]=\frac{1}{2\pi x^{\frac{3}{2}}}\sum_{v=0}^{\infty}\frac{\Gamma(-v+\frac{1}{2})}{v!}\left(\frac{-1}{4^3x^2}\right)^v$$

$$+\frac{1}{2^4\pi x^{\frac{5}{2}}}\sum_{v=0}^{\infty}\frac{\Gamma(-v-\frac{1}{2})}{v!}\left(\frac{-1}{4^3x^2}\right)^v=\frac{1}{2\sqrt{\pi}x^{\frac{3}{2}}}\left(\ {}_0F_1\left[\frac{1}{4^3x^2}|^{-}_{(\frac{1}{2})}\right]-\frac{1}{4x}\ {}_0F_1\left[\frac{1}{4^3x^2}|^{-}_{(\frac{3}{2})}\right]\right)$$

Note that $exp\left(\frac{-1}{4x}\right)=\ {}_0F_1\left[\frac{1}{4^3x^2}|^{-}_{(\frac{1}{2})}\right]-\frac{1}{4x}\ {}_0F_1\left[\frac{1}{4^3x^2}|^{-}_{(\frac{3}{2})}\right]$ and by setting $\alpha=\left(\frac{2}{8}\right)^{\frac{1}{2}}=\frac{1}{2}$ in (2.4), another representation can be born in the sum of faster convergent series.

4 Lévy smashing on the family of gamma density functions and Lévy-smashed gamma stochastic process

Mellin transform of a density function in statistics shows its statistical structure and this technique can be used as a tool to blend two independently distributed random variables. In this section, we show what Lévy effect could be and how we should understand it. To start with, consider the Lévy density function $\frac{1}{2\pi i}\oint_L \frac{\Gamma(\frac{1}{\alpha}-\frac{s}{\alpha})}{\alpha\Gamma(1-s)}x^{-s}ds$ and the gamma density functions $\frac{x^{\gamma-1}}{\Gamma(\gamma)}e^{-x}$ where $0<\alpha<1,\ 0<\gamma$ and $0<x<\infty$. $\frac{1}{2\pi i}\oint_L \frac{\Gamma(\frac{1}{\alpha}-\frac{s}{\alpha})}{\alpha\Gamma(1-s)}x^{-s}ds$ is the one-sided Lévy density function found in [5] constructed in a different way in [2]. We will use the Mellin transformation of the form $\int_t h_1(\frac{x}{t})h_2(t)\frac{dt}{t}$, where $h_1(x)$ and $h_2(x)$ are certain density functions. Then we have

$$\begin{aligned}
f_{(\alpha)}(x) &= \int_0^\infty \frac{1}{2\pi i}\oint_L \frac{\Gamma(\frac{1}{\alpha}-\frac{s}{\alpha})}{\alpha\Gamma(1-s)}x^{-s}t^s ds\, \frac{t^{\gamma-1}}{\Gamma(\gamma)}e^{-t}\frac{dt}{t} = \frac{1}{2\pi i}\oint_L \frac{\Gamma(\frac{1}{\alpha}-\frac{s}{\alpha})}{\alpha\Gamma(1-s)}x^{-s}\int_0^\infty \frac{t^{s+\gamma-1-1}}{\Gamma(\gamma)}e^{-t}dtds\\
&= \frac{1}{2\pi i}\oint_L \frac{\Gamma(\frac{1}{\alpha}-\frac{s}{\alpha})}{\alpha\Gamma(1-s)}\frac{\Gamma(s+\gamma-1)}{\Gamma(\gamma)}x^{-s}ds \ \text{ by transformation } s=1-\alpha s_1,\\
&= \frac{1}{2\pi i}\oint_L \frac{\Gamma(s_1)}{\Gamma(\alpha s_1)}\frac{\Gamma(\gamma-\alpha s_1)}{\Gamma(\gamma)}x^{\alpha s_1-1}ds_1 \qquad (4.1)
\end{aligned}$$

and its Laplace transform

$$\begin{aligned}
L_f(y) &= \int_0^\infty e^{-yx}\frac{1}{2\pi i}\oint_L \frac{\Gamma(s)}{\Gamma(\alpha s)}\frac{\Gamma(\gamma-\alpha s)}{\Gamma(\gamma)}x^{\alpha s-1}dsdx = \frac{1}{2\pi i}\oint_L \frac{\Gamma(s)\Gamma(\gamma-\alpha s)}{\Gamma(\gamma)}y^{-\alpha s}ds\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\gamma+\alpha k)}{k!\ \Gamma(\gamma)}(y)^{\alpha k}. \qquad (4.2)
\end{aligned}$$

When $\alpha=1$ in (4.1), (4.1) becomes gamma density $\frac{x^{\gamma-1}}{\Gamma(\gamma)}e^{-x}$. So $f_{(1)}(x)$ is a one-sided function concentrated at $x=1$ for $\mathbb{R}^+$. To understand its effect, put $\alpha=\frac{1}{2}$ in (4.1), then we have

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_L \frac{\Gamma(s_1)}{\Gamma(\frac{1}{2}s_1)}\frac{\Gamma(\gamma-\frac{1}{2}s_1)}{\Gamma(\gamma)}x^{\frac{1}{2}s_1-1}ds_1,\ s_1=2s \qquad (4.3)\\
&= \frac{1}{2\pi i}\oint_L \frac{\Gamma(2s)}{\Gamma(s)}\frac{\Gamma(\gamma-s)}{\Gamma(\gamma)}x^{s-1}2ds = \frac{1}{2\pi i}\oint_L \frac{2^{2s-1}\Gamma(s)\Gamma(s+\frac{1}{2})}{\sqrt{\pi}\Gamma(s)}\frac{\Gamma(\gamma-s)}{\Gamma(\gamma)}x^{s-1}2ds \qquad (4.4)\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\gamma+\frac{1}{2}+k)}{k!\sqrt{\pi}\Gamma(\gamma)}x^{-k-\frac{3}{2}}4^{-k-\frac{1}{2}} = \frac{\Gamma(\gamma+\frac{1}{2})}{2\sqrt{\pi}\Gamma(\gamma)x^{\frac{3}{2}}}\left(1+\frac{1}{4x}\right)^{-\gamma-\frac{1}{2}} \qquad (4.5)\\
&= \frac{4\Gamma(\gamma+\frac{1}{2})(4x)^{\gamma-1}}{\sqrt{\pi}\Gamma(\gamma)}(1+4x)^{-\gamma-\frac{1}{2}} \qquad (4.6)
\end{aligned}$$

Fig. 1 shows the impact on the family of gamma functions.

	gamma family		Lévy smashed gamma family
(a) $\gamma=1$	e^{-x}	$\leftrightarrow$	$2(1+4x)^{-3/2}$
(b) $\gamma=2$	xe^{-x}	$\leftrightarrow$	$12x(1+4x)^{-5/2}$
(c) $\gamma=3$	$\frac{1}{2}x^2e^{-x}$	$\leftrightarrow$	$60x^2(1+4x)^{-7/2}$
(d) $\gamma=4$	$\frac{1}{3!}x^3e^{-x}$	$\leftrightarrow$	$280x^3(1+4x)^{-9/2}$

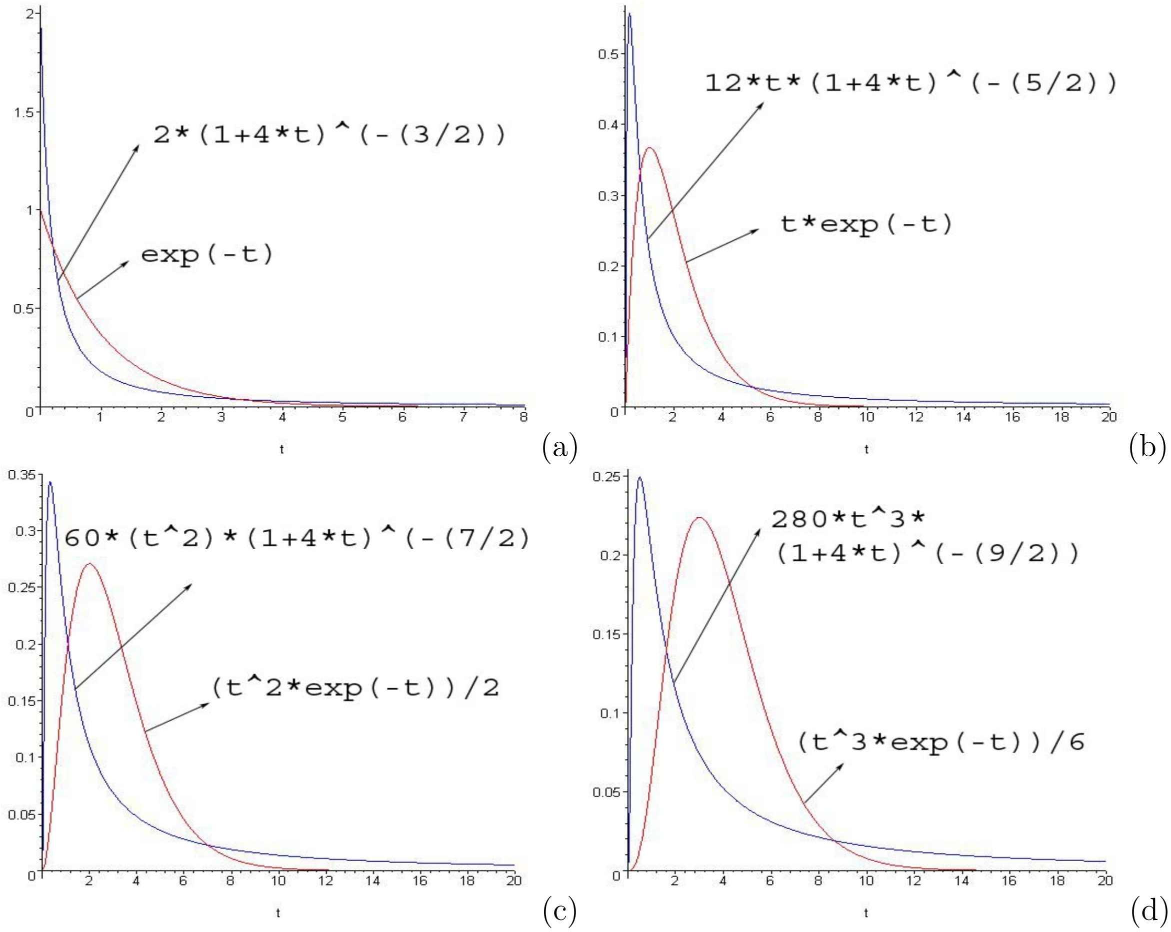

Figure 1.

$f_{(\alpha)}(x) = \frac{1}{2\pi i}\oint_L \frac{\Gamma(s)}{\Gamma(\alpha s)}\frac{\Gamma(\gamma-\alpha s)}{\Gamma(\gamma)}x^{\alpha s-1}ds$ is absolutely convergent series for all x since $\mu = \alpha - \alpha + 1 = 1 > 0$, see [7]. From the fig. 1, $f_{(\alpha)}(x)$ will be called as Lévy-smashed gamma density functions especially when the parameter $0 < \alpha < 1$. Note that α can be any positive real number.

The stochastic process $X(t), t > 0$ with $X(0) = 0$ and having stationary and independent increments, where $X(t)$ has the density function $\frac{1}{2\pi i}\oint_L \frac{\Gamma(s)}{\Gamma(\alpha s)}\frac{\Gamma(t-\alpha s)}{\Gamma(t)}x^{\alpha s-1}ds,\ 0 < \alpha \leq 1$, will be called Lévy-smashed gamma stochastic process. The Lévy-smashed gamma stochastic process $X(t)$ has the distribution function,for $t > 0, \quad 0 < \alpha < 1$, $F_{(\alpha,t)}(x) = \frac{1}{2\pi i}\oint_L \frac{\Gamma(s)}{\Gamma(1+\alpha s)}\frac{\Gamma(t-\alpha s)}{\Gamma(t)}x^{\alpha s}ds$. From [1] and [9], we can prove that the Lévy-smashed gamma distribution with parameter α is attracted to the stable distribution with exponent $\alpha, \quad 0 < \alpha < 1$. Namely,

$$\lim_{n\to\infty} L_f(\frac{y}{n}) = \lim_{n\to\infty}\int_0^\infty e^{-yx}\frac{1}{2\pi i}\oint_L \frac{\Gamma(s)}{\Gamma(\alpha s)}\frac{\Gamma(n-\alpha s)}{\Gamma(n)}n^{\alpha s}x^{\alpha s-1}dsdx$$
$$= \lim_{n\to\infty}\frac{1}{2\pi i}\oint_L \frac{\Gamma(s)\Gamma(n-\alpha s)}{\Gamma(n)}n^{\alpha s}y^{-\alpha s}ds = \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(n+\alpha k)}{k!\ \Gamma(n)}\left(\frac{y}{n}\right)^{\alpha k} = e^{-y^\alpha}$$

5 Remarks

In [3], they consider the signalling problem for the standard diffusion equation $\frac{\partial}{\partial t}u(x,t) = D\frac{\partial^2}{\partial x^2}u(x,t)$ with the conditions $u(x,0^+) = 0\ x > 0; u(0^+,t) = h(t), u(+\infty,t) = 0,\ t > 0$. And they say "$\cdots$ Then the solution is as follows $u(x,t) = \int_0^t \mathcal{G}_s^d(x,\tau)h(t-\tau)d\tau$, where $\mathcal{G}_s^d(x,t) = \frac{x}{2\sqrt{\pi D}}t^{-\frac{3}{2}}exp{-}\frac{x^2}{4Dt}$. Here $\mathcal{G}_s^d(x,t)$ represents the fundamental solution (or Green function) of the signalling problem, since it corresponds to $h(t) = \delta(t)$. We note that

$$\mathcal{G}_s^d(x,t) = p_{LS}(t;\mu) := \frac{\sqrt{\mu}}{\sqrt{2\pi}t^{\frac{3}{2}}}exp(-\frac{\mu}{2t}),\ t \geq 0,\ \mu = \frac{x^2}{2D} \tag{5.1}$$

where $p_{LS}(t;\mu)$ denotes the one-sided Lévy-Smirnov pdf spread out over all non-negative t (the time variable). The Lévy-Smirnov pdf has all moments of integer order infinite, since it decays at infinity as $t^{-\frac{3}{2}}$. However, we note that the absolute moments of real order ν are finite only if $0 \leq \nu < \frac{1}{2}$. In particular, for this pdf the mean is infinite, for which we can take the median as expected value. From $\mathcal{P}_{LS}(t_{med};\mu) = \frac{1}{2}$, it turns out that $t_{med} \approx 2\mu$, since the complementary error function gets the value $\frac{1}{2}$ as its argument is approximatively $\frac{1}{2}$. $\cdots$".

With the inspiration from the above paragraph, take the Lévy density function with the parameter $\alpha = \frac{1}{2}$, then the density function is well known to be $\frac{1}{2\sqrt{\pi}t^{\frac{3}{2}}}exp(-\frac{1}{4t})$. Put this in the mellin convolution formula $\int_t h_1(\frac{x}{t})h_2(t)\frac{dt}{t}$, then it becomes $\int_0^\infty \frac{\sqrt{t}}{2\sqrt{\pi}x^{\frac{3}{2}}}exp\left(-\frac{t}{4x}\right)\ h_2(t)dt$. $\frac{\sqrt{t}}{2\sqrt{\pi}x^{\frac{3}{2}}}exp\left(-\frac{t}{4x}\right)$ has the same form with (5.1) when $t = \mu$. Therefore the cases of Lévy smashing treated in section 4 can be thought of as superstatistics in Physics and Bayesian statistical analysis, subordination in statistics, namely,

$$f_{(\frac{1}{2})}(x) = \int_0^\infty \frac{1}{2\pi i}\oint_L \frac{\Gamma(2-2s)}{\frac{1}{2}\Gamma(1-s)}x^{-s}y^s ds\ \frac{y^{t-1}}{\Gamma(t)}e^{-y}\frac{dy}{y} = \int_0^\infty \frac{\sqrt{y}}{2\sqrt{\pi}x^{\frac{3}{2}}}exp\left(-\frac{y}{4x}\right)\ \frac{y^{t-1}}{\Gamma(t)}e^{-y}dy$$
$$= \int_0^\infty \frac{\sqrt{y}}{2\sqrt{\pi}x^{\frac{3}{2}}}\sum_{k=0}^\infty \frac{(-1)^k y^k}{k!\ (4x)^k}\ \frac{y^{t-1}}{\Gamma(t)}e^{-y}dy = \frac{1}{2\sqrt{\pi}\Gamma(t)x^{\frac{3}{2}}}\sum_{k=0}^\infty \frac{(-1)^k}{k!\ (4x)^k}\ \int_0^\infty y^{t-1+k+\frac{1}{2}}e^{-y}dy$$
$$\frac{1}{2\sqrt{\pi}\Gamma(t)x^{\frac{3}{2}}}\sum_{k=0}^\infty \frac{(-1)^k\Gamma(t+k+\frac{1}{2})}{k!\ (4x)^k} = \frac{4\Gamma(t+\frac{1}{2})(4x)^{t-1}}{\sqrt{\pi}\Gamma(t)}(1+4x)^{-t-\frac{1}{2}}$$

But in this paper, the concept of Lévy smashing is totally different from the point of view of superstatistics in Physics and Bayesian statistical analysis, subordination in statistics.

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