# Variational problem for the Frenkel and the Bargmann-Michel-Telegdi (BMT) equations

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**Abstract**

We propose Lagrangian formulation for the particle with value of spin fixed within the classical theory. The Lagrangian turns out to be invariant under non-abelian group of local symmetries. As the gauge-invariant variables for description of spin we can take either the Frenkel tensor or the BMT vector. Fixation of spin within the classical theory implies $O(\hbar)$-corrections to the corresponding equations of motion.

## 1 Introduction

Classical theories of spin are widely used (see [1-4] and references therein) in analysis of spin dynamics in various circumstances and are known to agree with the calculations based on the Dirac theory. The spin variables of the Frenkel and BMT theories obey the first-order equations of motion. On this reason, construction of the corresponding action functional represents rather nontrivial problem. Various sets of auxiliary variables have been suggested and discussed in attempts to solve the problem [5-11]. The present model is based on the recently developed construction of spin surface [12]. This represents an essentially unique $SO(n)$-invariant surface of $2n$-dimensional vector space which can be parameterized by generators of $SO(n)$-group[1]. In [13] it has been demonstrated that $SO(3)$ spin surface leads to a reasonable model of non-relativistic spin. $SO(2,3)$ spin surface implies the model of Dirac electron [14], and represents

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[1] More exactly, $(2n-3)$-dimensional spin surface has natural structure of fiber bundle. Its base can be parameterized by $SO(n)$-generators.

an example of pseudoclassical mechanics [15]. Here we demonstrate that $SO(1,3)$ spin surface can be used to construct variational problem for unified description of both the Frenkel and BMT theories of relativistic spin.

In the Frenkel theory [5] we include the three-dimensional spin-vector $S^i$, $(S^i)^2=\frac{3\hbar^2}{4}$, into the antisymmetric tensor $J^{\mu\nu}=-J^{\nu\mu}$. This required to obey the constraint

$$J^{\mu\nu}u_\nu=0, \tag{1}$$

where $u_\nu$ represents four-velocity of the particle. In the rest-frame, $u_\nu=(u_0,0,0,0)$, this implies $J^{0i}=0$, so only three components of the Frenkel tensor survive, they are $J^{ij}=\epsilon^{ijk}S^k$. Besides, we can impose the covariant constraint

$$J^{\mu\nu}J_{\mu\nu}=\frac{3\hbar^2}{2}. \tag{2}$$

As in the rest frame $J^{\mu\nu}J_{\mu\nu}=2(S^i)^2$, this implies the right value of three-dimensional spin, as well as the right number of spin degrees of freedom.

Frenkel tensor is equivalent to the four-vector[2] $S^\mu\equiv\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu J_{\alpha\beta}$, the latter obeys

$$S^\mu u_\mu=0. \tag{3}$$

This has been taken by Bargmann, Michel and Telegdi as the basic quantity in their description of spin [16]. In terms of the BMT-vector, spin can be fixed fixed by the constraint[3]

$$(S^\mu)^2=-\frac{1}{2}u^2J^2+(Ju)^2=-\frac{3\hbar^2}{4}u^2. \tag{4}$$

Equations for the BMT-vector can be fixed [16] from the requirements of relativistic covariance, the right non-relativistic limit and from the compatibility with above mentioned constraints. Using the proper time as the evolution parameter, they read[4]

$$\dot S^\mu=-\frac{\mu e}{mc^2}\left[(FS)^\mu+(SFu)u^\mu\right]-(\dot uS)u^\mu, \tag{5}$$

[2]We use the Minkowski metric $\eta^{\mu\nu}=(-,+,+,+)$ and the Levi-Civita symbol with $\epsilon^{0123}=+1$.

[3]We point out that $S^\mu$, being the Casimir operator of the Poincare group, has fixed value for the Poincare IRREPs as well.

[4]Our $\mu=\frac{g}{2}$ of BMT, and the sign of our charge $e$ is the negative of theirs.

where $\mu$ stands for the anomalous magnetic moment, $m\dot{u}^\mu = f^\mu$, and $f^\mu$ is four-force.

We are interested in to formulate a variational problem for the Frenkel and BMT classical spin theories. As compare with the previous attempts [5-11], we look for the action functional which, besides of the transversality constraints (1), (3), implies also the value-of-spin constraints (2), (4). We point out that mainly due to the absence of variational problem, canonical quantization of the Frenkel and BMT theories is not developed to date. We hope the present work may be a step towards this direction.

We construct the Frenkel tensor starting from angular momentum

$$J^{\mu\nu} = 2(\omega^\mu \pi^\nu - \omega^\nu \pi^\mu), \tag{6}$$

of the spin "phase" space with the coordinates $\omega^\mu$ and the conjugate momenta $\pi^\mu$. To achieve this, we restrict dynamics of the basic variables on the spin surface determined by $SO(1,3)$-invariant equations

$$\pi^2 = a_2, \qquad \omega^2 = a_5, \qquad \omega\pi = 0. \tag{7}$$

As $J^{\mu\nu}J_{\mu\nu} = 8(\omega^2\pi^2 - (\omega\pi)^2) = 8a_2a_5$, an appropriate choice of the numbers $a_2$ and $a_5$ in Eq. (7) fixes the value of spin. Besides, we impose the constraints

$$p\omega = 0, \qquad p\pi = 0, \tag{8}$$

where $p^\mu$ stands for conjugate momentum to the world-line coordinate $x^\mu$. Eqs. (8) guarantee the transversality (1) of the Frenkel tensor. The set (7), (8) contains one first-class constraint (see below). Taking into account that each second-class constraint rules out one phase-space variable, whereas each first-class constraint rules out two variables, we have the right number of spin degrees of freedom, $8 - (4 + 2) = 2$.

Dynamics of the position variable $x^\mu(\tau)$ is restricted by the standard mass-shell condition

$$p^2 + m^2c^2 = 0. \tag{9}$$

Our next task is to formulate the variational problem which implies these constraints. Since they are written for the phase-space variables, it is natural to start from construction of an action functional in the Hamiltonian formalism. We introduce the canonical pairs $(g_i, \pi_{gi})$, $i = 1, 2, 3, 4, 5$, of auxiliary variables associated with the constraints. Then the Hamiltonian action can be taken in the form

$$S_H = \int d\tau \; p_\mu \dot{x}^\mu + \pi_\mu \dot{\omega}^\mu + \pi_{gi} \dot{g}_i - H, \tag{10}$$

$$H = \frac{1}{2} g_1 (p^2 + m^2 c^2) + \frac{1}{2} g_2 (\pi^2 + a_2) + g_3 (p\pi) + \\ g_4 (p\omega) + \frac{1}{2} g_5 (\omega^2 + a_5) + \lambda_{gi} \pi_{gi}. \tag{11}$$

We have denoted by $\lambda_{gi}$ the Lagrangian multipliers for the primary constraints $\pi_{gi} = 0$. Variation of the action with respect to $g_i$ implies[5] the desired constraints (7), (8) and (9).

## 2 Lagrangian of a theory with quadratic constraints

Lagrangian of a given Hamiltonian theory with constraints can be restored within the extended Lagrangian formalism [17]. Our constraints (7), (8) and (9) are either linear or quadratic with respect to momenta. For this case, the general formalism can be simplified as follows. Consider mechanics with the configuration-space variables $Q^a(\tau)$, $g_{ab}(\tau) = g_{ba}$, $h^a{}_b(\tau)$ and $k_{ab}(\tau) = k_{ba}$ and with the Lagrangian action

$$S = \int d\tau \frac{1}{2} g_{ab} DQ^a DQ^b - \frac{1}{2} k_{ab} q^a Q^b - \frac{1}{2} M(\tilde{g}, h, k). \tag{12}$$

We have denoted $DQ^a \equiv \dot{Q}^a - h^a{}_b Q^b$, and $\tilde{g}^{ab}$ is the inverse matrix of $g_{ab}$ This action can be used to produce any desired quadratic constraints of the variables $Q, P$. Indeed, denoting the conjugate momenta as $P_a$, $\pi_g$, $\pi_h$ and $\pi_k$, the equations for $P_a$ can be solved

$$P_a = \frac{\partial L}{\partial \dot{Q}^a} = g_{ab} DQ^b, \;\; \Rightarrow \;\; \dot{Q}^a = \tilde{g}^{ab} P_b + h^a{}_b Q^b, \tag{13}$$

while equations for the remaining momenta turn out to be the primary constraints $\pi_g = \pi_h = \pi_k = 0$. The Hamiltonian reads

$$H = \frac{1}{2} \tilde{g}^{ab} P_a P_b + P_a h^a{}_b Q^b + \frac{1}{2} k_{ab} Q^a Q^b + \frac{1}{2} M +$$

[5] The equation $\omega\pi = 0$ appears as the third-stage constraint, see Eq. (19) below.

$$\lambda_g \pi_g + \lambda_k \pi_k + \lambda_h \pi_h. \tag{14}$$

Then preservation in time of the primary constraints implies the quadratic constraints $P_a P_b + \frac{\partial M}{\partial \tilde{g}^{ab}} = 0$, and so on.

Comparing the Hamiltonian of our interest (11) with the expression (14), let us take $Q^a = (x^\mu, \omega^\nu)$, $P_a = (p^\mu, \pi^\nu)$,

$$\tilde{g}^{ab} = \begin{pmatrix} g_1 & g_3 \\ g_3 & g_2 \end{pmatrix}, \; h^a{}_b = \begin{pmatrix} 0 & g_4 \\ 0 & 0 \end{pmatrix}, \; k_{ab} = \begin{pmatrix} g_5 & 0 \\ 0 & 0 \end{pmatrix},$$

where $g_1 = g_1 \eta^{\mu\nu}$ and so on. Besides, we take the "mass" term in the form $M = g_1 m^2 c^2 + g_2 a_2 + g_5 a_5$. With this choice, the equation (14) turns into the desired Hamiltonian (11). So the corresponding Lagrangian action reads from (12) as follows

$$S = \int d\tau \frac{1}{2A}\left[g_2 (Dx)^2 - 2g_3 (Dx\dot{\omega}) + g_1 \dot{\omega}^2\right] - \frac{1}{2} g_1 m^2 c^2 - \frac{1}{2} g_2 a_2 - \frac{1}{2} g_5 (\omega^2 + a_5). \tag{15}$$

We have denoted $A = \det \tilde{g} = g_1 g_2 - g_3^2$, $Dx^\mu = \dot{x}^\mu - g_4 \omega^\mu$.

## 3 Free theory

Equations for the canonical momenta $p^\mu$ and $\pi^\mu$ of the theory (15)

$$p^\mu = \frac{g_2}{A} Dx^\mu - \frac{g_1}{A} \dot{\omega}^\mu, \quad \pi^\mu = -\frac{g_3}{A} Dx^\mu + \frac{g_1}{A} \dot{\omega}^\mu, \tag{16}$$

can be resolved as follows

$$\dot{x}^\mu = g_1 p^\mu + g_3 \pi^\mu + g_4 \omega^\mu, \quad \dot{\omega}^\mu = g_3 p^\mu + g_2 \pi^\mu, \tag{17}$$

while equations for the remaining momenta imply the primary constraints, $\pi_{gi} = 0$. Using these equations in the expression $p\dot{x} + \pi\dot{\omega} - L$, we immediately obtain the Hamiltonian (11). Preservation in time of the primary constraints implies the following chains of higher-stage constraints:

$$\pi_{g1} = 0 \;\Rightarrow\; p^2 + m^2 c^2 = 0. \tag{18}$$

$$\left.\begin{array}{l} \pi_{g2} = 0, \;\; \Rightarrow \;\; \pi^2 + a_2 = 0 \\ \pi_{g5} = 0, \;\; \Rightarrow \;\; \omega^2 + a_5 = 0 \end{array}\right\} \Rightarrow \pi\omega = 0, \Rightarrow$$

$$g_5 = \frac{a_2}{a_5} g_2, \quad \Rightarrow \quad \lambda_{g5} = \frac{a_2}{a_5} \lambda_{g2}. \tag{19}$$

$$\begin{array}{l} \pi_{g3} = 0 \;\Rightarrow\; p\pi = 0, \;\Rightarrow\; g_4 = 0, \;\Rightarrow\; \lambda_{g4} = 0. \\ \pi_{g4} = 0 \;\Rightarrow\; p\omega = 0, \;\Rightarrow\; g_3 = 0, \;\Rightarrow\; \lambda_{g3} = 0. \end{array} \tag{20}$$

The constraints $p^2 + m^2c^2 = 0$ and $\pi^2 + a_2 + \frac{a_2}{a_5}(\omega^2 + a_5) = 0$ form the first-class subset. This indicates that the action (10) is invariant under the two-parametric group of local transformations. It is composed by the standard reparametrizations as well as by the following transformations with the parameter $\gamma(\tau)$:

$$\begin{array}{c} \delta\omega^\mu = \gamma g_2 \pi^\mu, \qquad \delta\pi^\mu = -\gamma g_5 \omega^\mu, \\ \delta g_2 = (\gamma g_2)^{\cdot}, \quad \delta g_5 = (\gamma g_5)^{\cdot}, \quad \delta g_3 = -\gamma g_4 g_2, \\ \delta g_4 = \gamma g_3 g_5, \quad \delta\lambda_{gi} = (\delta g_i)^{\cdot}. \end{array} \tag{21}$$

Note that $x^\mu$, $J^{\mu\nu}$ and $S^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} p_\nu J_{\alpha\beta}$ are $\gamma$-invariant quantities.

Besides the constraints, the action implies the Hamiltonian equations

$$\begin{array}{ll} \dot{x}^\mu = g_1 p^\mu, & \dot{p}^\mu = 0, \\ \dot{\omega}^\mu = g_2 \pi^\mu, & \dot{\pi}^\mu = -g_2 \frac{a_2}{a_5} \omega^\mu. \end{array} \tag{22}$$

Obtaining these equations, we have used the constraints (19) and (20). The functions $g_1(\tau)$ and $g_2(\tau)$ can not be determined neither with the constraints nor with the dynamical equations. It implies the functional ambiguity in solutions to the equations of motion (22): besides the integration constants, solution depends on these arbitrary functions. The ambiguity of $x^\mu$ due to $g_1$ reflects the reparametrization invariance, while the ambiguity of $\omega^\mu$ and $\pi^\mu$ due to $g_2$ is related with the $\gamma$-symmetry. According to the general theory of singular systems [18, 19, 20], the variables with ambiguous dynamics do not represent the observable quantities. So, our next task is to find candidates for observables, which are variables with unambiguous dynamics. Equivalently, we can look for the gauge-invariant variables. As the physical variables of the spin-sector, we can take either the Frenkel tensor or the BMT-vector, both turn out to be $\gamma$-invariant quantities. The ambiguity related with reparametrizations can be removed in the standard way: we assume that the functions $x^\mu(\tau)$ represent the physical variables $x^i(t)$ in the parametric form.

As it should be, dynamics of the physical variables is unambiguous

$$\frac{dx^i}{dt} = c\frac{p^i}{p_0}, \quad \frac{dp^i}{dt} = 0, \quad \frac{dJ^{\mu\nu}}{dt} = \frac{dS^\mu}{dt} = 0. \tag{23}$$

According to the equations (18)-(20), the variables obey also the constraints $p^2 + m^2c^2 = 0$, $J^{\mu\nu}p_\mu = 0$, $J^2 = 8a_2a_5$, $S^\mu p_\mu = 0$, $S^2 = 4m^2c^2a_2a_5$.

## 4 Interaction with uniform electromagnetic field

Let us consider the spinning particle with electric charge $e$ and the anomalous magnetic moment $\mu$. We take the Hamiltonian of interacting theory in the form

$$H = \frac{1}{2}g_1(\mathcal{P}^2 + \frac{e\mu}{2c}F_{\mu\nu}J^{\mu\nu} + m^2c^2) + \frac{1}{2}g_2(\pi^2 + a_2) +$$
$$g_3(\mathcal{P}\pi) + g_4(\mathcal{P}\omega) + \frac{1}{2}g_5(\omega^2 + a_5) + \lambda_{gi}\pi_{gi}. \tag{24}$$

We have denoted $\mathcal{P}^\mu \equiv p^\mu + \frac{e}{c}A^\mu$. In contrast to $p^\mu$, the $U(1)$-invariant quantities $\mathcal{P}^\mu$ have non-vanishing Poisson brackets. We restrict ourselves to the case of uniform electromagnetic field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \text{const}$, then $\{\mathcal{P}^\mu, \mathcal{P}^\nu\} = -\frac{e}{c}F^{\mu\nu}$.

We point out that the $\gamma$-symmetry survives in the interacting theory even for nonuniform field.

The Hamiltonian (24) implies the constraints (19), the mass-shell condition

$$\mathcal{P}^2 + \frac{e\mu}{2c}F_{\mu\nu}J^{\mu\nu} + m^2c^2 = 0, \tag{25}$$

as well as the chains

$$\mathcal{P}\pi = 0, \;\Rightarrow\; g_4 = -g_1\frac{e(\mu-1)}{c^3M^2}(\pi F\mathcal{P}), \;\Rightarrow\; \lambda_{g4} \sim \lambda g_1.$$
$$\mathcal{P}\omega = 0, \;\Rightarrow\; g_3 = g_1\frac{e(\mu-1)}{c^3M^2}(\omega F\mathcal{P}), \;\Rightarrow\; \lambda_{g3} \sim \lambda g_1. \tag{26}$$

We have denoted $M^2 = m^2 + \frac{e(2\mu+1)}{4c^3}F_{\mu\nu}J^{\mu\nu}$. The constraints imply the useful consequence

$$g_3(\pi F\mathcal{P}) + g_4(\omega F\mathcal{P}) = 0. \tag{27}$$

This equation can be used to verify that the quantities $F_{\mu\nu}J^{\mu\nu}$, $M^2$ and $\mathcal{P}^2$ represent the integrals of motion.

Hamiltonian equations for the basic variables read

$$\dot{x}^\mu = g_1 u^\mu, \quad \dot{\mathcal{P}}^\mu = -g_1\frac{e}{c}(Fu)^\mu, \tag{28}$$

$$\begin{aligned} \dot{\omega}^\mu &= -g_1\frac{e\mu}{c}(F\omega)^\mu + g_2\pi^\mu + g_3\mathcal{P}^\mu, \\ \dot{\pi}^\mu &= -g_1\frac{e\mu}{c}(F\pi)^\mu - \frac{a_2}{a_5}g_2\omega^\mu - g_4\mathcal{P}^\mu, \end{aligned} \tag{29}$$

where the four-velocity $u^\mu$ is (see Eq. (26))

$$u^\mu = \mathcal{P}^\mu + \frac{g_3}{g_1}\pi^\mu + \frac{g_4}{g_1}\omega^\mu = \mathcal{P}^\mu - \frac{e(\mu-1)}{2c^3M^2}(JF\mathcal{P})^\mu. \tag{30}$$

Hence the interaction leads to modification of the Lorentz-force equation. Only for the "classical" value of anomalous momentum, $\mu = 1$, the constraints (26) would be the same as in the free theory, $g_3 = g_4 = 0$. Then the four-velocity coincides with $\mathcal{P}$. When $\mu \neq 0$, the difference between $u$ and $\mathcal{P}$ is proportional to $\frac{J}{c^3} \sim \frac{\hbar}{c^3}$. All the basic variables have ambiguous evolution. $x^\mu$ and $\mathcal{P}^\mu$ have one-parametric ambiguity due to $g_1$ while $\omega$ and $\pi$ have two-parametric ambiguity due to $g_1$ and $g_2$.

The quantities $x^\mu$, $\mathcal{P}^\mu$ and the Frenkel tensor $J^{\mu\nu}$ are $\gamma$-invariants. Their equations of motion form a closed system

$$\dot{x}^\mu = g_1\left[\mathcal{P}^\mu - \frac{e(\mu-1)}{2c^3M^2}(JF\mathcal{P})^\mu\right], \quad \dot{\mathcal{P}}^\mu = -\frac{e}{c}(F\dot{x})^\mu, \tag{31}$$

$$\dot{J}^{\mu\nu} = -g_1\frac{e}{c}\left[\mu F^{[\mu}_{\alpha}J^{\alpha\nu]} - \frac{\mu-1}{c^2M^2}\mathcal{P}^{[\mu}J^{\nu]\alpha}(F\mathcal{P})_\alpha\right]. \tag{32}$$

The remaining ambiguity due to $g_1$ presented in these equations reflects the reparametrization symmetry of the theory. Assuming that the functions $x^\mu(\tau)$, $p^\mu(\tau)$ and $J^{\mu\nu}(\tau)$ represent the physical variables $x^i(t)$, $p^\mu(t)$ and $J^{\mu\nu}(t)$ in the parametric form, their equations read $\frac{dx^i}{dt} = c\frac{u^i}{u^0}$, $\frac{dp^\mu}{dt} = -e\frac{(Fu)^\mu}{u^0}$, $\frac{dJ^{\mu\nu}}{dt} = c\frac{\dot{J}^{\mu\nu}}{g_1u^0}$. As it should be, they have unambiguous dynamics.

Since $J^{\mu\nu}\mathcal{P}_\nu = 0$, the Frenkel tensor is equivalent to the BMT-vector constructed as follows:

$$S^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\mathcal{P}_\nu J_{\alpha\beta} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu J_{\alpha\beta}. \tag{33}$$

So the physical dynamics can be described using $S^\mu$ instead of $J^{\mu\nu}$. Using the identities

$$J^{\mu\nu} = \frac{1}{\mathcal{P}^2}\epsilon^{\mu\nu\alpha\beta}\mathcal{P}_\alpha S_\beta, \quad \epsilon^{\mu\nu\alpha\beta}J_{\alpha\beta} = \frac{2}{\mathcal{P}^2}S^{[\mu}\mathcal{P}^{\nu]}, \tag{34}$$

to represent $J^{\mu\nu}$ through $S^\mu$ in Eq. (31), we obtain the closed system of equations for $\gamma$-invariant quantities

$$\dot{x}^\mu = g_1\left[\mathcal{P}^\mu - \frac{e(\mu-1)}{2c^3M^2}\epsilon^{\mu\nu\alpha\beta}(F\mathcal{P})_\nu\mathcal{P}_\alpha S_\beta\right],$$

$$\dot{\mathcal{P}}^\mu = -\frac{e}{c}(F\dot{x})^\mu, \tag{35}$$

$$\dot{S}^\mu = -g_1\frac{e\mu}{c}\left[(FS)^\mu + \frac{1}{\mathcal{P}^2}(SF\mathcal{P})\mathcal{P}^\mu\right] - \frac{1}{\mathcal{P}^2}(\dot{\mathcal{P}}S)\mathcal{P}^\mu. \tag{36}$$

These equations are written in an arbitrary parametrization of the world-line. The choice of proper time as the evolution parameter corresponds to $g_1 = mc$.

## 5 Conclusion

In this work we have specified the construction of spin surface [12] for the case of $SO(1,3)$-group. On this base, we have constructed the Lagrangian action (15) which describe the particle with fixed value of spin interacting with uniform electromagnetic field. Due to the constraints (7), (8), the number of physical degrees of freedom in the spin-sector is equal to 2, as it should be. The basic spin-space coordinates $\omega^\mu$, $\pi^\nu$ are gauge non-invariant variables, hence they do not correspond to the observable quantities. We can take the anti-symmetric tensor (6) as an observable quantity. For an appropriate choice of the parameters $a_2$, $a_3$, this obeys both the transversality constraint (1) and the value-of-spin constraint (2). Its dynamics is governed by the Frenkel-type equation (32). Equivalently, we can take the vector (33) as an observable quantity. This is subject to the constraints (3), (4) and obeys the Bargmann-Michel-Telegdi equations of motion (36).

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