An Operational Meaning of Discord in terms of Teleportation Fidelity

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Quantum discord is a prominent measure of quantum correlations, playing an important role in expanding its horizon beyond entanglement. Here we provide an operational meaning of (geometric) discord, which quantifies the amount of non-classical correlation of an arbitrary quantum system in terms of its minimal distance from the set of classical states, in terms of teleportation fidelity for general two qubit and $d \otimes d$ dimensional isotropic and Werner states. A critical value of the discord is found beyond which the two qubit state must violate the Bell inequality. This is illustrated by an open system model of a dissipative two qubit. For the $d \otimes d$ dimensional states the lower bound of discord is shown to be obtainable from an experimentally measurable witness operator.

Introduction.-Quantum correlations occupy a central position in the quest for understanding and harvesting the power of quantum mechanics. This point of view has been highlighted in recent times by numerous developments in the field of quantum information. Entanglement [1], till about a decade back, was considered synonymous with quantum correlations. This was a natural outcome of the quest to understand the role of nonlocality in quantum mechanics, having a historical lineage from Einstein-Podolsky-Rosen [2], to Bell's inequality [3], leading to refinements resulting in the Bell-CHSH (Clauser-Horn-Shimony-Holt) inequalities [4]. With the advent of quantum discord [5, 6], the difference between the quantum generalizations of two classically equivalent formulations of mutual information, realization dawned that quantum correlations are bigger than entanglement. Thus, for example, separable states, having by definition zero entanglement, could have non-zero discord. Also, in the DQC1 model [7], entanglement is negligible but there is sufficient amount of quantum discord for a speed up over the best known classical algorithms.

Quantum discord for two qubit states with maximally mixed marginals was analytically obtained in [8], while in [9] an algorithm to calculate quantum discord for general two qubit states was developed. In general, it is very difficult to obtain an analytical formula for quantum discord because it involves an optimization over local measurements, requiring numerical methods. To overcome this difficulty, another measure of quantum correlation called geometric discord was introduced in [10] which quantifies the amount of non-classical correlation of an arbitrary quantum composite system in terms of its minimal distance from the set of classical states. An analytical expression for geometric discord for the two qubit case was also found. This was generalized in [11] to the case of a $d \otimes d'$ dimensional system. There is now an abundance of measures of quantum correlations such as quantum work deficit [12], measurement induced disturbance [13] and dissonance [14].

The understanding of a particular facet of a complex entity such as quantum correlations is greatly accentuated by the development of various operational tasks to which it can be put to use. This has been the case, particularly, for entanglement which was used for developing various useful aspects of quantum information processing such as teleportation [15], remote state preparation [16], quantum cryptography [17], and quantum dense coding [18]. Discord, likewise, has found use in explaining local broadcasting [19], quantum state merging [20] and remote state preparation [21]. Here we provide an operational meaning of (geometric) discord in terms of teleportation, the canonical model of quantum information and communication. Teleportation is particularly important, due not only to its operational aspect and experimental realization [22] but also because of the fundamental role it plays in sharpening understanding of Bell's inequality and entanglement [23, 24].

In this letter, we establish a connection between geometric discord and teleportation fidelity for general two qubit and $d \otimes d$ dimensional isotropic states. A critical value of the discord is found beyond which the two qubit state must violate the Bell-CHSH inequality. This is illustrated by an open system model of a dissipative two qubit [25]. In addition, for $d \otimes d$ dimensional isotropic and Werner states, we develop lower bounds of geometric discord in terms of experimentally measurable witness operators.

Maximum and Minimum value of Geometric Discord.- Any arbitrary two qubit mixed state can be written as $\rho = \frac{1}{4}(\mathbb{I}_2 \otimes \mathbb{I}_2 + \vec{x}.\vec{\sigma} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \vec{y}.\vec{\sigma} + \sum_{i,j=1}^{3} t_{ij}\sigma_i \otimes \sigma_j)$. Here $\mathbb{I}_2$ is the two dimensional identity matrix, $x_i = \mathrm{Tr}(\rho(\sigma_i \otimes \mathbb{I}_2))$, $y_i = \mathrm{Tr}(\rho(\mathbb{I}_2 \otimes \sigma_i))$ are components of local Bloch vectors $\vec{x}$ and $\vec{y}$, respectively, while $\{t_{ij}\} \equiv T = \mathrm{Tr}(\rho(\sigma_i \otimes \sigma_j))$ denotes the correlation matrix and $\sigma_i' s (i = 1, 2, 3)$ are the Pauli matrices. The geometric discord, normalized with respect to teleportation fidelity, is defined as [10] $D_G(\rho) = \frac{4}{3} min_{\chi \in \Omega_0} \|\rho - \chi\|_2^2$, where Ω_0 denotes the set of all zero discord states and $\|.\|_2$ denotes the Hilbert-Schmidt norm and is defined as $\|A\|_2 = \sqrt{\mathrm{Tr}(AA^{\dagger})}$. For the case of two qubits, geometric discord was shown [10] to be $D_G(\rho) = \frac{1}{3}[\|\vec{x}\|^2 + \|T\|^2 - \lambda_{max}(\vec{x}\vec{x}^{\dagger} + TT^{\dagger})]$. Here $\lambda_{max}(\vec{x}\vec{x}^{\dagger} + TT^{\dagger})$ is the maximum eigenvalue of the matrix $\vec{x}\vec{x}^{\dagger} + TT^{\dagger}$. To proceed, we make use of a very useful theorem by Weyl [26], which connects the eigenvalues of the sum of Hermitian matrices to those of the individual matrices and is made use of in, for example, understanding the stability of the spectrum of a Hermitian matrix with respect to perturbations. For convenience, we present the theorem below.

Theorem: Let $X, Y \in M_n$ be Hermitian matrices and let the eigenvalues $\lambda_i(X), \lambda_i(Y)$ and $\lambda_i(X+Y)$ be arranged in an increasing order. For each $k = 1, 2, ...n$, we have

$$\lambda_k(X) + \lambda_1(Y) \leq \lambda_k(X+Y) \leq \lambda_k(X) + \lambda_n(Y), \quad (1)$$

where M_n denotes the set of $n \times n$ Hermitian matrices and $\lambda_1(Y), \lambda_n(Y)$ denotes the minimum and maximum eigenvalues of Y, respectively. In particular, for $k = n$, the inequality [Eq. (1)] reduces to $\lambda_{max}(X) + \lambda_{min}(Y) \leq \lambda_{max}(X+Y) \leq \lambda_{max}(X) + \lambda_{max}(Y)$. If we identify the Hermitian matrices $\vec{x}\vec{x}^\dagger$ and $TT^\dagger$ with X and Y, respectively, then this inequality gives $\lambda_{max}(\vec{x}\vec{x}^\dagger) + \lambda_{min}(TT^\dagger) \leq \lambda_{max}(\vec{x}\vec{x}^\dagger + TT^\dagger) \leq \lambda_{max}(\vec{x}\vec{x}^\dagger) + \lambda_{max}(TT^\dagger)$. Using this and the form of geometric discord for two qubits, we have

$$\frac{1}{3}[\|\vec{x}\|^2 + \|T\|^2 - \lambda_{max}(\vec{x}\vec{x}^\dagger) - \lambda_{max}(TT^\dagger)] \leq D_G(\rho) \leq \frac{1}{3}[\|\vec{x}\|^2 + \|T\|^2 - \lambda_{max}(\vec{x}\vec{x}^\dagger) - \lambda_{min}(TT^\dagger)]. \quad (2)$$

From these inequalities, the maximum and minimum value of $D_G(\rho)$ can be seen to be

$$D_G^{min}(\rho) = \frac{1}{3}[\|T\|^2 - \lambda_{max}(TT^\dagger)], \quad (3)$$

$$D_G^{max}(\rho) = \frac{1}{3}[\|T\|^2 - \lambda_{min}(TT^\dagger)]. \quad (4)$$

These results would be needed to connect quantum discord with Bell's inequality and teleportation. If all the eigenvalues of the matrix $TT^\dagger$ are equal then $\lambda_{max}(TT^\dagger) = \lambda_{min}(TT^\dagger)$ and we have

$$D_G^{min}(\rho) = D_G^{max}(\rho) = D_G(\rho). \quad (5)$$

An example where such an equality is realized is the Werner state [27], a point to which we will return later. The above obtained bounds can be used to calculate the maximum value of quantum discord for separable states in $2 \otimes 2$ systems. Any separable state in a $2 \otimes 2$ system can be expressed as $\rho_{sep} = \sum_k p_k \frac{1}{4}(I \otimes I + \vec{x}^k.\vec{\sigma} \otimes I + I \otimes \vec{y}^k.\vec{\sigma} + \sum_i x_i^k y_i^k \sigma_i \otimes \sigma_i)$, where, $x_i^k, y_i^k \in \mathbb{R}, |\vec{x}^k| \leq 1, |\vec{y}^k| \leq 1$. Thus, for separable states, the correlation matrix T is the product of the two local Bloch vectors $\vec{x}$ and $\vec{y}$, that is, $T = \vec{x}^\dagger \vec{y}$, where $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$. The maximum quantum discord D_G^{max}, for separable states, is $D_G^{max}(\rho_{sep}) = \frac{1}{3}(|x_1|^2 + |x_2|^2 + |x_3|^2)(|y_1|^2 + |y_2|^2 + |y_3|^2)$. Since $|\vec{x}^k| \leq 1, |\vec{y}^k| \leq 1$, so $D_G^{max}(\rho_{sep}) \leq \frac{1}{3}$. It follows that for separable states ρ_{sep}, we have the inequality

$$0 \leq D_G(\rho_{sep}) \leq D_G^{max}(\rho_{sep}) \leq \frac{1}{3}. \quad (6)$$

Let us consider a state described by the density operator $\rho_1 = \frac{1}{4}[I \otimes I + \sigma_x \otimes I + I \otimes \sigma_x + \sigma_x \otimes \sigma_x]$. It is separable since the partial transpose with respect to one of the qubits is a valid density matrix, that is, $\rho_1^{T_A} = \rho_1$. In this case, the maximum value of discord is given by $D_G^{max}(\rho_1) = \frac{1}{3}$. Thus we show that there exist separable states for which the upper bound of the maximum discord is achieved.

Relation between Quantum Discord, Bell's inequality and Teleportation Fidelity.- We now establish a relation between maximum discord and teleportation fidelity, which also gives an operational meaning of quantum discord. To achieve our goal, let us consider

$$D_G^{max}(\rho) = \frac{1}{3}[\|T\|^2 - \lambda_{min}(TT^\dagger)] = \frac{1}{3}M(\rho), \quad (7)$$

where $M(\rho) = max_{i>j}(u_i + u_j)$, u_i, u_j are the eigenvalues of $TT^\dagger$.

If the state described by the density matrix ρ satisfies the Bell-CHSH inequality then it follows that

$$D_G^{max}(\rho) \leq \frac{1}{3}. \quad (8)$$

The above result holds for all separable states as is evident from Eq. (6). But there may also exist some entangled states that satisfy it.

Theorem-1: Any two qubit mixed state ρ is entangled if $D_G^{max}(\rho) > \frac{1}{3}$.

Proof: Any two qubit mixed state ρ is entangled if it violates the Bell inequality. The Bell inequality is violated iff $M(\rho) > 1$ [28]. Therefore the theorem follows from Eq. (7).

To establish an operational meaning of discord, we use a series of connections, that is, the relation of discord with negativity and then negativity with teleportation fidelity. The relation between an entanglement measure known as negativity ($N(\rho^{ent})$) and quantum discord $D_G(\rho^{ent})$ for an entangled state ρ^{ent} is given by [29] $N^2(\rho^{ent}) \leq D_G(\rho^{ent})$. Let $S = \{\rho_{CHSH}^{ent} : D_G^{max}(\rho_{CHSH}^{ent}) \leq \frac{1}{3}\}$ denotes the set of entangled states which satisfy the Bell-CHSH inequality. Therefore the states which belong to the set S must satisfy the inequality $N^2(\rho_{CHSH}^{ent}) \leq D_G(\rho_{CHSH}^{ent}) \leq D_G^{max}(\rho_{CHSH}^{ent}) \leq \frac{1}{3}$. For any two qubit mixed state ρ, teleportation fidelity $F(\rho)$ is related to negativity as [30] $3F(\rho) - 2 \leq N(\rho)$. Since this inequality holds for all ρ_{CHSH}^{ent}, we have $(3F(\rho_{CHSH}^{ent}) - 2)^2 \leq D_G^{max}(\rho_{CHSH}^{ent}) \leq \frac{1}{3}$. From Eq. (8) and the relations, discussed above, connecting negativity with discord and teleportation fidelity, it follows that $\frac{2}{3} < F(\rho_{CHSH}^{ent}) \leq \frac{2}{3} + \frac{1}{3\sqrt{3}}$. This is what is expected for states satisfying Bell's inequality, but at the same time useful for teleportation, bringing out the consistency of our results. Thus we obtain a bound of quantum correlation measured by discord for those entangled states which satisfy Bell's inequality but are still useful for teleportation. This result can be expressed in the form of the following theorem.

Theorem-2: If the entangled state ρ satisfies Bell's inequality but is useful for teleportation then $D_G^{max}(\rho)$ must satisfy the inequality

$$(3F(\rho)-2)^2 \leq D_G^{max}(\rho) \leq \frac{1}{3}. \qquad (9)$$

As $u_i \leq 1 \; for \;\; i=1,2,3$ and $U(\rho)=\sum_{i=1}^{3}\sqrt{u_i}$, $M(\rho)$ and $U(\rho)$ are related as $M(\rho) \leq U(\rho)$ [24], while the relation between $U(\rho)$ and teleportation fidelity $F(\rho)$ is $F(\rho)=\frac{1}{2}[1+\frac{1}{3}U(\rho)]$ [24]. Using these relations, Eq. (7) can be seen to reduce to $D_G^{max}(\rho) \leq 2F(\rho)-1$. Since this inequality holds for any mixed two qubit state, states not useful for teleportation must satisfy

$$0 \leq D_G^{max}(\rho) \leq 2F(\rho)-1, \quad F(\rho) \leq \frac{2}{3}. \qquad (10)$$

Theorem-3: A two qubit state ρ violates Bell-CHSH inequality and is useful for teleportation iff

$$\frac{1}{3} < D_G^{max}(\rho) \leq 2F(\rho)-1, \quad F(\rho) > \frac{2}{3}. \qquad (11)$$

Proof: It follows from **Theorem-1** and the inequality, shown above, connecting the maximum value of discord to the teleportation fidelity.

Eqs. (9), (10) and (11) are our principal results for the two qubit states, providing the relationship between discord and teleportation fidelity and covers the regimes where Bell's inequality is satisfied but the states may or may not be useful for teleportation as well as where the inequality gets violated, the critical value of discord above which Bell's inequality gets violated being 1/3.

Illustrations: (a). Let us consider a Werner state $\rho_W = p|\psi^-\rangle\langle\psi^-| + (1-p)\frac{I}{4}, \;\; 0 \leq p \leq 1$, where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is the singlet state. The correlation matrix T for ρ_W is given by $T = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$. As all the eigenvalues of $TT^\dagger$ are equal, we have $D_G^{min}(\rho_W) = D_G^{max}(\rho_W) = D_G(\rho_W)$, thus realizing Eq. (5). The quantum discord is $D_G(\rho_W) = \frac{2}{3}p^2$. From Eq. (8), it is clear that ρ_W violates Bell inequality iff $p > \frac{1}{\sqrt{2}}$. Also, using $F(\rho_W) = (1+p)/2$ [31] in Eq. (9), it is easy to see that Werner states satisfying Bell's inequality, but still useful for teleportation have the parameter p in the range $\frac{1}{3} < p \leq \frac{1}{3} + \frac{2}{3\sqrt{3}}$. This is known in the literature, providing a nice consistency check of our results. Finally, it follows from Eq. (11) that ρ_W is always useful for teleportation iff $\frac{1}{3} < D_G(\rho_W) \leq p, \quad p > \frac{1}{3}$.

(b). Next we consider an open quantum system model for two qubit mixed states [25], which could be thought of as a quantum channel used for studying quantum correlations. Open quantum system is the study of the evolution of a system of interest, such as a qubit, taking into account the effect of its surroundings alternatively called reservoir or bath. The evolution is mixed, in general, with decoherence and dissipation appearing as natural outcomes. We consider two qubits interacting with a bath, modeled as an electromagnetic field in a squeezed thermal state, via the dipole interaction. The system-reservoir coupling constant is dependent upon the position of the qubit, leading to interesting dynamical consequences. Basically this allows a classification of the dynamics into two regimes: the independent decoherence regime, where the inter qubit distances are such that each qubit sees an individual bath or the collective decoherence regime, where the qubits are close enough to justify a collective interaction with the bath. In Fig. 1, the evolution of Bell's inequality, teleportation fidelity and maximum value of geometric discord with respect to time is shown. It is very satisfactory that a dynamical model, which could be envisaged in an experimental setup [32], satisfies all the inequalities developed above connecting discord to teleportation fidelity, that is, Eqs. (9), (10) and (11), as well as the critical value of discord (= 1/3) above which the Bell-CHSH inequality gets violated.

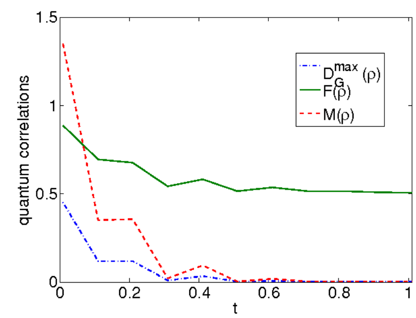

FIG. 1: (Color online) The figure depicts quantum correlations; Bell's inequality $M(\rho)$, teleportation fidelity $F(\rho)$ and maximum value of geometric discord $D_G^{max}(\rho)$, with respect to the time of evolution t. Here temperature (in units where $\hbar \equiv k_B = 1$) $T = 10$, inter qubit distance $r_{12} = 0.01$ (collective decoherence) and bath squeezing parameter $r = -1$. With time, states violating Bell's inequality start satisfying it, a natural consequence of the degradation of quantum correlations due to open system effects. At $t = 0.12$, Bell's inequality is violated, it can be seen that Eq. (11) gets satisfied. Again at $t = 0.2$, we have the case where Bell's inequality is satisfied but the states are useful for teleportation, as seen by the inequality Eq. (9) being satisfied. Finally, for longer evolutions, such as $t = 0.6$, the states are no longer useful for teleportation, and the figure satisfies Eq. (10).

Relation between discord, teleportation fidelity and witness operator in $d \otimes d$ system. Here, we study $d \otimes d$ systems and generalize the relation between quantum discord and teleportation fidelity for states with symmetry, in particular, the isotropic and Werner states and also derive experimentally achievable bounds of discord for them, using witness operators. A witness operator is a Hermitian operator which can detect entangled states and can be realized experimentally. Recently it has been shown that the witness operator also takes part in discriminating states useful for teleportation [33].

The isotropic state is defined by $\rho_f = \frac{1-f}{d^2-1}(I - |\phi^+\rangle\langle\phi^+|) + f|\phi^+\rangle\langle\phi^+|$, where $|\phi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$, f is the singlet fraction and d is the dimension of an individual component of the bipartite system. Its negativity is [34] $N(\rho_f) = \frac{fd-1}{d-1}$ and zero, for $f > \frac{1}{d}$, or $f < \frac{1}{d}$, respectively. For an isotropic state ρ_f, the following statements are equivalent, (a). ρ_f is separable iff ρ_f is PPT, (b). ρ_f is PPT iff $0 \leq f \leq \frac{1}{d}$. Using the relation between negativity and discord as well as the connection of teleportation fidelity with singlet fraction $F(\rho_f) = \frac{df(\rho_f)+1}{d+1}$ [35], we have

$$(\frac{(d+1)F(\rho_f) - 2}{d-1})^2 \leq D_G(\rho_f), \quad F(\rho_f) > \frac{2}{d+1}. \quad (12)$$

Eq. (12) provides a theoretical lower bound of discord as a function of the teleportation fidelity. A question then naturally arises, can we also have a lower bound which could be achieved in an experiment? The answer is in the affirmative. The lower bound of quantum discord for isotropic states can be achieved by witness operators. Witness operators act as a hyperplane separating entangled and separable states and can be divided into two different classes: decomposable and non-decomposable witness operators. Although non-decomposable witness operators detect both negative partial transpose (NPT) and positive partial transpose (PPT) entangled states, decomposable witness operators can only detect NPT entangled states. Recently, a generalized form of optimal teleportation witness operator was proposed and showed to be a decomposable entanglement witness operator [36].

Let us consider a decomposable witness operator of the form $W_f = \frac{1}{d}I - |\phi^+\rangle\langle\phi^+|$. Its expectation value in the state ρ_f is

$$\mathrm{Tr}(W_f\rho_f) = \frac{1}{d} - f, \quad f > \frac{1}{d}. \quad (13)$$

Thus, every entangled isotropic state is detected by W_f. Making use of Eq. (13) and the negativity of an isotropic state, the lower bound of discord, as a function of the witness operator W_f, is

$$\frac{d^2}{(d-1)^2}(-\mathrm{Tr}(W_f\rho_f))^2 \leq D_G(\rho_f). \quad (14)$$

Let us now consider the Werner state in a $d \otimes d$ dimensional Hilbert space where it is defined as [34] $\rho_x = \frac{2(1-x)}{d(d+1)}(\sum_{k=0}^{d-1}|kk\rangle\langle kk| + \sum_{i<j}|\Psi_{ij}^+\rangle\langle\Psi_{ij}^+|) + \frac{2x}{d(d-1)}\sum_{i<j}|\Psi_{ij}^-\rangle\langle\Psi_{ij}^-|$. Here $|\Psi_{ij}^{\pm}\rangle = \frac{1}{\sqrt{2}}(|ij\rangle \pm |ji\rangle)$ and $x = \mathrm{tr}(\rho_x\sum_{i<j}|\Psi_{ij}^-\rangle\langle\Psi_{ij}^-|)$. Analogous to an isotropic state, the following statements are equivalent for Werner state, (a). ρ_x is separable iff ρ_x is PPT, (b). ρ_x is PPT iff $0 \leq x \leq \frac{1}{2}$. The negativity of Werner state is $N(\rho_x) = \frac{2}{d}\frac{2x-1}{d-1}$ [34], while it is related to the singlet fraction as $f(\rho_x) \leq \frac{1+2N(\rho_x)}{d}$ [37]. If the state ρ_x is entangled, that is, $N(\rho_x) \neq 0$, then it is clear that every entangled state is useful for teleportation. In this case, the lower bound of discord for the Werner state, in terms of the witness operator $W_x = (|\Psi\rangle\langle\Psi|)^{T_A}$, where $|\Psi\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$, is given by

$$\frac{4}{(d-1)^2}(-\mathrm{Tr}(W_x\rho_x))^2 \leq D_G(\rho_x). \quad (15)$$

Thus for the cases of the isotropic as well as the Werner states, a lower bound of discord is obtained by two different routes. The first one, in theme with our approach for the $2 \otimes 2$ dimensional systems, goes about establishing a relation between discord and teleportation fidelity by connecting the concepts of negativity, singlet fraction, teleportation fidelity and discord; while the second approach relies upon the construction of decomposable witness operators.

For an experimental realization of the witness operation it is necessary to decompose the witness into operators that can be measured locally, that is, a decomposition into projectors of the form $W = \sum_{i=1}^{k} c_i|e_i\rangle\langle e_i| \otimes |f_i\rangle\langle f_i|$. The decomposition of the witness operator W_f for two qubit systems has been shown in [33]. Moreover, the witness operator W_x can easily be decomposed into local Pauli operators $\sigma_i, i = 1, 2, 3$ and local Gell-Mann matrices for two qubits and two qutrits systems, respectively. It can be also extended to generalized Gell-Mann matrices, which are standard $SU(d)$ generators, for d-dimensional systems.

Conclusions. In this work we have provided an operational meaning of discord by connecting it to teleportation fidelity. In $2 \otimes 2$ systems, we make use of a theorem of Weyl for Hermitian matrices which proves to be the key threading together the various links to provide the connection between discord and teleportation fidelity. The results are seen to be consistent when applied to the Werner state and a dynamically generated two qubit open system model. This study is further extended to a higher dimensional $d \otimes d$ isotropic system. We also obtain lower bounds of discord, for $d \otimes d$ isotropic and Wener states, in terms of appropriate witness operators, and discuss how these can be achieved experimentally. We hope this work motivates further research into providing an operational meaning of discord in general higher dimensional systems thereby harnessing its potential use in quantum information processing.

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